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Resonant double phase equations

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A B S T R A C T

We consider a double phase Dirichlet equation with a reaction which is asymptotically as $x \to \pm\infty$, resonant with respect to the first eigenvalue of a related eigenvalue problem. Using variational tools together with Morse theoretic arguments, we prove the existence of at least two bounded nontrivial solutions for the problem.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following double phase problem

$$-\Delta_p^a u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ 2 < p. \tag{1}$$

We denote by Δ_p^a the weighted p-Laplace differential operator, which is defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

The special feature of this operator is that the function $a(\cdot)$ is not bounded away from zero. Hence the integrand $\theta_0(z,t) = \frac{1}{p}a(z)t^p + \frac{1}{2}t^2$ $(t \geqslant 0)$ in the energy functional corresponding to this differential operator,

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exhibits unbalanced growth, namely we have

$$\frac{1}{2}t^2 \leqslant \theta_0(z,t) \leqslant c_0(1+t^p) \text{ for a.a. } z \in \Omega, \text{ all } t \geqslant 0, \text{ some } c_0 > 0.$$

Such integral functionals were first investigated by Marcellini [1,2] and Zhikov [3,4], in the context of problems originating in nonlinear elasticity theory. We also refer to the recent contributions of Marcellini [5] and Eleuteri, Marcellini & Mascolo [6,7] to the regularity of weak solutions for nonlinear elliptic PDEs with nonstandard growth. The difficulty that we face when dealing with such differential operators, is that there is no global regularity theory for the solutions. Recently there have been some local regularity results for local minimizers of such functionals produced by Mingione et al.; see Baroni, Colombo & Mingione [8], Colombo & Mingione [9]. However, a global regularity theory for general solutions, remains elusive. So, we cannot apply many of the techniques used in the context of (p,q)-equations; see, for example, Papageorgiou, Vetro & Vetro [10] and Papageorgiou & Zhang [11].

Recently there have been some existence and multiplicity results for double phase equations. We mention the works of Colasuonno & Squassina [12], Gasinski & Winkert [13], Ge, Lv & Lu [14], Liu & Dai [15], Papageorgiou, Rădulescu & Repovš [16,17], Papageorgiou, Vetro & Vetro [18]. These works either deal with problems which have a $(p-1)$-superlinear reaction (see [14–18]) or examine parametric problems (see [12,13]). We refer to Marcellini [5] and Mingione and Rădulescu [19] for overviews of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

The feature of this paper is that we consider resonant problems. To the best of our knowledge, this is the first work dealing with resonant double phase problems. Finally, we also mention the work of Bahrouni, Rădulescu & Repovš [20], where the reader can find applications of double phase equations to transonic flow problems and the paper of Liu & Papageorgiou [21], where the authors, under symmetry conditions on the reaction, produce a whole sequence of nodal solutions converging to zero.

In this paper, using a combination of variational and Morse theoretic techniques (critical groups), we show that problem (1) admits at least two nontrivial bounded solutions, when resonance occurs (see Section 5).

2. Mathematical preliminaries

The unbalanced growth of the integrand corresponding to the differential operator, dictates that the appropriate functional framework for the study of double phase problems, is provided by the Musielak–Orlicz–Sobolev spaces.

We introduce the following hypotheses on the function $a(z)$.

H_0 : $a : \overline{\Omega} \mapsto \mathbb{R}$ is Lipschitz continuous (that is, $a \in C^{0,1}(\overline{\Omega})$), $a(z) > 0$ for all $z \in \Omega$, $a|_{\partial\Omega} = 0$, and $a \in A_p =$ the p-Muckenhoupt class (see [22, p.145]).

We stress that $a(\cdot)$ is not assumed to be bounded away from zero. This leads to the unbalanced growth of the corresponding integrand

$$\theta(z,t) = a(z)t^p + t^2 \text{ for all } z \in \overline{\Omega}, \text{ all } t \geqslant 0.$$

This in turn requires the use of Musielak–Orlicz–Sobolev spaces, which we introduce below. In order to have useful embeddings, for the relevant spaces, we need an additional restriction on the exponents $2 < p$.

H_1 : $\frac{p}{2} < 1 + \frac{1}{N}$.

Remark 1. This hypothesis implies that $p < 2^*$ and leads to local regularity results for double phase problems (see Marcellini [5]).

In addition to the integrand $\theta(z,t)$, we also consider the integrand

$$\varphi(z,t) = a(z)t^p \text{ for all } z \in \overline{\Omega}, \text{ all } t \geqslant 0.$$

Both integrands θ and φ are uniformly convex in $t \geqslant 0$ (see Diening, Harjulehto, Hästö & Růžička [22, Remark 2.4.6, p. 41]). Also we have $\varphi \leqslant \theta$.

