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LARGE AND BOUNDED SOLUTIONS FOR A CLASS OF NONLINEAR SCHRÖDINGER STATIONARY SYSTEMS

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In the present paper, we are concerned with entire radially symmetric solutions of nonlinear Schrödinger elliptic systems in anisotropic media. In terms of the growth of the variable potential functions, we establish conditions such that the solutions are either bounded or blow up at infinity.

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1. Introduction and Auxiliary Results

Existence and nonexistence of solutions for the nonlinear Schrödinger-type system

$$(\mathcal{S})\begin{cases}\Delta u = F(x,u,v), & x \in \mathbb{R}^n,\\ \Delta v = G(x,u,v), & x \in \mathbb{R}^n,\end{cases}$$

have been intensively studied in the last few years. The interest in systems of nonlinear Schrödinger equations is motivated by applications to nonlinear optics. More precisely, coupled nonlinear Schrödinger systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical

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fibers and Kerr-like photorefractive media, see [1,17]. We also refer to [4,6,8,9,12, 13,16,18–21] and to the references therein for some recent results on the qualitative analysis of the solutions to systems of this type. Most of these results concern either the existence and the nonexistence of *bounded* positive solutions, or the existence and the nonexistence of *large* solutions. In the literature, a *large* solution means a couple (u, v) of positive smooth functions satisfying $(\mathcal{S})$ and such that both $u(x)$ and $v(x)$ tend to infinity as $|x| \to \infty$. We would like to quote some references in which systems of boundary blow-up solutions related to $(\mathcal{S})$ were analyzed. Lotka–Volterra type systems were considered in [10,11] (competitive type), [7] (predator-prey type) and [15] (cooperative type), while in [14] the objective was a competitive system not of Lotka–Volterra type.

Throughout this paper we assume that $F(x, u, v) = p(x)g(v)$, $G(x, u, v) = q(x)f(u)$, where p and q are smooth potentials, while the nonlinearities f and g are nonnegative Lipschitz continuous functions on each interval $[\varepsilon, \infty)$ (with $\varepsilon > 0$). In particular, this framework includes the sublinear case. In all the results, we establish in this paper we study only positive solutions, especially because of the physical meaning of the corresponding unknowns. One of our main purposes of this paper is to establish necessary and sufficient conditions for the existence of large solutions. The existence of bounded positive solutions is also studied in this paper, provided that f and g are nondecreasing and the Green potential of p and q are continuous and bounded in $\mathbb{R}^n$, $n \geq 3$. We recall that the Green potential of a nonnegative measurable function φ is defined on $\mathbb{R}^n$ by

$$V\varphi(x) = c_n \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}} dy, \quad \text{where } c_n = \frac{\Gamma\left(\frac{n}{2} - 1\right)}{4\pi^{\frac{n}{2}}}, \quad n \geq 3.$$

Moreover, $V\varphi$ is a lower semicontinuous function.

In order to discuss the existence of positive radial solutions to this class of nonlinear systems, we are concerned with the following system of nonlinear differential equations

$$(\mathcal{P}) \begin{cases} \dfrac{1}{A}(Au')' - p(t)g(v) = 0, \quad t \in (0, \infty), \\ \dfrac{1}{B}(Bv')' - q(t)f(u) = 0, \quad t \in (0, \infty), \\ u(0) = a > 0, \quad v(0) = b > 0, \\ \lim\limits_{t \to 0} A(t)u'(t) = 0, \quad \lim\limits_{t \to 0} B(t)v'(t) = 0, \end{cases}$$

where the continuous functions $A, B : [0, \infty) \to [0, \infty)$ are differentiable and positive on $(0, \infty)$ and satisfy the following growth hypotheses:

$$\int_0^1 \frac{1}{A(t)} \left(\int_0^t A(s) ds \right) dt < \infty$$

and

$$\int_0^1 \frac{1}{B(t)} \left(\int_0^t B(s)ds \right) dt < \infty.$$

In particular, these assumptions are fulfilled if A and B are nondecreasing.

