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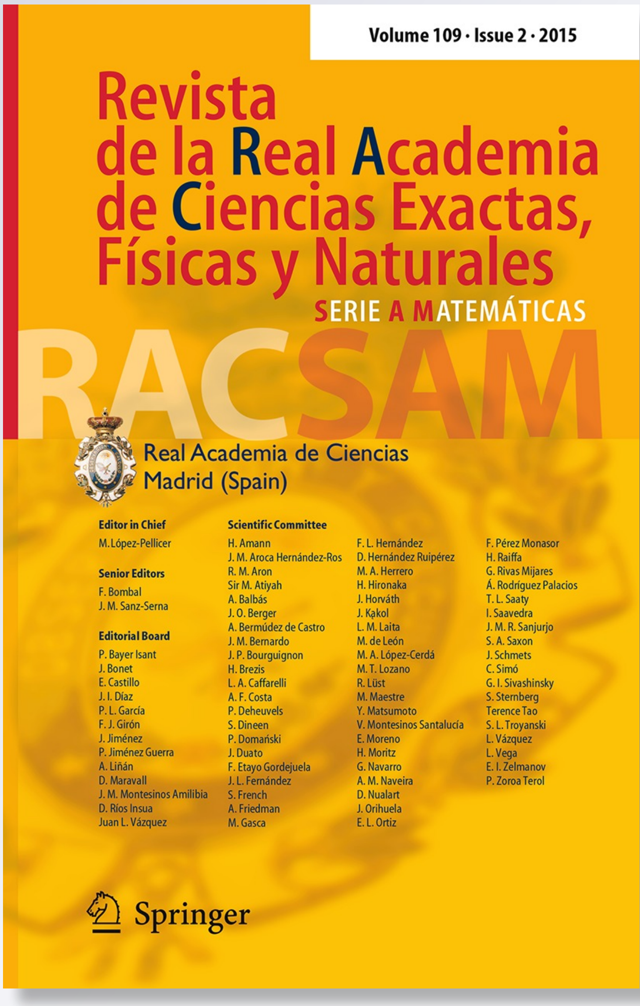

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ORIGINAL PAPER

The variational analysis of a nonlinear anisotropic problem with no-flux boundary condition

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Abstract We study the nonlinear degenerate anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u(x) = \text{constant} & \text{on } \partial\Omega \\ \sum_{i=1}^{N} \int_{\partial\Omega} a_i(x, \partial_{x_i} u)\nu_i d\sigma = 0, & \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The constant value of the boundary data is not specified, whereas the zero integral term corresponds to a no-flux boundary condition. In the case when $|u|^{q(x)-2}u$ "dominates" the left-hand side, we show that a nontrivial solution exists for all positive values of λ. If the term $|u|^{q(x)-2}u$ is dominated by the left-hand side, we prove that a solution exists either for small or for large values of $\lambda > 0$. The proofs combine variational arguments with energy estimates.

Keywords Degenerate anisotropic Sobolev spaces · Variable exponent · Boundary value condition · Variational methods

Mathematics Subject Classification 35J62 · 35J70 · 46E35

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1 Introduction

In this paper we are concerned with the existence of weak nontrivial solutions for a class of anisotropic equations in bounded domains of the Euclidean space. Our main results in the present paper continue and extend the work by Boureanu and Udrea [10].

Anisotropic operators appear in several places in the literature. Recent relevant applications include models in physics [8,11,12,16,17], biology [3,4], and image processing (see, for instance, the monograph by Weickert [30]). By definition, anisotropic operators involve directional derivatives with distinct weights. Relevant references on the theory of anisotropic Sobolev spaces are Besov [7], Kruzhkov and Kolodii [22], Kruzhkov and Korolev [23], Nikolskii [24], Rakosnik [25,26], Troisi [28], and Ven-tuan [29]. We also refer to Fragala, Gazzola, Kawohl [18] and El Hamidi, Vétois [19] as basic references in the treatment of nonlinear anisotropic problems and to the book by Antontsev, Diaz, Shmarev [2] as a source of valuable energy methods in the qualitative analysis of nonlinear boundary value problems.

No-flux problems were studied for the first time by Berestycki and Brezis [5], in relationship with models arising in plasma physics. As stated in [5], these problems stem "from a model describing the equilibrium of a plasma confined in a toroidal cavity (a Tokamak machine)"; see also [6].

2 Statement of the problem

The purpose of this work is to analyze the existence of weak solutions of the anisotropic problem

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u(x) = \text{constant} & \text{on } \partial\Omega \\ \displaystyle\sum_{i=1}^N \int_{\partial\Omega} a_i(x, \partial_{x_i} u)\nu_i d\sigma = 0, & \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary, $b \in L^\infty(\Omega)$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling some adequate hypotheses.