Now, let $k : \Omega \times [0,+\infty) \mapsto [0,+\infty)$ be a Carathéodory function (that is, for all $t \geqslant 0$ the mapping $z \mapsto k(z,t)$ is measurable and for a.a. $z \in \Omega$ the function $t \mapsto k(z,t)$ is continuous). We assume the following conditions on k:

- $k(z,t) > 0$ for a.a. $z \in \Omega$, all $t > 0$ and $k(z,0) = 0$ for a.a. $z \in \Omega$.
- For a.a. $z \in \Omega$, we have
$$\lim_{t\to 0^+} \frac{k(z,t)}{t} = 0 \text{ and } \lim_{t\to+\infty} \frac{k(z,t)}{t} = +\infty.$$
- For all $t > 0$, $k(\cdot,t) \in L^1(\Omega)$.
- For a.a. $z \in \Omega$, $k(z,\cdot)$ satisfies the Δ_2-condition, that is, there exist $\hat{c} > 0$ and $\beta \in L^1(\Omega)$ such that
$$k(z,2t) \leqslant \hat{c}k(z,t) + \beta(z) \text{ for a.a. } z \in \Omega, \text{ all } t \geqslant 0.$$

An integrand $k(z,t)$ satisfying all the above conditions, is said to be a "generalized N-function" (N stands for "nice"). By $\mathrm{N}(\Omega)$ we denote the family of such integrands. Clearly we have $\theta, \varphi \in \mathrm{N}(\Omega)$.

Also let $M(\Omega)$ be the space of all measurable functions $u : \Omega \mapsto \mathbb{R}$. As usual, we identify two such functions when they differ only on a Lebesgue-null set. Given $k \in \mathrm{N}(\Omega)$, the "Musielak–Orlicz space" $L^k(\Omega)$ is defined by

$$L^k(\Omega) = \left\{u \in M(\Omega) : \rho_k(u) < +\infty\right\},$$

where

$$\rho_k(u) = \int_\Omega k(z,|u|)dz \text{ (the modular function).}$$

We equip $L^k(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_k$ defined by

$$\|u\|_k = \inf\left\{\lambda > 0 : \rho_k\left(\frac{u}{\lambda}\right) \leqslant 1\right\}.$$

Then $L^k(\Omega)$ is a Banach space and if $k \leqslant \hat{k}$, then

$$L^{\hat{k}}(\Omega) \hookrightarrow L^k(\Omega) \text{ continuously.}$$

Moreover, if $k(z,\cdot)$ is uniformly convex, then the Banach space $L^k(\Omega)$ is uniformly convex (thus reflexive); see Diening, Harjulehto, Hästö & Růžička [22, Theorem 2.4.14 and Remark 2.4.15, p. 44] and Musielak [23, Corollary 11.7, p. 77].

There is a close relation between the norm $\|\cdot\|_k$ and the modular function $\rho_k(\cdot)$.

Proposition 1.

(a) If $u \in L^k(\Omega)$, then $\rho_k(u) < 1$ (resp. $= 1, > 1$) $\Longleftrightarrow \|u\|_k < 1$ (resp. $= 1, > 1$).
(b) $\|u\|_k \to 0 \Longleftrightarrow \rho_k(u) \to 0$ and $\|u\|_k \to +\infty \Longleftrightarrow \rho_k(u) \to +\infty$.

The related "Musielak–Orlicz–Sobolev space", $W^{1,k}(\Omega)$ is defined by

$$W^{1,k}(\Omega) = \left\{u \in L^k(\Omega) : |Du| \in L^k(\Omega)\right\}.$$

We equip $W^{1,k}(\Omega)$ with the norm

$$\|u\|_{1,k} = \|u\|_k + \|Du\|_k,$$

where $\|Du\|_k = \||Du|\|_k$. Similarly $\rho_k(Du) = \rho_k(|Du|)$. Also

$$W_0^{1,k}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,k}}.$$

Then $W^{1,k}(\Omega)$ and $W_0^{1,k}(\Omega)$ are both Banach spaces and if $k(z,\cdot)$ is uniformly convex, then they are uniformly convex Banach spaces, hence reflexive. Moreover, if $k \leqslant \hat{k}$, then

$$W^{1,\hat{k}}(\Omega) \hookrightarrow W^{1,k}(\Omega) \text{ and } W_0^{1,\hat{k}}(\Omega) \hookrightarrow W_0^{1,k}(\Omega) \text{ continuously.}$$

Under some additional regularity and growth conditions on k, which are satisfied in the case of the double integrand $\theta(z,t)$, we have that the Poincaré inequality holds for the space $W_0^{1,k}(\Omega)$ (see Harjulehto & Hästö [24, p. 100]) and so we can use the norm

$$\|u\|_{1,k} = \|Du\|_k \text{ for all } u \in W_0^{1,k}(\Omega).$$

For $x \in \mathbb{R}$, let $x^+ = \max\{x,0\}$, $x^- = \max\{-x,0\}$. Then for $u \in M(\Omega)$ we set $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. If $u \in W_0^{1,k}(\Omega)$, then $u^\pm \in W_0^{1,k}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

Following the proof of Theorem 6.3.5 of Harjulehto & Hästö [24, p. 142], we obtain the following compact embedding result.