Define the operators K and S on the set of nonnegative measurable functions on $[0, \infty)$ by

$$K\varphi(t) = \int_0^t \frac{1}{A(s)} \left(\int_0^s A(r)p(r)\varphi(r)dr \right) ds$$

and

$$S\varphi(t) = \int_0^t \frac{1}{B(s)} \left(\int_0^s B(r)q(r)\varphi(r)dr \right) ds.$$

By induction, it follows that for all $t \geq 0$ and $m \in \mathbb{N}$,

$$K^m 1(t) \leq \frac{[K1(t)]^m}{m!} \quad \text{and} \quad S^m 1(t) \leq \frac{[S1(t)]^m}{m!}, \tag{1.1}$$

where $K^j := K^{j-1} \circ K$ and $S^j := S^{j-1} \circ S$, for any integer $j \geq 2$. Indeed, put

$$h(t) = K1(t) = \int_0^t \left(\frac{1}{A(s)} \int_0^s A(r)p(r)dr \right) ds$$

and assume that (1.1) holds for some integer m. Then,

$$\begin{aligned} K^{m+1}1(t) = K(K^m 1)(t) &\leq \frac{1}{m!} K([h]^m)(t) \\ &= \frac{1}{m!} \int_0^t \frac{1}{A(s)} \left(\int_0^s A(r)p(r)[h(r)]^m dr \right) ds \\ &\leq \frac{1}{m!} \int_0^t [h(s)]^m \left(\frac{1}{A(s)} \int_0^s A(r)p(r)dr \right) ds \\ &= \frac{1}{m!} \int_0^t [h(s)]^m h'(s)ds = \frac{1}{(m+1)!} [h(t)]^{m+1}. \end{aligned}$$

Now, since $K1$ and $S1$ are nondecreasing, it follows that

$$(K \circ S)1(t) = K(S1)(t) = \int_0^t \frac{1}{A(s)} \left(\int_0^s A(r)p(r)S1(r)dr \right) ds \leq S1(t)K1(t)$$

and similarly

$$(S \circ K)1(t) \leq K1(t)S1(t).$$

Hence, by induction we obtain for each $t \geq 0$ and $m \in \mathbb{N}$,

$$(K \circ S)^m 1(t) \leq \frac{[K1(t)]^m}{m!} \frac{[S1(t)]^m}{m!} \quad \text{and} \quad (S \circ K)^m 1(t) \leq \frac{[K1(t)]^m}{m!} \frac{[S1(t)]^m}{m!}. \tag{1.2}$$

2. Main Results

We are first concerned with the existence and the uniqueness of a nonnegative solution of the system $(\mathcal{P})$. For this purpose, we assume that p, q, f, and g satisfy the following hypotheses.

(H_1) The functions $p, q, f, g : [0, \infty) \to [0, \infty)$ are continuous.
(H_2) For all $c > 0$, there exits $\beta > 0$ such that for all $x, y \in [c, \infty)$

$$|f(x) - f(y)| \leq \beta |x - y|$$

and

$$|g(x) - g(y)| \leq \beta |x - y|.$$

Remark 2.1. Under the hypotheses (H_1) and (H_2) there exist $\lambda, \mu > 0$ such that for each $x \geq 0$ we have

$$0 \leq f(x) \leq \lambda x + \mu,$$

and

$$0 \leq g(x) \leq \lambda x + \mu.$$

Our first existence result is the following.

Theorem 2.2. *Under the hypotheses* (H_1) *and* (H_2), *the problem* $(\mathcal{P})$ *has a unique solution* (u, v) *satisfying* $u, v \in C([0, \infty)) \cap C^1((0, \infty))$ *and* $u, v > 0$. *Moreover,*

$$\lambda a + \mu \leq \lambda u(t) + \mu \leq [\lambda a + \mu + (\lambda^2 b + \lambda\mu) K1(t)] \exp(\lambda^2 K(S1)(t))$$

and

$$\lambda b + \mu \leq \lambda v(t) + \mu \leq [\lambda b + \mu + (\lambda^2 a + \lambda\mu) S1(t)] \exp(\lambda^2 S(K1)(t)).$$

Next, we are concerned with the existence and the nonexistence of large or bounded solutions to the following system of nonlinear elliptic equations

$$(\mathcal{Q}) \begin{cases} \Delta u = p(x) g(v), & x \in \mathbb{R}^n \quad (n \geq 3) \\ \Delta v = q(x) f(u), & x \in \mathbb{R}^n. \end{cases}$$

Definition 2.3. Let $u, v : \mathbb{R}^n \to \mathbb{R}_+$ be continuous functions. We say that (u, v) is a large solution of the problem $(\mathcal{Q})$ if (u, v) satisfies $(\mathcal{Q})$ in the sense of distributions and $\lim_{|x| \to \infty} [u(x) + v(x)] = \infty$.