We notice that the constant value of the boundary data in problem (2.1) is not specified and corresponds to the one-dimensional case $u(0) = u(1)$, whereas the requirement in one-dimension that $u'(0) = u'(1)$, corresponds to the *no-flux* boundary integral term, in the case that $a_i(x, \xi) = \xi$ for all $i = 1, \ldots, N$. We also point out that Zou, Li, Liu and Lv [31,32] studied the existence of solutions for the problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \{u > 0\}, \\ -\Delta u = 0 & \text{in } \{u \leq 0\}, \\ u = c \text{ (a negative constant)} & \text{on } \partial\Omega \\ \displaystyle -\int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma = I \text{ (a given positive constant)}, & \end{cases}$$

where Ω is bounded, open and connected subset of $\mathbb{R}^2$ with regular boundary with the outward unit normal n. Problems of this type are related to plasma fusion and plasma confinement in Tokamak devices. The set $\{u > 0\}$ represents the region filled by the plasma, the set $\{u < 0\}$ represents the vacuum region, and the set $\{u = 0\}$ corresponds to to the free boundary that separate the plasma and the vacuum. The case studied in problem (2.1) corresponds to *nonresonant surfaces*, namely no-flux surfaces on which the wave number of the perturbation parallel to the equilibrium magnetic field is zero. Unfortunately, we are not able to give some *a posteriori* information on the constant value of the solution on the boundary in problem (2.1).

The differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(\cdot)$-Laplace type operator, $\overrightarrow{p}(x) = (p_1(x), p_2(x), \ldots, p_N(x))$. For $i = 1, \ldots, N$, $p_i(x)$ and $q(x)$ are continuous functions on $\overline{\Omega}$, while $a_i(x, \eta)$ is the continuous derivative with respect to η of the mapping $A_i : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $A_i = A_i(x, \eta)$, that is, $a_i(x, \eta) = \frac{\partial}{\partial \eta} A_i(x, \eta)$.

Throughout this paper we assume that the following hypotheses are fulfilled:

($\mathbf{A_0}$) $A_i(x, 0) = 0$ for a.e. $x \in \Omega$.

($\mathbf{A_1}$) There exists a positive constant $\overline{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x, \eta)| \leq \overline{c}_i (1 + |\eta|^{p_i(x)-1}),$$

for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$.

($\mathbf{A_2}$) The inequalities

$$|\eta|^{p_i(x)} \leq a_i(x, \eta)\eta \leq p_i(x) A_i(x, \eta),$$

hold for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}^N$.

($\mathbf{A_3}$) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{\eta + \xi}{2}\right) \leq \frac{1}{2} A_i(x, \eta) + \frac{1}{2} A_i(x, \xi) - k_i |\eta - \xi|^{p_i(x)},$$

for all $x \in \overline{\Omega}$ and $\eta, \xi \in \mathbb{R}^N$, with equality if and only if $\eta = \xi$.

($\mathbf{A_4}$) $a_i(x, 0) = 0$ for all $x \in \partial\Omega$.

($\mathbf{B}$) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

The operator presented above is the *anisotropic* $\overrightarrow{p}(x)$*-Laplace operator* because when we take

$$a_i(x, \eta) = |\eta|^{p_i(x)-2} \eta,$$

for all $i \in \{1, \ldots, N\}$, we have $A_i(x, \eta) = \frac{1}{p_i(x)} |\eta|^{p_i(x)}$ for all $i \in \{1, \ldots, N\}$. Therefore

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} (|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u).$$

There are many other operators deriving from $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$. Indeed, if we take

$$a_i(x, \eta) = (1 + |\eta|^2)^{\frac{(p_i(x)-2)}{2}} \eta,$$

for all $i \in \{1, \ldots, N\}$, we have $A_i(x, \eta) = \frac{1}{p_i(x)} [(1 + |\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1, \ldots, N\}$ and we obtain the *anisotropic variable mean curvature operator*

$$\sum_{i=1}^{N} \partial_{x_i} \left[(1 + |\partial_{x_i} u|^2)^{\frac{(p_i(x)-2)}{2}} \partial_{x_i} u \right].$$

A feature of the present paper is that we do not assume a zero Dirichlet boundary condition, but we work in the anisotropic variable exponent space of functions that are constant on the boundary and fulfill a no-flux boundary integral condition.

3 Abstract setting

In this section, we recall some definitions and basic properties of the spaces with variable exponent together with some results that are needed in the sequel.

For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega});\ 1 < \min_{x \in \Omega} h(x) \right\}.$$

Define

$$h^+ = \max\{h(x);\ x \in \Omega\}, \qquad h^- = \min\{h(x);\ x \in \Omega\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is a measurable real} - \text{valued function such that} \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

endowed with the *Luxemburg norm*

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1 \right\}.$$

Then $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable and reflexive Banach space [21, Theorem 2.5, Corollary 2.7]. Also, by Theorem 2.8 in [21], the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous, provided that Ω is bounded and $p_1, p_2 \in C_+(\overline{\Omega})$ are such that $p_1 \leq p_2$ in Ω.

The *isotropic Sobolev space* $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ \partial_{x_i} u \in L^{p(x)}(\Omega),\ i \in \{1, \ldots, N\}\}.$$

If equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p(x)}(\Omega)},$$

then $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ is a separable and reflexive Banach space (see [21, Theorem 1.3]).

