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Superlinear, Noncoercive Asymmetric Robin Problems with Indefinite, Unbounded Potential

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Abstract. We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential. The reaction term exhibits an asymmetric behavior, namely it is superlinear in the positive direction but without satisfying the Ambrosetti–Rabinowitz condition and it is sublinear but noncoercive in the negative direction. Using variational methods together with suitable truncation and perturbation techniques and Morse theory (critical groups), we prove a multiplicity theorem producing three nontrivial smooth solutions two of which have constant sign (one positive and the other negative).

Keywords. Superlinear reaction term, asymmetric nonlinearity, constant sign solutions, critical groups, C-condition, mountain pass theorem, indefinite potential, Robin boundary condition

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

In this problem, $\xi \in L^s(\Omega)$ (for $s > N$) is a potential function which may change sign (indefinite potential) and $f(z, x)$ is a Carathéodory reaction term

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(that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous), which exhibits asymmetric growth as $x \to \pm\infty$. More precisely, we assume that $f(z,\cdot)$ is superlinear near $+\infty$ but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (the AR-condition for short), while $f(z,\cdot)$ is sublinear near $-\infty$, but the energy (Euler) functional of the problem is not coercive in that direction. This means that the quotient $\frac{f(z,x)}{x}$ asymptotically as $x \to -\infty$ stays above the principle eigenvalue of $u \mapsto -\Delta u + \xi(z)u$, $u \in H^1(\Omega)$, with the Robin boundary condition. So, our work here complements the recent one by the authors, see Papageorgiou & Rădulescu [15], where the energy functional is coercive in the negative direction. The geometries of the two problems differ drastically and as expected, the present noncoercive setting is more complicated and requires more delicate arguments.

We mention that asymmetric boundary value problems were also investigated by Arcoya & Villegas [3], de Figueiredo & Ruf [6], Motreanu, Motreanu & Papageorgiou [10], Perera [18] and Recova & Rumbos [19]. Arcoya & Villegas [3] and de Figueiredo & Ruf [6], deal with Neumann problems (that is, $\beta \equiv 0$) and in addition, de Figueiredo & Ruf [6] assume that $N = 1$ (that is, they consider ordinary differential equations). Both works prove existence theorems. Multiplicity results can be found in the papers of Motreanu, Motreanu & Papageorgiou [10] (Dirichlet problems), Perera [18] (Neumann problems with $N = 1$) and Recova & Rumbos [19] (Dirichlet problems). In all the aforementioned works the potential function is zero (that is, $\xi \equiv 0$) and the superlinearity in the positive direction is expressed via the standard AR-condition.

Using variational methods based on the critical point theory, together with suitable truncation and perturbation techniques and Morse theory (critical groups) we establish the existence of up to three nontrivial smooth solutions.

2. Mathematical background

In this section, we review the main mathematical tools which will use in this work.

So, let X be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X).

Definition 2.1. Let $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:
"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ, which is needed since the ambient space need not be locally compact (X is usually infinite dimensional). This condition is the main tool in proving a deformation theorem, from which one can derive the minimax theory of the critical values of φ. One of the main results in this theory, is the so-called "mountain pass theorem" due to Ambrosetti & Rabinowitz [2]. Here the theorem is formulated in a slightly more general form (see Gasinski & Papageorgiou [7, p. 648]).

Theorem 2.2. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the* C*-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = m_\rho$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geq m_\rho$ and c is a critical value of φ.

In the study of problem (1), we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" spaces $L^p(\partial\Omega)$ $(1 \leq p \leq \infty)$. In what follows, $|| \cdot ||$ denotes the norm of $H^1(\Omega)$ defined by

$$||u|| = \left[||u||_2^2 + ||Du||_2^2\right]^{\frac{1}{2}} \quad \text{for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will be using the following open subset of C_+,

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N–1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leq p \leq \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map

$$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega),$$

known as the "trace map", such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. So, we understand the trace map $\gamma_0(u)$ as representing the "boundary values" of a Sobolev function u. We know that the linear map γ_0 is compact into $L^q(\Omega)$ with $q \in \left[1, \frac{2(N-1)}{N-2}\right)$ if $N \geq 3$ and into $L^q(\Omega)$ with $q \in \left[1, +\infty\right)$ if $N = 1, 2$.

In the sequel, for the sake of notational simplicity, we will drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We will also make use of the spectrum of the operator $H^1(\Omega) \ni u \mapsto -\Delta u + \xi(z)u$, with Robin boundary condition. So, we consider the following linear eigenvalue problem:

$$-\Delta u(z) + \xi(z)u(z) = \hat{\lambda}u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \tag{2}$$

In this boundary value problem, we assume that

- $\xi \in L^{\frac{N}{2}}(\Omega)$ if $N \geq 3$, $\xi \in L^r(\Omega)$ with $r \in (1, \infty)$ if $N = 2$ and $\xi \in L^1(\Omega)$ if $N = 1$;
- $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

By $\frac{\partial u}{\partial n}$ we denote the usual normal derivative of $u \in H^1(\Omega)$ defined by

$$\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N} \quad \text{on } \partial\Omega,$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = ||Du||_2^2 + \int_\Omega \xi(z)|u|^2 dz + \int_{\partial\Omega} \beta(z)|u|^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

From Papageorgiou & Rădulescu [15, 16], we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu||u||_2^2 \geq c_0||u||^2 \quad \text{for all } u \in H^1(\Omega) \text{ with } c_0 > 0. \tag{3}$$

Using (3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we can have a complete description of the spectrum of (2). So, we have a strictly increasing sequence $\{\hat{\lambda}_k\}_{k\geq 1}$ of eigenvalues such that $\hat{\lambda}_k \to +\infty$. By $E(\hat{\lambda}_k)$, $k \in \mathbb{N}$, we denote the corresponding eigenspace. We have the following orthogonal direct sum decomposition:

$$H^1(\Omega) = \overline{\underset{k\geq 1}{\oplus} E(\hat{\lambda}_k)}.$$

These eigenvalues have the following properties (see Papageorgiou & Rădulescu [14], where $\xi = 0$ and [15, 16])

- It holds
$$\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{||u||_2^2} : u \in H^1(\Omega), u \neq 0\right]$$
and the infimum is realized on $E(\hat{\lambda}_1)$; hence the elements of $E(\hat{\lambda}_1)$ have fixed sign;

- $\hat{\lambda}_1$ is simple (that is, $\dim E(\hat{\lambda}_1) = 1$);
- for $n \geq 2$, we have

$$\hat{\lambda}_n = \inf\left[\frac{\gamma(u)}{||u||_2^2} : u \in \overline{\underset{k\geq n}{\oplus} E(\hat{\lambda}_k)}, u \neq 0\right] = \sup\left[\frac{\gamma(u)}{||u||_2^2} : u \in \overset{n}{\underset{k=1}{\oplus}} E(\hat{\lambda}_k), u \neq 0\right]; \quad (4)$$

in (4) both the infimum and the supremum are realized on $E(\hat{\lambda}_n)$ and the eigenfunctions corresponding to $\hat{\lambda}_n$ are all nodal (that is, sign changing).

Let $\hat{u}_1$ denote the L^2-normalized (that is, $||\hat{u}_1||_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1$. If $\xi \in L^s(\Omega)$ with $s > N$, then using the regularity result of Wang [20, Lemmata 5.1 and 5.2], we have that $\hat{u}_1 \in C_+\backslash\{0\}$. Moreover, Harnack's inequality (see, for example, Motreanu, Motreanu & Papageorgiou [11, Corollary 8.17, p. 212]), we have $\hat{u}_1(z) > 0$ for all $z \in \Omega$. Suppose that $\xi^+ \in L^\infty(\Omega)$. Then the boundary point theorem (see, for example, Gasinski & Papageorgiou [7, Theorem 6.2.8, p. 738]), implies that $\hat{u}_1 \in D_+$. Moreover, when $\xi \in L^s(\Omega)$ with $s > \frac{N}{2}$, then the eigenspaces $E(\hat{\lambda}_k)$, $k \in \mathbb{N}$, have the so-called "Unique Continuation Property" (the "UCP" for short). This means that

"if $u \in E(\hat{\lambda}_k)$ and u vanishes on a set of positive Lebesgue measure, then $u \equiv 0$ (see de Figueiredo & Gossez [5])".

Using these properties, we can have the following useful inequalities.