We establish the following necessary and sufficient condition for the existence of large solutions, under the following additional hypothesis:

(H_3) $\inf_{t \geq a} f(t) \geq \delta > 0$ and $\inf_{t \geq b} g(t) \geq \delta > 0$.

Theorem 2.4. *Let p and q be radial functions and assume that the hypotheses* (H_1)–(H_3) *are satisfied. Then, the problem* $(\mathcal{Q})$ *has a large solution on $\mathbb{R}^n$ if and only if*

$$\int_0^\infty r(p(r)+q(r))dr=\infty. \tag{2.1}$$

We point out that condition (2.1) is closely related to assumption (2) in [6].

The following result deals with the existence of a positive bounded solution for the problem $(\mathcal{Q})$. Assume that $p, q \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ are nonnegative functions and denote by Vp and Vq their Green potentials. We refer to [2, 3] for related applications of Green-type potentials to nonlinear PDEs.

Theorem 2.5. *Let f, g be nonnegative nondecreasing continuous functions in $[0,\infty)$. Assume that Vp and Vq are continuous, bounded in $\mathbb{R}^n$ and the following hypothesis is satisfied:*

(H_4) *there exist $a>0$, $b>0$ such that $a-g(b)\|Vp\|_\infty>0$ and $b-f(a)\|Vq\|_\infty >0$.*

Then, the system $(\mathcal{Q})$ has a positive bounded continuous solution (u,v) satisfying, for each $x\in\mathbb{R}^n$,

$$a-g(b)Vp(x)\le u(x)\le a \quad \text{and} \quad b-f(a)Vq(x)\le v(x)\le b.$$

In the last part of this paper, we establish a nonexistence result, provided that f and g satisfy

(H_5) $\forall\, a>0, \exists\, c>0$ such that $\forall\,\xi\in[0,a]$ we have $f(\xi)\ge c\xi$ and $g(\xi)\ge c\xi$.

Theorem 2.6. *Assume that p and q are two nonnegative continuous functions on $\mathbb{R}^n$ satisfying $\int_0^\infty r\min_{|x|=r}[p(x)+q(x)]dr=\infty$ and the functions f, g satisfy* (H_5). *Then, the system $(\mathcal{Q})$ has no positive bounded solution.*

3. Proof of Theorem 2.2

Let $(u_m)_{m\ge 0}$ and $(v_m)_{m\ge 0}$ be sequences of positive continuous functions defined on $[0,\infty)$ by

$$\begin{cases} u_0(t)=a,\\ v_m(t)=b+\displaystyle\int_0^t \frac{1}{B(s)}\left(\int_0^s B(r)q(r)f(u_m(r))dr\right)ds=b+S(f\circ u_m)(t),\\ u_{m+1}(t)=a+\displaystyle\int_0^t \frac{1}{A(s)}\left(\int_0^s A(r)p(r)g(v_m(r))dr\right)ds=a+K(g\circ v_m)(t). \end{cases}$$

Thus, for all $t\ge 0$ and $m\in\mathbb{N}$, $u_m(t)\ge a$ and $v_m(t)\ge b$. Now, since $\min(a,b)>0$, then by (H_2) there exists $\beta>0$ such that for all $t\ge 0$

$$\begin{cases} |f(u_{m+1}(t))-f(u_m(t))|\le\beta|u_{m+1}(t)-u_m(t)|,\\ |g(v_{m+1}(t))-g(v_m(t))|\le\beta|v_{m+1}(t)-v_m(t)|. \end{cases}$$

