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ROBIN PROBLEMS WITH INDEFINITE AND UNBOUNDED POTENTIAL, RESONANT AT $-\infty$, SUPERLINEAR AT $+\infty$

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Abstract. We consider a semilinear Robin problem with an indefinite and unbounded potential and a reaction which exhibits asymmetric behavior as $x \to \pm\infty$. More precisely it is sublinear near $-\infty$ with possible resonance with respect to the principal eigenvalue of the negative Robin Laplacian and it is superlinear at $+\infty$. Resonance is also allowed at zero with respect to any nonprincipal eigenvalue. We prove two multiplicity results. In the first one, we obtain two nontrivial solutions and in the second, under stronger regularity conditions on the reaction, we produce three nontrivial solutions. Our work generalizes the recent one by Recova-Rumbos (Nonlin. Anal. 112 (2015), 181–198).

1. Introduction. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem:

$$
\left\{
\begin{array}{ll}
-\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega\,, \\
\dfrac{\partial u}{\partial n} + \beta(z)u(z) = 0 & \text{on } \partial\Omega\,.
\end{array}
\right.
\tag{1}
$$

In (1) the potential function $\xi(\cdot)$ is indefinite (that is, sign changing) and unbounded. The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \longmapsto f(z, x)$ is measurable and $x \longmapsto f(z, x)$ is continuous for a.a. $z \in \Omega$), which exhibits asymmetric behavior as $x \to \pm\infty$. More precisely, we assume that $f(z, \cdot)$ is sublinear as $x \to -\infty$ allowing for resonance to occur with respect to the principal eigenvalue $\hat{\lambda}_1$ of $u \longmapsto -\Delta u + \xi(z)u$ with Robin boundary condition, while $f(z, \cdot)$ is superlinear as $x \to +\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (AR-condition for short). Instead we employ a weaker condition which incorporates in our framework also superlinear reactions with slower growth near $+\infty$.

Our goal is to prove multiplicity theorems for problem (1). We have two such results. In the first theorem we produce two nontrivial solutions. In the second result, by strengthening the regularity on $f(z, \cdot)$ (we assume that for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$), we produce three nontrivial solutions. Our approach combines variational tools (critical point theory) with elements from Morse theory (critical groups).

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The starting point for this paper has been a recent work of Recova and Rumbos [24], who examined a similar asymmetric semilinear Dirichlet problem with no potential (that is, $\xi \equiv 0$) and with a reaction belonging in $C^1(\overline{\Omega} \times \mathbb{R})$ and satisfying more restrictive hypotheses (see hypotheses $(L_1)-(L_7)$ of [24]) and in particular for the superlinearity in the positive direction, they employ the classical in such cases AR-condition (see hypotheses (L_3) and (L_4) in [24]). In the past, there have been some other works on such asymmetric problems. We mention the papers of Arcoya and Villegas [3], de Figueiredo and Ruf [8], Motreanu, Motreanu and Papageorgiou [13] and Perera [23]. All these works consider equations with zero potential (that is, $\xi \equiv 0$) and the unilateral superlinearity of the reaction is expressed in terms of the AR-condition. In Arcoya and Villegas [3] and de Figueiredo and Ruf [8] the reaction is jointly continuous, in Perera [23] it is jointly C^1 and in Motreanu, Motreanu and Papageorgiou [13] it is assumed that $f(z,\cdot) \in C^1(\mathbb{R})$. Note that in de Figueiredo and Ruf [8] and Perera [23] it is assumed that $N = 1$ (that is, the boundary value problem is an ordinary differential equation). We mention that Arcoya and Villegas [3] and de Figueiredo and Ruf [8] prove only existence theorems, while Motreanu, Motreanu and Papageorgiou [13] and Perera [23] obtain multiplicity results producing two nontrivial solutions. Motreanu, Motreanu and Papageorgiou [13] consider Dirichlet problems, while the others deal with Neumann boundary conditions. We mention that our formulation here covers also the Neumann case (when $\beta \equiv 0$).

Finally we mention that semilinear problems with indefinite and unbounded potential were studied recently by Kyritsi and Papageorgiou [10], Papageorgiou and Papalini [17] (Dirichlet problems) and Papageorgiou and Rădulescu [18], [20], Papageorgiou and Smyrlis [22] (Neumann problems).

2. Mathematical background. Let X be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness condition on the functional φ. So, the necessary compactness requirement is transferred from the space X, which is not in general locally compact (being infinite dimensional) to the functional φ. The C-condition is more general than the more common Palais-Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem", due to Ambrosetti and Rabinowitz [2]. Here we state the result in a slightly more general form (see, for example, Gasinski and Papageorgiou [9, p. 648]).

THEOREM 1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = r] = m_r$$

and $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geqslant m_r$ and c is a critical value of φ.

The function spaces which we will use in the study of problem (1) are

- the Sobolev space $H^1(\Omega)$;
- the Banach space $C^1(\overline{\Omega})$;
- the "boundary" Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$.

By $\|\cdot\|$ we will denote the norm of the Sobolev space $H^1(\Omega)$ defined by

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \text{ for all } u \in H^1(\Omega)\,.$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior which includes

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

The Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$ are defined as follows. On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Having this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$. From the trace theorem, we know that there exists a unique linear continuous map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map" such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in C^1(\overline{\Omega})\,.$$

So, the trace map gives meaning to the notion of boundary values of a Sobolev function. We know that γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in \left[1, \frac{2(N-1)}{N-2}\right)$ if $N \geqslant 3$ and for all $q \geqslant 1$ if $N = 1, 2$. Moreover, we have

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \text{ and } \ker\gamma_0 = H_0^1(\Omega)\,.$$

In the rest of this paper, for the sake of notational simplicity, we drop the use of the trace map γ_0. All the restrictions on $\partial\Omega$ of functions from $H^1(\Omega)$ are understood in the sense of traces.

For $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in H^1(\Omega)$, we define

$$u^{\pm}(\cdot) = u(\cdot)^{\pm}\,.$$

We know that

$$u^{\pm} \in H^1(\Omega) \text{ and } u = u^+ - u^-,\ |u| = u^+ + u^-\,.$$

Also by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in H^1(\Omega)$$

(the Nemytski or superposition operator corresponding to h).

By 2^* we denote the critical Sobolev exponent defined by

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geqslant 3 \\ +\infty, & \text{if } N = 1, 2\,. \end{cases}$$

We impose the following conditions on the potential function $\xi(\cdot)$ and on the boundary weight $\beta(\cdot)$.

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geqslant 0$.

Evidently if $\beta \equiv 0$, then (1) reduces to the Neumann problem.

We introduce the C^1-functional $\vartheta : H^1(\Omega) \to \mathbb{R}$ defined by

$$\vartheta(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2\,d\sigma \text{ for all } u \in H^1(\Omega)\,.$$

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_0(z, x)| \leqslant a_0(z)(1 + |x|^{q-1}) \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R}\,,$$

with $a_0 \in L^\infty(\Omega)$ and $q \in (1, 2^*)$. We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and consider the C^1-functional $\varphi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\vartheta(u) - \int_\Omega F_0(z, u)\,dz \text{ for all } u \in H^1(\Omega)\,.$$

As in Papageorgiou and Rădulescu [19], using the regularity result of Wang [26], we have the following property.

PROPOSITION 2. *Assume that hypotheses $H(\xi)$, $H(\beta)$ hold and $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leqslant \rho_0\,.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is also a local $H^1(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \textit{ for all } h \in H^1(\Omega) \textit{ with } \|h\| \leqslant \rho_1\,.$$

We will also need the spectrum of $u \longmapsto -\Delta u + \xi(z)u$ with the Robin boundary condition. So, we consider the following linear eigenvalue problem

$$\text{(2)} \qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda} u(z) & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

This eigenvalue problem for $\beta \equiv 0$ (Neumann problem) was studied in Papageorgiou and Rădulescu [18] and Papageorgiou and Smyrlis [22] and for the p-Laplacian by Mugnai and Papageorgiou [16]. For the Robin problem with no potential (that is, $\xi \equiv 0$), we have the work of Papageorgiou and Rădulescu [19] and for the p-Laplacian we refer to Papageorgiou and Rădulescu [19], [21]. An analogous study can be conducted for problem (2) and leads to similar results.