Lemma 2.3. (a) *If $k \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\lambda}_k$ for almost all $z \in \Omega$, $\eta \not\equiv \hat{\lambda}_k$ then there exists $\hat{c} > 0$ such that*

$$\gamma(u) - \int_\Omega \eta(z)|u|^2 dz \geq \hat{c}||u||^2 \quad \text{for all } u \in \overline{\underset{i\geq k}{\oplus} E(\hat{\lambda}_i)}.$$

(b) *If $k \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \hat{\lambda}_k$ for almost all $z \in \Omega$, $\eta \not\equiv \hat{\lambda}_k$, then there exists $\bar{c} > 0$ such that*

$$\gamma(u) - \int_\Omega \eta(z)|u|^2 dz \leq -\bar{c}||u||^2 \quad \text{for all } u \in \overset{k}{\underset{i=1}{\oplus}} E(\hat{\lambda}_i).$$

There is also a weighted version of problem (2) with a weight $m \in L^\infty(\Omega)$, $m \geq 0$, $m \not\equiv 0$. So, we consider the following linear eigenvalue problem

$$-\Delta u(z) + \xi(z)u(z) = \tilde{\lambda} m(z)u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega.$$

As was the case with (2), this eigenvalue problem has an increasing sequence $\{\tilde{\lambda}_k(m)\}_{k\geq 1}$ of distinct eigenvalues such that $\tilde{\lambda}_k(m) \to +\infty$ as $k \to \infty$. As a consequence of the UCP, we have the following strict monotonicity property for the function $m \mapsto \tilde{\lambda}_k(m)$, $k \in \mathbb{N}$.

Lemma 2.4. *If* $m_1, m_2 \in L^\infty(\Omega)$, $0 \le m_1 \le m_2$, $m_1 \not\equiv 0$, $m_2 \not\equiv m_1$, *then for all* $k \in \mathbb{N}$ *we have* $\tilde{\lambda}_k(m_2) < \tilde{\lambda}_k(m_1)$.

Next let us recall some basic definitions and facts from Morse theory (critical groups) which we will need in the sequel.

So, let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \le c\}, & \dot{\varphi}^c &= \{u \in X : \varphi(u) < c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, & K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $k \in -\mathbb{N}$, then $H_k(Y_1, Y_2) = 0$. Suppose that $u_0 \in K_\varphi^c$ is isolated. Then the critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology theory, implies that the above definition is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, Gasinski & Papageorgiou [7, p. 628]), implies that the above definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that the critical set K_φ is finite. We introduce the following polynomials in $t \in \mathbb{R}$:

$$\begin{aligned} M(t, u) &= \sum_{k \ge 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \ge 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

The "Morse relation" says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{5}$$

with $Q(t) = \sum_{k \ge 0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Suppose that H is a Hilbert space, $\varphi \in C^2(H,\mathbb{R})$ and $u_0 \in K_\varphi$. The "Morse index" of u_0 is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u_0) \in \mathcal{L}(H)$ is negative definite.

Finally let us fix our notation and terminology. So, by $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}_N} dz \quad \text{for all } u, h \in H^1(\Omega).$$

We say that a Banach space X has the "Kadec–Klee property", if the following is true:

$$\text{“} u_n \xrightarrow{w} u \text{ in } X \quad \text{and} \quad ||u_n|| \to ||u|| \Rightarrow u_n \to u \text{ in } X \text{”}.$$

Locally uniformly convex Banach spaces, in particular Hilbert spaces, have the Kadec–Klee property.

Let $x \in \mathbb{R}$. We set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in H^1(\Omega)$, we define

$$u^\pm(\cdot) = u(\cdot)^\pm.$$

We know that $u^\pm \in H^1(\Omega),\ |u| = u^+ + u^-,\ u = u^+ - u^-$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, given a measurable function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function $f(z,x)$), we set

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega)$$

(the Nemytskii map corresponding to the function f). Evidently $z \mapsto N_f(u)(z) = f(z, u(z))$ is measurable. Finally, we define

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1, 2 \end{cases} \quad \text{(the critical Sobolev exponent)}$$

and $k_0 = \inf[k \in \mathbb{N} : \hat{\lambda}_k \geq 0]$.

3. Multiplicity theorem

In this section we prove a multiplicity theorem for problem (1), producing three nontrivial smooth solutions, two of which have constant sign. Throughout this section the hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$, are the following:

- $\mathrm{H}(\xi) : \xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.
- $\mathrm{H}(\beta) : \beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 3.1. When $\beta \equiv 0$, we recover the Neumann problem.

To produce the two constant sign solutions, we will use the following conditions on the reaction term $f(z,x)$:

$\mathrm{H}_1:$ $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for almost all $z\in\Omega$ and

(i) $|f(z,x)|\le a(z)(1+|x|^{r-1})$ for almost all $z\in\Omega$, all $x\in\mathbb{R}$, with $a\in L^\infty(\Omega)_+$, $2<r<2^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$ and $e(z,x)=f(z,x)x-2F(z,x)$, then

  (a) $\lim_{x\to+\infty}\frac{F(z,x)}{x^2}=+\infty$ uniformly for almost all $z\in\Omega$

  (b) $e(z,x)\le e(z,y)+\tau(z)$ for almost all $z\in\Omega$, all $0\le x\le y$ with $\tau\in L^1(\Omega)_+$.

(iii) there exist $m\ge\max\{k_0,2\}$, functions $\eta,\hat{\eta}\in L^\infty(\Omega)$ and constant $\hat{c}>0$ such that

  (a) $\hat{\lambda}_m\le\eta(z)\le\hat{\eta}(z)\le\hat{\lambda}_{m+1}$ for almost all $z\in\Omega$, $\hat{\lambda}_m\not\equiv\eta, \hat{\lambda}_{m+1}\not\equiv\hat{\eta}$,

  (b) $\eta(z)\le\liminf_{x\to-\infty}\frac{f(z,x)}{x}\le\limsup_{x\to-\infty}\frac{f(z,x)}{x}\le\hat{\eta}(z)$ uniformly for almost all $z\in\Omega$,

  (c) $-\hat{c}\le e(z,x)$ for almost all $z\in\Omega$, all $x\le 0$.

(iv) there exist functions $\vartheta_0,\vartheta\in L^\infty(\Omega)$ such that

  (a) $\vartheta_0(z)\le\vartheta(z)\le\hat{\lambda}_1$ for almost all $z\in\Omega, \vartheta\not\equiv\hat{\lambda}_1$,

  (b) $\vartheta_0(z)\le\liminf_{x\to 0}\frac{f(z,x)}{x}\le\limsup_{x\to 0}\frac{f(z,x)}{x}\le\vartheta(z)$ uniformly for almost all $z\in\Omega$.

Remark 3.2. Hypothesis H_1(ii) says that the primitive $F(z,\cdot)$ is superquadratic near $+\infty$. In fact the two conditions in hypothesis H_1(ii) imply that the reaction term $f(z,\cdot)$ is superlinear near $+\infty$. However, we stress that the superlinearity of $f(z,\cdot)$ is not expressed using the usual for such problems AR-condition. We recall that the AR-condition says that there exist $p>2$ and $M>0$ such that

$$0<pF(z,x)\le f(z,x)x \quad \text{for almost all } z\in\Omega,\ \text{all } x\ge M, \tag{6a}$$

$$0<\operatorname*{ess\,inf}_{\Omega}F(\cdot,M) \tag{6b}$$

(see Ambrosetti & Rabinowitz [2] and Mugnai [12]). Integrating (6a) and using (6b), we obtain

$$c_1x^p\le F(z,x) \quad \text{for almost all } z\in\Omega,\ \text{all } x\ge M,\ \text{with } c_1>0. \tag{7}$$

From (7) and the second condition in hypothesis H_1(ii) we have that $f(z,\cdot)$ has as least $(p-1)$-polynomial growth in the positive semiaxis (in particular then we have that $f(z,\cdot)$ is superlinear). Here instead of the AR-condition (see (6a), (6b)), we use a quasimonotonicity condition on the function $e(z,\cdot)$ (see hypothesis H_1(ii)). This quasimonotonicity condition is satisfied, if there exists $M > 0$ such that for almost all $z \in \Omega$ the function $x \mapsto \frac{f(z,x)}{x}$ is nondecreasing on $[M, +\infty)$. Slightly more restrictive version of this condition was used by Li & Yang [9] who conduct a detailed study of this condition and of its implications. Hypothesis H_1(iii) implies that $f(z, \cdot)$ is sublinear near $-\infty$ and we have nonuniform nonresonance with respect to any nonprincipal, nonnegative spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$. So, the problem is indefinite (noncoercive) in both directions and this distinguishes the present situation from that of Papageorgiou & Rădulescu [15]. The geometry of the two problems is different.

Recall that $\gamma : H^1(\Omega) \to \mathbb{R}$ is defined by

$$\gamma(u) = ||Du||_2^2 + \int_\Omega \xi(z)|u|^2 dz + \int_{\partial\Omega} \beta(z)|u|^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z, u(z))dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently $\varphi \in C^1(H^1(\Omega))$. Next we show that φ satisfies the C-condition and this permits the consideration of the critical groups of φ at infinity.