Let $c = \max(\lambda b + \mu, \lambda f(a))$. Then, using (1.1) and (1.2), we show by induction that

$$|u_{m+1}(t) - u_m(t)| \leq c\beta^{2m}\left[\frac{[K1(t)]^{(m+1)}}{(m+1)!}\frac{[S1(t)]^m}{m!} + \frac{[K1(t)]^{(m+1)}}{(m+1)!}\frac{[S1(t)]^{(m+1)}}{(m+1)!}\right],$$

and

$$\begin{aligned}&|v_{m+1}(t) - v_m(t)| \\ &\quad \leq c\beta^{2m+1}\left[\frac{[K1(t)]^{(m+1)}}{(m+1)!}\frac{[S1(t)]^{(m+1)}}{(m+1)!} + \frac{[K1(t)]^{(m+1)}}{(m+1)!}\frac{[S1(t)]^{(m+2)}}{(m+2)!}\right].\end{aligned}$$

Therefore, the sequences $(u_m)_{m\geq 0}$ and $(v_m)_{m\geq 0}$ converge locally uniformly to functions u and v that satisfy for each $t \geq 0$,

$$u(t) = a + \int_0^t \frac{1}{A(s)}\left(\int_0^s A(r)p(r)g(v(r))dr\right)ds$$

$$v(t) = b + \int_0^t \frac{1}{B(s)}\left(\int_0^s B(r)q(r)f(u(r))dr\right)ds.$$

Hence, $u, v \in C([0,\infty)) \cap C^1((0,\infty))$ and (u, v) is a solution of $(\mathcal{P})$.

Now, we prove the uniqueness of the solution. Indeed, let (u, v) and $(\widetilde{u}, \widetilde{v})$ be two solutions of the problem $(\mathcal{P})$. Then, for each $R \in (0,\infty)$ and $t \in [0, R]$, we have

$$|u(t) - \widetilde{u}(t)| \leq \beta K(|v - \widetilde{v}|)(t) \quad \text{and} \quad |v(t) - \widetilde{v}(t)| \leq \beta S(|u - \widetilde{u}|)(t).$$

By induction, we deduce that for each $m \geq 0$,

$$|u(t) - \widetilde{u}(t)| \leq \beta^{2m}(K \circ S)^m(|u - \widetilde{u}|)(t).$$

Since $K \circ S$ is a nondecreasing operator, it follows from (1.2) that for each $m \geq 0$

$$\begin{aligned}|u(t) - \widetilde{u}(t)| &\leq \beta^{2m}(K \circ S)^m 1(R) \sup_{r\in[0,R]}(|u(r) - \widetilde{u}(r)|, \\ &\leq \frac{[\beta^2 K1(R)S1(R)]^m}{(m!)^2} \sup_{r\in[0,R]}(|u(r) - \widetilde{u}(r)|).\end{aligned}$$

Now, letting $m \to \infty$, we deduce that $|u(t) - \widetilde{u}(t)| = 0$, for all $t \in [0, R]$. So $u = \widetilde{u}$ on $[0,\infty)$ and $v = \widetilde{v}$ on $[0,\infty)$.

Finally, we obtain for each $t > 0$

$$\begin{aligned}u'(t) &= \frac{1}{A(t)}\int_0^t A(r)p(r)g(v(r))dr \\ &\leq \frac{1}{A(t)}\int_0^t A(r)p(r)(\lambda v(r) + \mu)dr \\ &\leq \frac{\lambda}{A(t)}\int_0^t A(r)p(r)(b + S \circ f(u(r)))dr + \mu(K1)'(t) \\ &\leq \frac{\lambda(\lambda u(t) + \mu)}{A(t)}\int_0^t A(r)p(r)S1(r)dr + (\lambda b + \mu)(K1)'(t) \\ &\leq \lambda^2 u(t)K(S1)'(t) + \lambda\mu K(S1)'(t) + (\lambda b + \mu)(K1)'(t).\end{aligned}$$

Hence,

$$u'(t) - \lambda^2 u(t) K(S1)'(t) \le \lambda\mu K(S1)'(t) + (\lambda b + \mu)(K1)'(t).$$

For $t \ge 0$, let $b(t) = \lambda^2 K(S1)'(t)$ and $c(t) = \lambda\mu K(S1)'(t) + (\lambda b + \mu)(K1)'(t)$. Then,

$$u'(t) - b(t)u(t) \le c(t).$$

It follows that

$$\left[u(t)\exp\left(-\int_0^t b(s)ds\right)\right]' \le c(t)\exp\left(-\int_0^t b(s)ds\right)$$

and consequently

$$u(t)\exp\left(-\int_0^t b(s)ds\right) - a \le \int_0^t c(r)\exp\left(-\int_0^r b(s)ds\right) dr.$$