So, problem (2) has a smallest eigenvalue $\hat{\lambda}_1 > -\infty$, which is given by

$$\text{(3)} \qquad \hat{\lambda}_1 = \inf\left[\frac{\vartheta(u)}{\|u\|_2^2} : u \in H^1(\Omega),\ u \not\equiv 0\right].$$

We can find $\hat{\xi} > \max\{-\hat{\lambda}_1, 1\}$ such that

$$\text{(4)} \qquad \vartheta(u) + \hat{\xi}\|u\|_2^2 \geqslant c_0\|u\|^2 \ \text{ for all } u \in H^1(\Omega) \text{ and some } c_0 > 0\,.$$

Using (4) and the spectral theory of compact self-adjoint linear operators, exactly as in [18] and [22], we can generate the spectrum of (2), which consists of a sequence $\{\hat{\lambda}_k\}_{k\geqslant 1}$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$ as $k \to \infty$. Also, there is a corresponding sequence $\{\hat{u}_k\}_{k\geqslant 1} \subseteq C^1(\overline{\Omega})$ of eigenfunctions which form an orthogonal basis of $H^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. By $E(\hat{\lambda}_k)$ $k \geqslant 1$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. We have that $E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega})$ and it is finite dimensional. Moreover, from de Figueiredo and Gossez [7], we know that $E(\hat{\lambda}_m)$ has the “unique continuation property” (the UCP for short), that is, if $u \in E(\hat{\lambda}_m)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$.

We have

$$H^1(\Omega) = \overline{\bigoplus_{k\geqslant 1} E(\hat{\lambda}_k)}\,.$$

Using these eigenspaces, we can obtain the classical variational characterizations of $\hat{\lambda}_m$ for $m \geqslant 2$, namely

$$\begin{aligned} \hat{\lambda}_m &= \inf\left[\frac{\vartheta(u)}{\|u\|_2^2} : u \in \widehat{H}_m = \overline{\bigoplus_{k\geqslant m} E(\hat{\lambda}_k)},\ u \neq 0\right] \\ \text{(5)} \qquad &= \sup\left[\frac{\vartheta(u)}{\|u\|_2^2} : u \in \overline{H}_m = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k),\ u \neq 0\right]. \end{aligned}$$

The infimum in (3) and both the infimum and the supremum in (5), are realized on the corresponding eigenspace $E(\hat{\lambda}_m)$ $m \geqslant 1$. Using Picone’s identity (see, for example, Gasinski and Papageorgiou [9, p. 785]), we show that $\hat{\lambda}_1$ is simple. Moreover, from (3) it is clear that the elements of $E(\hat{\lambda}_1)$ do not change sign. Let $\hat{u}_1 \in H^1(\Omega)$ be the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1$. We have $\hat{u}_1 \in C_+ \setminus \{0\}$. Moreover, using hypothesis $H(\xi)$ and the maximum principle, we have $\hat{u}_1 \in \operatorname{int} C_+$. As in Gasinski and Papageorgiou [9, p. 743], via Picone's identity, we show that every nonprincipal eigenvalue $\hat{\lambda}_m$ $(m \geqslant 2)$ has nodal (sign changing) eigenfunctions.

From (3), (5) and the UCP, we derive easily the following useful inequalities.

PROPOSITION 3. *(a) If $\eta \in L^\infty(\Omega)$ and $\eta(z) \leqslant \hat{\lambda}_m$ for a.a. $z \in \Omega$ with $m \geqslant 1$ and the inequality is strict on a set of positive measure, then $\vartheta(u) - \int_\Omega \eta(z)u^2\,dz \geqslant c_1\|u\|^2$ for some $c_1 > 0$, all $u \in \widehat{H}_m = \overline{\bigoplus_{k\geqslant m} E(\hat{\lambda}_k)}$.*
(b) If $\eta \in L^\infty(\Omega)$ and $\eta(z) \geqslant \hat{\lambda}_m$ for a.a. $z \in \Omega$ with $m \geqslant 1$ and the inequality is strict on a set of positive measure, then $\vartheta(u) - \int_\Omega \eta(z)u^2\,dz \leqslant -c_2\|u\|^2$ for some $c_2 > 0$, all $u \in \overline{H}_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k)$.

We will also use the following weighted linear eigenvalue problem

$$-\Delta u(z) = \tilde{\lambda} m(z)u(z) \text{ in } \Omega,\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega\,. \tag{6}$$

Here $m \in L^s(\Omega)$ with $s > N$ and is in general indefinite. As in de Figueiredo [6] (see also Gasinski and Papageorgiou [9, Section 6.1]), we show that (6) has in general a double sequence of eigenvalues $\{(\hat{\lambda}_\pm)_k\}_{k\geqslant 1}$ such that

$$-\infty \leftarrow \hat{\lambda}_{-k} < \cdots < \hat{\lambda}_{-2} < \hat{\lambda}_{-1} \leqslant 0 \leqslant \hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_k \to \infty \text{ as } k \to \infty\,.$$

If $m^+ = 0$ there are no positive eigenvalues, while if $m^- = 0$, there are no negative eigenvalues. If $m^+ \neq 0$ (resp. $m^- \neq 0$), then $\hat{\lambda}_1$ (resp. $\hat{\lambda}_{-1}$) is simple. Moreover, we have

$$\tilde{\lambda}_1 = \inf\left[\|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2\,d\sigma :\ u \in H^1(\Omega),\ \int_\Omega m(z)u^2\,dz = 1\right] \tag{7}$$

and

$$\tilde{\lambda}_{-1} = \inf\left[\|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2\,d\sigma :\ u \in H^1(\Omega),\ \int_\Omega m(z)u^2\,dz = -1\right] \tag{8}$$

with both infima realized on the corresponding one dimensional eigenspace.

Next let us recall some basic definitions and facts from Morse theory (critical groups). For details we refer to Motreanu, Motreanu and Papageorgiou [14].

So, let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X :\ \varphi'(u) = 0\},\ K_\varphi^c = \{u \in K_\varphi :\ \varphi(u) = c\} \text{ and } \varphi^c = \{u \in X :\ \varphi(u) \leqslant c\}\,.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, we denote by $H_k(Y_1, Y_2)$ the kth-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Recall that for every integer $k < 0$, we have $H_k(Y_1, Y_2) = 0$. Let $u_0 \in K_\varphi^c$ be an isolated critical point. The critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{u_0\}) \text{ for every integer } k \geqslant 0\,.$$

Here U is an open neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for every integer } k \geqslant 0\,.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [9, p. 628]) implies that this definition of critical groups at infinity, is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that K_φ is finite. We introduce the following quantities

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \text{ for all } t \in \mathbb{R},\ \text{all } u \in K_\varphi\,,$$

$$P(t, \infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}\,.$$

The *Morse relation* says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t)\,, \tag{9}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Suppose that $X = Y \oplus V$ with $\dim Y < \infty$ and $\varphi \in C^1(X, \mathbb{R})$. We say that φ has a local linking at $u = 0$, if there exists $r > 0$ such that

$$\begin{aligned} &\varphi(y) \leqslant 0 \ \text{ for all } y \in Y,\ \|y\| \leqslant r \\ &\varphi(v) > 0 \ \text{ for all } v \in V,\ 0 < \|v\| \leqslant r\,. \end{aligned}$$

In this case we have

$$C_k(\varphi, 0) \neq 0 \ \text{ for } k = \dim Y < \infty\,.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and for some integer $k \geqslant 0$, we have

$$C_k(\varphi, 0) \neq 0 \ \text{ and } \ C_k(\varphi, \infty) = 0\,,$$

then there exists $u \in K_\varphi$, $u \neq 0$ such that

$$\textit{either} \ \ [\varphi(u) < 0 \text{ and } C_{k-1}(\varphi, u) \neq 0] \ \ \textit{or} \ \ [\varphi(u) > 0 \text{ and } C_{k+1}(\varphi, u) \neq 0]\,.$$

In what follows we denote by $A \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$ the continuous linear operator defined by

$$\langle A(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in H^1(\Omega)\,.$$

Also, we say that a Banach space X has the *Kadec-Klee property*, if the following is true: "If $u_n \xrightarrow{w} u$ in X and $\|u_n\| \to \|u\|$, then $u_n \to u$ in X".

We know (see [9]) that locally uniformly convex Banach spaces, in particular Hilbert spaces, have the Kadec-Klee property.