Proposition 3.3. *If hypotheses* $\mathrm{H}(\xi)$,$\mathrm{H}(\beta)$ *and* H_1 *hold, then the functional* φ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{8}$$

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{9}$$

We have

$$\begin{aligned}
\frac{1}{2}\gamma(u_n^+) &= \frac{1}{2}\gamma(u_n) - \frac{1}{2}\gamma(u_n^-) \\
&= \frac{1}{2}\gamma(u_n) - \frac{1}{2}\gamma(u_n^-) + \int_\Omega F(z, u_n)dz - \int_\Omega F(z, u_n)dz \\
&= \varphi(u_n) - \frac{1}{2}\gamma(u_n^-) + \int_\Omega F(z, u_n)dz \\
&\leq M_1 + \frac{1}{2}\left[\int_\Omega 2F(z, u_n)dz - \gamma(u_n^-)\right] \quad \text{for all } n \in \mathbb{N} \text{ (see (8))}.
\end{aligned} \tag{10}$$

From (9) we have

$$\begin{aligned}
&\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma - \int_\Omega f(z, u_n)h dz\right| \\
&\leq \frac{\epsilon_n ||h||}{1 + ||u_n||} \quad \text{for all } h \in H^1(\Omega), \text{ with } \epsilon_n \to 0^+.
\end{aligned} \tag{11}$$

In (11) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\begin{aligned} &|\gamma(u_n^-) - \int_\Omega f(z, -u_n^-)(-u_n^-)dz| \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow & -\gamma(u_n^-) \leq \epsilon_n - \int_\Omega f(z, -u_n^-)(-u_n^-) \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{12}$$

We return to (10) and use (12). We obtain

$$\frac{1}{2}\gamma(u_n^+) \leq M_2 + \frac{1}{2}\left[\int_\Omega (2F(z, u_n) - f(z, -u_n^-)(-u_n^-))dz\right] \tag{13}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$. We have

$$2F(z, u_n) = 2F(z, u_n^+) + 2F(z, -u_n^-) \leq 2F(z, u_n^+) + f(z, -u_n^-)(-u_n^-) + \hat{c} \tag{14}$$

for almost all $z \in \Omega$, all $n \in \mathbb{N}$ (see hypothesis H_1(iii)). We use (14) in (13) and obtain

$$\begin{aligned} &\gamma(u_n^+) \leq M_3 + \int_\Omega 2F(z, u_n^+)dz \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow\ & \varphi(u_n^+) \leq M_3 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{15}$$

In (11) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$-\gamma(u_n^+) + \int_\Omega f(z, u_n^+)u_n^+ dz \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

We add (15) and (16) and have

$$\begin{aligned} &\int_\Omega [f(z, u_n^+)u_n^+ - 2F(z, u_n^+)]dz \leq M_4 \quad \text{for some } M_4 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow & \int_\Omega e(z, u_n^+)dz \leq M_4 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{17}$$

We will use (17) to prove the following Claim.

Claim 1. $\{u_n^+\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded.

We argue indirectly. So, suppose that Claim 1 is not true. By passing to a subsequence if necessary, we may assume that $||u_n^+|| \to +\infty$. Let $y_n = \frac{u_n^+}{||u_n^+||}$ for all $n \in \mathbb{N}$. Then $||y_n|| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega) \tag{18}$$

(recall that $s > N$ (see hypothesis $\mathrm{H}(\xi)$) and note that $\frac{2s}{s-1} < 2^*$).

Suppose that $y \neq 0$ and let $\Omega_0 = \{z \in \Omega : y(z) = 0\}$. Then $|\Omega \backslash \Omega_0|_N > 0$ and we have

$$u_n^+(z) \to +\infty \quad \text{for almost all } z \in \Omega \backslash \Omega_0.$$

Hypothesis H_1(ii) implies that

$$\frac{F(z, u_n^+(z))}{||u_n^+||^2} = \frac{F(z, u_n^+(z))}{u_n^+(z)^2} y_n(z)^2 \to +\infty \quad \text{for almost all } z \in \Omega \backslash \Omega_0.$$

Using Fatou's lemma (hypotheses H_1(i), (ii) permit its use), we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^2} dz = +\infty. \tag{19}$$

Since $e(z, 0) = 0$ for almost all $z \in \Omega$, from hypothesis H_1(ii) we have

$$\begin{aligned} 2F(z, u_n^+) &\leq f(z, u_n^+)u_n^+ + \tau(z) \quad \text{for almost all } z \in \Omega, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \int_\Omega 2F(z, u_n^+)dz &\leq \int_\Omega f(z, u_n^+)u_n^+ dz + ||\tau||_1 \\ &\leq M_5 + \gamma(u_n^+) \quad \text{for some } M_5 > 0, \text{ all } n \in \mathbb{N} \text{ (see (16))}, \qquad (20) \\ \Rightarrow \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^2} dz &\leq \frac{M_5}{2||u_n^+||^2} + \frac{1}{2}\gamma(y_n) \\ &\leq M_6 \quad \text{for some } M_6 > 0, \text{ all } n \in \mathbb{N}. \end{aligned}$$

Comparing (19) and (20) we reach a contradiction. Therefore we can not have that $y \neq 0$. So, we must have $y \equiv 0$. Let $q > 0$ and consider the functions

$$v_n = (2q)^{\frac{1}{2}} y_n \in H^1(\Omega) \quad \text{for all } n \in \mathbb{N}.$$

Since $y = 0$, we have

$$v_n \to 0 \text{ in } L^{\frac{2s}{s-1}}(\Omega) \quad \text{(see (18))} \Rightarrow \int_\Omega F(z, v_n)dz \to 0. \tag{21}$$

Recall that $||u_n^+|| \to \infty$. So, we can find $n_0 \in \mathbb{N}$ such that

$$0 < (2q)^{\frac{1}{2}} \frac{1}{||u_n^+||} \leq 1 \quad \text{for all } n \geq n_0. \tag{22}$$

Let $t_n \in [0, 1]$ be such that

$$\varphi(t_n u_n^+) = \max\{\varphi(tu_n^+) : 0 \leq t \leq 1\}. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned} \varphi(t_n u_n^+) &\geq \varphi(v_n) \\ &= q\gamma(y_n) - \int_\Omega F(z, v_n)dz \\ &\geq qc_0 - q\mu||y_n||_2^2 - \int_\Omega F(z, v_n)dz \quad \text{for all } n \geq n_0 \end{aligned} \tag{24}$$

(see (3) and recall $||y_n|| = 1$ for all $n \in \mathbb{N}$). We know that $y_n \to 0$ in $L^2(\Omega)$ (see (18) and recall that $y = 0$). This fact and (21) imply that

$$\varphi(t_n u_n^+) \geq \frac{1}{2} q c_0 \quad \text{for all } n \geq n_1 \geq n_0. \tag{25}$$

Because $q > 0$ is arbitrary, from (25) we infer that

$$\varphi(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{26}$$

We know that

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(u_n^+) \leq M_3 \text{ for all } n \in \mathbb{N} \quad \text{(see (15))}. \tag{27}$$

From (23), (26) and (27) it follows that

$$t_n \in (0,1) \quad \text{for all } n \geq n_2. \tag{28}$$

Therefore we have

$$\begin{aligned}
\frac{d}{dt}\varphi(tu_n^+)\bigg|_{t=t_n} &= 0 \quad \text{for all } n \geq n_2, \\
\Rightarrow \left\langle \varphi'(t_n u_n^+), u_n^+ \right\rangle &= 0 \quad \text{for all } n \geq n_2 \quad \text{(by the chain rule)}, \\
\Rightarrow \left\langle \varphi'(t_n u_n^+), t_n u_n^+ \right\rangle &= 0 \quad \text{for all } n \geq n_2 \quad \text{(see (28))}, \\
\Rightarrow \gamma(t_n u_n^+) &= \int_\Omega f(z, t_n u_n^+)(t_n u_n^+) dz \quad \text{for all } n \geq n_2.
\end{aligned} \tag{29}$$

From (28) and hypothesis H_1(ii), we have

$$\int_\Omega e(z, t_n u_n^+) dz \leq \int_\Omega e(z, u_n^+) dz + ||\tau||_1 \leq M_4 + ||\tau||_1 \quad \text{for all } n \geq n_2 \tag{30}$$

(see (17)). From (30) and (29) it follows that

$$2\varphi(t_n u_n^+) \leq M_6 \quad \text{for some } M_6 > 0, \text{ all } n \geq n_2. \tag{31}$$

Comparing (26) and (31), again we reach a contradiction. This proves Claim 1.