Therefore,

$$\begin{aligned} u(t) &\le \left[a + \int_0^t c(r)\exp\left(-\int_0^r b(s)ds\right) dr\right]\exp\left(\int_0^t b(s)ds\right) \\ &\le \left[a + \int_0^t (\lambda\mu K(S1)'(r) + (\lambda b + \mu)(K1)'(r))\exp(-\lambda^2(K(S1)(r)))dr\right] \\ &\quad \times \exp(\lambda^2 K(S1)(t)) \\ &\le \left[a + \frac{\mu}{\lambda}(1 - \exp(-\lambda^2 K(S1)(t))) + (\lambda b + \mu)K1(t)\right]\exp(\lambda^2 K(S1)(t)). \end{aligned}$$

Finally, we obtain

$$\lambda a + \mu \le \lambda u(t) + \mu \le [\lambda a + \mu + (\lambda^2 b + \lambda\mu)K1(t)]\exp(\lambda^2 K(S1)(t)).$$

Similarly, we prove that

$$\lambda b + \mu \le \lambda v(t) + \mu \le [\lambda b + \mu + (\lambda^2 a + \lambda\mu)S1(t)]\exp(\lambda^2 S(K1)(t)).$$

This completes the proof of Theorem 2.2. □

Corollary 3.1. (1) *Assume that*

$$\int_0^\infty \left(\frac{1}{A(s)}\int_0^s A(r)p(r)dr\right) ds < \infty$$

and

$$\int_0^\infty \left(\frac{1}{B(s)}\int_0^s B(r)q(r)dr\right) ds < \infty.$$

Then, u and v are bounded.

(2) *Assume, moreover, that condition* (H$_3$) *holds. Then, for each $t \ge 0$ we have $u(t) \ge a + \delta K1(t)$ and $v(t) \ge b + \delta S1(t)$. In particular, u and v are bounded if and only if*

$$\int_0^\infty \left(\frac{1}{A(s)}\int_0^s A(r)p(r)dr\right) ds < \infty$$

and

$$\int_0^\infty \left(\frac{1}{B(s)} \int_0^s B(r)q(r)dr \right) ds < \infty.$$

Example 3.1. For $\gamma \geq 0$ and $\nu \geq 0$, we consider the nonnegative functions A and B defined on $[0,\infty)$ by

$$A(t) = t^\gamma \quad \text{and} \quad B(t) = t^\nu.$$

Let $p, q : [0,\infty) \to [0,\infty)$ be two continuous nonnegative functions and let $\delta, \theta \in [0,1]$. Then, the following problem

$$\begin{cases} u''(t) + \dfrac{\gamma}{t} u'(t) = p(t) v^\delta(t), \\ v''(t) + \dfrac{\nu}{t} v'(t) = q(t) u^\theta(t), \\ u(0) = a > 0, \quad v(0) = b > 0, \\ u'(0) = v'(0) = 0, \end{cases}$$

has a unique positive solution (u,v) with $u, v \in C([0,\infty)) \cap C^2((0,\infty))$. Moreover, u and v are bounded if and only if $\gamma > 1$, $\nu > 1$, $\int_0^\infty tp(t)dt < \infty$ and $\int_0^\infty tq(t)dt < \infty$.

Corollary 3.2. *Let* p, q, f *and* g *satisfying* (H_1)–(H_2) *and let* $\rho, \theta : (0,\infty) \to \mathbb{R}$ *be two continuous functions. Then, the following problem*

$$(*) \begin{cases} \Delta u + \rho(|x|) x \cdot \nabla u = p(|x|) g(v), \quad x \in \mathbb{R}^n \quad (n \geq 3), \\ \Delta v + \theta(|x|) x \cdot \nabla v = q(|x|) f(u), \quad x \in \mathbb{R}^n, \end{cases}$$

has infinitely many positive radial solutions (u,v).