3. Two nontrivial solutions. In this section, we prove a multiplicity theorem producing two nontrivial solutions for problem (1). For this purpose, we introduce the following hypotheses on the reaction $f(z,x)$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1+|x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $2 < r < 2^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega\,,$$

and if $\varrho(z,x) = f(z,x)x - 2F(z,x)$, then there exists $\tau \in L^1(\Omega)_+$ such that

$$\varrho(z,x) \leqslant \varrho(z,y) + \tau(z) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leqslant x \leqslant y\,; \tag{10}$$

(iii) there exists a real number ξ_0 such that $\xi_0 \leqslant \liminf_{x\to-\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to-\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_1$ uniformly for a.a. $z \in \Omega$ and

$$\lim_{x\to-\infty} \big[f(z,x)x - 2F(z,x)\big] = +\infty \text{ uniformly for a.a. } z \in \Omega\,;$$

(iv) there exist an integer $m \geqslant 2$, $\eta \in L^\infty(\Omega)$ and $\delta_0 > 0$ such that

$$\eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for a.a. } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_{m+1}$$

$$\limsup_{x\to 0} \frac{f(z,x)}{x} \leqslant \eta(z) \text{ uniformly for a.a. } z \in \Omega$$

$$f(z,x)x \geqslant \hat{\lambda}_m x^2 \text{ for a.a. } z \in \Omega,\ \text{all } |x| \leqslant \delta_0\,.$$

REMARK. Hypothesis H_1(ii) implies that for a.a. $z \in \Omega$ the reaction $f(z,\cdot)$ is superlinear near $+\infty$, but without satisfying the usual in such cases AR-condition. We recall that the AR-condition (unilateral version) says that there exist $q > 2$ and $M > 0$ such that

$$\begin{aligned} &(a) \quad 0 < qF(z,x) \leqslant f(z,x)x \text{ for a.a. } z \in \Omega,\ \text{all } x \geqslant M\,,\\ &(b) \quad \operatorname*{ess\,inf}_{\Omega} F(\cdot, M) > 0 \end{aligned} \tag{11}$$

(see Ambrosetti and Rabinowitz [2] and Mugnai [15]). Integrating (11)-(a) and using (11)-(b), we obtain the weaker condition

$$c_3 x^q \leqslant F(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } x \geqslant M \text{ and some } c_3 > 0\,. \tag{12}$$

From (12) we infer the much weaker condition

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega \text{ (recall } q > 2)\,. \tag{13}$$

The AR-condition ensures that the C-condition holds for the energy functional of problem (1). From (12) we see that the AR-condition implies that the primitive $F(z,\cdot)$ has at least q-polynomial growth near $+\infty$. In this way, we exclude from consideration superlinear reactions which exhibit “slower” growth near $+\infty$. For example, consider the function

$$f(x) = x \ln x \quad \text{for all } x \geqslant M > 1\,.$$

To be able to treat also such reactions, instead of the AR-reaction, we employ (10) and (13). Note that (10) which is a quasimonotonicity condition, is satisfied if for a.a. $z \in \Omega$ $\varrho(z,\cdot)$ is nondecreasing on $[M,+\infty)$ (in which case we can take $\tau \equiv 0$). This property is equivalent to saying that for a.a. $z \in \Omega$, the mapping $x \longmapsto \frac{f(z,x)}{x}$ is nondecreasing on $[M,+\infty)$, see Li and Yang [11] and Miyagaki and Souto [12].

Hypothesis H_1 (iii) implies that in the negative direction, $f(z,\cdot)$ is sublinear and can be resonant with respect to the principal eigenvalue $\hat{\lambda}_1$. Resonance can also occur at zero with respect to a nonprincipal eigenvalue $\hat{\lambda}_m$ (see hypothesis H_1 (iv)). So, we are dealing with a problem which can be resonant at both $-\infty$ and zero (double resonance).

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\vartheta(u) - \int_\Omega F(z,u(z))\,dz \quad \text{for all } u \in H^1(\Omega)\,.$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

PROPOSITION 4. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then the energy functional φ satisfies the C-condition.*

PROOF. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1\,, \tag{14}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty\,. \tag{15}$$

From (15) we have for all $n \geqslant 1$, all $h \in H^1(\Omega)$

$$\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h\,dz + \int_{\partial\Omega} \beta(z)u_n h\,d\sigma - \int_\Omega f(z,u_n)h\,dz\right| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|}\,, \tag{16}$$

with $\varepsilon \to 0^+$ as $n \to \infty$.

First we show that $\{u_n^-\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. Arguing by contradiction, suppose that for at least a subsequence, we have $\|u_n^-\| \to +\infty$ as $n \to \infty$. We set $y_n = \frac{u_n^-}{\|u_n^-\|}$ for all $n \geqslant 1$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \geqslant 1$. So, by passing to a suitable subsequence if necessary and using the Sobolev embedding theorem and the trace theorem, we have

$$y_n \to y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega),\ y \geqslant 0\,. \tag{17}$$

In (16) we choose $h = -u_n^- \in H^1(\Omega)$ and obtain

$$\vartheta(u_n^-) - \int_\Omega f(z,u_n)(-u_n^-)\,dz \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1\,,$$

$$\Rightarrow \vartheta(y_n) - \int_\Omega \frac{N_f(-u_n^-)}{\|u_n^-\|}(-y_n)\,dz \leqslant \frac{\varepsilon_n}{\|u_n^-\|^2} \quad \text{for all } n \geqslant 1\,. \tag{18}$$

We have that $u_n^-(z) \to +\infty$ for a.a. $z \in \{y > 0\}$. Hypotheses H_1 (i), (iii), (iv) imply that

$$|f(z,x)| \leqslant c_4|x| \quad \text{for a.a. } z \in \Omega,\ \text{all } x \leqslant 0 \text{ and some } c_4 > 0$$
$$\Rightarrow \left\{\frac{N_f(-u_n^-)}{\|u_n^-\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \quad \text{is bounded.}$$

Using this fact and hypothesis H_1 (iii), we have up to a subsequence

$$\frac{N_f(-u_n^-)}{\|u_n^-\|} \xrightarrow{w} -ky \quad \text{in } L^2(\Omega),\ k(z) \leqslant \hat{\lambda}_1 \quad \text{for a.a. } z \in \Omega \tag{19}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 16). So, if in (18) we pass to the limit as $n \to \infty$ and use (17) and (19), then

$$\vartheta(y) \leqslant \int_\Omega k(z)y^2\,dz\,. \tag{20}$$

If $k \not\equiv \hat{\lambda}_1$, then from (20) and Proposition 3, we have

$$c_1\|y\|^2 \leqslant 0 \Rightarrow y = 0\,.$$

Then from (18) we see that

$$\|Dy_n\|_2 \to 0 \Rightarrow y_n \to 0 \quad \text{in } H^1(\Omega) \text{ (see (17))},$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

If $k(z) = \hat{\lambda}_1$ for a.a. $z \in \Omega$, then from (20) and (3) we have

$$\vartheta(y) = \hat{\lambda}_1\|y\|_2^2 \Rightarrow y = t\hat{u}_1 \text{ with } t \geqslant 0 \text{ (recall } y \geqslant 0,\ \text{see (17))}.$$

If $t = 0$, then $y = 0$ and as above we reach a contradiction. Hence $t > 0$ and we have $y \in \operatorname{int} C_+$ (recall $\hat{u}_1 \in \operatorname{int} C_+$). Therefore

$$u_n^-(z) \to +\infty \quad \text{for a.a. } z \in \Omega$$
$$\Rightarrow f(z,-u_n^-(z))(-u_n^-(z)) - 2F(z,-u_n^-(z)) \to +\infty \quad \text{for a.a. } z \in \Omega \tag{21}$$

(see hypothesis H_1 (iii)).

Using hypothesis H_1 (iii), Fatou's lemma and (21), we have

$$\int_\Omega \left[f(z,-u_n^-)(-u_n^-) - 2F(z,-u_n^-)\right]dz \to +\infty \quad \text{as } n \to \infty\,. \tag{22}$$

From (14) we have

$$\vartheta(u_n) - \int_\Omega 2F(z,u_n)\,dz \leqslant 2M_1 \quad \text{for all } n \geqslant 1\,. \tag{23}$$

Also, if in (16) we choose $h = u_n \in H^1(\Omega)$, then

$$-\vartheta(u_n) + \int_\Omega f(z,u_n)u_n\,dz \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1\,. \tag{24}$$

Adding (23) and (24), we obtain for all $n \geqslant 1$

$$\int_\Omega \big[f(z,u_n)u_n - 2F(z,u_n)\big]\,dz \leqslant M_2 \text{ for some } M_2 > 0,\ \text{all } n \geqslant 1$$

(25)

$$\Rightarrow \int_\Omega \big[f(z,-u_n^-)(-u_n^-) - 2F(z,-u_n^-)\big]\,dz + \int_\Omega \big[f(z,u_n^+)u_n^+ - 2F(z,u_n^+)\big]\,dz \leqslant M_2\,.$$

Note that hypothesis H_1 (ii) implies that

$$-\tau(z) \leqslant f(z,x)x - 2F(z,x) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant 0\,. \tag{26}$$

Using (26) in (25), we obtain

$$\int_\Omega \big[f(z,-u_n^-)(-u_n^-) - 2F(z,-u_n^-)\big]\,dz \leqslant M_3 \quad \text{for some } M_3 > 0,\ \text{all } n \geqslant 1\,. \tag{27}$$