Claim 2. $\{u_n^-\}_{n \geq 1} \subseteq H^1(\Omega)$ is bounded.

Again we proceed by contradiction. So, we assume that at least for a subsequence, we have $||u_n^-|| \to \infty$. We set $w_n = \frac{u_n^-}{||u_n^-||}$, $n \in \mathbb{N}$. Then $||w_n|| = 1$, $w_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$w_n \xrightarrow{w} w \text{ in } H^1(\Omega) \quad \text{and} \quad w_n \to w \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega), \quad w \geq 0. \tag{32}$$

From (11) and Claim 1 it follows that

$$\left|\langle A(-u_n^-), h\rangle + \int_\Omega \xi(z)(-u_n^-)hdz + \int_{\partial\Omega}\beta(z)(-u_n^-)hd\sigma - \int_\Omega f(z,-u_n^-)hdz\right|$$
$$\leq M_7||h|| \quad \text{for some } M_7>0, \text{all } h\in H^1(\Omega),\ \text{all } n\in\mathbb{N},$$
$$\Rightarrow \left|\langle A(-w_n), h\rangle + \int_\Omega \xi(z)(-w_n)hdz + \int_{\partial\Omega}\beta(z)(-w_n)hd\sigma - \int_\Omega \frac{N_f(-u_n^-)}{||u_n^-||}hdz\right|$$
$$\leq \frac{M_7||h||}{||u_n^-||} \quad \text{for all } n\in\mathbb{N},\ \text{all } h\in H^1(\Omega). \tag{33}$$

Hypotheses H_1(i),(iii) imply that $|f(z,x)|\leq c_2(1+|x|)$ for almost all $z\in\Omega$, all $x\leq 0$, with $c_2>0$, Hence, $\left\{\frac{N_f(-u_n^-)}{||u_n^-||}\right\}_{n\geq 1}\subseteq L^2(\Omega)$ is bounded. So, we may assume that

$$\frac{N_f(-u_n^-)}{||u_n^-||}\xrightarrow{w} g \quad \text{in } L^2(\Omega). \tag{34}$$

Using hypothesis H_1(iii) we have

$$g=-\eta_0 w \quad \text{with} \quad \eta(z)\leq\eta_0(z)\leq\hat{\eta}(z) \quad \text{for almost all } z\in\Omega \tag{35}$$

(see Aizicovici, Papageorgiou & Staicu [1, proof of Proposition 14]). In (33) we pass to the limit as $n\to\infty$ and use (32), (34), (35). We obtain

$$\langle A(w),h\rangle + \int_\Omega \xi(z)whdz + \int_{\partial\Omega}\beta(z)whd\sigma = \int_\Omega \eta_0 whdz \quad \text{for all } h\in H^1(\Omega),$$
$$\Rightarrow \begin{cases} -\Delta w(z)+\xi(z)w(z)=\eta_0(z)w(z) & \text{for almost all } z\in\Omega,\\ \dfrac{\partial w}{\partial n}+\beta(z)w=0 & \text{on } \partial\Omega \end{cases} \tag{36}$$

(see Papageorgiou & Rădulescu [14]).

From (35), hypothesis H_1(iii) and Lemma 2.4, we have

$$\tilde{\lambda}_m(\eta_0)<\tilde{\lambda}_m(\hat{\lambda}_m)=1 \quad \text{and} \quad 1=\tilde{\lambda}_{m+1}(\hat{\lambda}_{m+1})<\tilde{\lambda}_{m+1}(\eta_0). \tag{37}$$

From (36) and (37) it follows that $w\equiv 0$. On the other hand, if in (33) we choose $h=w_n-w\in H^1(\Omega)$, pass to the limit as $n\to\infty$ and use (32), (34), then

$$\lim_{n\to\infty}\langle A(w_n),w_n-w\rangle=0 \ \Rightarrow\ ||Dw_n||_2\to||Dw||_2 \ \Rightarrow\ w_n\to w \text{ in } H^1(\Omega)$$

(by the Kadec–Klee property, see (32)), and $||w||=1$ follows, a contradiction. This proves Claim 2.

From Claims 1 and 2, we deduce that $\{u_n\}_{n\geq 1}\subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$u_n\xrightarrow{w} u \text{ in } H^1(\Omega) \quad \text{and} \quad u_n\to u \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{38}$$

In (11) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (38). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \ \Rightarrow\ u_n \to u \text{ in } H^1(\Omega)$$

(as before via the Kadec–Klee property), and φ satisfies the C-condition. This completes the proof of Proposition 3.3. □

Next we consider the following truncation-perturbation of the reaction term $f(z,\cdot)$:

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) + \mu x & \text{if } 0 \le x \end{cases} \tag{39}$$

(here $\mu > 0$ is as in (3)). This is a Carathéodory function. We set

$$\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)ds$$

and consider the C^1-functional $\hat{\varphi}_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}||u||^2 - \int_\Omega \hat{F}_+(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Proposition 3.4. *If hypotheses* H(ξ), H(β) *and* H_1 *hold, then the functional* $\hat{\varphi}_+$ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\ge1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\hat{\varphi}_+(u_n)| \le M_7 \quad \text{for some } M_7 > 0, \text{ all } n \in \mathbb{N}, \tag{40}$$

$$(1 + ||u_n||)\hat{\varphi}_+'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{41}$$

From (41) we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma + \mu\int_\Omega u_n h dz - \int_\Omega \hat{f}_+(z,u_n)hdz\right|$$

$$\le \frac{\epsilon_n ||h||}{1 + ||u_n||} \quad \text{for all } h \in H^1(\Omega), \text{ with } \epsilon_n \to 0^+. \tag{42}$$

In (42) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\gamma(u_n^-) + \mu||u_n^-||_2^2 \le \epsilon_n \quad \text{for all } n \in \mathbb{N} \quad \text{(see (39))},$$

$$\Rightarrow c_0||u_n^-||^2 \le \epsilon_n \quad \text{for all } n \in \mathbb{N} \quad \text{(see (3))},$$

$$\Rightarrow u_n^- \to 0 \quad \text{in } H^1(\Omega) \text{ as } n \to \infty. \tag{43}$$

From (40) and (43), we have

$$\gamma(u_n^+) - \int_\Omega 2F(z,u_n^+)dz \le M_8 \quad \text{for some } M_8 > 0, \text{ all } n \in \mathbb{N}. \tag{44}$$

Also, in (38) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$-\gamma(u_n^+) + \int_\Omega f(z, u_n^+)u_n^+ dz \leq M_9 \quad \text{for some } M_9 > 0, \text{ all } n \in \mathbb{N} \quad \text{(see (39))}. \tag{45}$$

We add (44) and (45) and obtain

$$\int_\Omega e(z, u_n^+)dz \leq M_{10} \quad \text{for some } M_{10} > 0, \text{ all } n \in \mathbb{N}. \tag{46}$$

Using (46) and reasoning as in the proof of Proposition 3.3 (see Claim 1 in that proof), we infer that

$$\{u_n^+\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.} \tag{47}$$

From (43) and (47) it follows that $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{48}$$

In (42) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (44). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \;\Rightarrow\; u_n \to u \text{ in } H^1(\Omega)$$

(as before via the Kadec–Klee property), and $\hat{\varphi}_+$ satisfies the C-condition. The proof of Proposition 3.4 is complete. ☐

Proposition 3.5. *If hypotheses* H(ξ), H(β) *and* H_1 *hold, then* $u = 0$ *is a local minimizer for the functionals* φ *and* $\hat{\varphi}_+$.

Proof. We do the proof for the functional φ, the proof for $\hat{\varphi}_+$ being similar. Using hypotheses H_1(i),(iv), we see that given $\epsilon > 0$, we can find $c_3 = c_3(\epsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{2}(\vartheta(z) + \epsilon)x^2 + c_3|x|^r \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{49}$$

Then for all $u \in H^1(\Omega)$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\gamma(u) - \int_\Omega F(z, u)dz \\ &\geq \frac{1}{2}\left[\gamma(u) - \int_\Omega \vartheta(z)u^2 dz\right] - \frac{\epsilon}{2}||u||^2 - c_4||u||^r \quad \text{for some } c_4 > 0 \text{ (see (49))} \\ &\geq \frac{1}{2}[c_5 - \epsilon]||u||^2 - c_4||u||^r \quad \text{for some } c_5 > 0 \quad \text{(see Lemma 2.3)}.\end{aligned} \tag{50}$$

Choosing $\epsilon \in (0, c_5)$, from (50) we conclude that

$$\varphi(u) \geq c_6||u||^2 - c_4||u||^r \quad \text{for some } c_6 > 0, \text{ all } u \in H^1(\Omega). \tag{51}$$

Since $r > 2$, from (51) we see that if $\rho \in (0, 1)$ is small, then $\varphi(u) > 0 = \varphi(0)$ for all $u \in H^1(\Omega)$, $0 < ||u|| \leq \rho$, and $u = 0$ is a (strict) local minimizer for φ.