The application $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$ called the $p(x)$-*modular* of $L^{p(x)}(\Omega)$ space, is useful in handling the Lebesgue space with variable exponent. Indeed, cf. [15, Theorems 1.3 and 1.4], if $u \in L^{p(x)}(\Omega)$ then

$$\|u\|_{L^{p(x)}(\Omega)} < 1\ (=1;\ > 1)\ \Leftrightarrow\ \rho_{p(x)}(u) < 1\ (=1;\ > 1); \tag{3.1}$$

$$\|u\|_{L^{p(x)}(\Omega)} > 1\ \Rightarrow\ \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}; \tag{3.2}$$

$$\|u\|_{L^{p(x)}(\Omega)} < 1\ \Rightarrow\ \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}; \tag{3.3}$$

$$\|u\|_{L^{p(x)}(\Omega)} \to 0\ (\to \infty)\ \Leftrightarrow\ \rho_{p(x)}(u) \to 0\ (\to \infty). \tag{3.4}$$

In addition, if $(u_n) \subset L^{p(x)}(\Omega)$, then

$$\lim_{n\to\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$$
$$\Leftrightarrow u_n \text{ converges to } u \text{ in measure and } \lim_{n\to\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u).$$

Finally, we introduce a natural generalization of the function space $W^{1,p(x)}(\Omega)$ that will enable us to study with sufficient accuracy problem (2.1). For this purpose, let us denote by $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\overrightarrow{p}(x)=(p_1(x),p_2(x),...,p_N(x))$ with $p_i(x)\in C_+(\overline{\Omega}),\ i\in\{1,\dots,N\}$ and we put

$$p_M(x)=\max\{p_1(x),\dots,p_N(x)\},\qquad p_m(x)=\min\{p_1(x),\dots,p_N(x)\}.$$

The *anisotropic Sobolev space with variable exponent* is

$$W^{1,\overrightarrow{p}(x)}(\Omega)=\{u\in L^{p_M(x)}(\Omega):\partial_{x_i}u\in L^{p_i(x)}(\Omega)\quad\text{for all}\quad i\in\{1,\dots,N\}\}.$$

This space is endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(x)}(\Omega)}=\|u\|_{L^{p_M(x)}(\Omega)}+\sum_{i=1}^N\|\partial_{x_i}u\|_{L^{p_i(x)}(\Omega)}.$$

The space $(W^{1,\overrightarrow{p}(x)}(\Omega),\|\cdot\|_{W^{1,\overrightarrow{p}(x)}(\Omega)})$ is a reflexive Banach space (see [14, Theorems 2.1 and 2.2]). Set

$$X=\left\{u\in W^{1,\overrightarrow{p}(x)}(\Omega):u|_{\partial\Omega}\equiv\text{constant}\right\}.$$

Since X is a closed subset of $W^{1,\overrightarrow{p}(x)}(\Omega)$, it follows that X is a reflexive Banach space.

We also recall (see [20, Theorem 2.2]) that if $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary and $q\in C_+(\overline{\Omega})$ satisfies $q(x)<\frac{Np_m^-}{N-p_m^-}$ for all $x\in\overline{\Omega}$, then the embedding $W^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact.

In the sequel, we use c_i and $\widetilde{c_i}$ to denote general nonnegative or positive constants (the exact value may change from line to line).

4 The first domination case

We start by giving the definition of weak solution of problem (2.1).

Definition 4.1 A function $u\in X$ that verifies

$$\int_\Omega\left\{\sum_{i=1}^N a_i(x,\partial_{x_i}u)\partial_{x_i}\varphi+b(x)|u|^{p_M(x)-2}u\varphi-\lambda|u|^{q(x)-2}u\varphi\right\}dx=0,$$

for all $\varphi\in X$ is called a weak solution of problem (2.1).

We associate to problem (2.1) the energy functional $I_\lambda:X\to\mathbb{R}$ defined by

$$I_\lambda(u)=\int_\Omega\left\{\sum_{i=1}^N A_i(x,\partial_{x_i}u)+\frac{b(x)}{p_M(x)}|u|^{p_M(x)}-\frac{\lambda}{q(x)}|u|^{q(x)}\right\}dx.$$

Then I_λ is well-defined and $I_\lambda\in C^1(X,\mathbb{R})$ with

$$\langle I'_\lambda(u),\varphi\rangle=\int_\Omega\left\{\sum_{i=1}^N a_i(x,\partial_{x_i}u)\partial_{x_i}\varphi+b(x)|u|^{p_M(x)-2}u\varphi-\lambda|u|^{q(x)-2}u\varphi\right\}dx,$$

for all $u,\varphi\in X$. Hence any critical point $u\in X$ of I_λ is a weak solution of problem (2.1).

The next result shows that if the variable exponent $q(\cdot)$ "dominates" $p(\cdot)$ then the solution exists for all positive values of the parameter λ. In other words, the right-hand side of problem (2.1) is "stronger" than the other side, even if $\lambda>0$ is small.

Theorem 4.2 *Assume that the function $q \in C_+(\overline{\Omega})$ verifies the hypothesis*

$$p_M^+ < q^- \leq q^+ < \frac{Np_m^-}{N - p_m^-}.$$

Then for any $\lambda > 0$ problem (2.1) possesses a nontrivial weak solution.

We first prove two auxiliary results.

Lemma 4.3 *There exist $\eta > 0$ and $\alpha > 0$ such that $I_\lambda(u) \geq \alpha > 0$ for any $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \eta$.*

Proof First, we point out that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^-} + |u(x)|^{q^+}, \qquad \text{for all} \quad x \in \overline{\Omega}.$$

By the above inequality, ($\mathbf{A_2}$) and ($\mathbf{B}$), we find

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)} |u|^{p_M(x)} - \frac{\lambda}{q(x)} |u|^{q(x)} \right\} dx \\ &\geq \frac{1}{p_M^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \, dx + \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)} \, dx - \frac{\lambda}{q^-} \left(\|u\|_{L^{q^-}(\Omega)}^{q^-} + \|u\|_{L^{q^+}(\Omega)}^{q^+} \right). \end{aligned} \tag{4.1}$$

From the hypotheses of Theorem 4.2, $W^{1,\vec{p}(x)}(\Omega)$ is continuously embedded in $L^{q^-}(\Omega)$ and $L^{q^+}(\Omega)$. Then, there exist two positive constants c_1 and c_2 such that for all $u \in X$

$$|u(x)|_{L^{q^-}(\Omega)} \leq c_1 \|u\|_{W^{1,\vec{p}(x)}(\Omega)} \quad \text{and} \quad |u(x)|_{L^{q^+}(\Omega)} \leq c_2 \|u\|_{W^{1,\vec{p}(x)}(\Omega)}. \tag{4.2}$$

Here, we let $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} < 1$, so $\|u\|_{L^{p_M(x)}(\Omega)} < 1$ and $\|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} < 1$, $i \in \{1, \ldots, N\}$.