Lemma 2. *If hypotheses* H_0 *hold, then* $W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega)$ *compactly.*

Proof. Since we will use convolutions (mollifications) and $a(z) = 0$ for all $z \in \partial\Omega$, we extend $a(\cdot)$ to all of $\mathbb{R}^N$ by setting $a(z) = 0$ for all $z \in \mathbb{R}^N \setminus \overline{\Omega}$.

Suppose $u_n \xrightarrow{w} u$ in $W_0^{1,\varphi}(\Omega)$. Let $y_n = u_n - u$. We have $y_n \xrightarrow{w} 0$ in $W_0^{1,\varphi}(\Omega)$. Hence we can find $\widehat{c}_1 > 0$ such that

$$\|y_n\|_{1,\varphi} \leqslant \widehat{c}_1, \ \forall n \in \mathbb{N}. \tag{2}$$

We know that $C_c^\infty(\Omega)$ is dense in $W_0^{1,\varphi}(\Omega)$. Hence we can find $\theta_n \in C_c^\infty(\Omega)$ such that

$$\begin{aligned} &\|u_n - \theta_n\|_{1,\varphi} \leqslant \frac{1}{n}, \ \forall n \in \mathbb{N}, \\ &\Rightarrow \theta_n \xrightarrow{w} 0 \text{ in } W_0^{1,\varphi}(\Omega). \end{aligned} \tag{3}$$

Let $\{\eta_\varepsilon\}_{\varepsilon\in(0,1]}$ be the standard mollifier. We have

$$\begin{aligned} \|\theta_n\|_\varphi &\leqslant \|\theta_n - \eta_\varepsilon * \theta_n + \eta_\varepsilon * \theta_n\|_\varphi \\ &\leqslant \|\theta_n - \eta_\varepsilon * \theta_n\|_\varphi + \|\eta_\varepsilon * \theta_n\|_\varphi. \end{aligned} \tag{4}$$

Note that

$$\begin{aligned} &(\eta_\varepsilon * \theta_n)(s) - \theta_n(s) \\ = &\int_{\mathbb{R}^N} \eta_\varepsilon(z)\left(\theta_n(s-z) - \theta_n(s)\right) dz \ \left(\text{recall that } \int_{\mathbb{R}^N} \eta_\varepsilon(z) dz = 1\right) \\ = &\int_{\mathbb{R}^N} \eta_\varepsilon(z) \int_0^1 \frac{d}{dt}\theta_n(s-tz) dt dz \\ = &\int_{\mathbb{R}^N} \eta_\varepsilon(z) \int_0^1 (D\theta_n(s-tz), z)_{\mathbb{R}^N} \, dt dz \ \text{(by the chain rule)} \\ = &\int_0^1 \int_{\mathbb{R}^N} \eta_\varepsilon(z)\,(D\theta_n(s-tz), z)_{\mathbb{R}^N} \, dz dt \ \text{(Fubini's theorem)} \\ = &\int_0^1 \int_{\mathbb{R}^N} \eta_{\varepsilon t}(z) \left(D\theta_n(s-z), \frac{z}{t}\right)_{\mathbb{R}^N} dz dt \ \text{(change of variables).} \end{aligned} \tag{5}$$

We know that

$$\eta_{\varepsilon t}(z) \neq 0 \text{ if } \frac{1}{t}|z| \leqslant \varepsilon. \tag{6}$$

Then from (5) and (6) we have

$$\begin{aligned}
&\left|\left(\eta_{\varepsilon} * \theta_n\right)(s) - \theta_n(s)\right| \\
&\leqslant \varepsilon \int_0^1 \eta_{\varepsilon}(z)|D\theta_n(s - tz)|dzdt \\
&= \varepsilon \int_0^1 \left(\eta_{\varepsilon t} * |D\theta_n|\right)(s)dt, \\
\Rightarrow \ \left\|\left(\eta_{\varepsilon} * \theta_n\right) - \theta_n\right\|_{\varphi} &\leqslant \widehat{c}_2\varepsilon \int_0^1 \|\eta\|_1 \|D\theta_n\|_{\varphi}dt \\
&\quad \text{for some } \widehat{c}_2 > 0 \text{ (see Harjulehto \& Hästö [24, p. 91])} \\
&= \widehat{c}_2\varepsilon\|D\theta_n\|_{\varphi} \text{ (recall that } \|\eta\|_1 = 1). \qquad (7)
\end{aligned}$$

Returning to (4) and using (7), we have

$$\|\theta_n\|_{\varphi} \leqslant \widehat{c}_2\varepsilon\|D\theta_n\|_{\varphi} + \|\eta_{\varepsilon} * \theta_n\|_{\varphi}, \ \forall n \in \mathbb{N}. \tag{8}$$