We argue similarly for the functional $\hat{\varphi}_+$. ☐

Proposition 3.6. *If hypotheses* H(ξ), H(β) *and* H$_1$ *hold, then* $K_{\hat{\varphi}_+} \subseteq D_+ \cup \{0\}$.

Proof. Let $u \in K_{\hat{\varphi}_+} \backslash \{0\}$. Then $\hat{\varphi}'_+(u) = 0$ implies

$$\langle A(u), h\rangle + \int_\Omega \xi(z) uh dz + \int_{\partial\Omega} \beta(z) uh d\sigma + \mu \int_\Omega uh dz = \int_\Omega \hat{f}_+(z,u) h dz \quad (52)$$

for all $h \in H^1(\Omega)$. In (48) we choose $h = -u^- \in H^1(\Omega)$. Then

$$\gamma(u^-) + \mu ||u^-||_2^2 = 0 \text{ (see (39))} \ \Rightarrow \ c_0 ||u^-||^2 \leq 0 \text{ (see (3))} \ \Rightarrow \ u \geq 0, \ u \neq 0.$$

Then (52) becomes

$$\langle A(u), h\rangle + \int_\Omega \xi(z) uh dz + \int_{\partial\Omega} \beta(z) uh d\sigma = \int_\Omega f(z,u) h dz$$
for all $h \in H^1(\Omega)$ (see (39)),
$$\Rightarrow \begin{cases} -\Delta u(z) + \xi(z) u(z) = f(z, u(z)) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z) u = 0 & \text{on } \partial\Omega \end{cases} \quad (53)$$

(see Papageorgiou & Rădulescu [14]).

Let $a_0(z) = \frac{f(z,u(z))}{1+u(z)}$. Then $|a_0(z)| = \frac{|f(z,u(z))|}{1+u(z)} \leq \frac{c_7(1+u(z)^{r-1})}{1+u(z)}$ for some $c_7 > 0$. Hence,

$$|a_0(z)| \leq \frac{c_8(1+u(z))^{r-1}}{1+u(z)} = c_8(1+u(z))^{r-2} \leq c_9(1+u(z)^{r-2}) \quad \text{for some } c_8, c_9 > 0,$$

and $a_0 \in L^{\frac{N}{2}}(\Omega)$. Note that

$$(r-2)\frac{N}{2} \leq \left(\frac{2N}{N-2} - 2\right)\frac{N}{2} = \frac{2N}{N-2} = 2^* \quad \text{if } N \geq 3.$$

Also by hypothesis H(ξ), $\xi \in L^s(\Omega)$ with $s > N$. Hence from (53) and Wang [20, Lemma 5.1], we have

$$u \in L^\infty(\Omega).$$

Then from (53) and using hypotheses H(ξ) and H$_1$(i), we have $\Delta u \in L^s(\Omega)$. Using the Calderon–Zygmund estimates (see Wang [20, Lemma 5.2]) and the Sobolev embedding theorem, we infer that

$$u \in C^{1,\alpha}(\overline{\Omega}) \quad \text{with } \alpha = 1 - \frac{N}{s} \in (0,1).$$

Let $\rho = ||u||_\infty$. We can find $\hat{\xi}_\rho > 0$ such that

$$f(z,x)x + \hat{\xi}_\rho x^2 \geq 0 \quad \text{for almost all } z \in \Omega, \ \text{all } x \in \mathbb{R}$$

(see hypotheses H_1(i),(iv)). Then from (53) and hypotheses $\mathrm{H}(\xi), H_1(\mathrm{i})$, we have

$$\Delta u(z) \leq (c_{10} + ||\xi^+||_\infty)u(z) \quad \text{for almost all } z \in \Omega.$$

Use the strong maximum principle (see Gasinski & Papageorgiou [7, p. 738]) to deduce $u \in D_+$. The proof of Proposition 3.6 is complete. □

Now working with the functional $\hat{\varphi}_+$ and using variational arguments we are ready to produce a positive solution.

Proposition 3.7. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(\beta)$ *and* H_1 *hold, then problem* (1) *admits a positive solution* $u_0 \in D_+$.

Proof. We may assume that $K_{\hat{\varphi}_+}$ is finite. Otherwise Proposition 3.6 and (39) imply that we have an infinity of positive smooth solutions for problem (1).

Because of Proposition 3.5, we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}_+(0) = 0 < \inf\{\hat{\varphi}_+(u) : ||u|| = \rho\} = \hat{m}_\rho^+ \tag{54}$$

(see Aizicovici, Papageorgiou & Staicu [1, proof of Proposition 29]).

If $u \in D_+$, then hypothesis H_1(ii) implies that

$$\hat{\varphi}_+(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{55}$$

Finally from Proposition 3.4 we know that

$$\hat{\varphi}_+ \text{ satisfies the C-condition.} \tag{56}$$

Relations (54)–(56) permit the use of Theorem 2.2 (the mountain pass theorem). So, we can find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_+} \quad \text{and} \quad \hat{m}_\rho^+ \leq \hat{\varphi}_+(u_0). \tag{57}$$

From (57), Proposition 3.6, (54) and (39), we conclude that $u_0 \in D_+$ is a positive solution of problem (1). □

Next we will produce a negative smooth solution. To this end, we consider the following truncation-perturbation of the reaction term $f(z,\cdot)$:

$$\hat{f}_-(z,x) = \begin{cases} f(z,x) + \mu|x|^{p-2}x & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x \end{cases} \tag{58}$$

(as before $\mu > 0$ is the constant from (3)). This is a Carathéodory function. We set $\hat{F}_-(z,x) = \int_0^x \hat{f}_-(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega \hat{F}_-(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Reasoning as in the proof of Proposition 3.5, we show the following result.

Proposition 3.8. *If hypotheses* H(ξ), H(β) *and* H_1 *hold, then* $u = 0$ *is a local minimizer for the functional* $\hat{\varphi}_-$.

Next we show that the functional $\hat{\varphi}_-$ satisfies the C-condition.

Proposition 3.9. *If hypotheses* H(ξ), H(β) *and* H_1 *hold, then the functional* $\hat{\varphi}_-$ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ be a sequence such that $\{\hat{\varphi}_-(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\hat{\varphi}'_-(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{59}$$

From (59), we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma + \mu \int_\Omega u_n^+ h dz - \int_\Omega f(z, -u_n^-)h dz\right|$$
$$\leq \frac{\epsilon||h||}{1 + ||u_n||} \quad \text{for all } h \in H^1(\Omega) \text{ with } \epsilon_n \to 0^+ \quad \text{(see (58))}. \tag{60}$$

In (60) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u_n^+) + \mu||u_n^+||_2^2 \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}, \\ &\Rightarrow c_0||u_n^+||^2 \leq \epsilon_n \quad \text{for all } n \in \mathbb{N} \quad \text{(see (3))}, \\ &\Rightarrow u_n^+ \to 0 \quad \text{in } H^1(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{61}$$

Arguing as in the proof of Proposition 3.3 (see Claim 2 in that proof), using hypothesis H_1(iii), we show that

$$\{u_n^-\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.} \tag{62}$$

From (61) and (62), we have that $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded, from which as before via the Kadec–Klee property, we conclude that $\hat{\varphi}_-$ satisfies the C-condition. $\square$

We can also characterize the critical points of $\hat{\varphi}_-$ as we did for those of $\hat{\varphi}_+$ (see Proposition 3.6).

Proposition 3.10. *If hypotheses* H(ξ), H(β) *and* H_1 *hold, then* $K_{\hat{\varphi}} \subseteq (-D_+) \cup \{0\}$.

Proof. Let $u \in K_{\hat{\varphi}_-} \backslash \{0\}$. We have

$$\hat{\varphi}'_-(u) = 0 \;\Rightarrow \begin{cases} \langle A(u), h\rangle + \displaystyle\int_\Omega \xi(z)uh dz + \int_{\partial\Omega} \beta(z)uh d\sigma + \mu \int_\Omega u^+ h dz \\ = \displaystyle\int_\Omega f(z, -u^-)h dz \quad \text{for all } h \in H^1(\Omega) \quad \text{(see (58))}. \end{cases} \tag{63}$$

In (63) we choose $h = u^+ \in H^1(\Omega)$. Then

$$\gamma(u^+) + \mu||u^+||_2^2 = 0 \ \Rightarrow \ c_0||u^+||^2 \leq 0 \text{ (see (3))} \ \Rightarrow \ u \leq 0, u \neq 0.$$

So, relation (63) becomes

$$\langle A(u), h\rangle + \int_\Omega \xi(z)uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma = \int_\Omega f(z,u)hdz \quad \text{for all } h \in H^1(\Omega),$$

$$\Rightarrow \begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z,u(z)) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega \end{cases} \tag{64}$$

(see Papageorgiou & Rădulescu [14]).

