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HOMOCLINIC SOLUTIONS OF DIFFERENCE EQUATIONS WITH VARIABLE EXPONENTS

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Dedicated to Professor Gheorghe Moroşanu
on the occasion of his 60-th birthday

Abstract. We study the existence of homoclinic solutions for a class of non-homogeneous difference equation with periodic coefficients. Our proofs rely on the critical point theory combined with adequate variational techniques, which are mainly based on the mountain-pass lemma.

1. Introduction

Partial difference equations usually describe the evolution of certain phenomena over the course of time. Elementary but relevant examples of partial difference equations are concerned with heat diffusion, heat control, temperature distribution, population growth, cellular neural networks, etc. We are concerned in this paper with a class of partial difference equations involving a nonhomogeneous operator. Our interest for problems of this type is motivated by major applications of differential and difference operators to various applied fields, such as electrorheological (smart) fluids, space technology, robotics, image processing, etc.

Let $T > 0$ be a given natural number and let $p(\,\cdot\,), q(\,\cdot\,)\colon \mathbb{Z} \to (1,\infty)$, $V(\,\cdot\,)\colon \mathbb{Z} \to \mathbb{R}$ be three T-periodic functions and $f(k,t)\colon \mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ be a continuous function in $t\in\mathbb{R}$ and T-periodic in k. This paper is devoted to the study

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of the difference non-homogeneous equations of type

$$\begin{cases} \Delta^2_{p(k-1)} u(k-1) - V(k)|u(k)|^{q(k)-2}u(k) + f(k,u(k)) = 0 \text{ for } k \in \mathbb{Z}, \\ u(k) \to 0 \text{ as } |k| \to \infty, \end{cases} \tag{1.1}$$

where $\Delta^2_{p(\,\cdot\,)}$ stands for the $p(\,\cdot\,)$-Laplace difference operator, that is,

$$\Delta^2_{p(k-1)} u(k-1) = |\Delta u(k)|^{p(k)-2}\Delta u(k) - |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1), \tag{1.2}$$

for each $k \in \mathbb{Z}$. We have denoted by Δ the difference operator, which is defined by

$$\Delta u(k) = u(k+1) - u(k), \quad \text{for each } k \in \mathbb{Z}.$$

The goal of the present paper is to establish the existence of nontrivial *homoclinic solutions* for problem (1.1). In order to explain the notion of *homoclinic solution* we go back to the definition of *homoclinic orbit*, which was introduced by H. Poincaré [14] in the context of Hamiltonian systems. More exactly, Poincaré called a trajectory $x(t)$ a *homoclinic orbit* (or *doubly asymptotic* trajectory) if it is asymptotic to a constant as $|t| \to \infty$. Since we are seeking solutions $u(k)$ for problem (1.1) satisfying $\lim_{|k|\to\infty} u(k) = 0$, in accord with the above discussion, we are interested in finding nontrivial homoclinic solutions for problem (1.1).

Throughout the years the study of existence of homoclinic orbits by means of variational methods has captured a special attention (see e.g. [5], [13], [8], [15]–[17] and the references therein). Particularly, we point out the very recent advances in this field obtained by C. Li [10] and A. Cabada, C. Li and S. Tersian [4]. More exactly, in [10] a unified approach to the existence of homoclinic orbits for some classes of ODE's with periodic potentials is established while in [4] the authors extend the ideas from [10] to the case of homoclinic orbits for discrete p-Laplacian type equations. Motivated by the studies in [10] and [4], we focus in the present paper on the case of non-homogeneous difference equations.

An important feature of this paper is the presence of the nonconstant potential $p(\,\cdot\,)$. The study of difference equations involving non-homogeneous difference operators of type (1.2) was initiated by M. Mihăilescu, V. Rădulescu and S. Tersian in [12], where some eigenvalue problems were investigated. After our best knowledge the present paper represents a first attempt in finding homoclinic solutions for non-homogeneous difference equations.

Finally, we remember that boundary value problems involving difference operators have been intensively studied in the last decade. In this context we point out the results obtained in the papers of R. P. Agarwal, K. Perera and D. O'Regan [1], A. Cabada, A. Iannizzotto and S. Tersian [3], H. Fang and D. Zhao [7], M. Ma and Z. Guo [11], A. Kristály, M. Mihăilescu, V. Rădulescu and S. Tersian [9].