Comparing (22) and (27), we reach a contradiction. This proves that

$$\{u_n^-\}_{n\geqslant 1} \subseteq H^1(\Omega) \quad \text{is bounded}\,. \tag{28}$$

From (14) and (28), we have

$$\vartheta(u_n^+) - \int_\Omega 2F(z,u_n^+)\,dz \leqslant M_4 \quad \text{for some } M_4 > 0,\ \text{all } n \geqslant 1\,. \tag{29}$$

Also, in (16) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$-\vartheta(u_n^+) + \int_\Omega f(z,u_n^+)u_n^+\,dz \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1\,. \tag{30}$$

We add (29) and (30) and obtain

$$\int_\Omega \big[f(z,u_n^+)u_n^+ - 2F(z,u_n^+)\big]\,dz \leqslant M_5 \quad \text{for some } M_5 > 0,\ \text{all } n \geqslant 1$$

$$\Rightarrow \int_\Omega \varrho(z,u_n^+)\,dz \leqslant M_5 \quad \text{for all } n \geqslant 1\,. \tag{31}$$

CLAIM. $\{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega)$ *is bounded.*

We argue indirectly. So, suppose that the Claim is not true. We may assume that

$$\|u_n^+\| \to \infty \text{ as } n \to \infty\,.$$

Let $v_n = \frac{u_n^+}{\|u_n^+\|}$ $n \geqslant 1$. Then $\|v_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } H^1(\Omega) \text{ and } v_n \to v \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega)\,. \tag{32}$$

First suppose that $v \neq 0$ and let $\Omega_0 = [v = 0]$. Then

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \Omega \setminus \Omega_0\,.$$

From hypothesis H_1 (ii), we have

$$\frac{F(z,u_n^+(z))}{\|u_n^+\|^2} = \frac{F(z,u_n^+(z))}{u_n^+(z)^2} v_n(z)^2 \to +\infty \text{ for a.a. } z \in \Omega \setminus \Omega_0\,.$$

Then, by virtue of hypothesis H_1 (ii) and Fatou's lemma, we have

$$\lim_{n\to\infty}\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}\,dz = +\infty\,. \tag{33}$$

On the other hand, from (14) and (28), we have

$$\begin{aligned}
&\int_\Omega F(z,u_n^+)\,dz \leqslant M_6 + |\vartheta(u_n^+)| \quad \text{for all } n\geqslant 1,\ \text{with } M_6 = 2M_1\\
&\Rightarrow \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}\,dz \leqslant \frac{M_6}{\|u_n^+\|^2} + |\vartheta(v_n)| \quad \text{for all } n\geqslant 1\\
&\Rightarrow \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2}\,dz \leqslant M_7 \quad \text{for some } M_7>0,\ \text{all } n\geqslant 1 \qquad (34)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{note that } \{\vartheta(v_n)\}_{n\geqslant 1}\subseteq \mathbb{R} \text{ is bounded}).
\end{aligned}$$

Comparing (33) and (34) we reach a contradiction.

Next assume $v = 0$. Let $\gamma > 0$ and define

$$\begin{aligned}
&w_n = (2\gamma)^{1/2} v_n \in H^1(\Omega) \quad \text{for all } n\geqslant 1\\
&\Rightarrow w_n \to 0 \quad \text{in } L^r(\Omega) \text{ (see (24) and recall } v=0).
\end{aligned}$$

Then from the theorem of Krasnoselskii (see, for example, Gasinski and Papageorgiou [9, p. 407]), we have

$$\int_\Omega F(z,w_n)\,dz \to 0 \quad \text{as } n\to\infty\,. \tag{35}$$

Since $\|u_n^+\|\to\infty$, we can find $n_0\in\mathbb{N}$ such that

$$0 < (2\gamma)^{1/2}\frac{1}{\|u_n^+\|} \leqslant 1 \quad \text{for all } n\geqslant n_0\,. \tag{36}$$

Let $t_n\in[0,1]$ be such that

$$\varphi(t_n u_n^+) = \max_{0\leqslant t\leqslant 1}\varphi(t u_n^+), \quad n\geqslant 1\,.$$

From (36), we have

$$\begin{aligned}
\varphi(t_n u_n^+) &\geqslant \varphi(w_n)\\
&= 2\gamma\vartheta(v_n) - \int_\Omega F(z,w_n)\,dz \quad \text{for all } n\geqslant 1\,.
\end{aligned}\tag{37}$$

From (4), we have

$$\vartheta(u) \geqslant c_0\|u\|^2 - \hat{\xi}\|u\|_2^2 \quad \text{for all } u\in H^1(\Omega)\,. \tag{38}$$

Using (38) in (37) and recalling that $\|v_n\| = 1$ for all $n\geqslant 1$, we obtain

$$\varphi(t_n u_n^+) \geqslant 2\gamma c_0 - 2\gamma\hat{\xi}\|v_n\|_2^2 - \int_\Omega F(z,w_n)\,dz \quad \text{for all } n\geqslant 1\,.$$

Since $\|v_n\|_2 \to 0$ (see (32) and recall that $v = 0$) and $\int_\Omega F(z, w_n)\, dz \to 0$ (see (35)), given $\delta > 0$, we can find $n_1 \in \mathbb{N}$, $n_1 \geqslant n_0$ such that

$$\varphi(t_n u_n^+) \geqslant 2\gamma c_0 - \delta \quad \text{for all } n \geqslant n_1 .$$

Recall that $\gamma > 0$ and $\delta > 0$ are arbitrary. So, let $\gamma \to +\infty$ and $\delta \to 0^+$ to conclude that

$$\varphi(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty . \tag{39}$$

We have

$$\varphi(0) = 0 \text{ and } \varphi(u_n^+) \leqslant M_8 \text{ for some } M_8 > 0,\ \text{all } n \geqslant 1$$
$$\text{(see (14) and (28))}.$$

From (39) it follows that we can find $n_2 \in \mathbb{N}$ such that $t_n \in (0, 1)$ for all $n \geqslant n_2$. Hence

$$\begin{aligned}
&\frac{d}{dt}\varphi(tu_n^+)|_{t=t_n} = 0 \quad \text{for all } n \geqslant n_2\\
&\Rightarrow \langle \varphi'(t_n u_n^+), u_n^+\rangle = 0 \quad \text{for all } n \geqslant n_2 \text{ (by the chain rule)}\\
&\Rightarrow \langle \varphi'(t_n u_n^+), t_n u_n^+\rangle = 0 \quad \text{for all } n \geqslant n_2 \text{ (recall } t_n \in (0, 1) \text{ for all } n \geqslant n_2)\\
&\Rightarrow \vartheta(t_n u_n^+) = \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)\, dz \text{ for all } n \geqslant n_2 . \qquad (40)
\end{aligned}$$

Hypothesis H_1 (ii) implies that

$$\int_\Omega \varrho(z, t_n u_n^+)\, dz \leqslant \int_\Omega \varrho(z, u_n^+)\, dz + \|\tau\|_1 \quad \text{for all } n \geqslant n_2 \tag{41}$$
$$\text{(recall } t_n \in (0, 1) \text{ for all } n \geqslant n_2) .$$

Using (41) in (40) and recalling the definition of the function $\varrho(z, x)$, we obtain

$$2\varphi(t_n u_n^+) \leqslant M_9 \quad \text{for some } M_9 > 0,\ \text{all } n \geqslant n_2 . \tag{42}$$

Comparing (39) and (42), we reach a contradiction. So, the Claim is true.

From the Claim and (28) it follows that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \ \text{ in } H^1(\Omega) \ \text{ and } u_n \to u \ \text{ in } L^{\frac{2s}{s-1}} \ \text{ and in } L^2(\partial\Omega) . \tag{43}$$

In (16) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (43). Then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0\\
&\Rightarrow \|Du_n\|_2 \to \|Du\|_2\\
&\Rightarrow u_n \to u \ \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property (see (43))}\\
&\Rightarrow \varphi \ \text{ satisfies the C-condition.}
\end{aligned}$$

This completes the proof of Proposition 4. □

In addition to the energy functional φ, we will also consider its "negative" truncation-perturbation. More precisely, let $\hat{\xi} > 0$ be as in (4) and define the following Carathéodory function

$$\hat{f}_-(z,x) = \begin{cases} f(z,x) + \hat{\xi}x & \text{if } x < 0 \\ 0 & \text{if } 0 \leqslant x\,. \end{cases}$$

We set $\widehat{F}_-(z,x) = \int_0^x \hat{f}_-(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi}_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{2}\vartheta(u) + \frac{\hat{\xi}}{2}\|u\|_2^2 - \int_\Omega \widehat{F}_-(z,u)\,dz \ \text{ for all } u \in H^1(\Omega)\,.$$

A careful reading of the proof of Proposition 4, leads to the following result.

