Nonlinear Analysis 201 (2020) 111812

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the nonlinear Schrödinger equation on the Poincaré ball model

Giovanni Molica Bisci [a,*], Vicenţiu D. Rădulescu [b,c]

[a] Dipartimento di Scienze Pure e Applicate (DiSPeA), Università degli Studi di Urbino Carlo Bo, Piazza della Repubblica 13, 61029 Urbino, Italy
[b] Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland
[c] Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

ARTICLE INFO

Article history:
Received 11 December 2019
Accepted 9 February 2020
Communicated by Enrico Valdinoci

This paper is dedicated with esteem to Professor Shair Ahmad on the occasion of his 85th anniversary

MSC:
primary 58J32
35J60
49K10
secondary 35A01
35R01

Keywords:
Elliptic problem on manifolds
Hyperbolic space
Poincaré model
Variational methods
Multiple solutions

ABSTRACT

In this paper, we prove the existence of a sequence of nonnegative (weak) solutions for the following problem

$$\begin{cases} -\Delta_H u + u = \lambda\alpha(\sigma)f(u) & \text{in } \mathbb{B}^N \\ u \in H^{1,2}(\mathbb{B}^N), \end{cases}$$

where Δ_H denotes the Laplace–Beltrami operator on the Poincaré ball model $\mathbb{B}^N$ (with $N \geq 3$) of the hyperbolic space $\mathbb{H}^N$, $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is a nonnegative and not identically zero radially symmetric potential, f is a suitable continuous function, and λ is a positive real parameter. The analysis developed in this paper combines a compactness embedding result due to Skrzypczak and Tintarev [(2013), Theorem 1.3 and Proposition 3.1], some group-theoretical arguments on the Poincaré ball model $\mathbb{B}^N$, and variational methods for smooth functionals defined on the Sobolev space $H^{1,2}(\mathbb{B}^N)$ associated to the homogeneous Hadamard manifold $\mathbb{B}^N$.

1. Introduction

Hyperbolic geometry was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry. In the theory of hyperbolic geometry there are different models for the hyperbolic space $\mathbb{H}^N$. Each model has its own metric, geodesics, isometries, and related properties. In order to understand the relationships among these models, it is helpful to understand the geometric properties of the connecting maps. Two of them are the central or stereographic projection from a sphere to a plane; see, e.g., the monograph [5] and references therein.

* Corresponding author.
E-mail addresses: giovanni.molicabisci@uniurb.it (G. Molica Bisci), radulescu@inf.ucv.ro (V.D. Rădulescu).

https://doi.org/10.1016/j.na.2020.111812
0362-546X/

Recently, some eigenvalue problems on the hyperbolic space framework have been studied; see, for instance, the papers [6,15,24,31], as well as [45,50,51]. Motivated by this wide interest on the current literature, in this paper we deal with the following elliptic problem

$$\begin{cases} -\Delta_H u + u = \lambda\alpha(\sigma)f(u) & \text{in } \mathbb{B}^N \\ u \in H^{1,2}(\mathbb{B}^N), \end{cases} \tag{1.1}$$

settled on the Poincaré ball model $\mathbb{B}^N$, with $N \geq 3$.

Here, Δ_H denotes the Laplace–Beltrami operator on $\mathbb{B}^N$, $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is a nonnegative and nontrivial radially symmetric potential, f is a continuous function, and λ is a positive real parameter.

As usual, according to the notations of Section 2, a weak solution of problem (1.1) is any function $u \in H^{1,2}(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ such that

$$\int_{\mathbb{B}^N} \langle \nabla_H u(\sigma), \nabla_H \varphi(\sigma)\rangle_\sigma \, d\mu + \int_{\mathbb{B}^N} u(\sigma)\varphi(\sigma)d\mu \\ - \lambda \int_{\mathbb{B}^N} \alpha(\sigma)f(u(\sigma))\varphi(\sigma)d\mu = 0,$$

for every $\varphi \in H^{1,2}(\mathbb{B}^N)$.

Let $N \geq 3$ and define the family of subgroups of the special orthogonal group $SO(N)$ given by

$$\mathscr{F} := \left\{ \mathscr{G} \subseteq SO(N) : \mathscr{G} := \prod_{j=1}^{\ell} SO(N_j), \text{ where } N_j \geq 2,\ j = 1, \ldots, \ell, \text{ and } \sum_{j=1}^{\ell} N_j = N \right\},$$

where $SO(N_j)$ denotes here the special orthogonal group in dimension N_j, for every $j = 1, \ldots, \ell$. Now, take $\mathscr{G} \in \mathscr{F}$ and let $\cdot : \mathscr{G} \times \mathbb{B}^N \to \mathbb{B}^N$ be the natural multiplicative action of the group $\mathscr{G}$ on $\mathbb{B}^N$. We briefly recall that a function $u \in H^{1,2}(\mathbb{B}^N)$ is said to be $\mathscr{G}$–invariant if

$$u(g \cdot \sigma) = u(\sigma), \quad \text{in } \mathbb{B}^N$$

for every $g \in \mathscr{G}$; see Section 3 for details.

A special case of the main result given in Theorem 5 of Section 4 reads as follows.

Theorem 1. *Let $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ be a nonnegative and not identically zero radially symmetric map with respect to the origin $\sigma_0 \in \mathbb{B}^N$. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

(f_0') *There are two real sequences $(\xi_k)_k$ and $(\zeta_k)_k$, with $\lim_{k\to\infty} \zeta_k = 0$ such that $0 \leq \xi_k < \zeta_k$ for every $k \in \mathbb{N}$, and*

$$F(\xi_k) = \sup_{t\in[0,\zeta_k]} F(t),$$

where $F(t) := \int_0^t f(s)ds$;

(f_1')

$$-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^2} \leq \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty.$$

Then, for every $\mathscr{G} \in \mathscr{F}$ and $\lambda > 0$, there exists a sequence $(u_k^{\mathscr{G}})_k \subset H^{1,2}(\mathbb{B}^N)$ of nonnegative and not identically zero $\mathscr{G}$–invariant weak solutions of problem (1.1) such that

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0. \tag{1.2}$$

Our approach in order to prove Theorem 5 is based on variational techniques; in the sequel, we will describe it briefly; see the classical book of Brezis [8] as general reference for this topics. More precisely, it is well known that $H^{1,2}(\mathbb{B}^N)$ cannot be compactly embedded into $L^\nu(\mathbb{B}^N)$, $\nu > 1$, due to the unboundedness of the hyperbolic space. However, by a Lions-type result, the fixed point space of $H^{1,2}(\mathbb{B}^N)$ under the action of $\mathscr{G} \in \mathscr{F}$, denoted $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, is compactly embedded into $L^\nu(\mathbb{B}^N)$ whenever $\nu \in (2, 2^*)$; see Skrzypczak and Tintarev [49].

Instead of (1.1) we study an auxiliary variational problem whose solutions also solve problem (1.1) in the weak sense. If J_λ is the C^1 energy functional associated to the aforementioned auxiliary problem, thanks to a compactness result due to Skrzypczak and Tintarev [49], the restriction of J_λ to $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, denoted by $J_{\mathscr{G},\lambda}$, is weakly sequentially lower semicontinuous and its critical points are critical points of J_λ as well, due to the principle of symmetric criticality of Palais; see [40, Theorem 5.4].

The crucial step in our arguments is the construction of an appropriate sequence of weakly closed subsets $(\mathbb{E}^{\mathscr{G}}_k)_k$ of $H^{1,2}(\mathbb{B}^N)$, proving that the relative minima of $J_{\mathscr{G},\lambda}$ on these sets are actually local minima of $J_{\mathscr{G},\lambda}$ on $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, so $\mathscr{G}$–invariant weak solutions of problem (1.1). Subsequently, a suitable subsequence of critical points of $J_{\mathscr{G},\lambda}$ can be extracted from the aforementioned local minima having the property (1.2). We emphasize that the crucial step described above can be achieved by using the continuity of the superposition operator due to Marcus and Mizel [25, Theorem 1, p. 219] settled in the hyperbolic context instead of the classical Euclidean framework; see [16, Proposition 2.5, p. 24] for additional comments and remarks.

Problem (1.1) is a reasonably useful generalization of most studied elliptic problems with oscillating nonlinearities, which naturally arise in different branches of mathematics. More precisely, the main result given in Theorem 5 complement some results obtained on bounded Euclidean domains where elliptic problems with oscillatory nonlinearities have been considered. For instance, among others, Dirichlet problems were studied by Anello and Cordaro in [1] and Molica Bisci and Pizzimenti in [32], while Neumann type problems have been considered by Anello and Cordaro in [2]. We point out that some almost straightforward computations in [1,2] are adapted here to the hyperbolic setting. Anyway, due to the non-compact framework, our abstract procedure, as well as the setting of the main results, is different from the results contained in [1,2], where elliptic problems on bounded smooth domains have been studied.