2. Main result

Throughout this paper we denote

$$p^+ := \sup_{k\in\mathbb{Z}} p(k), \quad p^- := \inf_{k\in\mathbb{Z}} p(k), \quad q^+ := \sup_{k\in\mathbb{Z}} q(k), \quad q^- := \inf_{k\in\mathbb{Z}} q(k),$$

and we assume that

$$1 < q^- \le q^+ < p^- \le p^+. \tag{2.1}$$

We also assume that the T-periodic function V satisfies the supplementary conditions:

(V1) $0 < V_0 := \min\{V(0), \dots, V(T-1)\}$;
(V2) $V_0 < q^+$,

while the continuous function $f = f(k,t)\colon \mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ which is assumed to be T-periodic in k verifies

(F1) there exist $\alpha > p^+$ and $r > 0$ such that

$$\alpha F(k,t) := \alpha \int_0^t f(k,s)\,ds \le t f(k,t), \quad \text{for all } k\in\mathbb{Z},\ t\neq 0,$$

and $F(k,t) > 0$ for all $k\in\mathbb{Z}$, $t\ge r$;

(F2) $f(k,t) = o(|t|^{q^+-1})$ as $|t|\to 0$.

In order to present our main result, we introduce for each $p(\,\cdot\,)\colon \mathbb{Z}\to(1,\infty)$ the space

$$\ell^{p(\,\cdot\,)} := \Big\{u\colon \mathbb{Z}\to\mathbb{R};\ \rho_{p(\,\cdot\,)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty\Big\}.$$

On $\ell^{p(\,\cdot\,)}$ we introduce the Luxemburg norm

$$|u|_{p(\,\cdot\,)} := \inf\Big\{\mu > 0;\ \sum_{k\in\mathbb{Z}} \Big|\frac{u(k)}{\mu}\Big|^{p(k)} \le 1\Big\}.$$

It is easy to check that the following relations hold true

$$|u|_{p(\,\cdot\,)} < 1 \ \Rightarrow\ |u|_{p(\,\cdot\,)}^{p^+} \le \rho_{p(\,\cdot\,)}(u) \le |u|_{p(\,\cdot\,)}^{p^-}, \tag{2.2}$$

$$|u|_{p(\,\cdot\,)} > 1 \ \Rightarrow\ |u|_{p(\,\cdot\,)}^{p^-} \le \rho_{p(\,\cdot\,)}(u) \le |u|_{p(\,\cdot\,)}^{p^+}, \tag{2.3}$$

$$|u|_{p(\,\cdot\,)} \to 0 \ \Leftrightarrow\ \rho_{p(\,\cdot\,)}(u) \to 0. \tag{2.4}$$

We also consider the space

$$\ell^\infty = \Big\{u\colon \mathbb{Z}\to\mathbb{R};\ |u|_\infty := \sup_{k\in\mathbb{Z}} |u(k)| < \infty\Big\}.$$

We start with the following embedding property.

Proposition 2.1. *Assume condition* (2.1) *is fulfilled. Then* $\ell^{q(\,\cdot\,)} \subset \ell^{p(\,\cdot\,)}$.

Proof. Indeed, if $\sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty$ then there exists $S > 0$ such that

$$|u(k)|^{q(k)} \leq 1, \quad \text{for all } |k| > S.$$

It follows that

$$|u(k)| \leq 1, \quad \text{for all } |k| > S.$$

By relation (2.1) we infer that $q(k) < p(k)$ for all $k \in \mathbb{Z}$. That fact and the above inequality assure that

$$|u(k)|^{p(k)} \leq |u(k)|^{q(k)}, \quad \text{for all } |k| > S,$$

and the conclusion of Proposition 2.1 is obvious now. □

By Proposition 2.1, relation (2.1) and the hypotheses on functions V and f we infer that the natural space where we should seek homoclinic solutions for (1.1) is $\ell^{q(\,\cdot\,)}$. Thus, we say that $u \in \ell^{q(\,\cdot\,)}$ is a *homoclinic solution* for (1.1) if

$$\sum_{k\in\mathbb{Z}} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) \\ + \sum_{k\in\mathbb{Z}} V(k)|u(k)|^{q(k)-2}u(k)v(k) - \sum_{k\in\mathbb{Z}} f(k,u(k))v(k) = 0,$$

for all $v \in \ell^{q(\,\cdot\,)}$ and $\lim_{|k|\to\infty} u(k) = 0$.