Hypotheses H_1(i),(iii),(iv), imply that

$$|f(z,x)| \leq c_{11}|x| \quad \text{for almost all } z \in \Omega, \text{ all } x \leq 0, \text{ with } c_{11} > 0. \tag{65}$$

We set

$$\hat{m}(z) = \begin{cases} \frac{f(z,u(z))}{u(z)} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0. \end{cases}$$

From (65) we have that $\hat{m} \in L^\infty(\Omega)$ and from (64) we have

$$-\Delta u(z) = (\hat{m} - \xi)(z)u(z) \quad \text{for almost all } z \in \Omega. \tag{66}$$

We have $(\hat{m} - \xi)(\cdot) \in L^s(\Omega)$ (see hypothesis $\mathrm{H}(\xi)$). So, Wang [20, Lemma 5.1] implies that $u \in L^\infty(\Omega)$ and so from (66) we have $\Delta u \in L^s(\Omega)$. As before, by the Calderon–Zygmund estimates (see Wang [20, Lemma 5.2]) we have

$$u \in W^{2,s}(\Omega) \ \Rightarrow \ u \in C^{1,\alpha}(\overline{\Omega}) \quad \text{with } \alpha = 1 - \frac{N}{s}$$

(see the Sobolev embedding theorem). From (64), (65) and hypothesis $\mathrm{H}(\xi)$ we have

$$\Delta(-u)(z) \leq (c_{12} + ||\xi^+||_\infty)(-u)(z) \quad \text{for almost all } z \in \Omega \ \Rightarrow \ u \in (-D_+)$$

(by the strong maximum principle, see Gasinski & Papageorgiou [7, p. 738]). $\square$

Now we are ready to produce the negative solution.

Proposition 3.11. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(\beta)$ *and* H_1 *hold, then problem* (1) *admits a negative solution* $v_0 \in (-D_+)$.

Proof. We may assume that $K_{\hat{\varphi}_-}$ is finite. Otherwise Proposition 3.10 and (66) imply that we already have an infinity of distinct negative smooth solutions.

Because of Proposition 3.8, we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}_-(0) = 0 < \inf\{\hat{\varphi}_-(u) : ||u|| = \rho\} = \hat{m}_\rho^- \tag{67}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29). Hypotheses H_1(i),(iii) imply that given $\epsilon > 0$, we can find $c_{13} = c_{13}(\epsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{2}(\eta(z) - \epsilon)x^2 - c_{13} \quad \text{for almost all } z \in \Omega, \text{ all } x \leq 0.$$

Then for $t > 0$ we have

$$\begin{aligned} \hat{\varphi}_-(t(-\hat{u}_1)) &= \frac{t^2}{2}\gamma(\hat{u}_1) - \int_\Omega F(z, t(-\hat{u}_1))dz \quad \text{(see (58))} \\ &\leq \frac{t^2}{2}\left[\gamma(\hat{u}_1) - \int_\Omega \eta(z)\hat{u}_1^2 dz + \epsilon\right] + c_{13}|\Omega|_N \end{aligned} \tag{68}$$

(see (67) and recall that $||\hat{u}_1||_2 = 1$).

Note that $\gamma(\hat{u}_1) - \int_\Omega \eta(z)\hat{u}_1^2 dz = \int_\Omega (\hat{\lambda}_1 - \eta(z))\hat{u}_1^2 dz = \zeta < 0$ (see hypothesis H_1(iii) and recall $\hat{u}_1 \in \operatorname{int} C_+$) So, from (68) we have

$$\hat{\varphi}_-(t(-\hat{u}_1)) \leq \frac{t^2}{2}[\zeta + \epsilon] + c_{13}|\Omega|_N. \tag{69}$$

Choosing $\epsilon \in (0, -\zeta)$, from (69) we infer that

$$\hat{\varphi}_-(t(-\hat{u}_1)) \to -\infty \quad \text{as } t \to +\infty. \tag{70}$$

From (67), (70) and Proposition 3.9, we see that we can use Theorem 2.2 (the mountain pass theorem). So, we can find $v_0 \in H^1(\Omega)$ such that

$$v_0 \in K_{\hat{\varphi}_-} \quad \text{and} \quad \hat{m}_\rho^- \leq \hat{\varphi}_-(v_0). \tag{71}$$

From (67), (71) and Proposition 3.10, we have that $v_0 \in -D_+$ is a negative smooth solution of problem (1). □

We will produce a third nontrivial smooth solution using tools from Morse theory (critical groups). To do this we will need eventually to strengthen the regularity of $f(z, \cdot)$. However, for the moment, we do not need this extra condition. We start by computing the critical groups at infinity of the energy functional φ.

Proposition 3.12. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(\beta)$, H_1 *hold and* $\varphi(K_\varphi)$ *is bounded below, then* $C_k(\varphi, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. We set $\varphi_c = \varphi|_{C^1(\overline{\Omega})}$. As in the proof of Proposition 3.6 (see also the proof of Proposition 3.10), we can show that

$$K_{\varphi_c} = K_\varphi = K.$$

Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, from Palais [13, Theorem 16], we have

$$H_k(H^1(\Omega), \dot{\varphi}^a) = H_k(C^1(\overline{\Omega}), \dot{\varphi}_c^a) \quad \text{for all } k \in \mathbb{N}_0, \text{ all } a \in \mathbb{R}.$$

If $a \in \mathbb{R}$ is not a critical value of φ, then

$$H_k(H^1(\Omega), \varphi^a) = H_k(H^1(\Omega), \dot{\varphi}^a) \quad \text{and} \quad H_k(C^1(\overline{\Omega}), \varphi_c^a) = H_k(C^1(\overline{\Omega}), \dot{\varphi}_c^a)$$

for all $k \in \mathbb{N}_0$ (see Granas & Dugundji [8, p. 407]).

Let $a < \inf \varphi(K_\varphi)$ (recall that by hypothesis, $\varphi(K_\varphi)$ is bounded below). Then from the previous remarks and the definition of the critical groups at infinity (see Section 2), it suffices to show that

$$H_k(C^1(\overline{\Omega}), \varphi_c^a) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Let $C \subseteq \varphi_c^a$ be a compact set.

Claim 3. For $a < 0$ with $|a| > 0$ big, the set C is contractible in φ_c^a.

In what follows by $\langle \cdot, \cdot \rangle_c$ we denote the duality brackets for the pair $\left(C^1(\overline{\Omega})^*, C^1(\overline{\Omega})\right)$ and let $i : C^1(\overline{\Omega}) \to H^1(\Omega)$ be the continuous embedding. We have

$$\varphi_c = \varphi \circ i \ \Rightarrow \ \varphi_c'(u) = i^* \varphi'(u) \quad \text{for all } u \in C^1(\overline{\Omega}). \tag{72}$$

Let $u \in \varphi_c^a$ and $t > 0$. Then we have

$$\begin{aligned}
\frac{d}{dt}\varphi_c(tu) &= \langle \varphi_c'(tu), u \rangle_c \quad \text{(by the chain rule)} \\
&= \langle \varphi'(tu), u \rangle \quad \text{(see (72))} \\
&= \frac{1}{t} \langle \varphi'(tu), tu \rangle \\
&= \frac{1}{t} \left[\gamma(tu) - \int_\Omega f(z, tu)(tu) dz \right] \\
&= \frac{1}{t} \left[\gamma(tu) - \int_\Omega f(z, tu^+)(tu^+) dz - \int_\Omega f(z, -tu^-)(-tu^-) dz \right] \\
&\le \frac{1}{t} \left[\gamma(tu) - \int_\Omega 2F(z, tu^+) dz - \int_\Omega 2F(z, -tu^-) dz + c_{14} \right] \\
&\quad \text{for some } c_{14} > 0 \quad \text{(see hypotheses } H_1\text{(ii), (iii))} \\
&= \frac{1}{t} \left[\gamma(tu) - \int_\Omega 2F(z, tu) dz + c_{14} \right] \\
&= \frac{1}{t} \left[2\varphi_c(tu) + c_{14} \right].
\end{aligned}$$