Proof. Let u and v be two radial functions. Then, (u,v) is a solution of $(*)$ if and only if

$$\begin{cases} u''(r) + \left[\dfrac{n-1}{r} + r\rho(r) \right] u'(r) = p(r) g(v(r)) \quad r > 0 \\ v''(t) + \left[\dfrac{n-1}{r} + r\theta(r) \right] v'(r) = q(r) f(u(r)) \quad r > 0 \end{cases}$$

or, equivalently,

$$\begin{cases} \dfrac{1}{A}(Au')' = p(r) g(v), \quad r > 0 \\ \dfrac{1}{B}(Bv')' = q(r) f(u), \quad r > 0, \end{cases}$$

where $A(r) = r^{n-1} \exp(\int_1^r s\rho(s)ds)$ and $B(r) = r^{n-1} \exp(\int_1^r s\theta(s)ds)$. So, the result follows from Theorem 2.2. $\square$

4. Large and Bounded Solutions

4.1. *Proof of Theorem 2.4*

If p and q satisfy the condition (2.1), then using Theorem 2.2 and Corollary 3.1 for $A(t) = B(t) = t^{n-1}$, we obtain a large radial solution for $(\mathcal{Q})$.

Now, we assume that

$$\int_0^\infty rp(r)dr < \infty \quad \text{and} \quad \int_0^\infty rq(r)dr < \infty$$

and let (u, v) be a solution for the problem $(\mathcal{Q})$. We define the functions $\tilde{u}$ and $\tilde{v}$ on $(0, \infty)$ by

$$\tilde{u}(t) = \int_{S^{n-1}} u(tw)d\sigma(w) \quad \text{and} \quad \tilde{v}(t) = \int_{S^{n-1}} v(tw)d\sigma(w),$$

where S^{n-1} is the united sphere in $\mathbb{R}^n$ and σ denotes the Lebesgue measure on S^{n-1}. Then, for all $t > 0$,

$$\begin{aligned}
\Delta\tilde{u}(t) &= \frac{1}{t^{n-1}}(t^{n-1}\tilde{u}')' = \int_{S^{n-1}} \Delta u(tw)d\sigma(w) = p(t)\int_{S^{n-1}} g(v(tw))d\sigma(w)\\
\Delta\tilde{v}(t) &= \frac{1}{t^{n-1}}(t^{n-1}\tilde{v}')' = \int_{S^{n-1}} \Delta v(tw)d\sigma(w) = q(t)\int_{S^{n-1}} f(u(tw))d\sigma(w)\\
\tilde{u}'(t) &= \frac{1}{t^{n-1}}\int_{B(0,t)} \Delta u(x)dx = \frac{1}{t^{n-1}}\int_{B(0,t)} p(|x|)g(v(x))dx \geq 0\\
\tilde{v}'(t) &= \frac{1}{t^{n-1}}\int_{B(0,t)} \Delta v(x)dx = \frac{1}{t^{n-1}}\int_{B(0,t)} q(|x|)f(u(x))dx \geq 0.
\end{aligned}$$

Thus, $\tilde{u}$ and $\tilde{v}$ are nondecreasing. Using now hypothesis (H_2), we obtain

$$\begin{aligned}
\tilde{u}'(t) = \frac{1}{t^{n-1}}\int_{B(0,t)} \Delta u(x)dx &\leq [\lambda\tilde{v}(t) + \mu]\frac{1}{t^{n-1}}\int_0^t r^{n-1}p(r)dr\\
&\leq [\lambda(\tilde{u}(t) + \tilde{v}(t)) + \mu]\frac{1}{t^{n-1}}\int_0^t r^{n-1}p(r)dr
\end{aligned}$$

and

$$\begin{aligned}
\tilde{v}'(t) = \frac{1}{t^{n-1}}\int_{B(0,t)} \Delta v(x)dx &\leq [\lambda\tilde{u}(t) + \mu]\frac{1}{t^{n-1}}\int_0^t r^{n-1}q(r)dr\\
&\leq [\lambda(\tilde{u}(t) + \tilde{v}(t)) + \mu]\frac{1}{t^{n-1}}\int_0^t r^{n-1}q(r)dr.
\end{aligned}$$

Hence,

$$\frac{\tilde{u}'(t) + \tilde{v}'(t)}{\lambda(\tilde{u}(t) + \tilde{v}(t)) + \mu} \leq \frac{1}{t^{n-1}}\int_0^t r^{n-1}(p(r) + q(r))dr.$$

Consequently,

$$\lambda(\tilde{u}(t)+\tilde{v}(t))+\mu \leq [\lambda(\tilde{u}(0)+\tilde{v}(0))+\mu]\exp\left(\frac{\lambda}{n-2}\int_0^\infty r(p(r)+q(r))dr\right).$$

This implies that $\tilde{u}+\tilde{v}$ is bounded and consequently (u,v) is not a large solution. This completes the proof. □

4.2. *Proof of Theorem 2.5*

We start with the following auxiliary result.