The main result of this paper is given by the following theorem.

Theorem 2.2. *Assume hypotheses* (2.1), (V1), (V2), (F1) *and* (F2) *are fulfilled. Then problem* (1.1) *possesses at least a nontrivial homoclinic solution. Moreover, given a nontrivial homoclinic solution u of problem* (1.1), *there exist two integers S_1 and S_2 with $S_1 \leq S_2$ such that for all $k > S_2$ and all $k < S_1$ the sequence $u(k)$ is strictly monotone.*

3. Auxiliary results

The basic idea in proving Theorem 2.2 is to consider the associate energetic functional of problem (1.1) and to show that it possesses a nontrivial critical point by using the mountain-pass lemma, see [2]. In order to do that we first introduce the following notations:

$$\varphi_{p(t)}(t) := |t|^{p(t)-2}t, \qquad \Phi_{p(t)}(t) := \frac{|t|^{p(t)}}{p(t)}.$$

Note that

$$\Delta^2_{p(k-1)}u(k-1) = \Delta(\varphi_{p(k-1)}(\Delta u(k-1))).$$

Next, we introduce the functional $A\colon \ell^{q(\,\cdot\,)} \to \mathbb{R}$ defined by

$$A(u) := \sum_{k\in\mathbb{Z}} \Phi_{p(k-1)}(\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} V(k)\Phi_{q(k)}(u(k)).$$

Now we define the energetic functional associate to problem (1.1) as $J\colon \ell^{q(\,\cdot\,)} \to \mathbb{R}$ defined by

$$J(u) := A(u) - \sum_{k\in\mathbb{Z}} F(k,u(k)).$$

Standard arguments show that $J \in C^1(\ell^{q(\,\cdot\,)},\mathbb{R})$ with the derivative given by

$$\begin{aligned}\langle J'(u),v\rangle = \sum_{k\in\mathbb{Z}}[&|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1)\\ &+ V(k)|u(k)|^{q(k)-2}u(k)v(k) - f(k,u(k))v(k)],\end{aligned}$$

for all u, $v \in \ell^{q(\,\cdot\,)}$. Thus, we found that the critical points of J correspond to the weak solutions of equation (1.1).

In order to facilitate further computations we point out that on $\ell^{q(\,\cdot\,)}$ we can introduce an equivalent norm with $|\cdot|_{q(\,\cdot\,)}$, namely

$$\|u\|_{q(\,\cdot\,)} := \inf\bigg\{\mu>0;\ \sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}\bigg|\frac{u(k)}{\mu}\bigg|^{q(k)} \le 1\bigg\}.$$

It is easy to check the following properties of the above norm

$$\|u\|_{q(\,\cdot\,)} < 1 \ \Rightarrow\ \|u\|^{q^+}_{q(\,\cdot\,)} \le \sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} \le \|u\|^{q^-}_{q(\,\cdot\,)}, \tag{3.1}$$

$$\|u\|_{q(\,\cdot\,)} > 1 \ \Rightarrow\ \|u\|^{q^-}_{q(\,\cdot\,)} \le \sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} \le \|u\|^{q^+}_{q(\,\cdot\,)}, \tag{3.2}$$

$$\|u\|_{q(\,\cdot\,)} \to 0 \ \Leftrightarrow\ \sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} \to 0. \tag{3.3}$$

The next lemma assures that J has a mountain-pass geometry.

Lemma 3.1. *Assume the hypotheses of Theorem* 2.2 *are fulfilled. Then there exists* $\varrho>0$ *and* $\nu>0$ *and* $e\in\ell^{q(\,\cdot\,)}$ *with* $\|e\|_{q(\,\cdot\,)}>\varrho$ *such that*

(a) $J(u)\ge\nu$ *for all* $u\in\ell^{q(\,\cdot\,)}$ *with* $\|u\|_{q(\,\cdot\,)}>\varrho$;
(b) $J(e)<0$.