Hence $\frac{d}{dt}\varphi_c(tu)|_{t=1} \leq 2\varphi_c(u) + c_{14} \leq 2a + c_{14}$ (recall that $u \in \varphi_c^a$). It follows that

$$a < -\frac{c_{14}}{2} \;\Rightarrow\; \frac{d}{dt}\varphi_c(tu)|_{t=1} < 0.$$

By virtue of hypothesis H_1(ii), if $\varphi_c(u) \in (a-1, a]$, then we can find a unique $\hat{\tau}(u) > 0$ such that

$$\varphi_c(\hat{\tau}(u)u) = a - 1.$$

If $u \in \varphi_c^{a-1}$, then we set $\hat{\tau}(u) = 1$. The implicit function theorem implies that $\hat{\tau} \in C(\varphi_c^a, (0,1])$. Consider the deformation $h_1 : [0,1] \times C \to \varphi_c^a$ defined by

$$h_1(t,u) = [(1-t) + t\hat{\tau}(u)]u.$$

We set $C_1 = h_1(1, C) \subseteq \varphi_c^{a-1}$. The set $C_1 \subseteq C^1(\overline{\Omega})$ is compact. So, we can find $M_{11} > 0$ such that

$$\left|\frac{\partial u}{\partial n}(z)\right| \leq M_{11} \quad \text{for all } z \in \partial\Omega, \text{ all } u \in C_1. \tag{73}$$

Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$ (the distance function from $\partial\Omega$). For $\epsilon, c > 0$ we consider the function

$$\hat{\zeta}_{\epsilon,c}(z) = \begin{cases} c\hat{d}(z) & \text{if } \hat{d}(z) \leq \epsilon \\ c\epsilon & \text{if } \hat{d}(z) > \epsilon \end{cases} \quad \text{for all } z \in \Omega.$$

We approximate $\hat{\zeta}_\epsilon$ by a $C^1(\overline{\Omega})$-function ζ_ϵ. Then because of (73) by choosing $c > 0$ big we can have $(u + \zeta_{\epsilon,c})^+ \neq 0$ for all $u \in C_1$. In addition, if $\epsilon > 0$ is small we have

$$\varphi_c(u + t\zeta_{\epsilon,c}) \leq a \quad \text{for all } (t,u) \in [0,1] \times C_1.$$

So, we can have a deformation $h_2 : [0,1] \times C_1 \to \varphi_c^a$ defined by

$$h_2(t,u) = u + t\zeta_{\epsilon,c} \quad \text{for } (t,u) \in [0,1] \times C_1.$$

We set $C_2 = h_2(1, C_1)$ and consider $u \in C_2$. Then $u^+ \neq 0$ and

$$\varphi_c(u) = \varphi_c(u^+) + \varphi_c(-u^-) \leq a.$$

From the previous considerations we know that $[1,\infty) \ni t \mapsto \varphi_c(tu)$ is decreasing. Since the set $C_2 = h_2(1, C_1) \subseteq C_1(\overline{\Omega})$ is compact, we can find $t_* \geq 1$ such that

$$\varphi_c(tu^+) \leq a \quad \text{for all } t \geq t_*, \text{ all } u \in C_2. \tag{74}$$

We introduce the deformation $h_3 : [0,1] \times C_2 \to \varphi_c^a$ defined by

$$h_3(t,u) = (1 - t + t^* t)u \quad \text{for all } (t,u) \in [0,1] \times C_2.$$

We set $C_3 = h_3(1, C_2)$. From (74) we have

$$\varphi_c(u^+) \le a \quad \text{for all } u \in C_3. \tag{75}$$

The set $C_3 = h_3(1, C_2) \subseteq C^1(\overline{\Omega})$ is compact. So, we can find $c_{15} > 0$ such that

$$\varphi_c(s(-u^-)) \le c_{15} \quad \text{for all } s \in [0,1], \text{ all } u \in C_3. \tag{76}$$

Since the function $t \to \varphi_c(tu^+)$ is decreasing on $[1, \infty)$, using (75) we see that we can find $t'_* \ge 1$ such that

$$\varphi_c(t'_* u^+) \le a - c_{15} \quad \text{for all } u \in C_3.$$

So, we define a deformation $h_4 : [0,1] \times C_3 \to \varphi_c^a$ by setting

$$h_4(t,u) = (1 - t + t'_* t)u^+ + u^- \quad \text{for all } (t,u) \in [0,1] \times C_3.$$

Let $C_4 = h_4(1, C_3)$. Then $C_4 \subseteq C^1(\overline{\Omega})$ is compact and

$$C_4 \subseteq \varphi_c^a \cap \{u \in C^1(\overline{\Omega}) : \varphi_c(u^+) \le a - c_{15}\} \quad (\text{see } (76)). \tag{77}$$

Finally we introduce the deformation $h_5 : [0,1] \times C_4 \to \varphi_c^a$ defined by

$$h_5(t,u) = u^+ + (1-t)(-u^-) \quad \text{for all } (t,u) \in [0,1] \times C_4.$$

We have

$$\varphi_c(h_5(t,u)) = \varphi_c(u^+ + (1-t)(-u^-)) = \varphi_c(u^+) + \varphi_c((1-t)(-u^-)) \le a - c_{15} + c_{15} = a$$

(see (76), (77)), which shows that this deformation is well-defined into the sublevel set φ_c^a. We set $C_5 = h_5(1, C_4)$. Then $C_5 \subseteq C^1(\overline{\Omega})$ is compact and we have

$$C_5 \subseteq \varphi_c^a, C_5 \subseteq C_+ \ \Rightarrow \ C_5 \subseteq \varphi_c^a \cap C_+ = C_+^a. \tag{78}$$

Let $\partial B_+^c = \{u \in C^1(\overline{\Omega}) : ||u||_{C^1(\overline{\Omega})} = 1\} \cap C_+$. From the first part of the proof we know that given $u \in \partial B_+^c$, we can find $\tau_0(u) > 0$ such that $\varphi_c(\tau_0(u)u) = a$ and

$$C_+^a = \{tu : u \in \partial B_+^c, t \ge \tau_0(u)\}.$$

Via the radial retraction, we see that the sets C_+^a and ∂B_+^c are homotopy equivalent. Consider the deformation $h_+ : [0,1] \times \partial B_+^c \to \partial B_+^c$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\hat{u}_1}{||(1-t)u + t\hat{u}_1||_{C^1(\overline{\Omega})}} \quad \text{for all } (t,u) \in [0,1] \times \partial B_+^c.$$

We have $h_+(1,u) = \frac{\hat{u}_1}{||\hat{u}_1||_{C^1(\overline{\Omega})}} \in \partial B_+^c$ (recall $\hat{u}_1 \in D_+$),

$\Rightarrow$ ∂B_+^c is contractible $\Rightarrow$ C_+^a is contractible $\Rightarrow$ C_5 is contractible (see (78)).

Therefore, by successive deformations we have passed from the initial set $C \subseteq \varphi_c^a$ to the set C_5 which is contractible. Hence C is contractible and this proves the Claim.

Let $* \in \varphi_c^a$. Since $a < \inf \varphi(K) = \inf \varphi_c(K)$, we have

$$H_k(\varphi_c^a, *) = H_k(\dot{\varphi}_c^a, *) \quad \text{for all } k \in \mathbb{N}_0. \tag{79}$$

The Banach space $C^1(\overline{\Omega})$ is separable. So, we can find a sequence $\{X_n\}_{n\geq 1}$ of finite dimensional subspaces of $C^1(\overline{\Omega})$ such that

$$C^1(\overline{\Omega}) = \overline{\bigcup_{n\geq 1} X_n}.$$

Without any loss of generality, we assume that

$$* \in \bar{B}_n^{X_n} = \{u \in X_n : ||u||_{C^1(\overline{\Omega})} \leq n\}$$

for all $n \in \mathbb{N}$ big enough. From Claim 3 and (79), we have

$$H_k(\dot{\varphi}_c^a, *) = H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a \cap \bar{B}_n^{X_n}) \quad \text{for all } k \in \mathbb{N}_0. \tag{80}$$

From Palais [13, Corollary, p. 5], we have

$$0 = H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a) = \lim_{\vec{n}} H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a \cap \bar{B}_n^{X_n}) \quad \text{for all } k \in \mathbb{N}_0, \tag{81}$$

with $\lim_{\vec{n}}$ denoting the inductive limit. From (79)–(81) it follows that

$$H_k(\varphi_c^a, *) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{82}$$

We consider the following triple of sets

$$\{*\} \subseteq \varphi_c^a \subseteq C^1(\overline{\Omega})$$

We consider the long exact sequence of singular homology groups corresponding to this triple (see, for example, Motreanu, Motreanu & Papageorgiou [11, Proposition 6.14, p. 143]). We have

$$\cdots \to H_k(\varphi_c^a, *) \xrightarrow{i_*} H_k(C^1(\overline{\Omega}), \varphi_c^a) \xrightarrow{\partial_*} H_{k-1}(\varphi_c^a, *) \to \cdots \tag{83}$$