From (3) we have

$$\left(\eta_{\varepsilon} * \theta_n\right)(s) = \int_{\mathbb{R}^N} \eta_{\varepsilon}(s - z)\theta_n(z)dz \to 0 \text{ as } n \to \infty. \tag{9}$$

We set $\overline{\Omega}_{-\varepsilon} = \left\{z \in \mathbb{R}^N : d(z, \Omega) \leqslant \varepsilon\right\}$. Then

$$\left(\eta_{\varepsilon} * \theta_n\right)(s) = 0 \text{ for all } s \in \mathbb{R}^N \setminus \overline{\Omega}_{-\varepsilon}. \tag{10}$$

So, we have

$$\begin{aligned}
\left|\left(\eta_{\varepsilon} * \theta_n\right)(s)\right| &\leqslant \int_{\mathbb{R}^N} \eta_{\varepsilon}(s - z)|\theta_n(z)|dz \\
&\leqslant \widehat{c}_3\|\eta_{\varepsilon}(s - \cdot)\|_{\varphi^*}\|\theta_n\|_{\varphi} \text{ for some } \widehat{c}_3 > 0 \\
&\quad \text{(by Hölder's inequality, see [24, p. 54])} \\
&\leqslant \widehat{c}_4\frac{1}{\varepsilon^N}\|\chi_{\overline{\Omega}_{-\varepsilon}}\|_{\varphi^*} \text{ for some } \widehat{c}_4 > 0 \text{ (see (10))} \\
&\leqslant \frac{\widehat{c}_5}{\varepsilon^N} \text{ for some } \widehat{c}_5 > 0, \text{ all } s \in \overline{\Omega}_{-\varepsilon}, \text{ all } n \in \mathbb{N}, \\
\Rightarrow \ \left|\left(\eta_{\varepsilon} * \theta_n\right)(s)\right| &\leqslant \frac{\widehat{c}_5}{\varepsilon^N}\chi_{\overline{\Omega}_{-\varepsilon}}(s), \forall s \in \mathbb{R}^N \text{ (see (10))}. \qquad (11)
\end{aligned}$$

From (9), (11) and the dominated convergence theorem (see [24, p. 45]), we have

$$\eta_{\varepsilon} * \theta_n \to 0 \text{ in } L^{\varphi}(\Omega). \tag{12}$$

Returning to (8), passing to the limit as $n \to \infty$ and using (2) and (12), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\theta_n\|_{\varphi} \leqslant \varepsilon\widehat{c}_6 \text{ for some } \widehat{c}_6 > 0, \\
&\Rightarrow \theta_n \to 0 \text{ in } L^{\varphi}(\Omega) \text{ (since } \varepsilon > 0 \text{ is arbitrary)}, \\
&\Rightarrow y_n \to 0 \text{ in } L^{\varphi}(\Omega), \\
&\Rightarrow W_0^{1,\varphi}(\Omega) \hookrightarrow L^{\varphi}(\Omega) \text{ compactly.}
\end{aligned}$$

The proof is now complete. $\square$

As we already mentioned in the proof of the multiplicity theorem (see Section 5), we also use Morse theoretic tools (critical groups). So, below we recall some basic definitions and facts about them. Let X be a Banach space and (Y_1, Y_2) a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$ by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Given $\psi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$, we introduce the following sets

$$\psi^c = \{u \in X : \psi(u) \leqslant c\}, \, K_\psi = \{u \in X : \psi'(u) = 0\}, \, K_\psi^c = \{u \in K_\psi : \psi(u) = c\}.$$

If $u \in K_\psi^c$ is isolated, then the critical groups of ψ at u are defined by

$$C_k(\psi, u) = H_k\left(\psi^c \cap U, \psi^c \cap U \setminus \{u\}\right) \text{ for all } k \geqslant 0,$$

with U being a neighborhood of u such that $K_\psi \cap \psi^c \cap U = \{u\}$. The excision property of singular homology, implies that the definition of critical groups is independent of the choice of the isolating neighborhood U.

If $\psi \in C^1(X, \mathbb{R})$ satisfies the C-condition (that is, every sequence $\{u_n\}_{n \geqslant 1} \subseteq X$ such that $\{\psi(u_n)\} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\psi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence), $\inf \psi(K_\psi) > -\infty$ and $c < \inf \psi(K_\psi)$, then the critical groups of ψ at infinity are defined by

$$C_k(\psi, \infty) = H_k(X, \psi^c) \text{ for all } k \geqslant 0.$$

On account of Corollary 5.3.13 of Papageorgiou, Rădulescu & Repovš [25, p. 392], we see that the above definition is independent of the choice of the level $c < \inf \psi(K_\psi)$.