Taking into account relations (3.3) and (4.2), the inequality (4.1) reduces to

$$\begin{aligned} I_\lambda(u) \geq\ & \frac{\min\{1, b_0\}}{p_M^+ (N+1)^{p_M^+ - 1}} \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{p_M^+} \\ & - \frac{\lambda}{q^-} \left[(c_1 \|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^-} + (c_2 \|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^+} \right] \\ =\ & \left(c_3 - c_4 \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{q^- - p_M^+} - c_5 \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{q^+ - p_M^+} \right) \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{p_M^+}, \end{aligned}$$

for any $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} < 1$. Since the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = c_3 - c_4 t^{q^- - p_M^+} - c_5 t^{q^+ - p_M^+}$$

is positive in a neighborhood of the origin, the conclusion of the lemma follows. □

Lemma 4.4 *There exists $e \in X$ with $\|e\|_{W^{1,\vec{p}(x)}(\Omega)} > \eta$ (where η is given in Lemma 4.3) such that $I_\lambda(e) < 0$.*

Proof From ($\mathbf{A_0}$) and ($\mathbf{A_1}$), we have

$$A_i(x, \eta) = \int_0^1 a_i(x, t\eta)\eta dt \leq c_6 \left(|\eta| + \frac{1}{p_i(x)} |\eta|^{p_i(x)} \right),$$

for all $x \in \Omega$ and $\eta \in \mathbb{R}^N$, where $c_6 = \max_{i\in\{1,\dots,N\}} \overline{c}_i$. Therefore

$$\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)dx \leq c_6 \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} u| + \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\right) dx. \tag{4.3}$$

Let $\varphi \in X$, $\varphi \neq 0$. For any $t > 1$, we find

$$\begin{aligned} I_\lambda(t\varphi) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\varphi)) + \frac{b(x)}{p_M(x)} |t\varphi|^{p_M(x)} - \frac{\lambda}{q(x)} |t\varphi|^{q(x)} \right\} dx \\ &\leq c_6 \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(t\varphi)| + \frac{|\partial_{x_i}(t\varphi)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{1}{p_M^+} \int_\Omega b(x) |t\varphi|^{p_M(x)} dx \\ &\quad - \lambda \int_\Omega \frac{1}{q(x)} |t\varphi|^{q(x)} dx \\ &\leq c_6 t^{p_M^+} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\varphi| + \frac{1}{p_m^-} |\partial_{x_i}\varphi|^{p_i(x)} \right) dx + \frac{t^{p_M^+}}{p_M^+} \int_\Omega b(x) |\varphi|^{p_M(x)} dx \\ &\quad - \frac{\lambda t^{q^-}}{q^+} \int_\Omega |\varphi|^{q(x)} dx. \end{aligned}$$

Since $p_M^+ < q^-$, we infer that $\lim_{t\to\infty} I_\lambda(t\varphi) = -\infty$. Then for $t > 1$ large enough, we can take $e = t\varphi$ such that $\|e\|_{W^{1,\overrightarrow{p}(x)}(\Omega)} > \eta$ and $I_\lambda(e) < 0$. □

Now, we prove the following useful property.

Lemma 4.5 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary. Assume that the sequence (u_n) converges weakly to u in $W^{1,\overrightarrow{p}(x)}(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u)\, dx \leq 0.$$

Then (u_n) converges strongly to u in $W^{1,\overrightarrow{p}(x)}(\Omega)$.

Proof We use the fact that $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{p_M(x)}(\Omega)$ compactly. Since $u_n \rightharpoonup u$ in $W^{1,\overrightarrow{p}(x)}(\Omega)$, we deduce that

$$u_n \to u \quad \text{in} \quad L^{p_M(x)}(\Omega). \tag{4.4}$$

Then, combining (4.4) and (3.4) we conclude that $u_n \to u$ in $W^{1,\overrightarrow{p}(x)}(\Omega)$. □

Proof of Theorem 4.2 By Lemmas 4.3–4.4 and the mountain pass theorem of Ambrosetti and Rabinowitz [1], we deduce the existence of a sequence $(u_n) \subset X$ such that

$$I_\lambda(u_n) \to c_7 > 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0 \text{ as} \quad n \to \infty. \tag{4.5}$$

We claim that (u_n) is bounded. Arguing by contradiction, we assume that, up to a subsequence still denoted by (u_n), we have $\|u_n\|_{W^{1,\overrightarrow{p}(x)}(\Omega)} \to \infty$ as $n \to \infty$.