PROPOSITION 5. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then the functional $\hat{\varphi}_-$ is coercive.*

PROOF. We argue by contradiction. So, suppose that $\hat{\varphi}_-$ is not coercive. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ and $M_{10} > 0$ such that

$$\|u_n\| \to \infty \ \text{ and } \hat{\varphi}_-(u_n) \leqslant M_{10} \ \text{ for all } n \geqslant 1\,. \tag{44}$$

From (44) we have

$$\frac{1}{2}\vartheta(u_n) + \frac{\hat{\xi}}{2}\|u_n\|_2^2 - \int_\Omega \widehat{F}_-(z,u_n)\,dz \leqslant M_{10} \ \text{ for all } n \geqslant 1\,. \tag{45}$$

Let $y_n = \frac{u_n}{\|u_n\|}$ $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$. Using the Sobolev embedding theorem and the trace theorem and by passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \ \text{ in } H^1(\Omega) \ \text{ and } y_n \to y \ \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega)\,. \tag{46}$$

From (45) we have

$$\frac{1}{2}\vartheta(y_n) + \frac{\hat{\xi}}{2}\|y_n^+\| - \int_\Omega \frac{F(z,-u_n^-)}{\|u_n\|^2}\,dz \leqslant \frac{M_{10}}{\|u_n\|^2} \ \text{ for all } n \geqslant 1\,. \tag{47}$$

Hypotheses H_1 (i), (iii) imply that

$$|F(z,x)| \leqslant c_5(1+|x|^2) \ \text{ for a.a. } z \in \Omega,\ \text{all } x \leqslant 0 \text{ and some } c_5 > 0\,. \tag{48}$$

From (46) and (48) it follows that

$$\left\{\frac{N_F(-u_n^-)}{\|u_n\|^2}\right\}_{n\geqslant 1} \subseteq L^1(\Omega) \ \text{ is uniformly integrable}\,.$$

Then by the Dunford-Pettis theorem and at least for a subsequence, we have

$$\frac{N_F(-u_n^-)}{\|u_n\|^2} \xrightarrow{w} \psi \ \text{ in } L^1(\Omega) \ \text{ as } n \to \infty\,. \tag{49}$$

Hypothesis H_1 (iii) implies that for some $\xi_0^* \in \mathbb{R}$, we have

$$-\infty < \xi_0^* \leqslant \liminf_{x \to -\infty} \frac{F(z,x)}{|x|^2} \leqslant \limsup_{x \to -\infty} \frac{F(z,x)}{|x|^2} \leqslant \frac{\hat{\lambda}_1}{2} \text{ uniformly for a.a. } z \in \Omega . \tag{50}$$

From (50) it follows that

$$\psi = \frac{1}{2} g (y^-)^2 \text{ with } \xi_0^* \leqslant g(z) \leqslant \hat{\lambda}_1 \text{ for a.a. } z \in \Omega \tag{51}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 16). So, if in (47) we pass to the limit as $n \to \infty$ and use (46), (49) and (51), then

$$\vartheta(y) + \hat{\xi} \|y^+\|_2^2 \leqslant \int_\Omega g(z)(y^-)^2 \, dz$$

$$\Rightarrow \vartheta(y^-) + \vartheta(y^+) + \hat{\xi} \|y^+\|_2^2 \leqslant \int_\Omega g(z)(y^-)^2 \, dz \tag{52}$$

$$\Rightarrow \vartheta(y^-) \leqslant \int_\Omega g(z)(y^-)^2 \, dz \quad \text{(see (4))}. \tag{53}$$

If $g \not\equiv \hat{\lambda}_1$, then from (53) and Proposition 3 we have

$$c_1 \|y^-\|^2 \leqslant 0 \Rightarrow y^- = 0 . \tag{54}$$

So, from (52) and (4) we have

$$c_0 \|y^+\|^2 \leqslant 0 \Rightarrow y^+ = 0, \text{ that is, } y = 0 \text{ (see (54))}.$$

Then from (47) we see that

$$\|Dy_n\|_2 \to 0$$

$$\Rightarrow y_n \to 0 \text{ in } H^1(\Omega), \quad \text{a contradiction since } \|y_n\| = 1 \text{ for all } n \geqslant 1 .$$

If $g(z) = \hat{\lambda}_1$ for a.a. $z \in \Omega$, then from (53) and (3) we have

$$y^- = k\hat{u}_1 \text{ with } k \geqslant 0 .$$

If $k = 0$, then $y = 0$ and so as above we reach a contradiction.

If $k > 0$, then $y^- \in \operatorname{int} C_+$ (since $\hat{u}_1 \in \operatorname{int} C_+$) and so $y = -y^- \in -\operatorname{int} C_+$. Then

$$u_n(z) = -u_n^-(z) \to -\infty \text{ for a.a. } z \in \Omega . \tag{55}$$

For a.a. $z \in \Omega$ and all $x < 0$, we have

$$\frac{d}{dx}\left(\frac{F(z,x)}{|x|^2}\right) = \frac{f(z,x)|x|^2 - 2xF(z,x)}{|x|^4}$$

$$= \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} . \tag{56}$$

Hypothesis H_1 (iii) implies that given any $\nu > 0$, we can find $M_{11} = M_{11}(\nu) > 0$ such that

$$f(z,x)x - 2F(z,x) \geqslant \nu \text{ for a.a. } z \in \Omega, \text{ all } x \leqslant -M_{11} . \tag{57}$$

Using (57) in (56), we obtain

$$\frac{d}{dx}\left(\frac{F(z,x)}{|x|^2}\right) \geqslant \frac{\nu}{|x|^2\nu} \text{ for a.a. } z \in \Omega,\ \text{all } x \leqslant -M_{11}$$
$$\Rightarrow \frac{F(z,y)}{|y|^2} - \frac{F(z,x)}{|x|^2} \geqslant -\frac{\nu}{2}\left[\frac{1}{|y|^2} - \frac{1}{|x|^2}\right] \tag{58}$$
$$\text{for a.a. } z \in \Omega,\ \text{all } y \leqslant x \leqslant -M_{11}\,.$$

Letting $y \to -\infty$ and using (50), we obtain

$$\frac{1}{2}\hat{\lambda}_1 - \frac{F(z,x)}{|x|^2} \geqslant \frac{\nu}{2}\frac{1}{|x|^2} \text{ for a.a. } z \in \Omega,\ \text{all } x \leqslant -M_{11}$$
$$\Rightarrow \hat{\lambda}_1|x|^2 - 2F(z,x) \geqslant \nu \text{ for a.a. } z \in \Omega,\ \text{all } x \leqslant -M_{11}\,.$$

Since $\nu > 0$ is arbitrary, we infer that

$$\hat{\lambda}_1|x|^2 - 2F(z,x) \to +\infty \text{ uniformly for a.a. } z \in \Omega \text{ as } x \to -\infty\,. \tag{59}$$

From (45), (4) and (3), we have

$$0 \leqslant c_0\|u_n^+\|^2 \leqslant M_{10} - \int_\Omega \left[\hat{\lambda}_1(u_n^-)^2 - 2F(z,-u_n^-)\right] dz \text{ for all } n \geqslant 1\,. \tag{60}$$

From (55), (59) and Fatou's lemma, we have

$$\int_\Omega \left[\hat{\lambda}_1(u_n^-)^2 - 2F(z,-u_n^-)\right] dz \to +\infty \text{ as } n \to \infty\,. \tag{61}$$

Then (60) and (61) lead to a contradiction. This proves the coercivity of the functional $\hat{\varphi}_-$. □

REMARK. If $H_+ = \{u \in H^1(\Omega) : u(z) \geqslant 0 \text{ for a.a. } z \in \Omega\}$, then clearly $\hat{\varphi}_-|_{-H_+} = \varphi|_{-H_+}$. So, it follows that $\varphi|_{-H_+}$ is coercive.

Next we compute the critical groups of φ at infinity.