Furthermore, the existence of infinitely many solutions of the following Dirichlet problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$ has been extensively studied. Most results assume that the nonlinearity f is odd in order to apply some variant of the classical Lusternik–Schnirelmann theory. Only a few papers deal with nonlinearities having no symmetry properties. Among them, the ones which are closest to the present article are certainly [1,38,47,48]. For instance, in [38], Omari and Zanolin proved that if

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = 0 \quad \text{and} \quad \limsup_{t\to 0^+} \frac{F(t)}{t^2} = \infty, \tag{1.4}$$

problem (1.3) has a sequence of not identically zero and nonnegative weak solutions, satisfying (1.2); see also [25,36,37,39] for related topics. We refer the interested reader to the results proved by Molica Bisci in [29], where a similar multiplicity property has been established for an elliptic problem defined on the Euclidean sphere $\mathbb{S}^N \hookrightarrow \mathbb{R}^{N+1}$ endowed with the Euclidean induced metric.

The Laplacian case was also studied using different methods, and the existence of infinitely many weak solutions, with the property that the L^2–norm of their gradient goes to infinity, was proved by Kavian in [22] and Struwe in [52,53]; see also the classical book of Rabinowitz [46].

The non–compact hyperbolic setting presents additional difficulties respect to the aforementioned cases and suitable geometrical and algebraic tools need to be exploited in order to get the main results. For instance, a crucial ingredient used along the proof of Theorem 5 is based on a careful analysis of the energy level on $(\mathbb{E}_k^{\mathscr{G}})_k$ of some $\mathscr{G}$–invariant functions whose simple prototype, for suitable $0 < r < \rho$ and $\varepsilon \in (0,1)$, is defined by

$$w_{\rho,r}^{\varepsilon}(\sigma) := \begin{cases} 0 & \text{if } \sigma \in \mathbb{B}^N \setminus A_r^{\rho} \\ 1 & \text{if } \sigma \in A_{\varepsilon r}^{\rho} \\ \dfrac{1}{(1-\varepsilon)r}\left(r - \left|\log\left(\dfrac{1+|\sigma|}{1-|\sigma|}\right) - \rho\right|\right) & \text{if } \sigma \in A_r^{\rho} \setminus A_{\varepsilon r}^{\rho}, \end{cases}$$

and whose support is contained in the annular domain A_r^{ρ} of $\mathbb{B}^N$; see Section 4 and [30] for related topics.

The Poincaré ball model is a significative model of Hadamard manifold, that is a complete, simply connected Riemannian manifold with nonpositive sectional curvature. The approach adopted here can be used in order to study the existence of multiple sequences of solutions for elliptic problems on Hadamard manifolds in presence of a compact topological group action. Since this approach differs to the above, we will treat it in a forthcoming manuscript. General results on complete Riemannian manifolds can be found in [35] and [42].

The paper is organized in the following way: in Section 2 we recall some notions and notations which will be used throughout the paper. In Section 3, in order to handle the lack of compactness of $\mathbb{B}^N$, a compactness argument will be used, based on the action of a suitable subgroup of the group of isometries of $\mathbb{B}^N$. More precisely, we shall adapt the main results of Skrzypczak and Tintarev [49, Theorem 1.3 and Proposition 3.1] to our setting concerning Sobolev spaces in the presence of group-symmetries. The last section is devoted to the proof of the main result stated in Theorem 5.

Some of the abstract tools used in this paper can be found in the recent monograph by Papageorgiou, Rădulescu and Repovš [41].

2. Abstract framework

In this section we briefly recall some notions from Riemannian geometry needed in the sequel and then illustrate the functional framework we will move in. We refer the reader to the following source [5] for detailed derivations of the geometric quantities, their motivation and further applications. As is well-known there are several models for the hyperbolic space $\mathbb{H}^N$, for instance, the Poincaré ball model $\mathbb{B}^N$. In particular, the Poincaré disk model, also called the conformal ball model, is a model of two-dimensional hyperbolic geometry in which the points of the geometry are inside the unit disk, and the straight lines consist of all segments of circles contained within that disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.

To be specific, let us set

$$\mathbb{B}^N := \{\sigma = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \ |\sigma| < 1\},$$

endowed with the Riemannian metric given by

$$g_{ij} := \frac{4}{(1-|\sigma|^2)^2}\delta_{ij} \quad (\sigma \in \mathbb{B}^N; i,j = 1,\ldots,N),$$

where $|\cdot|$ and δ_{ij} denote respectively the Euclidean distance and the usual Kronecker delta.

For every $i,j = 1,\ldots,N$, we also set

$$g^{ij} := (g_{ij})^{-1}, \quad \text{and} \quad \mathrm{g} := \det(g_{ij}).$$

In this setting the Laplace–Beltrami operator Δ_H is locally defined as follows

$$\Delta_H := \frac{1}{\sqrt{g}} \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sqrt{\mathrm{g}} \sum_{j=1}^{N} g^{ij} \frac{\partial}{\partial x_j}\right).$$

Now, as usual, let

$$d\mu := \sqrt{\mathrm{g}}dx = \frac{2^N}{(1-|\sigma|^2)^N}dx$$

be the Riemannian volume element in $\mathbb{B}^N$, where dx denotes standard Lebesgue measure in the Euclidean space $\mathbb{R}^N$.

Hence, if

$$d_H(\sigma) := 2\int_0^{|\sigma|}\frac{dt}{1-t^2} = \log\left(\frac{1+|\sigma|}{1-|\sigma|}\right),$$

denotes the geodesic distance of $\sigma \in \mathbb{B}^N$ from the origin $\sigma_0 \in \mathbb{B}^N$, a direct computation ensures that the operator Δ_H has the more convenient form

$$\Delta_H = \frac{1}{4}(1-|\sigma|^2)^2\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{N-2}{2}(1-|\sigma|^2)\sum_{i=1}^N x_i\frac{\partial}{\partial x_i}.$$

Finally, let (ϱ,θ) be polar geodesic coordinates in $\mathbb{B}^N \setminus \{0\}$. Then, in $\mathbb{B}^N \setminus \{0\}$ we have that

$$ds^2 = d\varrho^2 + (\sinh\varrho)^2 d\theta,$$

and

$$\Delta_H = \frac{\partial^2}{\partial\varrho^2} + (N-1)\coth\varrho\frac{\partial}{\partial\varrho} + \frac{1}{(\sinh\varrho)^2}\Delta_\theta,$$

being Δ_θ the Laplace–Beltrami operator on the sphere $\mathbb{S}_{N-1} \hookrightarrow \mathbb{R}^N$.

We also notice that the hyperbolic distance in the Poincaré ball model is given by the formula

$$d_H(\sigma_1,\sigma_2) = \mathrm{Arccosh}\left(1 + \frac{2|\sigma_2-\sigma_1|^2}{(1-|\sigma_1|^2)(1-|\sigma_2|^2)}\right),$$

for every $\sigma_1,\sigma_2 \in \mathbb{B}^N$.

Now, let $r \in (0,1)$ and denote by

$$B(r) := \{x \in \mathbb{B}^N : |x| < r\}$$

the Euclidean ball of radius r and centered at the origin. Moreover, let

$$B_H(y) := \{\sigma \in \mathbb{B}^N : d_H(\sigma) < y\}$$

be the geodesic ball of radius $y > 0$ and centered in $\sigma_0 \in \mathbb{B}^N$. One has

$$B(r) = B_H\left(\log\left(\frac{1+r}{1-r}\right)\right).$$

See [45] for additional comments and related facts.

Let $T_\sigma(\mathbb{B}^N)$ be the tangent space at $\sigma \in \mathbb{B}^N$ endowed by the inner product $\langle\cdot,\cdot\rangle_\sigma$ and by $T(\mathbb{B}^N) = \bigcup_{\sigma\in\mathbb{B}^N} T_\sigma(\mathbb{B}^N)$ the tangent bundle. When no confusion arises, if $X, Y \in T_\sigma(\mathbb{B}^N)$, we simply write $|X|$ and $\langle X,Y\rangle$ instead of the norm $|X|_\sigma$ and inner product $g_\sigma(X,Y) = \langle X,Y\rangle_\sigma$, respectively. If $C_0^\infty(\mathbb{B}^N)$ denotes, as customary, the space of real-valued compactly supported smooth functions on $\mathbb{B}^N$, we set

$$\|u\| := \left(\int_{\mathbb{B}^N}|\nabla_H u(\sigma)|^2 d\mu + \int_{\mathbb{B}^N}|u(\sigma)|^2 d\mu\right)^{1/2}, \tag{2.1}$$

for every $u \in C_0^\infty(\mathbb{B}^N)$, where ∇_H is the covariant derivative of u and $d\mu$ is the Riemannian measure on $\mathbb{B}^N$. In a direct form

$$\nabla_H = \left(\frac{(1-|\sigma|^2)}{2}\right)^2\nabla, \quad \text{and} \quad |\nabla_H u(\sigma)| = \left(\frac{(1-|\sigma|^2)}{2}\right)^2\sqrt{\langle\nabla u,\nabla u\rangle},$$

where ∇ denotes the Euclidean gradient.