Proof. (a) Condition (F2) implies that there exists $\delta\in(0,1)$ such that

$$F(k,t) \le \frac{V_0}{2q^+}|t|^{q^+} \le \frac{V_0}{2q^+}|t|^{q(k)}, \tag{3.4}$$

for all $|t|\le\delta$ and all $k\in\mathbb{Z}$. Define

$$\varrho := \bigg(\frac{V_0}{q^+}\bigg)^{1/q^-}\delta^{q^+/q^-}.$$

By condition (V2) we deduce that $\varrho \in (0,1)$. Then for any $\|u\|_{q(\,\cdot\,)} = \varrho$ relation (3.1) yields

$$\varrho^{q^-} = \frac{V_0}{q^+}\delta^{q^+} = \|u\|_{q(\,\cdot\,)}^{q^-} \geq \sum_{i\in\mathbb{Z}} \frac{V(i)}{q(i)}|u(i)|^{q(i)} \geq \frac{V_0}{q^+}|u(k)|^{q(k)},$$

for all $k \in \mathbb{Z}$. It follows that

$$1 > \delta^{q^+} \geq |u(k)|^{q(k)},$$

for all $k \in \mathbb{Z}$. Consequently, $|u(k)| < 1$ for every $k \in \mathbb{Z}$ and thus

$$|u(k)|^{q(k)} \geq |u(k)|^{q^+},$$

for all $k \in \mathbb{Z}$. The above inequalities show that actually $\delta \geq |u(k)|$, for all $k \in \mathbb{Z}$. Next, by (3.4) we deduce

$$\sum_{k\in\mathbb{Z}} F(k,u(k)) \leq \frac{V_0}{2q^+}\sum_{k\in\mathbb{Z}}|u(k)|^{q(k)} \leq \frac{1}{2}\sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)},$$

provided $\|u\|_{q(\,\cdot\,)} = \varrho$. Defining $\nu := \varrho^{q^+}/2$, we get by (3.1) that for each u with $\|u\|_{q(\,\cdot\,)} = \varrho$ it holds true the following estimates

$$\begin{aligned} J(u) = A(u) - \sum_{k\in\mathbb{Z}} F(k,u(k)) &\geq \sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} - \frac{1}{2}\sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} \\ &\geq \frac{1}{2}\sum_{k\in\mathbb{Z}}\frac{V(k)}{q(k)}|u(k)|^{q(k)} \geq \frac{1}{2}\|u\|_{q(\,\cdot\,)}^{q^+} = \nu. \end{aligned}$$

Thus, the first part of the lemma is verified.

(b) We infer by condition (F1) that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$F(k,t) \geq c_1|t|^\alpha - c_2,$$

for all $k \in \mathbb{Z}$ and all $t \in \mathbb{R}$. Define $v \in \ell^{q(\,\cdot\,)}$ by $v(0) = a > 0$ and $v(k) = 0$ if $k \neq 0$. We find

$$J(\eta v) = A(\eta v) - \sum_{k\in\mathbb{Z}} F(k,\eta v(k)) = \frac{2}{p(0)}\eta^{p(0)}a^{p(0)} + V(0)\frac{\eta^{q(0)}a^{q(0)}}{q(0)} - c_1\eta^\alpha a^\alpha + c_2,$$

for each $\eta > 0$. Since by relation (2.1) we have $\alpha > p^+ \geq p(0) \geq p^- > q^+ \geq q(0)$ the above relation shows that for any $\eta > 0$ sufficiently large we have $J(\eta v) < 0$.$\square$

In order to prove the next result, we recall that given $c \in \mathbb{R}$, we say that a sequence $u_n \subset \ell^{q(\,\cdot\,)}$ is said to satisfy the $(\mathrm{PS})_c$ *condition* if

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0.$$

Lemma 3.2. *Assume the hypotheses of Theorem* 2.2 *are fulfilled. Then, there exists* $c > 0$ *and a bounded* $(\mathrm{P})_c$ *sequence for* J *in* $\ell^{q(\,\cdot\,)}$.

Proof. Lemma 3.1 and the mountain-pass lemma imply that there exists a sequence $\{u_n\} \subset \ell^{q(\,\cdot\,)}$ such that

$$J(u_n) \to c, \quad J'(u_n) \to 0, \tag{3.5}$$

where $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ and

$$\Gamma := \{\gamma \in C([0,1],\ \ell^{q(\,\cdot\,)});\ \gamma(0) = 0,\ \gamma(1) = e\}$$

with e given in Lemma 3.1(b). We verify that $\{u_n\}$ is bounded in $\ell^{q(\,\cdot\,)}$. Indeed, assuming that $\|u_n\|_{q(\,\cdot\,)} > 1$ for each n we deduce by condition (F1) and relation (3.2) that

$$\begin{aligned}
\alpha J(u_n) &- \langle J'(u_n), u_n\rangle \\
&= \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{p(k)} - 1\right) |\Delta u_n(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{q(k)} - 1\right) V(k)|u_n(k)|^{q(k)} \\
&\quad - \sum_{k\in\mathbb{Z}} [\alpha F(k,u_n(k)) - f(k,u_n(k))u_n(k)] \geq (\alpha - q^+)\|u_n\|_{q(\,\cdot\,)}^{q^-},
\end{aligned}$$

for all n. The above estimates and condition (3.5) show that $\{u_n\}$ is bounded in $\ell^{q(\,\cdot\,)}$. $\square$