PROPOSITION 6. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then $C_k(\varphi,\infty) = 0$ for all $k \geqslant 0$.*

PROOF. Let

$$\partial B_1^+ = \{u \in H^1(\Omega) : \|u\| = 1,\ u^+ \neq 0\}\,.$$

We consider the function $h : [0,1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h(t,u) = \frac{(1-t)u + t\hat{u}_1}{\|(1-t)u + t\hat{u}_1\|} \text{ for all } (t,u) \in [0,1] \times \partial B_1^+\,.$$

We see that

$$h(0,\cdot) = \mathrm{id}|_{\partial B_1^+} \text{ and } h(1,\cdot) \equiv \frac{\hat{u}_1}{\|\hat{u}_1\|} \in \partial B_1^+\,.$$

Hence ∂B_1^+ is contractible in itself. Hypothesis H_1 (ii) implies that given any $\tilde{\xi} > 0$, we can find $M_{12} = M_{12}(\xi) > 0$ such that

$$F(z, x) \geqslant \frac{\tilde{\xi}}{2} x^2 \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant M_{12}\,. \tag{62}$$

Similarly hypothesis H_1 (iii) implies that we can find $c_6 > 0$ and $M_{12}^* > 0$ such that

$$F(z, x) \geqslant -\frac{c_6}{2} |x|^2 \text{ for a.a. } z \in \Omega, \text{ all } x \leqslant -M_{12}^*\,. \tag{63}$$

Finally, because of hypothesis H_1 (i), we have

$$|F(z, x)| \leqslant c_7 \text{ for a.a. } z \in \Omega, \text{ all } x \in [-M_{12}^*, M_{12}] \text{ and some } c_7 > 0\,. \tag{64}$$

Let $u \in \partial B_1^+$ and $t > 0$. We have

$$\begin{aligned}
\varphi(tu) &= \frac{t^2}{2}\vartheta(u) - \int_\Omega F(z, tu)\, dz \\
&= \frac{t^2}{2}\vartheta(u) - \int_{\{tu \geqslant M_{12}\}} F(z, tu)\, dz - \int_{\{tu \leqslant -M_{12}^*\}} F(z, tu)\, dz - \int_{I^*} F(z, tu)\, dz \\
&\qquad\qquad \text{with } I^* = (-M_{12}^*, M_{12}) \\
&\leqslant \frac{t^2}{2}\vartheta(u) - \frac{t^2}{2}\tilde{\xi} \int_{\{tu \geqslant M_{12}\}} u^2\, dz + \frac{t^2}{2} c_6 \int_{\{tu \leqslant -M_{12}^*\}} u^2\, dz + c_7 |\Omega|_N \\
&\qquad\qquad \text{(see (62), (63), (64))}
\end{aligned}$$

$$\leqslant \frac{t^2}{2}\left[c_8 - \tilde{\xi} \int_{\{tu \geqslant M_{12}\}} u^2\, dz\right] + c_7 |\Omega|_N \tag{65}$$

$$\text{(see hypotheses } H(\xi),\ H(\beta) \text{ and recall } \|u\| = 1)\,.$$

Because $u \in \partial B_1^+$, we can find $t^* > 0$ and $J > 0$ such that

$$\int_{\{tu \geqslant M_{12}\}} u^2\, dz \geqslant J \text{ for all } t \geqslant t^*\,.$$

Recall that $\tilde{\xi} > 0$ is arbitrary. So, choosing $\tilde{\xi} > 0$ big, from (65) we have

$$\tilde{\xi} J - c_8 > 0 \text{ (recall } \|u\| = 1)\,.$$

Therefore from (65) it follows that

$$\varphi(tu) \to -\infty \text{ as } t \to +\infty\,. \tag{66}$$

Hypothesis H_1 (ii) implies that

$$0 = \varrho(z, 0) \leqslant \varrho(z, x) + \tau(z) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0\,. \tag{67}$$

Also, hypotheses H_1 (ii), (iii) imply that we can find $c_9 > 0$ such that

$$-c_9 \leqslant \varrho(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \leqslant 0\,. \tag{68}$$

Using (67) and (68) that for all $u \in H^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega \varrho(z,u)\,dz &= \int_\Omega \varrho(z,u^+)\,dz + \int_\Omega \varrho(z,-u^-)\,dz\\ &\geqslant -\|\tau\|_1 - c_9|\Omega|_N\end{aligned}$$

$$\Rightarrow -c_{10} + 2\int_\Omega F(z,u)\,dz \leqslant \int_\Omega f(z,u)u\,dz \;\text{ with } c_{10} = \|\tau\|_1 + c_9|\Omega|_N > 0\,. \tag{69}$$

Then for $u \in \partial B_1^+$ and $t \geqslant 1$, we have

$$\begin{aligned}\frac{d}{dt}\varphi(tu) &= \langle\varphi'(tu),u\rangle \;\text{ (by the chain rule)}\\ &= \frac{1}{t}\langle\varphi'(tu),tu\rangle\\ &= \frac{1}{t}\left[\vartheta(tu) - \int_\Omega f(z,tu)(tu)\,dz\right]\\ &\leqslant \frac{1}{t}\,[2\varphi(tu)+c_{10}] \quad \text{(see (69))}.\end{aligned} \tag{70}$$

From (66) and (70) it follows that for $t \geqslant 1$ big, we have

$$\frac{d}{dt}\varphi(tu) < 0\,. \tag{71}$$

Recall that $H_+ = \{u \in H^1(\Omega) : u(z) \geqslant 0 \text{ for a.a. } z \in \Omega\}$ and that $\varphi|_{-H_+}$ is coercive (see Proposition 5 and the Remark following it). So, we can find $c_{11} > 0$ such that

$$\varphi|_{-H_+} \geqslant -c_{11}\,.$$

We choose $\lambda < \min\left\{-c_{10},\ -c_{11},\ \inf_{\partial B_1}\varphi\right\}$, where $\partial B_1 = \{u \in H^1(\Omega) : \|u\| = 1\}$. From (71) we infer that there exists a unique $k(u) > 1$ such that

$$\begin{aligned}&\varphi(tu) > \lambda \;\text{ for } t \in [0,k(u))\\ &\varphi(tu) = \lambda \;\text{ for } t = k(u)\\ &\varphi(tu) < \lambda \;\text{ for } t > k(u)\,.\end{aligned} \tag{72}$$

Moreover, from the implicit function theorem we have that $k \in C(\partial B_1^+, [1,\infty))$. The choice of λ and (72) imply that

$$\varphi^\lambda \subseteq \{tu : u \in \partial B_1^+,\ t \geqslant k(u)\}\,.$$

Let $D_+ = \{tu : u \in \partial B_1^+,\ t \geqslant 1\}$. Then $\varphi^\lambda \subseteq D_+$.

Consider the deformation $h^* : [0,1]\times D_+ \to D_+$ defined by

$$h^*(s,u) = \begin{cases}(1-s)tu + sk(u)u & \text{if } t \in [1,k(u)]\\ tu & \text{if } k(u) < t\,.\end{cases}$$

We have

$$h^*(0,D_+) \subseteq \varphi^\lambda \;\text{ and } h^*(s,\cdot)|_{\varphi^\lambda} = \mathrm{id}|_{\varphi^\lambda} \;\text{ for all } s \in [0,1] \text{ (see (72))}.$$

This means that φ^λ is a strong deformation retract of D_+. Using the radial retraction, we see that D_+ and ∂B_1^+ are homotopy equivalent (see Dugundji [5, Theorem 6.5, p. 325]). Therefore

$$H_k(H^1(\Omega), \varphi^\lambda) = H_k(H^1(\Omega), D_+) = H_k(H^1(\Omega), \partial B_1^+) \text{ for all } k \geqslant 0\,. \tag{73}$$

Recall that ∂B_1^+ is contractible in itself. Hence

$$\begin{aligned} &H_k(H^1(\Omega), \partial B_1^+) = 0 \text{ for all } k \geqslant 0 \\ \Rightarrow &H_k(H^1(\Omega), \varphi^\lambda) = 0 \text{ for all } k \geqslant 0 \text{ (see (73)).} \end{aligned} \tag{74}$$

Choosing $|\lambda|$ even bigger if necessary, from (74) we infer that

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geqslant 0 \text{ (see Section 2).}$$ □

REMARK. We mention that Wang [27] was the first to compute the critical groups at infinity for the energy functional of problems with Dirichlet boundary condition, zero potential (that is, $\xi \equiv 0$) and with a C^1-reaction $x \to f(x)$ which exhibits symmetric behavior as $x \to \pm\infty$ and is superlinear satisfying the AR-condition. Note that in the context of Dirichlet problems, the Poincaré inequality simplifies the argument.

Using Proposition 5 and the direct method, we can produce a negative solution for problem (1).