The space $H^{1,2}(\mathbb{B}^N)$ is defined to be the completion of $C_0^\infty(\mathbb{B}^N)$ with respect to the norm (2.1) and it turns out to be a Hilbert space equipped with the inner product

$$\langle u, v\rangle := \int_{\mathbb{B}^N} \langle \nabla_H u(\sigma), \nabla_H v(\sigma)\rangle \, d\mu + \int_{\mathbb{B}^N} u(\sigma)v(\sigma) d\mu, \tag{2.2}$$

for every $u, v \in H^{1,2}(\mathbb{B}^N)$.

Referring to Hoffman and Spruck [21], the Sobolev embedding $H^{1,2}(\mathbb{B}^N) \hookrightarrow L^\nu(\mathbb{B}^N)$ is continuous (but not compact) for every $\nu \in [2, 2^*]$, where $2^* = 2N/(N-2)$. In the light of this result, we indicate by c_ν the positive constant

$$c_\nu := \sup_{u\in H^{1,2}(\mathbb{B}^N)\setminus\{0\}} \frac{\left(\displaystyle\int_{\mathbb{B}^N} |u(\sigma)|^\nu d\mu\right)^{1/\nu}}{\left(\displaystyle\int_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu + \int_{\mathbb{B}^N} |u(\sigma)|^2 d\mu\right)^{1/2}},$$

hence $c_\nu := \sup_{u\in H^{1,2}(\mathbb{B}^N)\setminus\{0\}} \frac{\|u\|_\nu}{\|u\|}$, where $\|\cdot\|_\nu$ denote, as usual, the L^ν-norm on $\mathbb{B}^N$.

The following result will be crucial in the sequel.

Proposition 2. *Let $\varrho : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and $u \in H^{1,2}(\mathbb{B}^N)$. If $\varrho \circ u \in L^2(\mathbb{B}^N)$, then $\varrho \circ u \in H^{1,2}(\mathbb{B}^N)$ and*

$$|\nabla_H(\varrho \circ u)(\sigma)| = |\varrho'(u(\sigma))| \, |\nabla_H u(\sigma)|,$$

for a.e. $\sigma \in \mathbb{B}^N$.

See [16, Proposition 2.5, page 24] for a detailed proof valid on a smooth complete Riemannian manifold.

Since problem (1.1) is settled in the entire non–compact space $\mathbb{B}^N$, in the next section we will adopt a group theoretical approach to identify suitable symmetric subspaces of $H^{1,2}(\mathbb{B}^N)$ for which the compactness of the embedding in $L^\nu(\mathbb{B}^N)$ can be regained.

3. Isometric invariant functions

Let $SO(N)$ be the special orthogonal group in dimension $N \geq 3$ and let $\cdot : \mathscr{G} \times \mathbb{B}^N \to \mathbb{B}^N$ be the natural multiplicative action of the group $SO(N)$ on $\mathbb{B}^N$. Furthermore, let us consider the family of subgroups of $SO(N)$ given by

$$\mathscr{F} := \left\{ \mathscr{G} \subseteq SO(N) : \mathscr{G} := \prod_{j=1}^{\ell} SO(N_j), \text{ where } N_j \geq 2,\ j = 1, \dots, \ell, \text{ and } \sum_{j=1}^{\ell} N_j = N \right\},$$

where $SO(N_j)$ denotes here the special orthogonal group in dimension N_j, for every $j = 1, \dots, \ell$. The action $\circledast_{\mathscr{G}} : \mathscr{G} \times H^{1,2}(\mathbb{B}^N) \to H^{1,2}(\mathbb{B}^N)$ of a subgroup $\mathscr{G} \in \mathscr{F}$ on $H^{1,2}(\mathbb{B}^N)$ is given, as usual, by

$$g \circledast_{\mathscr{G}} u(\sigma) := u(g^{-1} \cdot \sigma), \quad \text{for a.e. } \sigma \in \mathbb{B}^N, \tag{3.1}$$

for every $g \in \mathscr{G}$ and $u \in H^{1,2}(\mathbb{B}^N)$.

Denote by

$$H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) := \left\{ u \in H^{1,2}(\mathbb{B}^N) : g \circledast_{\mathscr{G}} u = u \text{ for every } g \in \mathscr{G} \right\}$$

the subspace of $\mathscr{G}$–invariant functions of $H^{1,2}(\mathbb{B}^N)$.

By using a recent embedding result à la Lions due to Skrzypczak and Tintarev [49, Theorem 1.3 and Proposition 3.1] we have the following compactness argument.

Theorem 3. *Let $(\mathbb{B}^N, g_{ij})$ be the N–dimensional homogeneous Poincaré ball model and let $\mathscr{G} \in \mathscr{F}$. Then, the embedding*

$$H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) \hookrightarrow L^\nu(\mathbb{B}^N)$$

is compact for any $\nu \in (2, 2^*)$.

See [12] for related topics and additional comments and remarks.

Finally, let us recall the well known principle of symmetric criticality of Palais. A group $(\mathscr{H}, *)$ acts continuously on a real Banach space X by an application $(\tau, u) \mapsto \tau \circledast_{\mathscr{H}} u$ from $\mathscr{H} \times X$ to X if this map itself is continuous on $\mathscr{H} \times X$ and satisfies

(i_1) $id_{\mathscr{H}} \circledast_{\mathscr{H}} u = u$ for every $u \in X$, where $id_{\mathscr{H}} \in \mathscr{H}$ is the identity element of $\mathscr{H}$;
(i_2) $(\tau_1 * \tau_2) \circledast_{\mathscr{H}} u = \tau_1 \circledast_{\mathscr{H}} (\tau_2 \circledast_{\mathscr{H}} u)$ for every $\tau_1, \tau_2 \in \mathscr{H}$ and $u \in X$;
(i_3) $u \mapsto \tau \circledast_{\mathscr{H}} u$ is linear for every $\tau \in \mathscr{H}$.

Set

$$Fix_{\mathscr{H}}(X) := \{u \in X : \tau \circledast_{\mathscr{H}} u = u \text{ for every } \tau \in \mathscr{H}\}.$$

A functional $\mathcal{J} : X \to \mathbb{R}$ is said to be $\mathscr{H}$–invariant if

$$\mathcal{J}(\tau \circledast_{\mathscr{H}} u) = \mathcal{J}(u),$$

for every $u \in X$.

With the above notations, according to [40], the following result is valid.

Theorem 4. *Let X be a real Banach space, $\mathscr{H}$ be a compact topological group acting continuously on X by a map $\circledast_{\mathscr{H}} : \mathscr{H} \times X \to X$, and $\mathcal{J} : X \to \mathbb{R}$ be a $\mathscr{H}$–invariant C^1–function. If $u \in Fix_{\mathscr{H}}(X)$ is a critical point of the restriction $\mathcal{J}_{|Fix_{\mathscr{H}}(X)}$, then $u \in X$ is also a critical point of $\mathcal{J}$.*

For details and comments we refer to [11] and [9, Section 5]. See also [30,35,43] for related topics and results.

4. The main result

The main result reads as follows.

Theorem 5. *Let $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ be a nonnegative and not identically zero radially symmetric map with respect to the origin $\sigma_0 \in \mathbb{B}^N$. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:*

(f_0) *There are two real sequences $(\xi_k)_k$ and $(\zeta_k)_k$, with $\lim_{k\to\infty} \zeta_k = 0$ such that $0 \leq \xi_k < \zeta_k$ for every $k \in \mathbb{N}$, and*

$$F(\xi_k) = \sup_{t\in[\xi_k,\zeta_k]} F(t),$$

where $F(t) := \int_0^t f(s)ds$;

(f_1) *There exist a constant $M > 0$ and a sequence $(\eta_k)_k \subset (0, \infty)$, with $\lim_{k\to\infty} \eta_k = 0$ such that*

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty,$$

and

$$\inf_{t\in[0,\eta_k]} F(t) \geq -MF(\eta_k).$$

Then, for every $\mathscr{G} \in \mathscr{F}$ *and* $\lambda > 0$, *there exists a sequence* $(u_k^{\mathscr{G}})_k \subset H^{1,2}(\mathbb{B}^N)$ *of nonnegative and not identically zero* $\mathscr{G}$*–invariant weak solutions of problem* (1.1) *such that*

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0.$$

Proof. Fix $\lambda > 0$ and $t_0 > 0$. Since the term f is continuous, there exists $\kappa > 0$ such that

$$|f(t)| \leq \kappa,$$

for every $0 \leq t \leq t_0$. Moreover, conditions (f_0) and (f_1) yield $f(0) = 0$. Indeed, by (f_0) the function F attains its maximum in $[\xi_k, \zeta_k]$ at the point ξ_k. Then

$$\lim_{t\to 0^+} \frac{\displaystyle\int_{\xi_k}^{\xi_k+t} f(s)ds}{t} = f(\xi_k) \leq 0.$$

Since f is continuous, passing to the limit, as $k \to \infty$, one has $\lim_{k\to\infty} f(\xi_k) = f(0) \leq 0$. On the other hand, let $(\eta_k)_k$ be the real sequence that appears in (f_1). Since

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty, \tag{4.1}$$

one has $f(0) \geq 0$. Indeed, arguing by contradiction, assume that $f(0) < 0$. Exploiting again the continuity of f, it follows that $f(t) < 0$, for every $t \in (0, \delta)$, for some $\delta > 0$. Consequently, $F(t) < 0$ for every $t \in (0, \delta)$. Bearing in mind that $\eta_k \to 0^+$, as $k \to \infty$, one has

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} \leq 0. \tag{4.2}$$

Clearly, inequality (4.2) contradicts (4.1). In conclusion, the function f vanishes at zero.