Proof of Theorem 2.2. Assume $\{u_n\}$ is the sequence given by Lemma 3.2. Then for each $n \in \mathbb{N}$ the sequence $\{|u_n(k)|;\ k \in \mathbb{Z}\} \subset \ell^{q(\,\cdot\,)}$ is bounded and $|u_n(k)| \to 0$ as $|k| \to \infty$.

Assume that $\{|u_n(k)|\}_{k\in\mathbb{Z}}$ achieves its maximum in $k_n \in \mathbb{Z}$. Undoubtedly, there exists $j_n \in \mathbb{Z}$ such that

$$j_n T \leq k_n < (j_n + 1)T,$$

and define $w_n(k) := u_n(k - j_n T)$. Then $\{|w_n(k)|\}_{k\in\mathbb{Z}}$ attains its maximum in $i_n := k_n - j_n T \in [0,T]$.

The T-periodicity of $p(\,\cdot\,)$, $q(\,\cdot\,)$ and $V(\,\cdot\,)$ implies

$$\sum_{k\in\mathbb{Z}} \frac{V(k)}{q(k)}|u_n(k)|^{q(k)} = \sum_{k\in\mathbb{Z}} \frac{V(k)}{q(k)}|w_n(k)|^{q(k)} \quad \text{and} \quad J(u_n) = J(w_n).$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\ell^{q(\,\cdot\,)}$ the above estimates and relations (3.1) and (3.2) yield that $\{w_n\}_{n\in\mathbb{N}}$ is bounded in $\ell^{q(\,\cdot\,)}$, too. Then there exists $w \in \ell^{q(\,\cdot\,)}$ such that w_n converges weakly to w in $\ell^{q(\,\cdot\,)}$ as $n \to \infty$.

We claim that $w_n(k) \to w(k)$ as $n \to \infty$ for each $k \in \mathbb{Z}$. Indeed, defining the test function $v_m \in \ell^{q(\,\cdot\,)}$ by

$$v_m(j) := \begin{cases} 1 & \text{if } j = m, \\ 0 & \text{if } j \neq m, \end{cases}$$

and taking into account the weak convergence of w_n to w in $\ell^{q(\cdot)}$ we find

$$\lim_{n\to\infty} w_n(k) = \lim_{n\to\infty}\langle w_n, v_k\rangle = \langle w, v_k\rangle = w(k),$$

for all $k\in\mathbb{Z}$. The claim is clear now.

Next, we point out that for each $v\in\ell^{q(\cdot)}$ we have

$$|\langle J'(w_n), v\rangle| = |\langle J'(u_n), v(\cdot + j_nT)\rangle| \le \|J'(u_n)\|_\star \|v\|_{q(\cdot)},$$

which in view of relation (3.5) from Lemma 3.2 implies $J'(w_n)\to 0$ as $n\to\infty$. It follows that, for each $v\in\ell^{q(\cdot)}$, we have

$$\begin{aligned}\text{(3.6)}\quad \sum_{k\in\mathbb{Z}}[\varphi_{p(k-1)}(\Delta w_n(k-1))\Delta v(k-1)& \\ + V(k)\varphi_{q(k)}(w_n(k))v(k) - f(k, w_n(k))v(k) \to 0] &\to 0,\end{aligned}$$

as $n\to\infty$.