Suppose that K_ψ is finite. We introduce the following quantities

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\psi, u) t^k \text{ for all } t \in \mathbb{R}, \text{ all } u \in K_\psi,$$
$$P(t, \infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\psi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\psi} M(t, u) = P(t, \infty) + (1 + t) Q(t),$$

with $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally we mention that by $\left\{\widetilde{\lambda}_k(2)\right\}_{k \in \mathbb{N}_0}$ we denote the eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. We know that $\widetilde{\lambda}_1(2) > 0$ and $\widetilde{\lambda}_k(2) \to +\infty$ as $k \to \infty$. Also by $\widetilde{u}_1(2)$ we denote the positive, L^2-normalized (that is, $\|\widetilde{u}_1(2)\|_2 = 1$) eigenfunction corresponding to $\widetilde{\lambda}_1(2) > 0$. We know that $\widetilde{u}_1(2) \in \operatorname{int} C_+$, with $C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}$ (the positive (order) cone of $C_0^1(\overline{\Omega})$).

Throughout this work, by "solution" of problem (1) we understand a weak solution, that is a function $u \in W_0^{1,\theta}(\Omega)$ satisfying for all $h \in W_0^{1,\theta}(\Omega)$

$$\int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz = \int_\Omega f(z, u) h dz.$$

3. A maximum problem

In this section we prove a maximum principle for the weighted p-Laplacian Δ_p^a (that is, for $\varphi \in \mathrm{N}(\Omega)$). Related properties can be found in Zhang [26] (for anisotropic differential operators) and in Papageorgiou, Vetro & Vetro [18] (double phase differential operators, that is, for $\theta \in \mathrm{N}(\Omega)$).

Proposition 3. *If $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$, $u \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$, $u \geqslant 0$, $u \neq 0$ and we have (in the weak sense)*

$$-\Delta_p^a u(z) + \xi(z)u(z)^{p-1} \geqslant 0 \text{ in } \Omega,$$

then $u(z) > 0$ for a.a. $z \in \Omega$.

Proof. First we assume that $u \in C^1(\Omega)$. Arguing by contradiction, suppose that we can find $z_1, z_2 \in \Omega$ and an open ball $B_{2r}(z_2) = \left\{z \in \mathbb{R}^N : |z - z_2| < 2r\right\}$ $(r > 0)$ such that

$$z_1 \in \partial B_{2r}(z_2),\ u(z_1) = 0,\ u|_{B_{2r}(z_2)} > 0. \tag{13}$$

By moving the center z_2, we have $r > 0$ arbitrarily small. Then

$$u(z_1) = \min_{\Omega} u \text{ and } Du(z_1) = 0.$$

We define

$$m = \min\left\{u(z) : z \in \partial B_r(z_2)\right\} > 0 \text{ (see (13))}.$$

Using L'Hôpital's rule, we see that

$$m,\ \frac{m}{r} \to 0^+ \text{ as } r \to 0^+. \tag{14}$$

We introduce the following items

$$\begin{aligned} &R = \{z \in \Omega : r < |z - z_2| < 2r\} \text{ (an open ring in } B_{2r}(z_2)),\\ &\hat{m} = \sup\left\{|Da(z)| : z \in R\right\},\ m_0 = \min_{\overline{R}} a > 0. \end{aligned}$$

By hypothesis H_0, the weight function $a(\cdot)$ is Lipschitz continuous, hence by Rademacher's theorem (see Evans & Gariepy [27, p. 81]), $a(\cdot)$ is differentiable almost everywhere in Ω. Therefore $\hat{m}$ is well defined and $\hat{m} < +\infty$. We set

$$\tau = -\ln\frac{m}{r} + \frac{N-1}{r} + 3\frac{\hat{m}}{m_0}. \tag{15}$$

We consider the function

$$w(t) = \frac{m\left(e^{\frac{\tau t}{p-1}} - 1\right)}{e^{\frac{\tau r}{p-1}} - 1} \text{ for all } t \in [0, r].$$

For $r > 0$ small, we have (see (14))

$$\left\{\begin{array}{l} 0 < w(t),\ w'(t) < 1 \text{ for all } 0 < t \leqslant r,\\ w''(t) = \dfrac{\tau}{p-1}w'(t) \text{ for all } 0 \leqslant t \leqslant r. \end{array}\right\} \tag{16}$$

To simplify the presentation, without any loss of generality, we assume that $z_2 = 0$. Let $l = |z|$, $t = 2r - l$. For $t \in [r, 2r]$, we have

$$\begin{aligned} &y(l) = w(2r - l) = w(t),\\ \Rightarrow\ &y'(l) = -w'(t),\ y''(l) = w''(t). \end{aligned} \tag{17}$$

For $z \in \mathbb{R}$ with $|z| = l$ we write

$$\begin{aligned} &\hat{y}(z) = y(l),\\ \Rightarrow\ &\hat{y} \in C^2(R). \end{aligned}$$