Without loss of generality suppose that $\max\{\eta_k, \zeta_k\} \leq t_0$, for every $k \in \mathbb{N}$, and define the truncated (continuous) function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) := \begin{cases} f(t_0) & \text{if} \quad t > t_0 \\ f(t) & \text{if} \quad 0 \leq t \leq t_0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Thanks again on our assumptions on the nonlinear term f and on the weight α, by using the continuous embedding result due to Hoffman and Spruck [21], the energy functional $J_\lambda : H^{1,2}(\mathbb{B}^N) \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{2\lambda}\|u\|^2 - \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(\sigma)} h(t)dt \right) d\mu \tag{4.3}$$

is of class $C^1(H^{1,2}(\mathbb{B}^N))$. Now, let us consider the auxiliary problem

$$\begin{cases} -\Delta_H u + u = \lambda\alpha(\sigma)h(u) & \text{in } \mathbb{B}^N \\ u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N). \end{cases} \tag{4.4}$$

Set

$$J_{\mathscr{G},\lambda}(u) := \frac{1}{\lambda}\Phi(u) - \Psi(u), \quad \forall\, u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N),$$

where

$$\Phi(u) := \frac{1}{2}\|u\|^2,$$

and

$$\Psi(u) := \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(\sigma)} h(t)dt \right) d\mu,$$

for every $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$.

Let us fix $q \in (2, 2^*)$. Since h is bounded and $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$, bearing in mind that the embedding of $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$ into $L^q(\mathbb{B}^N)$ is compact, the functional $J_{\mathscr{G},\lambda}$ is well-defined, sequentially weakly lower semicontinuous and continuously Gâteaux derivable in the Sobolev space $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. Hence, the weak solutions of problem (4.4) are exactly the critical points of the C^1 functional $J_{\mathscr{G},\lambda}$. Indeed, a weak solution of problem (4.4) is any function $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ such that

$$\begin{aligned} \int_{\mathbb{B}^N} \langle \nabla_H u(\sigma), \nabla_H v(\sigma) \rangle \, d\mu + \int_{\mathbb{B}^N} u(\sigma)\varphi(\sigma) d\mu & \\ - \lambda \int_{\mathbb{B}^N} \alpha(\sigma) h(u(\sigma))\varphi(\sigma) d\mu = 0, & \end{aligned}$$

for every $\varphi \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$.

Fix $k \in \mathbb{N}$ and define

$$\mathbb{E}^{\mathscr{G}}_k := \left\{ u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) : 0 \leq u(\sigma) \leq \zeta_k \text{ a.e. in } \mathbb{B}^N \right\}.$$

Step 1 - The functional $J_{\mathscr{G},\lambda}$ is bounded from below on $\mathbb{E}^{\mathscr{G}}_k$ and its infimum on $\mathbb{E}^{\mathscr{G}}_k$ is attained at $u^{\mathscr{G}}_k \in \mathbb{E}^{\mathscr{G}}_k$.

Since the set $\mathbb{E}^{\mathscr{G}}_k$ is closed and convex in $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, the same set is weakly closed in $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. Moreover, for every $u \in \mathbb{E}^{\mathscr{G}}_k$, one has

$$J_{\mathscr{G},\lambda}(u) \geq -\Psi(u) \geq -\kappa \|\alpha\|_1 \zeta_k, \tag{4.5}$$

taking into account that

$$\begin{aligned} \Psi(u) &\leq \int_{\mathbb{B}^N} \alpha(\sigma) \left| \int_0^{u(\sigma)} h(t)dt \right| d\mu \\ &\leq \kappa \int_{\mathbb{B}^N} \alpha(\sigma) u(\sigma) d\mu \leq \kappa \int_{\mathbb{B}^N} \alpha(\sigma) \xi_k d\mu \\ &\leq \kappa \|\alpha\|_1 \zeta_k, \end{aligned}$$

for every $u \in \mathbb{E}^{\mathscr{G}}_k$.

Let $\alpha^{\mathscr{G}}_k := \inf_{u \in \mathbb{E}^{\mathscr{G}}_k} J_{\mathscr{G},\lambda}(u)$. For every $j \in \mathbb{N}$, there exists $v_j \in \mathbb{E}^{\mathscr{G}}_k$ such that

$$\alpha^{\mathscr{G}}_k \leq J_{\mathscr{G},\lambda}(v_j) < \alpha^{\mathscr{G}}_k + \frac{1}{j}.$$

Hence, it follows that

$$\begin{aligned} \Phi(v_j) &= \lambda(\Psi(v_j) + J_\lambda(v_j)) \\ &\leq \lambda \left(\int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(\sigma)} h(t)dt \right) d\mu + \alpha^{\mathscr{G}}_k + \frac{1}{j} \right) \\ &\leq \lambda \left(\kappa \|\alpha\|_1 \zeta_k + \alpha^{\mathscr{G}}_k + 1 \right). \end{aligned}$$

Then $(v_j)_j$ is norm bounded in $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. This implies that there exists a subsequence $(v_{j_l})_l$ weakly convergent to $u^{\mathscr{G}}_k \in \mathbb{E}^{\mathscr{G}}_k$, being $\mathbb{E}^{\mathscr{G}}_k$ weakly closed. At this point, we exploit the weak sequentially lower semicontinuity of $J_{\mathscr{G},\lambda}$ obtaining that

$$\alpha_k^{\mathscr{G}} = \inf_{u\in\mathbb{E}_{\mathscr{G},k}} J_{\mathscr{G},\lambda}(u) \leq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) \leq \liminf_{l\to\infty} J_{\mathscr{G},\lambda}(v_{j_l}) = \alpha_k^{\mathscr{G}}.$$

Hence $J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) = \alpha_k^{\mathscr{G}}$ as affirmed.

Step 2 - For every $k \in \mathbb{N}$ one has that $u_k^{\mathscr{G}}(\sigma) \in [0,\xi_k]$ for a.e. $\sigma \in \mathbb{B}^N$.

In fact, fix $k \in \mathbb{N}$ and define $\varrho_k : \mathbb{R} \to \mathbb{R}$ as follows

$$\varrho_k(t) = \begin{cases} \xi_k & \text{if} \quad t > \xi_k \\ t & \text{if} \quad 0 \leq t \leq \xi_k \\ 0 & \text{if} \quad t < 0, \end{cases}$$

and consider the superposition operator $T_k : H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) \to H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, such that $u \mapsto T_k u$, where

$$T_k u(\sigma) := \varrho_k(u(\sigma)), \quad \text{a.e. in } \mathbb{B}^N,$$

for every $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. We notice that, for every $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, $T_k u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. Indeed, since ϱ_k is Lipschitz continuous, with $\varrho_k(0) = 0$, then $T_k u \in H^{1,2}(\mathbb{B}^N)$; see Proposition 2 in Section 2. We claim that $T_k u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. Indeed, since $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, one has

$$\begin{aligned} g \circledast_{\mathscr{G}} T_k u(\sigma) &= T_k u(g^{-1}\cdot\sigma) = (\varrho_k \circ u)(g^{-1}\cdot\sigma) \\ &= \varrho_k(u(g^{-1}\cdot\sigma)) = \varrho_k(u(\sigma)) \\ &= T_k u(\sigma) \end{aligned}$$

for a.e. $\sigma \in \mathbb{B}^N$ and $g \in \mathscr{G}$. More precisely, for every $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, one has $T_k u \in \mathbb{E}_k^{\mathscr{G}}$ for every $k \in \mathbb{N}$.