Here i_* is the group homomorphism induced by the inclusion $i : (\varphi_c^a, *) \to (C^1(\overline{\Omega}), \varphi_c^a)$ and ∂_* is the boundary homomorphism. Exploiting the exactness of (83) and using (82), we infer that

$$\begin{aligned} &H_k(C^1(\overline{\Omega}), \varphi_c^a) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ &\Rightarrow C_k(\varphi_c, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{recall that } a < \inf \varphi_c(K) = \inf \varphi(K)), \\ &\Rightarrow C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{see the beginning of the proof}). \qquad \square \end{aligned}$$

As we already mentioned earlier, in order to produce a third nontrivial smooth solution, we need to strengthen the regularity of $f(z, \cdot)$. So, the new stronger conditions on the reaction term $f(z, x)$ are the following:

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $2 < r < 2^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$ and $e(z, x) = f(z, x)x - 2F(z, x)$, then

  (a) $\lim_{x\to+\infty} \frac{F(z,x)}{x^2} = +\infty$ uniformly for almost all $z \in \Omega$,

  (b) $e(z, x) \leq e(z, y) + \tau(z)$ for almost all $z \in \Omega$, all $0 \leq x \leq y$ with $\tau \in L^1(\Omega)_+$;

(iii) there exist $m \geq \max\{k_0, 2\}$, functions $\eta, \hat{\eta} \in L^\infty(\Omega)$ and a constant $\hat{c} > 0$ such that

  (a) $\hat{\lambda}_m \leq \eta(z) \leq \hat{\eta}(z) \leq \hat{\lambda}_{m+1}$ for almost all $z \in \Omega$, $\hat{\lambda}_m \not\equiv \eta, \hat{\lambda}_{m+1} \not\equiv \hat{\eta}$,

  (b) $\eta(z) \leq \liminf_{x\to-\infty} \frac{f(z,x)}{x} \leq \limsup_{x\to-\infty} \frac{f(z,x)}{x} \leq \hat{\eta}(z)$ uniformly for almost all $z \in \Omega$,

  (c) $-\hat{c} \leq e(z, x)$ for almost all $z \in \Omega$, all $x \leq 0$;

(iv) $f'_x(z, 0) = \lim_{x\to0} \frac{f(z,x)}{x}$ uniformly for almost all $z \in \Omega$ and $f'_x(z, 0) \leq \hat{\lambda}_1$ for almost all $z \in \Omega$, $f'_x(\cdot, 0) \not\equiv \hat{\lambda}_1$.

Let $u_0 \in D_+$ and $v_0 \in -D_+$ be the two nontrivial constant sign smooth solutions produced in Propositions 3.7 and 3.11 respectively.

Proposition 3.13. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(\beta)$, H_2 *hold and* K_φ *is finite, then* $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$.

Proof. From the proof of Proposition 3.7, we know that $u_0 \in D_+$ is a critical point of mountain pass type for the functional $\hat{\varphi}_+$. So, we have

$$C_1(\hat{\varphi}_+, u_0) \neq 0 \tag{84}$$

(see Motreanu, Motreanu & Papageorgiou [11, Corollary 6.81, p. 168]). Consider the homotopy

$$\hat{h}_+(t, u) = (1 - t)\varphi(u) + t\hat{\varphi}_+(u) \quad \text{for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Suppose that we could find $\{t_n\}_{n\geq1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq1} \subseteq H^1(\Omega)$ such that

$$t_n \to t, u_n \to u_0 \text{ in } H^1(\Omega) \quad \text{and} \quad (\hat{h}_+)'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}_0. \tag{85}$$

From the equation in (85), we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n h dz + \int_{\partial\Omega} \beta(z) u_n h d\sigma$$
$$= \int_\Omega f(z, u_n) h dz - t_n \int_\Omega f(z, -u_n^-) h dz + t_n \int_\Omega u_n^- h dz \quad \text{for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \begin{cases} -\Delta u_n(z) + \xi(z) u_n(z) = f(z, u_n(z)) - t_n f(z, -u_n^-(z)) + t_n u_n^-(z) \\ \qquad\qquad \text{for almost all } z \in \Omega, \\ \dfrac{\partial u_n}{\partial n} + \beta(z) u_n = 0 \quad \text{on } \partial\Omega \end{cases} \tag{86}$$

(see Papageorgiou & Rădulescu [14]).

From (86) and Wang [20] (see also Papageorgiou & Rădulescu [17]), we have that we can find $M_{12} > 0$ such that $||u_n||_\infty \leq M_{12}$ for all $n \in \mathbb{N}$. Then using (86) and the Calderon–Zygmund estimates (see Wang [20, Lemma 5.2]), as before, we can find $\alpha \in (0,1)$ and $M_{13} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leq M_{13} \quad \text{for all } n \in \mathbb{N}. \tag{87}$$

From (85), (87) and since $C^{1,\alpha}(\overline{\Omega})$ is embedded compactly into $C^1(\overline{\Omega})$, we have

$$u_n \to u_0 \text{ in } C^1(\overline{\Omega}) \;\Rightarrow\; u_n \in D_+ \text{ for all } n \geq n_0 \quad (\text{recall that } u_0 \in D_+),$$

and $\{u_n\}_{n \geq n_0} \subseteq D_+$ are all distinct positive solutions of (1), a contradiction to our hypothesis that K_φ is finite. So, (85) cannot occur and then by the homotopy invariance of critical groups (see Corvellec & Hantoute [4, Theorem 5.2]), we have

$$C_k(\varphi, u_0) = C_k(\hat{\varphi}_+, u_0) \text{ for all } k \in \mathbb{N}_0 \;\Rightarrow\; C_1(\varphi, u_0) \neq 0 \quad (\text{see (84)}).$$

Hypotheses H_2 imply that $\varphi \in C^2(H^1(\Omega))$ and

$$\langle \varphi''(u_0) v, h\rangle = \int_\Omega (Dv, Dh)_{\mathbb{R}^N} dz + \int_\Omega \xi(z) v h dz + \int_{\partial\Omega} \beta(z) v h d\sigma \\ - \int_\partial f_x'(z, u_0) v h dz \quad \text{for all } v, h \in H^1(\Omega). \tag{88}$$

Suppose that the Morse index of $u_0 \in K_\varphi$ is zero. Then we have

$$||Dv||_2^2 + \int_{\partial\Omega} \beta(z) v^2 d\sigma \geq \int_\Omega [f_x'(z, u_0) - \xi(z)] v^2 dz \quad \text{for all } v \in H^1(\Omega) \tag{89}$$

(see (88)). Assume that $[f_x'(\cdot, u_0(\cdot)) - \xi(\cdot)]^+ = 0$ and let $v \in \ker \varphi''(u_0)$. Then from (88) we have

$$||Dv||_2^2 + \int_{\partial\Omega} \beta(z) v^2 d\sigma \leq 0 \;\Rightarrow\; v = \text{constant} \quad (\text{see hypothesis } \mathrm{H}(\beta)).$$

Next assume that $[f'_x(\cdot, u_0(\cdot)) - \xi(\cdot)]^+ \neq 0$ and again let $v \in \ker \varphi''(u_0)$. From (88) and (89) we have

$$\tilde{\lambda}_1(\tilde{m}) = 1 \quad \text{where } \tilde{m} = f'_x(\cdot, u_0(\cdot)) - \xi(\cdot)$$

and we know that it is simple. So, it follows that $\dim \ker \varphi''(u_0) = 1$. Therefore we can use Motreanu, Motreanu & Papageorgiou [11, Corollary 6.102, p. 177] and conclude that

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Reasoning in a similar fashion and using this time the functional $\hat{\varphi}_-$, we show that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

This completes the proof of Proposition 3.13. □

Now we are ready for the multiplicity theorem (three nontrivial solutions theorem) for problem (1).

Theorem 3.14. *If hypotheses* H(ξ), H(β) *and* H_2 *hold, then problem* (1) *has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.7 and 3.11, we already have two nontrivial, constant sign smooth solutions

$$u_0 \in D_+ \quad \text{and} \quad v_0 \in -D_+.$$

Suppose that $K_\varphi = \{0, u_0, v_0\}$. Then from Proposition 3.5, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{90}$$

Also, from Proposition 3.13, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{91}$$

Finally from Proposition 3.12, we know that

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{92}$$

From (90)–(92) and the Morse relation with $t = -1$ (see (5)), we have

$$(-1)^0 + 2(-1)^1 = 0,$$

a contradiction. So, there exists $y_0 \in K_\varphi, y_0 \notin \{0, u_0, v_0\}$. Then this is the third nontrivial solution of problem (1) and the regularity theory (see Wang [20]) implies that $y_0 \in C^1(\overline{\Omega})$. □

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