We have

$$
\begin{aligned}
&\operatorname{div}\left(a(z)|D\hat{y}|^{p-2}D\hat{y}\right)\\
&= a(z)(p-1)w'(t)^{p-2}w''(t) - a(z)\frac{N-1}{r}w'(t)^{p-1}\\
&- w'(t)^{p-1}\sum_{k=1}^{N}\frac{\partial a}{\partial z_k}\frac{z_k}{r}\ \ (z=(z_k)_{k=1}^{N})\\
&\geqslant a(z)\left(\tau - \frac{N-1}{r}\right)w'(t)^{p-1} - 2\hat{m}w'(t)^{p-1}\\
&= a(z)\left(-\ln\frac{m}{r} + 3\frac{\hat{m}}{m_0}\right)w'(t) - 2\hat{m}w'(t)^{p-1}\\
&\geqslant \left(a(z)\left(-\ln\frac{m}{r}\right)+\hat{m}\right)w'(t)^{p-1}\\
&\geqslant \left(\eta\left(-\ln\frac{m}{r}\right)+\hat{m}\right)w'(t)^{p-1}\\
&\text{(for } r>0 \text{ small and with } 0<\eta\leqslant a(z) \text{ for all } z\in R,\ \text{see (14)).}
\end{aligned}
\tag{18}
$$

Note that $0 < w'(0) \leqslant w'(t)$ for all $t \in [0, r]$ and $w(\cdot)$ is increasing. So, for $r > 0$ even smaller if necessary, we will have

$$
\begin{aligned}
&\left(\eta\left(-\ln\frac{m}{r}\right)+\hat{m}\right)w'(t)^{p-1}\\
&\geqslant \left(\eta\left(-\ln\frac{m}{r}\right)+\hat{m}\right)w'(0)^{p-1}\\
&\geqslant \|\xi\|_\infty w(r)^{p-1}\\
&\geqslant \xi(z)w(t)^{p-1} \text{ for a.a. } z\in\Omega \text{ (see (14)).}
\end{aligned}
\tag{19}
$$

Using (19) in (18), we obtain that

$$\operatorname{div}\left(a(z)|D\hat{y}|^{p-2}D\hat{y}\right) \geqslant \xi(z)\hat{y} \text{ in } R.$$

This means that $\hat{y}$ is a lower solution of the equation

$$-\Delta_p^a u(z) + \xi(z)u(z)^{p-1} = 0 \text{ in } R.$$

From (17) we see that $\hat{y} \leqslant u$ on ∂R. Hence by the weak comparison principle (see Pucci & Serrin [28, Theorem 3.4.1, p. 61]), we have

$$\hat{y} \leqslant u \text{ in } R.$$

Then we have

$$
\begin{aligned}
\lim_{s\to 0^+}\frac{u(z_1+s(z_2-z_1))-u(z_1)}{s} &= \lim_{s\to 0^+}\frac{u(z_1+s(z_2-z_1))}{s} \text{ (see 3)}\\
&\geqslant \lim_{s\to 0^+}\frac{\hat{y}(z_1+s(z_2-z_1))-\hat{y}(z_1)}{s} = w'(0) > 0.
\end{aligned}
$$

But this contradicts the fact that $Du(z_1) = 0$.

So, we have proved that

$$u(z) > 0 \text{ for all } z\in\Omega, \text{ when } u\in C^1(\Omega).$$

Now remove the requirement that $u \in C^1(\Omega)$. For this purpose we introduce the following set

$$\hat{\Omega} = \left\{z\in\Omega:\ \text{there exists an open ball } B_r(z)\subseteq\Omega \text{ such that } u|_{B_r(z)} = 0 \text{ a.e.}\right\}.$$

Also let $\Omega_+ = \Omega \setminus \hat{\Omega}$.

Consider $z_0 \in \Omega_+$. We can find a ball $\overline{B}_{2r}(z_0) \subseteq \Omega$ such that $u|_{\partial B_{2r}(z_0)}$ is not identically zero. We consider the following nonhomogeneous Dirichlet problem

$$-\Delta_p^a w(z) + \xi(z)w(z)^{p-1} = 0 \text{ in } B_{2r}(z_0),\ w|_{\partial B_{2r}(z_0)} = u|_{\partial B_{2r}(z_0)}. \tag{20}$$

Since $a|_{\overline{B}_{2r}(z_0)} > 0$ (see hypothesis H_0), using Theorem 1 of Lieberman [29], we have that if $w \in W^{1,p}(B_{2r}(z_0))$ is a solution of (20), then

$$w \in C^1(\overline{B}_{2r}(z_0)),\ w \geqslant 0,\ w \neq 0.$$

Then we can argue as in the first part of the proof and infer that

$$w(z) \geqslant c_r > 0 \text{ for all } z \in \overline{B}_r(z_0). \tag{21}$$

By the weak comparison principle (see Pucci & Serrin [28, Theorem 3.4.1, p. 61]) we have

$$\begin{aligned} & w(z) \leqslant u(z) \text{ for a.a. } z \in \overline{B}_{2r}(z_0), \\ \Rightarrow\ & 0 < c_r \leqslant u(z) \text{ for a.a. } z \in \overline{B}_r(z_0) \text{ (see (21))}. \end{aligned}$$

The set $\hat{\Omega} \subseteq \Omega$ is a strict open subset of Ω (recall $u \neq 0$). Therefore, if $\hat{\Omega} \neq \emptyset$, then we can find $z_0 \in \Omega_+ \cap \partial\hat{\Omega}$. From the above argument, we infer that for $r > 0$ small we will have

$$u(z) > 0 \text{ for a.a. } z \in \hat{\Omega} \cap B_r(z_0),$$

which contradicts the definition of $\hat{\Omega}$. Therefore $\hat{\Omega} = \emptyset$ and so $\Omega = \Omega_+$. Then by a standard compactness argument, we conclude that $u(z) > 0$ for a.a. $z \in \Omega$.