PROPOSITION 7. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then problem* (1) *admits a negative solution $u_0 \in -\operatorname{int} C_+$, which is a local minimizer of the energy functional φ.*

PROOF. From Proposition 5, we know that $\hat{\varphi}_-$ is coercive. Moreover, using the Sobolev embedding theorem and the trace theorem, we see that $\hat{\varphi}_-$ is sequentially weakly lower semicontinuous. Then, by the Weierstrass theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$\hat{\varphi}_-(u_0) = \inf\big[\hat{\varphi}_-(u) : u \in H^1(\Omega)\big]\,. \tag{75}$$

Let $\delta_0 > 0$ be as in hypothesis H_1 (iv). We choose $t \in (0, 1]$ small such that $t\hat{u}_1(z) \in (0, \delta_0]$ for all $z \in \overline{\Omega}$ (recall $\hat{u}_1 \in \operatorname{int} C_+$). We have

$$\begin{aligned} \hat{\varphi}_-(-t\hat{u}_1) &= \frac{t^2}{2}\vartheta(\hat{u}_1) - \int_\Omega F(z, -t\hat{u}_1)\,dz \\ &\leqslant \frac{t^2}{2}\hat{\lambda}_1 - \frac{t^2}{2}\hat{\lambda}_m \text{ (see hypothesis } H_1\text{(iv) and recall } \|\hat{u}_1\|_2 = 1) \\ &= \frac{t^2}{2}\big[\hat{\lambda}_1 - \hat{\lambda}_m\big] < 0 \text{ (since } m \geqslant 2) \\ \Rightarrow\ &\hat{\varphi}_-(u_0) < 0 = \hat{\varphi}_-(0) \text{ (see (75)), hence } u_0 \neq 0\,. \end{aligned}$$

From (75) we have

$$\hat{\varphi}'_-(u_0) = 0\,,$$

$$\Rightarrow \langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \hat{\xi})u_0 h\,dz + \int_{\partial\Omega} \beta(z)u_0 h\,d\sigma = \int_\Omega \hat{f}_-(z, u_0)h\,dz \tag{76}$$

$$\text{for all } h \in H^1(\Omega)\,.$$

In (76) we choose $h = u_0^+ \in H^1(\Omega)$. Then

$$\begin{aligned}&\vartheta(u_0^+) + \hat{\xi}\|u_0^+\|_2^2 = 0\\ \Rightarrow &c_0\|u_0^+\|^2 \leqslant 0 \ \text{ (see (4)), hence } u_0 \leqslant 0,\ u_0 \neq 0\,.\end{aligned}$$

So, relation (76) becomes

$$\begin{aligned}&\langle A(u_0), h\rangle + \int_\Omega \xi(z)u_0 h\,dz + \int_{\partial\Omega}\beta(z)u_0 h\,d\sigma = \int_\Omega f(z,u_0)h\,dz \ \text{ for all } h \in H^1(\Omega)\\ &\Rightarrow -\Delta u_0(z) + \xi(z)u_0(z) = f(z,u_0(z)) \ \text{ for a.a. } z\in\Omega,\ \frac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 \text{ on } \partial\Omega\\ &\qquad\qquad\qquad\qquad\text{(see Papageorgiou and Rădulescu [21]).}\end{aligned}$$

So, u_0 is a negative solution of problem (1). We define

$$\chi(z) = \begin{cases} \dfrac{f(z,u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0\\ 0 & \text{otherwise}\,.\end{cases}$$

Hypotheses H_1 (i), (iii), (iv) imply that

$$\begin{aligned}&|f(z,x)| \leqslant c_{12}|x| \ \text{ for a.a. } z\in\Omega,\ \text{ all } x \leqslant 0 \ \text{ and some } c_{12} > 0\\ \Rightarrow &\left|\frac{f(z,u_0(z))}{u_0(z)}\right| \leqslant c_{12} \ \text{ for a.a. } z \in \{u_0 > 0\}\,.\end{aligned}$$

Hence $\chi \in L^\infty(\Omega)$. We have

$$-\Delta u_0(z) = (\chi(z) - \xi(z))u_0(z) \ \text{ for a.a. } z\in\Omega\,. \tag{77}$$

Note that $\chi - \xi \in L^s(\Omega)$ with $s > N$ (see hypothesis $H(\xi)$). Using Lemma 5.1 of Wang [26], we have $u_0 \in L^\infty(\Omega)$. Then (77) implies

$$\Delta u_0 \in L^s(\Omega)\,.$$

Using Lemma 5.2 of Wang [26] (the Agmon-Douglis-Nirenberg or Calderon-Zygmund estimates), we obtain that $u_0 \in W^{2,s}(\Omega)$. Since $s > N$ (see hypothesis $H(\xi)$) from the Sobolev embedding theorem, we have $W^{2,s}(\Omega) \hookrightarrow C^{1+\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s}$. Hence $u_0 \in -C_+ \setminus \{0\}$. From (77) we have

$$\begin{aligned}\Delta(-u_0)(z) &= (\xi(z) - \chi(z))(-u_0)(z)\\ &\leqslant (\|\xi^+\|_\infty + \|\chi\|_\infty)(-u_0)(z) \ \text{ for a.a. } z\in\Omega\end{aligned}$$

$\Rightarrow u_0 \in -\operatorname{int} C_+$

(by the maximum principle, see, for example, Gasinski and Papageorgiou [9, p. 738]).

Note that $\hat{\varphi}_-|_{-C_+} = \varphi|_{-C_+}$ and so we have that u_0 is a local $C^1(\overline{\Omega})$-minimizer of φ. Invoking Proposition 2, we conclude that u_0 is a local $H^1(\Omega)$-minimizer of φ. □

In order to generate a second nontrivial solution of problem (1), we need to examine the critical groups of φ at the origin.

PROPOSITION 8. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then $C_{d_m}(\varphi, 0) \neq 0$ where $d_m = \dim \overline{H}_m = \dim \bigoplus_{k=1}^{m} E(\hat{\lambda}_k) \geqslant 2$.*

PROOF. We know that

$$H^1(\Omega) = \overline{H}_m \oplus \widehat{H}_{m+1}\,, \tag{78}$$

where $\overline{H}_m = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k)$ and $\widehat{H}_{m+1} = \overline{H}_m^{\perp} = \overline{\bigoplus_{k\geqslant m+1} E(\hat{\lambda}_k)}$. The space $\overline{H}_m$ is finite dimensional. So, all norms are equivalent. Therefore, we can find $\delta_1 > 0$ such that

$$u \in \overline{H}_m,\ \|u\| \leqslant \delta_1 \ \Rightarrow\ |u(z)| \leqslant \delta_0 \text{ for all } z \in \overline{\Omega} \ \ (\text{recall that } \overline{H}_m \subseteq C(\overline{\Omega}))\,.$$

Using hypothesis H_1 (iv), we have

$$\varphi(u) \leqslant \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}\|u\|_2^2 \leqslant 0 \ \text{ for all } u \in \overline{H}_m \ \text{ with } \|u\| \leqslant \delta_1 \qquad (\text{see } (5)). \tag{79}$$

Hypotheses H_1 (i), (iv) imply that given $\varepsilon > 0$, we can find $c_{13} = c_{13}(\varepsilon) > 0$ such that

$$F(z, x) \leqslant \frac{1}{2}(\eta(z) + \varepsilon)x^2 + c_{13}|x|^r \ \text{ for a.a. } z \in \Omega,\ \text{ all } x \in \mathbb{R}\,. \tag{80}$$

Then for all $u \in \widehat{H}_{m+1}$, we have

$$\begin{aligned}
\varphi(u) &\geqslant \frac{1}{2}\vartheta(u) - \frac{1}{2}\int_{\Omega} \eta(z)u^2\,dz - \frac{\varepsilon}{2}\|u\|_2^2 - c_{14}\|u\|^r \ \text{ for some } c_{14} > 0 \ \ (\text{see } (80)) \\
&\geqslant \frac{1}{2}(c_1 - \varepsilon)\|u\|^2 - c_{14}\|u\|^r \ \ (\text{see Proposition 3}).
\end{aligned}$$

Choosing $\varepsilon \in (0, c_1)$, we see that

$$\varphi(u) \geqslant c_{15}\|u\|^2 - c_{14}\|u\|^r \ \text{ for some } c_{15} > 0\,.$$

Since $r > 2$, we can find $\delta_2 > 0$ such that

$$\varphi(u) > 0 \ \text{ for all } u \in \widehat{H}_{m+1} \ \text{ with } 0 < \|u\| \leqslant \delta_2\,. \tag{81}$$

If $\delta = \min\{\delta_1, \delta_2\}$, then from (78), (79) and (81), we see that φ has a local linking at the origin. Therefore $C_{d_m}(\varphi, 0) \neq 0$ with $d_m = \dim \overline{H}_m \geqslant 2$ (see [14, p. 171]). □

Now we are ready for our first multiplicity theorem in which we produce two nontrivial solutions.