Now, set $v^{\star}_{\mathscr{G},k} := T_k u_k^{\mathscr{G}}$ and let

$$X_k^{\mathscr{G}} := \{\sigma \in \mathbb{B}^N : u_k^{\mathscr{G}}(x) \notin [0,\xi_k]\}.$$

If the Riemann measure $\mu(X_k^{\mathscr{G}}) = 0$ our conclusion is achieved. Otherwise, suppose that $\mu(X_k^{\mathscr{G}}) > 0$. Then, one has

$$\xi_k < u_k^{\mathscr{G}}(\sigma) \leq \zeta_k,$$

as well as

$$v^{\star}_{\mathscr{G},k}(\sigma) = T_k u_k^{\mathscr{G}}(\sigma) = \xi_k, \tag{4.6}$$

for a.e. $\sigma \in X_k^{\mathscr{G}}$. However, hypothesis (f_0) gives

$$\int_0^{u_k^{\mathscr{G}}(\sigma)} h(t)dt \leq \sup_{t\in[\xi_k,\zeta_k]} \int_0^t h(s)ds = \int_0^{\xi_k} h(t)dt = \int_0^{v^{\star}_{\mathscr{G},k}(\sigma)} h(t)dt,$$

for a.e. $\sigma \in X_k^{\mathscr{G}}$. Hence

$$\int_0^{u_k^{\mathscr{G}}(\sigma)} h(t)dt \leq \int_0^{v^{\star}_{\mathscr{G},k}(\sigma)} h(t)dt, \tag{4.7}$$

and $|\nabla v^{\star}_{k,\mathscr{G}}(\sigma)| = 0$ for a.e. $\sigma \in X_k^{\mathscr{G}}$. By (4.7), it follows that

$$\begin{aligned} \|v^{\star}_{\mathscr{G},k}\|^2 - \|u_k^{\mathscr{G}}\|^2 &= \int_{\mathbb{B}^N} (|v^{\star}_{\mathscr{G},k}(\sigma)|^2 - |u_k^{\mathscr{G}}(\sigma)|^2)d\mu + \int_{\mathbb{B}^N} (|\nabla_H v^{\star}_{\mathscr{G},k}(\sigma)|^2 - |\nabla_H u_k^{\mathscr{G}}(\sigma)|^2)d\mu \\ &= \int_{X_k^{\mathscr{G}}} (\xi_k^2 - (u_k^{\mathscr{G}}(\sigma))^2)d\mu - \int_{X_k^{\mathscr{G}}} |\nabla_H u_k^{\mathscr{G}}(\sigma)|^2 d\mu \\ &\leq -\int_{X_k^{\mathscr{G}}} |u_k^{\mathscr{G}}(\sigma) - \xi_k|^2 d\mu - \int_{X_k^{\mathscr{G}}} |\nabla_H v^{\star}_{\mathscr{G},k}(\sigma) - \nabla_H u_k^{\mathscr{G}}(\sigma)|^2 d\mu \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{B}^N} |v^\star_{\mathscr{G},k}(\sigma) - u^{\mathscr{G}}_k(\sigma)| d\mu - \int_{\mathbb{B}^N} |\nabla_H v^\star_{\mathscr{G},k}(\sigma) - \nabla_H u^{\mathscr{G}}_k(\sigma)|^2 d\mu \\
&= -\|v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k\|^2.
\end{aligned}
$$

Hence, the above inequality ensures that

$$
\begin{aligned}
J_{\mathscr{G},\lambda}(v^\star_{\mathscr{G},k}) - J_{\mathscr{G},\lambda}(u^{\mathscr{G}}_k) &= \frac{1}{2\lambda}(\|v^\star_{\mathscr{G},k}\|^2 - \|u^{\mathscr{G}}_k\|^2) - \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_{u^{\mathscr{G}}_k}^{v^\star_{\mathscr{G},k}} h(t)dt\right) d\mu \\
&\leq -\frac{1}{2\lambda}\|v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k\|^2 - \int_{X^{\mathscr{G}}_k} \alpha(\sigma)\left(\int_{u^{\mathscr{G}}_k}^{v^\star_{\mathscr{G},k}} h(t)dt\right) d\mu \\
&\leq -\frac{1}{2\lambda}\|v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k\|^2.
\end{aligned}
$$

Since $v^\star_{\mathscr{G},k} \in \mathbb{E}^{\mathscr{G}}_k$, it follows that $J_{\mathscr{G},\lambda}(v^\star_{\mathscr{G},k}) \geq J_{\mathscr{G},\lambda}(u^{\mathscr{G}}_k)$. Then

$$
\|v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k\| = 0,
$$

that is

$$
\|v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k\| = \left(\int_{X^{\mathscr{G}}_k} |\nabla(v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k)(\sigma)|^2 d\mu + \int_{X^{\mathscr{G}}_k} |(v^\star_{\mathscr{G},k} - u^{\mathscr{G}}_k)(\sigma)|^2 d\mu\right)^{1/2} = 0.
$$

Since $\mu(X^{\mathscr{G}}_k) > 0$, one has $u^{\mathscr{G}}_k(\sigma) = v^\star_{\mathscr{G},k}(\sigma) \in [0, \xi_k]$ a.e. in $\mathbb{B}^N$. Thus, the claim is proved.
Step 3 - For every $k \in \mathbb{N}$ one has that $u^{\mathscr{G}}_k$ is a local minimum point of functional $J_{\mathscr{G},\lambda}$ in the Sobolev space $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$.

To this end, let us fix $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, and let

$$
Z_{\mathscr{G},k} := \{\sigma \in \mathbb{B}^N : u(\sigma) \notin [0, \xi_k]\},
$$

for every $k \in \mathbb{N}$.

Now, if T_k is the operator defined above, set

$$
v^\star_k(\sigma) := T_k u(\sigma) = \begin{cases} \xi_k & \text{if } \; u(\sigma) > \xi_k \\ u(\sigma) & \text{if } \; 0 \leq u(\sigma) \leq \xi_k \\ 0 & \text{if } \; u(\sigma) < 0, \end{cases} \tag{4.8}
$$

for a.e. $\sigma \in \mathbb{B}^N$.

The definition of T_k yields

$$
\int_{v^\star_k(\sigma)}^{u(\sigma)} h(t)dt = 0,
$$

if $\sigma \in \mathbb{B}^N \setminus Z_{\mathscr{G},k}$. Furthermore, if $\sigma \in Z_{\mathscr{G},k}$, then the following alternatives hold:

(a) If $u(\sigma) \leq 0$, then

$$
\int_{v^\star_k(\sigma)}^{u(\sigma)} h(t)dt = \int_0^{u(\sigma)} h(t)dt = 0.
$$

(b) If $\xi_k < u(\sigma) \leq \zeta_k$, one has

$$
\begin{aligned}
\int_{v^\star_k(\sigma)}^{u(\sigma)} h(t)dt &= \int_0^{u(\sigma)} h(t)dt - \int_0^{v^\star_k(\sigma)} h(t)dt \\
&= \int_0^{u(\sigma)} h(t)dt - \int_0^{\xi_k} h(t)dt \\
&= \int_0^{u(\sigma)} h(t)dt - \sup_{t\in[\xi_k,\zeta_k]} \int_0^t h(s)ds \\
&\leq 0.
\end{aligned}
$$

(c) If $u(\sigma) > \zeta_k$, it follows that

$$\begin{aligned}\int_{v_k^\star(\sigma)}^{u(\sigma)} h(t)dt &= \int_{\xi_k}^{u(\sigma)} h(t)dt \\ &\leq \left|\int_{\xi_k}^{u(\sigma)} h(t)dt\right| \leq \kappa(u(\sigma)-\xi_k).\end{aligned}$$

Hence, the constant

$$C := \frac{\kappa}{\|\alpha\|_\infty} \sup_{\xi\geq\zeta_k} \frac{\xi-\xi_k}{(\xi-\xi_k)^q},$$

is finite and we have that

$$\int_{v_k^\star(\sigma)}^{u(\sigma)} h(t)dt \leq C\|\alpha\|_\infty |u(\sigma)-v_k^\star(\sigma)|^q,$$

a.e. in $\mathbb{B}^N$. Then, we can write

$$\int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_{v_k^\star(\sigma)}^{u(\sigma)} h(t)dt\right) d\mu \leq C\gamma^q \|u-v_k^\star\|^q, \tag{4.9}$$

where

$$\gamma := \sup_{u\in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)\setminus\{0\}} \frac{\left(\int_{\mathbb{B}^N} |u(\sigma)|^q d\mu\right)^{1/q}}{\|u\|} < +\infty.$$

Moreover, one has

$$\begin{aligned}\|u\|^2-\|v_k^\star\|^2 &= \int_{\mathbb{B}^N} (|u(\sigma)|^2-|v_k^\star(\sigma)|^2)d\mu + \int_{\mathbb{B}^N} (|\nabla_H u(\sigma)|^2-|\nabla_H v_k^\star(\sigma)|^2)d\mu \\ &= \int_{Z^-_{\mathscr{G},k}} |u(\sigma)|^2 d\mu + \int_{Z^+_{\mathscr{G},k}} |u(\sigma)^2-\xi_k^2|d\mu + \int_{Z_{\mathscr{G},k}} |\nabla_H u(\sigma)|^2 d\mu \\ &\geq \int_{Z^-_{\mathscr{G},k}} |u(\sigma)-v_k^\star(\sigma)|^2 d\mu + \int_{Z^+_{\mathscr{G},k}} |u(\sigma)-\xi_k|^2 d\mu \\ &\quad + \int_{Z_{\mathscr{G},k}} |\nabla_H u(\sigma)-\nabla_H v_k^\star(\sigma)|^2 d\mu \\ &= \|u-v_k^\star\|^2,\end{aligned}$$

where

$$Z^-_{\mathscr{G},k} := \{\sigma\in Z_{\mathscr{G},k} : u(\sigma)<0\} \text{ and } Z^+_{\mathscr{G},k} := \{\sigma\in Z_{\mathscr{G},k} : u(\sigma)>0\}.$$