This proof is now complete. □

4. A weighted eigenvalue problem

In this section we study the following eigenvalue problem for the operator Δ_p^a.

$$-\Delta_p^a u(z) = \hat{\lambda} a(z)|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{22}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of the Dirichlet Δ_p^a operator, if problem (22) admits a nontrivial solution $\hat{u} \in W_0^{1,\varphi}(\Omega)$, called an "eigenfunction" corresponding to $\hat{\lambda}$.

In the next proposition, we show the existence of a smallest eigenvalue $\hat{\lambda}_1$ and determine a sign property for the corresponding eigenfunctions.

Proposition 4. *If hypotheses* H_0, H_1 *hold, then the eigenvalue problem* (22) *has a smallest eigenvalue and every corresponding eigenfunction* $\hat{u}_1 \in W_0^{1,\varphi}(\Omega)$ *satisfies* $\hat{u}_1 \in L^\infty(\Omega)$ *and either* $\hat{u}_1(z) > 0$ *for a.a.* $z \in \Omega$ *or* $\hat{u}_1(z) < 0$ *for a.a.* $z \in \Omega$ *(that is,* $\hat{u}_1(\cdot)$ *has fixed sign).*

Proof. Let

$$\hat{\lambda}_1 = \inf\left\{ \frac{\int_\Omega a(z)|Du|^p dz}{\int_\Omega a(z)|u|^p dz} : u \in W_0^{1,\varphi}(\Omega),\ u \neq 0 \right\}. \tag{23}$$

Exploiting the homogeneity of both integrals, we can write

$$\hat{\lambda}_1 = \inf\left\{ \rho_\varphi(Du) : u \in W_0^{1,\varphi}(\Omega),\ \rho_\varphi(u) = 1 \right\}.$$

Consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,\varphi}(\Omega)$ such that

$$\rho_\varphi(u_n) = 1 \text{ for all } n \in \mathbb{N}_0,\ \rho_\varphi(Du_n) \downarrow \hat{\lambda}_1 \text{ as } n \to \infty.$$

Evidently $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,\varphi}(\Omega)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \hat{u}_1 \text{ in } W_0^{1,\varphi}(\Omega). \tag{24}$$

From Lemma 2, we know that $W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega)$ compactly (see hypotheses H_1). Therefore from (24) it follows that

$$\begin{aligned} & u_n \to \hat{u}_1 \text{ in } L^\varphi(\Omega), \\ \Rightarrow\ & \rho_\varphi(\hat{u}_1) = 1. \end{aligned} \tag{25}$$

Note that the modular function $\rho_\varphi(\cdot)$ is continuous, convex, hence it is sequentially weakly lower semicontinuous. So, from (24), we have

$$\begin{aligned} & \rho_\varphi(D\hat{u}_1) \leqslant \liminf_{n\to\infty} \rho_\varphi(Du_n), \\ \Rightarrow\ & \rho_\varphi(D\hat{u}_1) \leqslant \hat{\lambda}_1 \text{ and } \rho_\varphi(\hat{u}_1) = 1, \text{ (see (25))}, \\ \Rightarrow\ & \rho_\varphi(D\hat{u}_1) = \hat{\lambda}_1,\ \rho_\varphi(\hat{u}_1) = 1, \\ \Rightarrow\ & \hat{\lambda}_1 > 0. \end{aligned}$$

From the Lagrange multiplier theorem (see Papageorgiou, Rădulescu & Repovš [25, Theorem 5.5.9, p. 422]), we have

$$-\Delta_p^a \hat{u}_1(z) = \hat{\lambda}_1 a(z)|\hat{u}_1(z)|^{p-2}\hat{u}_1(z) \text{ in } \Omega,\ \hat{u}_1|_{\partial\Omega} = 0. \tag{26}$$

Suppose that $\hat{u}_1^+ \neq 0$. Acting on (26) with $\hat{u}_1^+ \in W_0^{1,\varphi}(\Omega)$, we obtain

$$\begin{aligned} & \rho_\varphi(D\hat{u}_1^+) = \hat{\lambda}_1 \rho_\varphi(\hat{u}_1^+), \\ \Rightarrow\ & \hat{u}_1^+ \text{ realizes the infimum in (23)}, \\ \Rightarrow\ & \hat{u}_1^+ \text{ is an eigenfunction corresponding to } \hat{\lambda}_1 > 0. \end{aligned}$$

From Colasuonno & Squassina [12, pp. 1933–1934], we have that $\hat{u}_1^+ \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$. Also by Proposition 2 (the maximum principle) we have $\hat{u}_1^+ > 0$ for a.a. $z \in \Omega$. If $\hat{u}_1^+ = 0$, then $\hat{u}_1 = -\hat{u}_1^- \leqslant 0$ and so $-\hat{u}_1 \geqslant 0$ is also an eigenfunction corresponding to $\hat{\lambda}_1 > 0$ and we are back to the previous case and obtain $\hat{u}_1 < 0$ for a.a. $z \in \Omega$.