Consider $v\in\ell^{q(\cdot)}$ has compact support, hence there exist $a,b\in\mathbb{Z}$, $a<b$ such that $v\,(k)=0$ if $k\in\mathbb{Z}\setminus[a,b]$ and $v(k)\neq 0$ if $k\in\{a+1,b-1\}$. The set of compact support functions, denoted by $\ell_0^{q(\cdot)}$ is dense in $\ell^{q(\cdot)}$. That fact can be easily explained since for each $v\in\ell^{q(\cdot)}$ we can define $v_n\in\ell_0^{q(\cdot)}$ by $v_n(j)=0$ if $|j|\ge n+1$ and $v_n(j)=v(j)$ if $|j|\neq n$ and we have

$$\sum_{j\in\mathbb{Z}}\frac{V(j)}{q(j)}|v(j)-v_n(j)|\to 0 \quad \text{as } n\to\infty,$$

or, by relation (3.3), $\|v-v_n\|_{q(\cdot)}\to 0$ as $n\to\infty$.

Now, for each $v\in\ell_0^{q(\cdot)}$ in (3.6) taking into account the finite sums and the continuity of $f(k,\cdot)$ we obtain by passing to the limit as $n\to\infty$ that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}[\varphi_{p(k-1)}(\Delta w(k-1))\Delta v(k-1)& \\ + V(k)\varphi_{q(k)}(w(k))v(k) - f(k, w(k))v(k) \to 0] &\to 0.\end{aligned}$$

We found that w is a critical point of J and consequently a solution of (1.1).

We show that $w\neq 0$. Assume by contradiction the contrary, that means $w=0$. Then we have

$$|u_n|_\infty = |w_n|_\infty = \max\{|w_n(k)|;\ k\in\mathbb{Z}\}\to 0,$$

as $n\to\infty$. On the other hand, condition (F2) implies that for a given $\varepsilon>0$ there exists $\delta\in(0,1)$ such that

$$\text{(3.7)}\qquad \begin{cases} |F(k,t)|\le\varepsilon|t|^{q^+}, \\ |f(x,t)t|\le\varepsilon|t|^{q^+}, \end{cases}$$

for all $k \in \{0, \dots, T-1\}$ and all $|t| < \delta$. The above inequalities show that for every $k \in \{0, \dots, T-1\}$ there exists M_k such that for $n > M_k$ we have $|w_n(k)| < \delta$.

Since $i_n \in \{0, \dots, T-1\}$ it follows that, for $n > M := \max\{M_n;\ k \in \{0, \dots, T-1\}\}$ and every $k \in \mathbb{Z}$, we have

$$|w_n(k)| \le |w_n(i_n)| < \delta < 1.$$

That fact and relation (3.7) imply

$$|F(k, w_n(k))| \le \varepsilon |w_n(k)|^{q^+} \le \varepsilon |w_n(k)|^{q(k)},$$

and

$$|f(k, w_n(k)) w_n(k)| \le \varepsilon |w_n(k)|^{q^+} \le \varepsilon |w_n(k)|^{q(k)}.$$

We infer that for each $n > M$ and every $k \in \mathbb{Z}$ the following estimates hold true

$$\begin{aligned}
0 < q^- J(w_n) &= q^- \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |\Delta w_n(k-1)|^{p(k-1)} \\
&\quad + q^- \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |w_n(k)|^{q(k)} - q^- \sum_{k \in \mathbb{Z}} F(k, w_n(k)) \\
&\le \sum_{k \in \mathbb{Z}} |\Delta w_n(k-1)|^{p(k-1)} + \sum_{k \in \mathbb{Z}} V(k) |w_n(k)|^{q(k)} \\
&\quad - \sum_{k \in \mathbb{Z}} f(k, w_n(k)) w_n(k) - \sum_{k \in \mathbb{Z}} (q^- F(k, w_n(k)) - f(k, w_n(k)) w_n(k)) \\
&\le \langle J'(w_n), w_n \rangle + q^- \sum_{k \in \mathbb{Z}} F(k, w_n(k)) + \sum_{k \in \mathbb{Z}} |f(k, w_n(k)) w_n(k)| \\
&\le \langle J'(w_n), w_n \rangle + q^- \varepsilon \sum_{k \in \mathbb{Z}} |w_n(k)|^{q(k)} + \varepsilon \sum_{k \in \mathbb{Z}} |w_n(k)|^{q(k)} \\
&\le \langle J'(w_n), w_n \rangle + \left(q^- \varepsilon \frac{q^+}{V_0} + \varepsilon \frac{q^+}{V_0} \right) \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |w_n(k)|^{q(k)} \\
&\le \|J'(w_n)\|_* \|w_n\|_{q(\,\cdot\,)} + \varepsilon \frac{q^+(q^- + 1)}{V_0} \Big[\|w_n\|_{q(\,\cdot\,)}^{q^+} + \|w_n\|_{q(\,\cdot\,)}^{q^-} \Big].
\end{aligned}$$

Taking into account that $\|u_n\|_{q(\,\cdot\,)}$ is bounded, $J'(u_n) \to 0$ as $n \to \infty$ and $\varepsilon > 0$ is arbitrary we find by the above estimates a contradiction with $J(w_n) \to c > 0$ as $n \to \infty$. Thus, we have that w is a nontrivial solution of problem (1.1).