Lemma 4.1. *Let φ and ψ be two measurable functions such that $0 \leq \varphi \leq \psi$. If $V\psi$ is continuous then $V\varphi$ is also continuous.*

Proof. Let $\theta \geq 0$ such that $\psi = \varphi+\theta$. Since $V\varphi$ and $V\theta$ are lower semi-continuous and $V\varphi+V\theta = V\psi$ which is continuous, then $V\varphi$ is an upper and lower semi-continuous function. So, $V\varphi$ is a continuous function. □

We consider the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ defined by

$$\begin{cases} u_0 = a - V(pg(b)), \\ u_{k+1} = a - V(pg(v_k)), \\ v_k = b - V(qf(u_k)). \end{cases}$$

We claim that for each $k \geq 0$,

$$a - g(b)Vp \leq u_k \leq u_{k+1} \leq a \quad \text{and} \quad b - f(a)Vq \leq v_{k+1} \leq v_k \leq b.$$

Indeed, from hypothesis (H_4), we have $0 < u_0 = a - g(b)Vp \leq a$. So

$$b \geq v_0 = b - V(qf(u_0)) \geq b - V(qf(a)) > 0$$

$$u_1 - u_0 = -V[p(g(v_0) - g(b))] \geq 0$$

and

$$v_1 - v_0 = -V[q(f(u_1) - f(u_0))] \leq 0.$$

Let us assume that the claim holds for some $k \in \mathbb{N}$. Then, we have

$$v_{k+2} - v_{k+1} = -V[q(f(u_{k+1}) - f(u_k))] \leq 0$$

and

$$u_{k+2} - u_{k+1} = -V[p(g(v_{k+1}) - g(v_k))] \geq 0.$$

Moreover, for all integer k we have $u_k \leq a$, $v_k \leq b$, $u_{k+1} = a - V(pg(v_k)) \geq a - g(b)Vp$ and $v_{k+1} = b - V(qf(u_{k+1})) \geq b - f(a)Vq$. This completes the proof of the claim. Therefore, the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ converge, respectively, to two functions u and v satisfying $0 < a - g(b)Vp \leq u \leq a$ and $0 < b - f(a)Vq \leq v \leq b$.

By the dominated convergence theorem we deduce that u and v satisfies

$$\begin{cases} u = a - V(pg(v)) \\ v = b - V(qf(u)). \end{cases} \tag{4.1}$$

Now, using the fact that Vp and Vq are continuous and $V(pg(v)) \leq g(b)Vp, V(qf(u)) \leq f(a)Vq$, we deduce from Lemma 4.1 that u and v are continuous. Finally, we deduce from relation (4.1) that (u, v) is a weak bounded positive solution of problem $(\mathcal{Q})$. This completes the proof of Theorem 2.5.

Remark 4.2. The condition (H_4) is satisfied in the particular case where $f(t) = t^\alpha$, $g(t) = t^\beta$ with $0 \leq \alpha, \beta$ and $\alpha\beta \neq 1$. Hence, Theorem 2.5 generalizes Theorem 1 in [21].

Remark 4.3. Let p be a nonnegative function in $L^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $m > \frac{n}{2} \geq \frac{3}{2}$. Then, it follows from [5, pp. 64–66] that Vp is continuous in $\mathbb{R}^n$ and tends to zero at infinity.

For the next result we fix two nonnegative functions $p, q \in L^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $m > \frac{n}{2} \geq \frac{3}{2}$ and $a > 0$, $b > 0$. Put $\lambda_0 = \frac{a}{g(b)\|Vp\|_\infty}$ and $\mu_0 = \frac{b}{f(a)\|Vq\|_\infty}$.