Using relation (4.5), for n large enough, we have

$$
\begin{aligned}
1+c_7+\|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq I_\lambda(u_n)-\frac{1}{q^-}\langle I_\lambda'(u_n),u_n\rangle\\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)+\frac{1}{p_M(x)}b(x)|u_n|^{p_M(x)}-\frac{\lambda}{q(x)}|u_n|^{q(x)}\right]dx\\
&\quad-\frac{1}{q^-}\sum_{i=1}^{N}\int_\Omega\left[a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n+b(x)|u_n|^{p_M(x)}-\lambda|u_n|^{q(x)}\right]dx\\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)-\frac{1}{q^-}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\right]dx\\
&\quad+\left(\frac{1}{p_M^+}-\frac{1}{q^-}\right)\int_\Omega b(x)|u_n|^{p_M(x)}\,dx+\lambda\int_\Omega\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|u_n|^{q(x)}\,dx. \qquad (4.6)
\end{aligned}
$$

From $(\mathbf{A_2})$, for all $x\in\Omega$ and $i\in\{1,\ldots,N\}$ we have

$$a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n \leq p_i(x)A_i(x,\partial_{x_i}u_n)\leq p_M^+A_i(x,\partial_{x_i}u_n),$$

which implies

$$-\frac{1}{q^-}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\geq-\frac{p_M^+}{q^-}A_i(x,\partial_{x_i}u_n).$$

Inserting this inequality into relation (4.6) we obtain

$$
\begin{aligned}
1+c_7+\|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq \left(1-\frac{p_M^+}{q^-}\right)\sum_{i=1}^{N}\int_\Omega A_i(x,\partial_{x_i}u_n)\,dx\\
&\quad+b_0\left(\frac{1}{p_M^+}-\frac{1}{q^-}\right)\int_\Omega|u_n|^{p_M(x)}\,dx.
\end{aligned}
$$

Again from $(\mathbf{A_2})$ we have

$$A_i(x,\partial_{x_i}u_n)\geq\frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)}\geq\frac{1}{p_M^+}|\partial_{x_i}u_n|^{p_i(x)},$$

for all $x\in\Omega$ and $i\in\{1,\ldots,N\}$, thus

$$
\begin{aligned}
1+c_7+\|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq \left(\frac{1}{p_M^+}-\frac{1}{q^-}\right)\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\,dx\\
&\quad+b_0\left(\frac{1}{p_M^+}-\frac{1}{q^-}\right)\int_\Omega|u_n|^{p_M(x)}\,dx. \qquad (4.7)
\end{aligned}
$$

We denote

$$\ell_1=\{i\in\{1,\ldots,N\}:\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}\leq 1\}$$

and

$$\ell_2=\{i\in\{1,\ldots,N\}:\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}>1\}.$$

By (3.1), (3.2), (3.3) and Jensen's inequality (applied to the convex function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(t) = t^{p_m^-}$, $p_m^- > 1$), for n large enough we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx &= \sum_{i\in\ell_1} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx + \sum_{i\in\ell_2} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
&\geq \sum_{i\in\ell_1} \|\partial_{x_i} u_n\|^{p_M^+}_{L^{p_i(x)}(\Omega)} + \sum_{i\in\ell_2} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq \sum_{i=1}^{N} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - \sum_{i\in\ell_1} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq N\Big(\frac{\sum_{i=1}^N \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}}{N}\Big)^{p_m^-} - N. \qquad (4.8)
\end{aligned}$$

We analyze now the two cases corresponding to the value of $\|u\|_{L^{p_M(x)}(\Omega)}$.

Case 1: $\|u\|_{L^{p_M(x)}(\Omega)} \geq 1$. By (4.7) and (4.8) we have

$$\begin{aligned}
1 + c_7 + \|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq \Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big)\Big[\frac{1}{N^{p_m^- -1}}\Big(\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}\Big)^{p_m^-} - N\Big] \\
&\quad + b_0\Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big)\|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)}.
\end{aligned}$$

and thus

$$\begin{aligned}
1 + c_7 + \|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq \frac{1}{2^{p_m^-}}\Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big) \min\Big\{\frac{1}{N^{p_m^- -1}}, b_0\Big\}\|u_n\|^{p_m^-}_{W^{1,\vec{p}(x)}(\Omega)} \\
&\quad - N\Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big). \qquad (4.9)
\end{aligned}$$

Case 2: $\|u\|_{L^{p_M(x)}(\Omega)} < 1$. Then

$$\begin{aligned}
1 + c_7 + \|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} &\geq \Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big)\Big[\frac{1}{N^{p_m^- -1}}\Big(\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}\Big)^{p_m^-} - N\Big] \\
&\geq \Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big)\Big[\frac{1}{N^{p_m^- -1}}\Big(\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}\Big)^{p_m^-} \\
&\quad + \|u_n\|^{p_m^-}_{L^{p_M(x)}(\Omega)} \qquad - N - 1\Big].
\end{aligned}$$

We obtain

$$1 + c_7 + \|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} \geq \frac{1}{2^{p_m^-}}\Big(\frac{1}{p_M^+} - \frac{1}{q^-}\Big) \min\Big\{\frac{1}{N^{p_m^- -1}}, 1\Big\}\|u_n\|^{p_m^-}_{W^{1,\vec{p}(x)}(\Omega)} - \frac{N+1}{p_M^+} \qquad (4.10)$$

By (4.9) and (4.10), we deduce that there exist $\widetilde{c_1}, \widetilde{c_2} > 0$ such that

$$1 + c_7 + \|u_n\|_{W^{1,\vec{p}(x)}(\Omega)} \geq \widetilde{c_1}\|u_n\|^{p_m^-}_{W^{1,\vec{p}(x)}(\Omega)} - \widetilde{c_2}.$$

Dividing the above inequality by $\|u_n\|^{p_m^-}_{W^{1,\overrightarrow{p}(x)}(\Omega)}$ and passing to the limit as $n \to \infty$ we obtain a contradiction.