Next, let u be a nonzero homoclinic solution of problem (1.1). Assume that it attains positive local maximums and negative local minimums at infinitely many points k_n. In particular we can assume that $\{|k_n|\} \to \infty$. Consequently,

$$\Delta^2_{p(k_n - 1)} u(k_n - 1)\, u(k_n) \le 0 \quad \text{and} \quad u(k_n) \to 0,$$

as $n \to \infty$. Using that facts and multiplying in (1.1) by $u(k_n)/|u(k_n)|^{q(k_n)}$, we have

$$\frac{f(k_n,u(k_n))\,u(k_n)}{|u(k_n)|^{q(k_n)}} \geq \frac{\Delta^2_{p(k_n-1)}u(k_n-1)\,u(k_n)}{|u(k_n)|^{q(k_n)}} + \frac{f(k_n,u(k_n))\,u(k_n)}{|u(k_n)|^{q(k_n)}} = V(k_n) \geq V_0 > 0. \tag{3.8}$$

Using (3.8) and condition (F2) we deduce

$$0 = \lim_{n\to\infty} \frac{f(k_n,u(k_n))\,u(k_n)}{|u(k_n)|^{q(k_n)}} \geq V_0 > 0,$$

which represents a contradiction. Consequently u does not attain positive local maximums and negative local minimums at infinitely many points.

Assume now that function u vanishes at infinitely many points l_n. By condition (F2) we find that $\Delta^2_{p(l_n-1)}u(l_n-1) = 0$ and, consequently, $u(l_n-1)\,u(l_n+1) < 0$. Therefore it has an unbounded sequence of positive local maximums and negative local minimums, in contradiction with the previous assertion.

We proved that, for $|k|$ large enough, function u has constant sign and it is strictly monotone. □

Remark 3.3. Note, that the "homogeneous" problem

$$\begin{cases} \Delta^2_{p(k-1)}u(k-1) - V(k)|u(k)|^{q(k)-2}u(k) = 0 & \text{for } k \in \mathbb{Z}, \\ u(k) \to 0 & \text{as } |k| \to \infty, \end{cases}$$

has only the trivial solution. Indeed, if u is positive or negative, let k_0 be the point of his positive maximum or negative minimum. Then $\Delta^2_{p(k_0-1)}u(k_0-1)u(k_0) \leq 0$ and

$$0 = \Delta^2_{p(k_0-1)}u(k_0-1)u(k_0) - V(k_0)|u(k_0)|^{q(k_0)} < 0,$$

which is a contradiction. The same conclusion can be made if u is sign-changing.

Note that under assumptions of Theorem 2.2, for every $\lambda > 0$ one can prove, following the same approach, the existence of a nontrivial solution for the eigenvalue problem

$$\begin{cases} \Delta^2_{p(k-1)}u(k-1) - V(k)|u(k)|^{q(k)-2}u(k) + \lambda f(k,u(k)) = 0 & \text{for } k \in \mathbb{Z}, \\ u(k) \to 0 & \text{as } |k| \to \infty, \end{cases} \tag{3.9}$$

Above remark shows that $\lambda = 0$ is a bifurcation point of problem (3.9).

Remark 3.4. We can prove that if in addition to conditions (F1) and (F2) the following condition holds:

(F3) $f(k,t) \geq 0$ for any $t < 0$ and all $k \in \mathbb{Z}$,

the homoclinic solution of the problem (1.1) is positive.

Indeed, let u be a homoclinic solution of the problem and assume that (F3) holds. Suppose that there exists k_0 such that $u(k_0) < 0$ and let k_1 be such that

$u(k_1) = \min\{u(k),\ k \in \mathbb{Z}\} < 0$. In consequence $\Delta^2_{p(k_1-1)} u(k_1 - 1) \geq 0$, which implies that

$$f(k_1, u(k_1)) = -\Delta^2_{p(k_1-1)} u(k_1 - 1) + V(k_1)|u(k_1)|^{q(k_1)-2} u(k_1) < 0,$$

in contradiction with (F3). Then $u(k) \geq 0$ for every $k \in \mathbb{Z}$.