Taking into account the above computations, it follows that

$$\begin{aligned}J_{\mathscr{G},\lambda}(u)-J_{\mathscr{G},\lambda}(v_k^\star) &= \frac{1}{2\lambda}(\|u\|^2-\|u_k^\star\|^2) - \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_{v_k^\star(\sigma)}^{u(\sigma)} h(t)dt\right) d\mu \\ &= \frac{1}{2\lambda}\|u-v_k^\star\|^2 - \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_{v_k^\star(\sigma)}^{u(\sigma)} h(t)dt\right) d\mu \\ &\geq \frac{1}{2\lambda}\|u-v_k^\star\|^2 - C\gamma^q\|u-v_k^\star\|^q.\end{aligned}$$

Since $v_k^\star \in \mathbb{E}_k^{\mathscr{G}}$, it follows that $J_{\mathscr{G},\lambda}(v_k^\star) \geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}})$. Then, we have

$$J_{\mathscr{G},\lambda}(u) \geq J_{\mathscr{G},\lambda}(u_k^{\mathscr{G}}) + \|u-v_{\mathscr{G}}^\star\|^2 \left(\frac{1}{2\lambda} - C\gamma^q\|u-v_k^\star\|^{q-2}\right).$$

Now, for every $k \in \mathbb{N}$, the operator $T_k : H^{1,2}_{\mathscr{G}}(\mathbb{B}^N) \to \mathbb{E}^{\mathscr{G}}_k$ is continuous on account of Proposition 2 and [25, Theorem 1, page 219]. Since $q > 2$ and

$$\begin{aligned}\|u - v^{\star}_k\| &\leq \|u - u^{\mathscr{G}}_k\| + \|u^{\mathscr{G}}_k - v^{\star}_k\| \\ &= \|u - u^{\mathscr{G}}_k\| + \|T_k u^{\mathscr{G}}_k - v^{\star}_k\|,\end{aligned}$$

there exists $\beta > 0$ such that

$$\|u - v^{\star}_k\|^{q-2} \leq \frac{1}{4\lambda C \gamma^q},$$

for every $u \in H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$ with $\|u - u^{\mathscr{G}}_k\| < \beta$. Hence, if $\|u - u^{\mathscr{G}}_k\| < \beta$, it follows that

$$\begin{aligned}J_{\mathscr{G},\lambda}(u) &\geq J_{\mathscr{G},\lambda}(u^{\mathscr{G}}_k) + \frac{1}{4\lambda}\|u - v^{\star}_k\|^2 \\ &\geq J_{\mathscr{G},\lambda}(u^{\mathscr{G}}_k),\end{aligned}$$

that is, $u^{\mathscr{G}}_k$ is a local minimum of $J_{\mathscr{G},\lambda}$ in $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$.

Step 4 - If

$$\alpha^{\mathscr{G}}_k := \inf_{u \in \mathbb{E}^{\mathscr{G}}_k} J_{\mathscr{G},\lambda}(u), \tag{4.10}$$

then $\lim_{k\to\infty} \alpha^{\mathscr{G}}_k = \lim_{k\to\infty} \|u^{\mathscr{G}}_k\| = 0$.

Since $u^{\mathscr{G}}_k \in \mathbb{E}^{\mathscr{G}}_k$ and $\alpha^{\mathscr{G}}_k = J_{\mathscr{G},\lambda}(u^{\mathscr{G}}_k)$, one has

$$\begin{aligned}\int_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu + \int_{\mathbb{B}^N} |u(\sigma)|^2 d\mu &= 2\lambda \left(\Psi(u) + J_{\mathscr{G},\lambda}(u)\right) \\ &= 2\lambda \left(\int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(\sigma)} h(t)dt \right) d\mu + \alpha^{\mathscr{G}}_k \right) \\ &\leq 2\lambda \left(\kappa \|\alpha\|_1 \zeta_k + \alpha^{\mathscr{G}}_k \right).\end{aligned} \tag{4.11}$$

Now (4.5) holds and

$$-\kappa \|\alpha\|_1 \zeta_k \leq \alpha^{\mathscr{G}}_k = \inf_{u \in \mathbb{E}^{\mathscr{G}}_k} J_{\mathscr{G},\lambda}(u) \leq 0, \tag{4.12}$$

taking into account that the identically zero function $u_0 \equiv 0$ belongs to $\mathbb{E}^{\mathscr{G}}_k$ and $J_{\mathscr{G},\lambda}(0) = 0$. By (4.12), since $\zeta_k \to 0$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} \alpha^{\mathscr{G}}_k = 0. \tag{4.13}$$

Hence, inequality (4.11) yields

$$\lim_{k\to\infty} \|u^{\mathscr{G}}_k\| = 0.$$

Step 5 - Let $\alpha^{\mathscr{G}}_k$ be given as in (4.10). Then

$$\alpha^{\mathscr{G}}_k < 0,$$

for every $k \in \mathbb{N}$.

To prove this, let us fix $k \in \mathbb{N}$. We introduce a class of functions belonging to $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$ that will be crucial along the proof of the main step. Since $\alpha \in L^{\infty}(\mathbb{B}^N) \setminus \{0\}$ is nonnegative in $\mathbb{B}^N$, there are real numbers $\rho > r > 0$ and $\alpha_0 > 0$ such that

$$\operatorname*{essinf}_{\sigma \in A^{\rho}_r} \alpha(\sigma) \geq \alpha_0 > 0. \tag{4.14}$$

Furthermore, for every $0 < a < b$, define the following annular domain

$$A^b_a = B_H(a + b) \setminus B_H(b - a),$$

where
$$d_H(\sigma) := 2\int_0^{|\sigma|} \frac{dt}{1-t^2} = \log\left(\frac{1+|\sigma|}{1-|\sigma|}\right),$$
denotes the geodesic distance of the point $\sigma \in \mathbb{B}^N$ from the origin σ_0 of $\mathbb{B}^N$.

With the above notations, fix $\varepsilon \in (0,1)$ and set $w^\varepsilon_{\rho,r} \in H^{1,2}(\mathbb{B}^N)$ given by
$$w^\varepsilon_{\rho,r}(\sigma) := \begin{cases} 0 & \text{if } \sigma \in \mathbb{B}^N \setminus A^\rho_r \\ 1 & \text{if } \sigma \in A^\rho_{\varepsilon r} \\ \dfrac{r - |d_H(\sigma)-\rho|}{(1-\varepsilon)r} & \text{if } \sigma \in A^\rho_r \setminus A^\rho_{\varepsilon r}, \end{cases} \tag{4.15}$$
for every $\sigma \in \mathbb{B}^N$. Since the group $\mathscr{G}$ is a compact connected subgroup of the isometry group $\mathrm{Isom}_g(\mathbb{B}^N)$ such that $Fix_{\mathscr{G}}(\mathbb{B}^N) = \{\sigma_0\}$, one has that the function $w^\varepsilon_{\rho,r} \in H^{1,2}(\mathbb{B}^N)$, given in (4.15), belongs to $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$. Direct computations yield

(j_1) $\mathrm{supp}(w^\varepsilon_{\rho,r}) \subseteq A^\rho_r$;
(j_2) $\|w^\varepsilon_{\rho,r}\|_\infty \le 1$;
(j_3) $w^\varepsilon_{\rho,r}(\sigma) = 1$ for every $\sigma \in A^\rho_{\varepsilon r}$.

Set $g_\mu : (0,1) \to (0,\infty)$ be the real function defined by
$$g_\mu(\varepsilon) := \frac{\mu(A^\rho_{\varepsilon r})}{\mu(A^\rho_r \setminus A^\rho_{\varepsilon r})}, \quad \varepsilon \in (0,1),$$
where μ is the Riemann measure on $\mathbb{B}^N$. Clearly, if $\varepsilon \to 1^-$ then $g_\mu(\varepsilon) \to \infty$ as well as $g_\mu(\varepsilon) \to 0$ if $\varepsilon \to 0^+$. Thus, there exists $\varepsilon_0 \in (0,1)$ such that
$$\frac{\mu(A^\rho_{\varepsilon_0 r})}{\mu(A^\rho_r \setminus A^\rho_{\varepsilon_0 r})} = M+1,$$
where $M > 0$ is given in condition (f_1).