It follows that (u_n) is bounded in $W^{1,\overrightarrow{p}(x)}(\Omega)$. This information combined with the fact that $W^{1,\overrightarrow{p}(x)}(\Omega)$ is reflexive implies that there exists a subsequence, still denoted by (u_n), and $u_0 \in W^{1,\overrightarrow{p}(x)}(\Omega)$ such that (u_n) converges weakly to u_0 in $W^{1,\overrightarrow{p}(x)}(\Omega)$.

Using (4.5), we infer that

$$\lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u_0\rangle = 0,$$

more precisely,

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \Big[\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) + b(x)|u_n|^{p_M(x)-2} u_n(u_n - u_0) \\ - \lambda |u_n|^{q(x)-2} u_n(u_n - u_0)\Big] dx = 0. \qquad (4.11)\end{aligned}$$

Since the space $W^{1,\overrightarrow{p}(x)}(\Omega)$ is compactly embedded in $L^{p_M(x)}(\Omega)$ and $L^{q(x)}(\Omega)$, it follows that (u_n) converges strongly to u_0 in $L^{p_M(x)}(\Omega)$ and also in $L^{q(x)}(\Omega)$. Therefore

$$\begin{aligned}&\left| \int_\Omega b(x)|u_n|^{p_M(x)-2} u_n(u_n - u_0)\, dx \right| \\ &\quad \le 2\|b\|_{L^\infty(\Omega)} \left\| |u_n|^{p_M(x)-1} \right\|_{L^{\frac{p_M(x)}{p_M(x)-1}}(\Omega)} \|u_n - u_0\|_{L^{p_M(x)}(\Omega)}, \qquad (4.12)\end{aligned}$$

and

$$\left| \int_\Omega |u_n|^{q(x)-2} u_n(u_n - u_0)\, dx \right| \le 2 \left\| |u_n|^{q(x)-1} \right\|_{L^{\frac{q(x)}{q(x)-1}}(\Omega)} \|u_n - u_0\|_{L^{q(x)}(\Omega)}. \qquad (4.13)$$

By (4.12), (4.13) and (3.4), using the strong convergence of (u_n) to u_0 in $L^{p_M(x)}(\Omega)$ and $L^{q(x)}(\Omega)$ we deduce

$$\lim_{n\to\infty} \int_\Omega b(x)|u_n|^{p_M(x)-2} u_n(u_n - u_0)\, dx = 0,$$

and

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q(x)-2} u_n(u_n - u_0)\, dx = 0.$$

By the above relations, (4.11) reduces to

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)\, dx = 0.$$

Using Lemma 4.5, we deduce that (u_n) converges strongly to u_0 in $W^{1,\overrightarrow{p}(x)}(\Omega)$. Since X is a closed subspace of $W^{1,\overrightarrow{p}(x)}(\Omega)$ and $(u_n) \subset X$ we obtain that $u_0 \in X$. Then by relation (4.5)

$$I_\lambda(u_0) = c_7 > 0 \quad \text{and} \quad I'_\lambda(u_0) = 0,$$

that is, u_0 is a nontrivial weak solution for problem (2.1). $\square$

5 The second domination case

The next result establishes an interesting concentration property in neighborhoods of the origin and of the infinity. More precisely, under an additional assumption, we show that there are positive numbers λ^* and λ^{**} such that problem (2.1) has a solution provided that either $\lambda \in (0, \lambda^*)$ or $\lambda \in (\lambda^{**}, \infty)$. The existence of a "gap" between λ^* and λ^{**} still remains an interesting open problem.

Theorem 5.1 *In addition, we assume that $q \in C_+(\overline{\Omega})$ satisfies the hypothesis*

$$1 < q^- \leq q^+ < p_m^-.$$

Then the following properties hold.

(i) *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ problem (2.1) possesses a nontrivial weak solution.*
(ii) *There exists $\lambda^{**} > 0$ such that for any $\lambda > \lambda^{**}$ problem (2.1) possesses a nontrivial weak solution.*

The assumptions in Theorem 5.1 show that the weight $p(\cdot)$ is dominating with respect to the variable exponent $q(\cdot)$ that controls the right-hand side. The above results asserts that, in such a case, a solution exists either if $\lambda > 0$ is sufficiently small or for large values of λ.

First, applying Ekeland's variational principle [13], we show that there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ the functional I_λ has a nontrivial critical point. We start with two auxiliary results.