Suppose that $u(k_2) = 0$ for some $k_2 \in \mathbb{Z}$. Note, that if $u(k_2 + 1) = 0$ or $u(k_2 - 1) = 0$, the solution is identically zero by a recursion and $f(k, 0) = 0$. Hence $\Delta^2_{p(k_2-1)} u(k_2 - 1) > 0$ and we arrive to a contradiction. Hence, the solution u is positive.

We summarize in what follows the above remark.

Theorem 3.5. *Suppose that the functions* $V: \mathbb{Z} \to \mathbb{R}$ *and* $f: \mathbb{Z} \to \mathbb{R}$ *satisfy assumptions of Theorem* 2.2 *and* (F3). *Then, problem* (1.1) *has a positive homoclinic solution.*

Remark 3.6. In the case $q(k) = 2$ we can estimate the maximum of the positive solution u, provided the additional assumption:

(F4) Let $f(k, t)$ has the form $f(k, t) = tg(k, t)$, where $g(k, t)$ is T-periodic in k, $g(k, 0) = 0$ and for each k, $g(k, t)$ is increasing in t for $t > 0$.

Let $g^{-1}(k, t)$ be the inverse function of $g(k, t)$ for $t > 0$. We have that $g^{-1}(k, t)$ is increasing in t for $t > 0$. Let u be a positive homoclinic solution of the problem (1.1) and $u(k_0) > 0$ is its maximum. Note that, in view of the periodicity of coefficients, if $u(\cdot)$ is a solution of problem (1.1), then $u(\cdot + jT)$, $j \in \mathbb{Z}$ is also a solution of (1.1). Hence, we may assume that $k_0 \in [0, T - 1]$. We have $\Delta^2_{p(k_0-1)} u(k_0 - 1) \leq 0$ and

$$u(k_0) g(k_0, u(k_0)) - V(k_0) u(k_0) \geq 0,$$

and, by properties of g and V,

$$u(k_0) \geq g^{-1}(k_0, V_0).$$

Thus

$$\max\{u(k) : k \in [0, T - 1]\} \geq \min\{g^{-1}(k, V_0) : k \in [0, T - 1]\}. \tag{3.10}$$

Thus, we have obtained the following result.

Corllary 3.7. *Suppose that the functions* $V: \mathbb{Z} \to \mathbb{R}$ *and* $f: \mathbb{Z} \to \mathbb{R}$ *satisfy assumptions of Theorem* 3.5 *and* (F4). *Then the positive homoclinic solution of the equation*

$$\Delta^2_{p(k-1)} u(k - 1) - V(k) u(k) + u(k) g(k, u(k)) = 0$$

satisfies relation (3.10).

Example 3.8. Consider the equations

$$\Delta^2_{p(k-1)}u(k-1) - V(k)u(k) + au^2(k) + bu^3(k) = 0, \quad k \in \mathbb{Z}, \tag{3.11}$$

and

$$\Delta^2_{p(k-1)}u(k-1) - V(k)u(k) + au^2(k) + bu^3_+(k) = 0, \quad k \in \mathbb{Z}, \tag{3.12}$$

where $u_+(k) = \max\{u(k), 0\}$, $1 < p(k) < 3$, $p(k)$ and $V(k)$ are T-periodic, $V_0 < 2$, $a > 0$ and $b > 0$. All assumptions of Theorem 3.5 are satisfied for $f(k,t) = at^2 + bt^3_+$ and there exists a positive homoclinic solution u of equation (3.12), which is a homoclinic solution of problem (3.11). Moreover, we can estimate $\max\{u(k) : k \in \mathbb{Z}\}$. Let $u(k_0) > 0$ be the maximum of u. Then $\Delta^2_{p(k_0-1)}u(k_0-1) \le 0$ and by equation (3.11) we have

$$-V(k_0)u(k_0) + au^2(k_0) + bu^3(k_0) \ge 0,$$

which implies

$$au(k_0) + bu^2(k_0) - V_0 \ge 0.$$

Hence

$$\max\{u(k) : k \in \mathbb{Z}\} \ge \frac{\sqrt{a^2 + 4bV_0} - a}{2b}.$$

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