By the former condition of (f_1), there exists $j_0 \in \mathbb{N}$ such that $\eta_j \le \zeta_k$ and
$$\int_0^{\eta_j} h(t)dt > \frac{1}{\lambda}\frac{(M+1)\|w^{\varepsilon_0}_{\rho,r}\|}{\alpha_0 \mu(A^\rho_{\varepsilon_0 r})}\frac{\eta_j^2}{2},$$
for every $j \ge j_0$. On account of (j_1)–(j_3), the latter condition of (f_1) and (4.14) yields
$$\begin{aligned}
-\frac{\Psi(\eta_j w^{\varepsilon_0}_{\rho,r})}{\|\eta_j w^{\varepsilon_0}_{\rho,r}\|^2} &= -\frac{\displaystyle\int_{A^\rho_{\varepsilon_0 r}} \alpha(\sigma)\left(\int_0^{\eta_j} h(t)dt\right)d\mu}{\eta_j^2\|w^{\varepsilon_0}_{\rho,r}\|^2} - \frac{\displaystyle\int_{A^\rho_r\setminus A^\rho_{\varepsilon_0 r}} \alpha(\sigma)\left(\int_0^{\eta_j w^{\varepsilon_0}_{\rho,r}(\sigma)} h(t)dt\right)d\mu}{\eta_j^2\|w^{\varepsilon_0}_{\rho,r}\|^2} \\
&\le -\alpha_0\frac{\displaystyle\int_{A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} h(t)dt\right)d\mu + \int_{A^\rho_r\setminus A^\rho_{\varepsilon_0 r}} \inf_{t\in[0,\eta_j]}\left(\int_0^t h(s)ds\right)d\mu}{\eta_j^2\|w^{\varepsilon_0}_{\rho,r}\|^2} \\
&\le \alpha_0\frac{\displaystyle M\int_{A^\rho_r\setminus A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} h(t)dt\right)d\mu - \int_{A^\rho_{\varepsilon_0 r}} \left(\int_0^{\eta_j} h(t)dt\right)d\mu}{\eta_j^2\|w^{\varepsilon_0}_{\rho,r}\|^2} \\
&= -\left(\frac{\mu(A^\rho_{\varepsilon_0 r})}{M+1}\right)\frac{\alpha_0}{\|w^{\varepsilon_0}_{\rho,r}\|^2}\frac{\displaystyle\int_0^{\eta_j} h(t)dt}{\eta_j^2} \\
&< -\frac{1}{2\lambda},
\end{aligned}$$
for every $j \ge j_0$. Whence $\eta_j w^{\varepsilon_0}_{\rho,r} \in \mathbb{E}_{k,\mathscr{G}}$ and $J_{\mathscr{G},\lambda}(\eta_j w^{\varepsilon_0}_{\rho,r}) < 0$, for $j \ge j_0$. Hence $\alpha_k := \inf_{u\in\mathbb{E}_{k,\mathscr{G}}} J_{\mathscr{G},\lambda}(u) < 0$ as claimed.

Proof of Theorem 5 concluded. Taking into account that $\|u_k\|_\infty \to 0$, as $k \to \infty$, there exists a subsequence of $(u_k^{\mathscr{G}})_k \subset H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, still denoted by $(u_k)_k$, of pairwise distinct elements with

$$0 \le \|u_k^{\mathscr{G}}\|_\infty \le t_0,$$

that weakly solves problem (4.4). Since the fixed point set of $H^{1,2}(\mathbb{B}^N)$ under the action of the group $\mathscr{G}$ is exactly $H^{1,2}_{\mathscr{G}}(\mathbb{B}^N)$, the symmetric criticality principle recalled in Theorem 4 ensures that $(u_k^{\mathscr{G}})_k \subset H^{1,2}(\mathbb{B}^N)$ is a sequence of critical points for the C^1–functional J_λ, i.e. weak solutions of (1.1). $\square$

4.1. Further remarks and perspectives

We conclude this paper with several remarks and examples.

Remark 1. We notice that, in order to apply the Palais principle in Theorem 5, the functional J_λ defined in (4.3) needs to be $\mathscr{G}$–invariant. To prove this, let $u \in H^{1,2}(\mathbb{B}^N)$ and $g \in \mathscr{G}$ be fixed. Since $g \in \mathscr{G} \subseteq SO(N)$ is an isometry, on account of (3.1), it follows the chain rule

$$\nabla_H(g \circledast_{\mathscr{G}} u)(\sigma) = Dg_{g^{-1}\cdot\sigma} \nabla_H u(g^{-1} \cdot \sigma), \tag{4.16}$$

for a.e. $\sigma \in \mathbb{B}^N$, where $Dg_{g^{-1}\cdot\sigma} : T_{g^{-1}\cdot\sigma}(\mathbb{B}^N) \to T_\sigma(\mathbb{B}^N)$ denotes the differential of $g \in \mathscr{G}$ at the point $g^{-1}\cdot\sigma$. Setting $z := g^{-1} \cdot \sigma$, it follows that

$$\begin{aligned} \|g \circledast_{\mathscr{G}} u\|^2 &= \int_{\mathbb{B}^N} \left(|\nabla_H(g \circledast_{\mathscr{G}} u)(\sigma)|_\sigma^2 + |(g \circledast_{\mathscr{G}} u)(\sigma)|^2\right) d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} \left(|\nabla_H u(g^{-1}\cdot\sigma)|^2_{g^{-1}\cdot\sigma} + |u(g^{-1}\cdot\sigma)|^2\right) d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} \left(|\nabla_H u(z)|_z^2 + |u(z)|^2\right) d\mu(z) \\ &= \|u\|^2, \end{aligned} \tag{4.17}$$

where we have made use of (4.16) and the fact that the map $Dg_{g^{-1}\cdot\sigma}$ is inner product-preserving. Moreover, since $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is radially symmetric respect to the origin $\sigma_0 \in \mathbb{B}^N$, one has

$$\begin{aligned} \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{(g\circledast_{\mathscr{G}} u)(\sigma)} h(t)dt\right) d\mu(\sigma) &= \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(g^{-1}\cdot\sigma)} h(t)dt\right) d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} \alpha(z) \left(\int_0^{u(z)} h(t)dt\right) d\mu(z). \end{aligned} \tag{4.18}$$

Thus, by (4.17) and (4.18) we infer

$$J_\lambda(g \circledast_{\mathscr{G}} u) = \frac{1}{2\lambda}\|g \circledast_{\mathscr{G}} u\|^2 - \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{(g\circledast_{\mathscr{G}} u)(\sigma)} h(t)dt\right) d\mu = J_\lambda(u),$$

which proves the claim; see [12] and references therein for additional comments and remarks.

Remark 2. We point out that Theorem 1 in Introduction is a direct consequence of Theorem 5. Indeed if (f_0') holds, condition (f_0) is automatically verified. On the other hand, hypothesis (f_1') implies (f_1). To prove this, assume that condition (f_1') holds. Since $\limsup_{t\to 0^+} F(t)/t^2 = \infty$, there exists a sequence $(\eta_k)_k \subset (0, \infty)$ such that $\lim_{k\to\infty} \eta_k = 0$ and

$$\limsup_{k\to\infty} \frac{F(\eta_k)}{\eta_k^2} = \infty.$$

Moreover, owing to $\liminf_{t\to 0^+} F(t)/t^2 > -\infty$, there exist $M, \delta > 0$ such that $F(t) \geq -Mt^2$, for every $t \in (0,\delta)$. Since $\lim_{k\to\infty} \eta_k = 0$, there is $\nu \in \mathbb{N}$ such that $\eta_k \in (0,\delta)$ and $F(\eta_k) \geq -MF(\eta_k)$, for every $k \geq \nu$. Thus, condition (f_1) is verified as claimed.

The following model equation illustrates how our result can be applied.

Example 1. Let us consider the problem

$$\begin{cases} -\Delta_H u + u = \lambda \left(\dfrac{1-|\sigma|^2}{2}\right)^N f(u) & \text{in } \mathbb{B}^N \\ u \in H^{1,2}(\mathbb{B}^N), \end{cases} \tag{4.19}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t) := \begin{cases} 15\sqrt[3]{t^2}\sin\dfrac{1}{\sqrt[3]{t}} - 3\sqrt[3]{t}\cos\dfrac{1}{\sqrt[3]{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Owing to Theorem 5, for every $\mathscr{G} \in \mathscr{F}$ and $\lambda > 0$, there exists a sequence $(u_k^{\mathscr{G}})_k \subset H^{1,2}(\mathbb{B}^N)$ of nonnegative and nonzero $\mathscr{G}$–invariant weak solutions of problem (4.19) such that

$$\lim_{k\to\infty} \|u_k^{\mathscr{G}}\| = \lim_{k\to\infty} \|u_k^{\mathscr{G}}\|_\infty = 0.$$

Now, a direct computation ensures that

$$F(t) := \begin{cases} 9\sqrt[3]{t^5}\sin\dfrac{1}{\sqrt[3]{t}} & \text{if } t > 0 \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = -\infty.$$

Thus, Theorem 1 cannot be applied in this case; see [1, Example 3.1] for additional comments and remarks.

Remark 3. As a biproduct of the results contained in [10] the bottom of the spectrum of $-\Delta_H$ in $\mathbb{B}^N$ is given by

$$\lambda_1(-\Delta_H) = \inf_{H^{1,2}(\mathbb{B}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{B}^N} |\nabla u(\sigma)|^2 d\mu}{\displaystyle\int_{\mathbb{B}^N} |u(\sigma)|^2 d\mu} = \frac{(N-1)^2}{4},$$

see also [24] for related topics and direct applications. This variational characterization of the first eigenvalue $\lambda_1(-\Delta_H)$ ensures that, for every $\mu < \frac{(N-1)^2}{4}$, the norm

$$\|u\|_\mu := \left(\int_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu - \mu\int_{\mathbb{B}^N} |u(\sigma)|^2 d\mu\right)^{1/2}, \qquad u \in C_0^\infty(\mathbb{B}^N)$$

is equivalent to the $H^{1,2}$–norm displayed in (2.1). Taking into account the above remarks it is easily seen that the validity of the main results can be checked for the following semilinear problem

$$\begin{cases} -\Delta_H u - \mu u = \lambda\alpha(\sigma)f(u) & \text{in } \mathbb{B}^N \\ u \in H^{1,2}(\mathbb{B}^N), \end{cases} \tag{4.20}$$

where $\mu < \frac{(N-1)^2}{4}$.