Corollary 4.4. *Let f, g be two nonnegative nondecreasing continuous functions and $a, b > 0$. Then, for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$, the nonlinear elliptic system*

$$\begin{cases} \Delta u = \lambda p(x) g(v), \\ \Delta v = \mu q(x) f(u), \\ \lim\limits_{x \to \infty} u(x) = a, \\ \lim\limits_{x \to \infty} v(x) = b \end{cases}$$

has a positive bounded continuous solution (u, v) satisfying

$$a\left(1 - \frac{\lambda}{\lambda_0}\right) \leq u \leq a \quad \textit{and} \quad b\left(1 - \frac{\mu}{\mu_0}\right) \leq v \leq b.$$

Example 4.1. Let p, q be two nonnegative functions in $L^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $m > \frac{n}{2} \geq \frac{3}{2}$ and $a > 0$, $b > 0$. Let $\alpha \geq 0$ and $\beta \geq 0$. Then, there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$, the nonlinear elliptic system

$$\begin{cases} \Delta u = \lambda p(x) v^\beta, \\ \Delta v = \mu q(x) u^\alpha, \\ \lim\limits_{x \to \infty} u(x) = a, \\ \lim\limits_{x \to \infty} v(x) = b, \end{cases}$$

has a positive bounded solution (u, v).

5. A Nonexistence Result

In this section we are concerned with the proof of Theorem 2.6.

Let $M > 0$ and assume that (u, v) is a solution of $(\mathcal{Q})$ with $0 < u \le M$ and $0 < v \le M$. Let $\tilde{u}$ and $\tilde{v}$ be as in the proof of Theorem 2.4. Then, for each $t > 0$ we have

$$\begin{aligned}\tilde{u}'(t) &= \frac{1}{t^{n-1}}\int_{B(0,t)} p(x)g(v(x))dx \\ &= \frac{1}{t^{n-1}}\int_0^t s^{n-1}\int_{S^{n-1}} p(sw)g(v(sw))d\sigma(w)ds \ge 0 \end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\tilde{v}'(t) &= \frac{1}{t^{n-1}}\int_{B(0,t)} q(x)f(u(x))dx \\ &= \frac{1}{t^{n-1}}\int_0^t s^{n-1}\int_{S^{n-1}} q(sw)f(u(sw))d\sigma(w)ds \ge 0. \end{aligned} \tag{5.2}$$

Since $\tilde{u}$ and $\tilde{v}$ are nondecreasing, there exist $R > 0$ and $\varepsilon > 0$ such that $\tilde{u}(t) \ge \varepsilon$ and $\tilde{v}(t) \ge \varepsilon$ $(\forall\, t \ge R)$. Using relations (5.1), (5.2) and hypothesis (H_5), we obtain for all $t \ge R$,

$$\begin{aligned} M \ge \tilde{u}(t) &= \tilde{u}(0) + \int_0^t r^{1-n}\int_0^r s^{n-1}\int_{S^{n-1}} p(sw)g(v(sw))d\sigma(w)dsdr \\ &\ge \tilde{u}(0) + \int_0^t r^{1-n}\int_0^r s^{n-1}\left(\min_{|x|=s} p(x)\right)\int_{S^{n-1}} g(v(sw))d\sigma(w)dsdr \\ &\ge \tilde{u}(0) + c\int_0^t\int_0^r s^{n-1}\left(\min_{|x|=s} p(x)\right)\tilde{v}(s)dsdr \\ &\ge \tilde{u}(0) + c\varepsilon\int_R^t r^{1-n}\int_R^r s^{n-1}\left(\min_{|x|=s} p(x)\right)dsdr \\ &\ge \tilde{u}(0) + c\varepsilon\int_R^t s^{n-1}\left(\min_{|x|=s} p(x)\right)\left(\int_s^t r^{1-n}dr\right)ds. \end{aligned}$$

Similarly, we prove that

$$M \ge \tilde{v}(t) \ge \tilde{v}(0) + c\varepsilon\int_R^t s^{n-1}\left(\min_{|x|=s} q(x)\right)\left(\int_s^t r^{1-n}dr\right)ds.$$

Consequently,

$$2M \ge \tilde{u}(t) + \tilde{v}(t) \ge \tilde{u}(0) + \tilde{v}(0) + c\varepsilon\int_R^t s^{n-1}\min_{|x|=s}[p(x)+q(x)]\left(\int_s^t r^{1-n}dr\right)ds.$$

Or, from the hypothesis on p and q the right-hand side of this inequality tends to infinity as $t \to \infty$. This yields to a contradiction and achieves the proof. $\square$

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