Lemma 5.2 *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there are $\rho, a > 0$ such that $I_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \rho$.*

Proof Under the conditions of Theorem 5.1, $W^{1,\vec{p}(x)}(\Omega)$ is continuously embedded in $L^{q(x)}(\Omega)$. Thus, there exists a positive constant c_8 such that

$$\|u\|_{L^{q(x)}(\Omega)} \leq c_8 \|u\|_{W^{1,\vec{p}(x)}(\Omega)} \quad \text{for all} \quad u \in X. \tag{5.1}$$

Now, let us assume that $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} < \min\{1, \frac{1}{c_8}\}$, where c_8 is the positive constant from above. Then we have $\|u\|_{L^{q(x)}(\Omega)} < 1$. Using (3.3) we get

$$\int_\Omega |u|^{q(x)} dx \leq \|u\|_{L^{q(x)}(\Omega)}^{q^-} \quad \text{for all} \quad u \in X \quad \text{with} \quad \|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \rho \in (0, 1). \tag{5.2}$$

Relations (5.1) and (5.2) imply

$$\int_\Omega |u|^{q(x)} dx \leq c_8^{q^-} \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{q^-} \quad \text{for all} \quad u \in X \quad \text{with} \quad \|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \rho. \tag{5.3}$$

Using the hypothesis $(\mathbf{A_2})$, $(\mathbf{B})$ and relation (5.3), we deduce that for any $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \rho$, the following hold:

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)} |u|^{p_M(x)} - \frac{\lambda}{q(x)} |u|^{q(x)} \right\} dx \\
&\geq \frac{1}{p_M^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx + \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)}\, dx - \frac{\lambda}{q^-} c_8^{q^-} \|u\|^{q^-}_{W^{1,\vec{p}(x)}(\Omega)} \\
&\geq \frac{\min\{1, b_0\}}{p_M^+ (N+1)^{p_M^+ - 1}} \|u\|^{p_M^+}_{W^{1,\vec{p}(x)}(\Omega)} - \frac{\lambda}{q^-} c_8^{q^-} \|u\|^{q^-}_{W^{1,\vec{p}(x)}(\Omega)} \\
&= \frac{\min\{1, b_0\}}{p_M^+ (N+1)^{p_M^+ - 1}} \rho^{p_M^+} - \frac{\lambda}{q^-} c_8^{q^-} \rho^{q^-} \\
&= \rho^{q^-} \left(\frac{\min\{1, b_0\}}{p_M^+ (N+1)^{p_M^+ - 1}} \rho^{p_M^+ - q^-} - \frac{\lambda}{q^-} c_8^{q^-} \right).
\end{aligned} \tag{5.4}
$$

If we define

$$
\lambda^* = \frac{\min\{1, b_0\} q^-}{2 p_M^+ (N+1)^{p_M^+ - 1} c_8^{q^-}} \rho^{p_M^+ - q^-}, \tag{5.5}
$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \rho$, there exists $a = \frac{\min\{1, b_0\} \rho^{p_M^+}}{2 p_M^+ (N+1)^{p_M^+ - 1}}$ such that $I_\lambda(u) \geq a > 0$. □

Lemma 5.3 *Assume that $\lambda \in (0, \lambda^*)$, where λ^* is given by (5.5). Then there exists $\psi \in X$ such that $\psi \geq 0$, $\psi \neq 0$ and $I_\lambda(t\psi) < 0$ for all $t > 0$ small enough.*

Proof By the conditions of Theorem 5.1, $q^- < p_m^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p_m^-$. Since $q \in C(\overline{\Omega})$, there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. It follows that $q(x) < q^- + \epsilon_0 < p_m^-$ for all $x \in \Omega_0$.

Let $\psi \in X$ be such that $\operatorname{supp}(\psi) \supset \overline{\Omega_0}$, $\psi(x) = 1$ for all $x \in \overline{\Omega_0}$ and $0 \leq \psi \leq 1$ in Ω. Then by (4.3) for any $t \in (0, 1)$, we have

$$
\begin{aligned}
I_\lambda(t\psi) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\psi)) + \frac{b(x)}{p_M(x)} |t\psi|^{p_M(x)} - \frac{\lambda}{q(x)} |t\psi|^{q(x)} \right\} dx \\
&\leq c_6 \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(t\psi)| + \frac{|\partial_{x_i}(t\psi)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{1}{p_M^-} \int_\Omega b(x) |t\psi|^{p_M(x)} dx \\
&\quad - \lambda \int_\Omega \frac{1}{q(x)} |t\psi|^{q(x)} dx \\
&\leq c_6 t^{p_m^-} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{1}{p_m^-} |\partial_{x_i}\psi|^{p_i(x)} \right) dx + \frac{t^{p_M^-}}{p_M^-} \int_\Omega b(x) |\psi|^{p_M(x)} dx \\
&\quad - \frac{\lambda}{q^+} \int_{\Omega_0} t^{q(x)} |\psi|^{q(x)} dx \\
&\leq c_6 t^{p_m^-} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{1}{p_m^-} |\partial_{x_i}\psi|^{p_i(x)} \right) dx + \frac{t^{p_m^-}}{p_M^-} \int_\Omega b(x) |\psi|^{p_M(x)} dx \\
&\quad - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} \int_{\Omega_0} |\psi|^{q(x)} dx.
\end{aligned}
$$

So, $I_\lambda(t\psi) < 0$ for $t < \delta^{\frac{1}{p_m^- - q^- - \epsilon_0}}$, with

$$0 < \delta < \min\left\{1, \frac{\lambda}{q^+} \frac{\int_{\Omega_0} |\psi|^{q(x)} dx}{c_6 \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{1}{p_m^-}|\partial_{x_i}\psi|^{p_i(x)}\right) dx + \frac{1}{p_M^-}\int_\Omega b(x)|\psi|^{p_M(x)} dx}\right\}.$$