THEOREM 9. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then problem* (1) *has at least two nontrivial solutions*

$$u_0 \in -\operatorname{int} C_+ \ \textit{and}\ \hat{u} \in C^1(\overline{\Omega})\,.$$

PROOF. From Proposition 7 we already have a negative solution $u_0 \in -\operatorname{int} C_+$ which is a local minimizer of the energy functional φ. Hence

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geqslant 0\,. \tag{82}$$

We know that

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geqslant 0 \text{ (see Proposition 6)}$$
$$C_{d_m}(\varphi, 0) \neq 0 \text{ (see Proposition 8)}.$$

So, we can find $\hat{u} \in K_\varphi$ such that

$$\begin{aligned} &\varphi(\hat{u}) < 0 = \varphi(0) \text{ and } C_{d_m - 1}(\varphi, \hat{u}) \neq 0 \\ \text{or } &\varphi(\hat{u}) > 0 = \varphi(0) \text{ and } C_{d_m + 1}(\varphi, \hat{u}) \neq 0\,. \end{aligned} \tag{83}$$

Evidently $\hat{u} \neq 0$. Also since $d_m \geqslant 2$, we have

$$\begin{aligned} &1 \leqslant d_m - 1 < d_m + 1 \\ &\Rightarrow \hat{u} \neq u_0 \text{ (see (82) and (83))}. \end{aligned}$$

So, $\hat{u}$ is a second nontrivial solution of problem (1). As before, using the regularity result of Wang [26], we have $\hat{u} \in C^1(\overline{\Omega})$. □

REMARK. Our theorem improves Theorem 1 of Recova and Rumbos [24], which produces only one nontrivial solution for Dirichlet problems with zero potential and an asymmetric reaction satisfying more restrictive conditions (compare hypotheses H_1 with hypotheses $(L_1)-(L_7)$ in [24]).

4. Three nontrivial solutions. In this section, by strengthening the regularity of the reaction $f(z,\cdot)$, we can improve Theorem 9 and establish the existence of at least three nontrivial solutions for problem (1).

The new hypotheses on the reaction $f(z,x)$ are the following.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)(1+|x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $2 < r < 2^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\lim_{x \to +\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega$$

and if $\varrho(z,x) = f(z,x)x - 2F(z,x)$, then there exists $\tau \in L^1(\Omega)_+$ such that

$$\varrho(z,x) \leqslant \varrho(z,y) + \tau(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant y\,;$$

(iii) $-\infty < \xi_0 \leqslant \liminf\limits_{x \to -\infty} \dfrac{f(z,x)}{x} \leqslant \limsup\limits_{x \to -\infty} \dfrac{f(z,x)}{x} \leqslant \hat{\lambda}_1$ uniformly for a.a. $z \in \Omega$ and

$$\lim_{x \to -\infty} \big[f(z,x)x - 2F(z,x)\big] = +\infty \text{ uniformly for a.a. } z \in \Omega\,;$$

(iv) there exist an integer $m \geqslant 2$ and $\delta_0 > 0$ such that

$$f_x'(z,x) \leqslant \hat{\lambda}_{m+1} \text{ for a.a. } z \in \Omega,\ f_x'(\cdot,0) \not\equiv \hat{\lambda}_{m+1}$$
$$f(z,x)x \geqslant \hat{\lambda}_m x^2 \text{ for a.a. } z \in \Omega,\ \text{all } |x| \leqslant \delta_0\,.$$

THEOREM 10. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_2 hold. Then problem* (1) *admits at least three nontrivial solutions*

$$u_0 \in -\operatorname{int} C_+,\ \hat{u},\ \tilde{u} \in C^1(\overline{\Omega})\,.$$

PROOF. From Proposition 7 we know that there is a negative solution $u_0 \in -\operatorname{int} C_+$ which is a local minimizer of φ. We assume that K_φ is finite (otherwise we already have an infinity of nontrivial solutions which belong in $C^1(\overline{\Omega})$ (as before, using the result of Wang [26]) and so we are done). Then we can find $\varrho \in (0,1)$ small such that

$$\varphi(u_0) < \inf\left[\varphi(u) : \|u-u_0\| = \varrho\right] = m_\varrho \tag{84}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). Hypothesis H_2 (ii) implies that

$$\varphi(t\hat{u}_1) \to -\infty \quad \text{as } t \to +\infty\,. \tag{85}$$

From Proposition 4 we know that

$$\varphi \ \text{ satisfies the C-condition}\,. \tag{86}$$

Then (84), (85), (86) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_\varphi \ \text{ and } m_\varrho \leqslant \varphi(\hat{u})\,. \tag{87}$$

From (84) and (87) it follows that $\hat{u} \neq u_0$ and $\hat{u}$ is a solution of problem (1) with $\hat{u} \in C^1(\overline{\Omega})$ (as in the proof of Proposition 7, using the result of Wang [26]). Since $\hat{u}$ is a critical point of φ of mountain pass type, we have

$$C_1(\varphi, \hat{u}) \neq 0\,. \tag{88}$$

Hypotheses H_2 imply that $\varphi \in C^2(H^1(\Omega))$ and

$$\langle \varphi'(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)uh\,dz + \int_{\partial\Omega} \beta(z)uh\,d\sigma - \int_\Omega f(z,u)h\,dz$$
$$\text{for all } h \in H^1(\Omega)\,,$$
$$\Rightarrow \langle \varphi''(u)y, h\rangle = \int_\Omega (Dy, Dh)_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)yh\,dz + \int_{\partial\Omega} \beta(z)yh\,d\sigma - \int_\Omega f_x'(z,u)yh\,dz$$
$$\text{for all } y, h \in H^1(\Omega)\,.$$

Suppose that the spectrum $\sigma(\varphi''(\hat{u})) \subseteq [0,\infty)$. Then we have

$$\vartheta(y) \geqslant \int_\Omega m(z)y^2\,dz \ \text{ for all } y \in H^1(\Omega)\,, \tag{89}$$

with $m(z) = f'_x(z, \hat{u}(z))$, $m \in L^\infty(\Omega)$ (see hypothesis H_2 (i)). Suppose $u \in \ker \varphi''(\hat{u})$. Then

$$-\Delta u(z) = (m - \xi)(z)u(z) \text{ in } \Omega, \ \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega \,. \tag{90}$$

We have $(m-\xi)(\cdot) \in L^s(\Omega)$, $s > N$. If $(m-\xi)^+ = 0$, then using (90) and the Green identity, we have

$$\|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2 \, d\sigma \leqslant 0$$
$$\Rightarrow u \equiv c \in \mathbb{R}, \quad \text{that is } \dim \ker \varphi''(\hat{u}) = 1 \,.$$

If $(m - \xi)^+ \neq 0$, then we know that

$$\tilde{\lambda}_1 = \inf\left[\|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2 \, d\sigma \,:\, u \in H^1(\Omega), \ \int_\Omega (m - \xi)(z)u^2 \, dz = 1 \right] \tag{91}$$
(see (7)).

From (89) and (91) we see that

$$\tilde{\lambda}_1 \geqslant 0 \,. \tag{92}$$

From (90) and (92) we infer that

$$u = 0 \text{ or } \tilde{\lambda}_1 = 1 \,.$$

But we know that $\tilde{\lambda}_1$ is simple (see Section 2). So, it follows that

$$\dim \ker \varphi''(\hat{u}) \leqslant 1 \,.$$

Then using (88) and Proposition 2.5 of Bartsch [4], we conclude that

$$C_k(\varphi, \hat{u}) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geqslant 0 \,. \tag{93}$$

From Proposition 6, we know that

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geqslant 0 \,. \tag{94}$$

From the proof of Proposition 8, we know that φ has a local linking at the origin. Since $\varphi \in C^2(H^1(\Omega))$, using Proposition 2.3 of Su [25], we have

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geqslant 0 \text{ with } d_m = \dim \overline{H}_m \geqslant 2 \,. \tag{95}$$

Finally since $u_0 \in -\operatorname{int} C_+$ is a local minimizer of φ, we have

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geqslant 0 \,. \tag{96}$$

Suppose $K_\varphi = \{0, u_0, \hat{u}\}$. Then from (93), (94), (95), (96) and the Morse relation with $t = -1$ (see (9)), we have

$$(-1)^{d_m} + (-1)^0 + (-1)^1 = 0$$
$$\Rightarrow (-1)^{d_m} = 0 \,, \quad \text{a contradiction.}$$

So, there exists $\tilde{u} \in K_\varphi$, $\tilde{u} \notin \{0, u_0, \hat{u}\}$. Then $\tilde{u}$ is the third nontrivial solution of (1). As before we have $\tilde{u} \in C^1(\overline{\Omega})$. □

REMARK. Theorem 10 extends Theorem 1.2 of Recova and Rumbos [24], which is for Dirichlet problems with no potential (that is, $\xi \equiv 0$) and under more restrictive conditions on the reaction $f(z, x)$. In particular, the authors of [24] introduce also the extra hypothesis that there exists $x_0 > 0$ such that $f(z, x_0) = 0$ for all $z \in \Omega$. No such condition is used here.

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