Remark 4. Problem (1.1) is a reasonably useful generalization of most studied elliptic problems with subcritical nonlinearities, which naturally arise in different branches of mathematics. For instance, an important incentive to the study of Kirchhoff-type problems was recently provided in [3,4,13,14], and [26–28,44]. Along this direction, in [33] Theorem 5 has been proved for Kirchhoff equations on the hyperbolic space $\mathbb{B}^N$; see [17–20,31] as well as [7,23,34] for related topics.

Acknowledgments

The authors warmly thank the anonymous referees for their useful and nice comments on the paper. The first author was partially supported by the Italian MIUR project

Variational methods, with applications to problems in mathematical physics and geometry (2015KB9WPT 009).

References

[1] G. Anello, G. Cordaro, Infinitely many arbitrarily small positive solutions for the Dirichlet problem involving the p–Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002) 511–519, 3; Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 1, Corrigendum.

[2] G. Anello, G. Cordaro, Infinitely many positive solutions for the Neumann problem involving the p–Laplacian, Colloq. Math. 97 (2003) 221–231.

[3] G. Autuori, F. Colasuonno, P. Pucci, On the existence of stationary solutions for higher order p–Kirchhoff problems via variational methods, Commun. Contemp. Math. 16 (2014) 43, 1450002.

[4] G. Autuori, P. Pucci, C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, Arch. Ration. Mech. Anal. 196 (2010) 489–516.

[5] R. Benedetti, C. Petronio, Lectures on Hyperbolic Geometry, Springer, 1992.

[6] L.P. Bonorino, P.K. Klaser, Bounded λ–harmonic functions in domains of $\mathbb{H}^N$ with asymptotic boundary with fractional dimension, J. Geom. Anal. 28 (2018) 2503–2521.

[7] S. Bordoni, P. Pucci, Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group, Bull. Sci. Math. 146 (2018) 50–88.

[8] H. Brézis, Analyse fonctionelle. Théorie et applications, Masson, Paris, 1983.

[9] M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, Ann. Mat. Pura Appl. 195 (2016) 2099–2129.

[10] E.B. Davies, Heat Kernel and Spectral Theory, Cambridge University Press, 1989.

[11] D.C. de Morais Filho, M.A.S. Souto, J.M. do Ó, A compactness embedding lemma, a principle of symmetric criticality and applications to elliptic problems, Proyecciones 19 (2000) 1–17.

[12] Cs. Farkas, A. Kristály, Schrödinger–Maxwell systems on non–compact Riemannian manifolds, Nonlinear Anal. RWA 31 (2016) 473–491.

[13] A. Fiscella, A fractional Kirchhoff problem involving a singular term and a critical nonlinearity, Adv. Nonlinear Anal. 8 (1) (2019) 645–660.

[14] A. Fiscella, P. Pucci, B. Zhang, p–fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities, Adv. Nonlinear Anal. 8 (1) (2019) 1111–1131.

[15] D. Ganguly, K. Sandeep, Nondegeneracy of positive solutions of a semilinear elliptic problem in the hyperbolic space, Commun. Contemp. Math. 17 (01) (2015) 1450019.

[16] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, in: Courant Lecture Notes in Mathematics, New York, 1999.

[17] E. Hebey, Compactness and the Palais–Smale property for critical Kirchhoff equations in closed manifolds, Pacific J. Math. 280 (1) (2016) 41–50.

[18] E. Hebey, Multiplicity of solutions for critical Kirchhoff type equations, Comm. Partial Differential Equations 41 (6) (2016) 913–924.

[19] E. Hebey, Stationary kirchhoff equations with powers, Adv. Calc. Var. 11 (2) (2018) 139–160.

[20] E. Hebey, P.-D. Thizy, Stationary Kirchhoff systems in closed high dimensional manifolds, Commun. Contemp. Math. 18 (2) (2016) 53, 1550028.

[21] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, Comm. Pure Appl. Math. 27 (1974) 715–727.

[22] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, in: Mathématiques & Applications, Springer-Verlag, Paris, 1993.

[23] J.-L. Lions, On some equations in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, 1997), in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, 1978, pp. 284–346.

[24] G. Mancini, K. Sandeep, On a semilinear elliptic equation in $\mathbb{H}^N$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (4) (2008) 635–671.
[25] M. Marcus, V. Mizel, Every superposition operator mapping one Sobolev space into another is continuous, J. Funct. Anal. 33 (1979) 217–229.
[26] X. Mingqi, V.D. R˘ adulescu, B. Zhang, Fractional Kirchhoff problems with critical Trudinger–Moser nonlinearity, Calc. Var. Partial Differential Equations 58 (2) (2019) 27, Art. 57.
[27] X. Mingqi, V.D. Rădulescu, B. Zhang, Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions, Nonlinearity 31 (7) (2018) 3228–3250.
[28] X. Mingqi, V.D. Rădulescu, B. Zhang, A critical fractional Choquard–Kirchhoff problem with magnetic field, Commun. Contemp. Math. 21 (4) (2019) 36, 1850004.
[29] G. Molica Bisci, Variational problems on the sphere, in: Recent Trends in Nonlinear Partial Differential Equations II. Stationary Problems, Contemp. Math. 595 (2013) 273–291.
[30] G. Molica Bisci, A group–theoretical approach for nonlinear Schrödinger equations, Adv. Calc. Var. 54 (2015) 2985–3008.
[31] G. Molica Bisci, Kirchhoff–type problems on a geodesic ball of the hyperbolic space, Nonlinear Anal. 186 (2019) 55–73.
[32] G. Molica Bisci, P. Pizzimenti, Sequences of weak solutions for non–local elliptic problems with Dirichlet boundary condition, Proc. Edinb. Math. Soc. 57 (2014) 779–809.
[33] G. Molica Bisci, P. Pucci, Nonlinear problems with lack of compactness, in: Series in Nonlinear Analysis and Applications, De Gruyter, in press.
[34] G. Molica Bisci, P. Pucci, Critical Dirichlet problems on $\mathcal{H}$ domains of Carnot groups, Two nonlinear days in Urbino 2017, Electron. J. Differ. Equ. Conf. 25 (2018) 179–196, Special Volume dedicated to the memory of Anna Aloe.
[35] D. Mugnai, P. Pucci, Maximum principles for inhomogeneous elliptic inequalities on complete Riemannian manifolds, Adv. Nonlinear Stud. 9 (2009) 429–452.
[36] F.I. Njoku, P. Omari, F. Zanolin, Multiplicity of positive radial solutions of a quasilinear elliptic problem in a ball, Adv. Differential Equations 5 (2000) 1545–1570.
[37] F. Obersnel, P. Omari, Positive solutions of elliptic problems with locally oscillating nonlinearities, J. Math. Anal. Appl. 323 (2006) 913–929.
[38] P. Omari, F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential, Comm. Partial Differential Equations 21 (1996) 721–733.
[39] P. Omari, F. Zanolin, An elliptic problem with arbitrarily small positive solutions, Electron. J. Differ. Equ. Conf. 5 (2000) 301–308.
[40] R.S. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–30.
[41] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis–Theory and Methods, in: Springer Monographs in Mathematics, Springer, Cham, 2019.
[42] P. Pucci, M. Rigoli, Entire solutions of singular elliptic inequalities on complete manifolds, Discrete Contin. Dyn. Syst. 20 (2008) 115–137.
[43] P. Pucci, V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, Boll. Unione Mat. Ital. (9) (3) (2010) 543–584.
[44] P. Pucci, S. Saldi, Asymptotic stability for nonlinear damped Kirchhoff systems involving the fractional p–Laplacian operator, J. Differential Equations 263 (2017) 2375–2418.
[45] F. Punzo, On well–posedness of semilinear parabolic and elliptic problems in the hyperbolic space, J. Differential Equations 251 (7) (2011) 1972–1989.
[46] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS Reg. Conf. Ser. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.
[47] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.
[48] J. Saint Raymond, On the multiplicity of the solutions of the equation $-\Delta u = \lambda f(u)$, J. Differential Equations 180 (2002) 65–88.
[49] L. Skrzypczak, C. Tintarev, A geometric criterion for compactness of invariant subspaces, Arch. Math. 101 (2013) 259–268.
[50] S. Stapelkamp, Das Brezis–Nirenberg Problem in $\mathbb{H}^N$ (Ph.D. thesis), University of Basel, available at: http://edoc.unibas.ch/67.
[51] S. Stapelkamp, The Brézis–Nirenberg problem on $\mathbb{H}^N$. Existence and uniqueness of solutions, in: Elliptic and Parabolic Problems, Rolduc/Gaeta, 2001, World Sci. Publ., River Edge, NJ, 2002, pp. 283–290.
[52] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, Manuscripta Math. 32 (1980) 335–364.
[53] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, Springer–Verlag, Berlin, 1990.