□

Proof of Theorem 5.1 **(i)**. Let λ^* be defined as in (5.3) and $\lambda \in (0, \lambda^*)$. By Lemma 5.2, it follows that on the boundary of the ball centered at the origin and of radius ρ in X, we have

$$\inf_{\partial B_\rho(0)} I_\lambda(u) > 0.$$

On the other hand, by Lemma 5.3, there exists $\psi \in X$ such that

$$I_\lambda(t\psi) < 0 \quad \text{for} \quad t > 0 \quad \text{small enough.}$$

Moreover, for $u \in B_\rho(0)$,

$$I_\lambda(u) \geq \frac{\min\{1, b_0\}}{p_M^+ (N+1)^{p_M^+ - 1}} \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{p_M^+} - \frac{\lambda}{q^-} c_8^{q^-} \|u\|_{W^{1,\vec{p}(x)}(\Omega)}^{q^-}.$$

It follows that

$$-\infty < c_9 = \inf_{\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda - \inf_{B_\rho(0)} I_\lambda$. Applying Ekeland's variational principle [13] to the functional $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} I_\lambda + \varepsilon,$$
$$I_\lambda(u_\varepsilon) < I_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{W^{1,\vec{p}(x)}(\Omega)}, \quad u \neq u_\varepsilon.$$

Since

$$I_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} I_\lambda + \varepsilon \leq \inf_{B_\rho(0)} I_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $K_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $K_\lambda(u) = I_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{W^{1,\vec{p}(x)}(\Omega)}$. It is clear that u_ε is a minimum point of K_λ and thus

$$\frac{K_\lambda(u_\varepsilon + tv) - K_\lambda(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_\rho(0)$. The above relation yields

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\|_{W^{1,\vec{p}(x)}(\Omega)} \geq 0.$$

Letting $t \to 0$ it follows that $\langle I_\lambda'(u_\varepsilon), v\rangle + \varepsilon\|v\|_{W^{1,\vec{p}(x)}(\Omega)} > 0$, hence $\|I_\lambda'(u_\varepsilon)\|_{W^{1,\vec{p}(x)}(\Omega)} \leq \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_1(0)$ such that

$$I_\lambda(v_n) \to c_9 \quad \text{and} \quad I'_\lambda(v_n) \to 0. \tag{5.6}$$

It is clear that (v_n) is bounded in X. Actually, with similar arguments as those used in the end of Theorem 4.2, we can show that (v_n) converges strongly to u_1 in X. So, by (5.6)

$$I_\lambda(u_1) = c_9 < 0 \quad \text{and} \quad I'_\lambda(u_1) = 0,$$

that is, u_1 is a nontrivial weak solution for the problem (2.1). This completes the proof. □

Next, we want to construct a global minimizer of the functional I_λ. We start with the following auxiliary result.

Lemma 5.4 *The functional I_λ is coercive on X.*

Proof Using the hypothesis $(\mathbf{A_2})$, relation (4.2) we deduce for all $u \in X$,

$$\begin{aligned} I_\lambda(u) \geq \frac{1}{p_M^+} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)}\, dx \\ - \frac{\lambda}{q^-}\Big[(c_1\|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^-} + (c_2\|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^+}\Big]. \end{aligned} \tag{5.7}$$

Now we set $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} > 1$. Using the same techniques as in the proof of (4.7) combined with (5.7) we find that

$$I_\lambda(u) \geq \widetilde{c_1}\|u\|^{p_m^-}_{W^{1,\vec{p}(x)}(\Omega)} - \widetilde{c_2} - \frac{\lambda}{q^-}\Big[(c_1\|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^-} + (c_2\|u\|_{W^{1,\vec{p}(x)}(\Omega)})^{q^+}\Big],$$

for any $u \in X$ with $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} > 1$. Since $q^- \leq q^+ < p_m^-$, we infer that $I_\lambda(u) \to \infty$ as $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} \to \infty$, that is, I_λ is coercive. □

Proof of Theorem 5.1 **(ii)**. The same arguments as in the proof of Lemma 3 of [9] can be used in order to show that I_λ is weakly lower semicontinuous on X. By Lemma 5.4, the functional I_λ is also coercive on X. We know from [27] that there exists $u_\lambda \in X$, a global minimizer of I_λ and thus weak solution of problem (2.1).

We show that u_λ is nontrivial if λ is large enough. Letting $t_0 > 1$ be a constant and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$, we assume that $v_0 \in X$ is such that $v_0(x) = t_0$ for any $x \in \overline{\Omega_1}$ and $0 \leq v_0(x) \leq t_0$ in $\Omega \backslash \Omega_1$. We have

$$\begin{aligned} I_\lambda(v_0) &= \int_\Omega \Big\{ \sum_{i=1}^{N} A_i(x, \partial_{x_i}(v_0)) + \frac{b(x)}{p_M(x)}|v_0|^{p_M(x)} - \frac{\lambda}{q(x)}|v_0|^{q(x)} \Big\}\, dx \\ &\leq c_6 \sum_{i=1}^{N} \int_\Omega \Big(|\partial_{x_i}(v_0)| + \frac{|\partial_{x_i}(v_0)|^{p_i(x)}}{p_i(x)} \Big) dx + \frac{1}{p_M^-} \int_\Omega b(x)|v_0|^{p_M(x)}\, dx \\ &\quad - \lambda \int_\Omega \frac{1}{q(x)}|v_0|^{q(x)} dx \leq c_{10} - \frac{\lambda}{q^+} t_0^{q^-}|\Omega_1|. \end{aligned}$$

So there exists $\lambda^* > 0$ such that $J_\lambda(v_0) < 0$ for any $\lambda \in (\lambda^*, +\infty)$. It follows that for any $\lambda \geq \lambda^*$, u_λ is a nontrivial weak solution of problem (2.1) for λ large enough. □

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