This proof is now complete. □

Using these properties of $\hat{\lambda}_1 > 0$ and of the corresponding eigenfunctions, we can prove the following useful result.

Proposition 5. *If hypotheses* H_0, H_1 *hold,* $\eta \in L^\infty(\Omega)_+$ *and*

$$\eta(z) \leqslant \hat{\lambda}_1 a(z) \textit{ for a.a. } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_1 a,$$

then there exists $c^* > 0$ *such that*

$$c^*\|u\|_{1,\varphi}^p \leqslant \rho_\varphi(Du) - \int_\Omega \eta(z)|u|^p dz \textit{ for all } u \in W_0^{1,\varphi}(\Omega).$$

Proof. We consider the C^1-functional $\sigma : W_0^{1,\varphi}(\Omega) \mapsto \mathbb{R}$ defined by

$$\sigma(u) = \rho_\varphi(Du) - \int_\Omega \eta(z)|u|^p dz \text{ for all } u \in W_0^{1,\varphi}(\Omega).$$

On account of (23) and using the hypothesis on $\eta(\cdot)$, we have

$$0 \leqslant \sigma(u) \text{ for all } u \in W_0^{1,\varphi}(\Omega).$$

Arguing by contradiction, suppose that the assertion of the proposition is not true. Then we can find $\{u_n\}_{n\in\mathbb{N}_0} \subseteq W_0^{1,\varphi}(\Omega)$ such that

$$0 \leqslant \sigma(u_n) = \rho_\varphi(Du_n) - \int_\Omega \eta(z)|u_n|^p dz < \frac{1}{n}\|u_n\|_{1,\varphi}^p \text{ for all } n \in \mathbb{N}_0. \tag{27}$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,\varphi}}$, $n \in \mathbb{N}_0$. Then $\|y_n\|_{1,\varphi} = 1$ for all $n \in \mathbb{N}_0$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\varphi}(\Omega) \text{ and } y_n \to y \text{ in } L^\varphi(\Omega) \tag{28}$$

(recall that $W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega)$ compactly). From (27), we have

$$0 \leqslant \rho_\varphi(Dy_n) - \int_\Omega \eta(z)|y_n|^p dz < \frac{1}{n} \text{ for all } n \in \mathbb{N}_0. \tag{29}$$

On account of (28) and by passing to a subsequence if necessary, we may assume that

$$y_n(z) \to y(z) \text{ for a.a. } z \in \Omega,\ |y_n(z)| \leqslant h(z) \text{ for a.a. } z \in \Omega,\ \text{ all } n \in \mathbb{N}_0 \tag{30}$$

with $h \in L^\varphi(\Omega)_+$ (see Diening, Harjulehto, Hästö & Růžička [22, Lemma 2.3.15, p. 37] and Colasuonno & Squassina [12, Lemma 2.16]).

We have

$$0 \leqslant \eta(z)|y_n|^p \leqslant \hat{\lambda}_1 a(z)|y_n|^p \leqslant \hat{\lambda}_1 a(z)h(z) \text{ for a.a. } z \in \Omega,\ \text{ all } n \in \mathbb{N}_0 \text{ (see (30))}. \tag{31}$$

From (30), (31) and the Lebesgue dominated convergence theorem, we have

$$\int_\Omega \eta(z)|y_n|^p dz \to \int_\Omega \eta(z)|y|^p dz \text{ as } n \to \infty. \tag{32}$$

Also, from the sequential weak lower semicontinuity of the modular function $\rho_\varphi(\cdot)$ and (28), we have

$$\rho_\varphi(Dy) \leqslant \liminf_{n\to\infty} \rho_\varphi(Dy_n). \tag{33}$$

We return to (29), pass to the limit as $n \to \infty$ and use (32) and (33). We obtain

$$\rho_\varphi(Dy) \leqslant \int_\Omega \eta(z)|y|^p dz. \tag{34}$$

If $y = 0$, then we have

$$\begin{aligned} &\rho_\varphi(Dy_n) \to 0 \text{ (see (29) and (32))}, \\ \Rightarrow\ &\|y_n\|_{1,\varphi} \to 0 \text{ (see Proposition 1}(b)). \end{aligned}$$

But this contradicts the fact that $\|y_n\|_{1,\varphi} = 1$ for all $n \in \mathbb{N}_0$.

If $y \neq 0$, then from (34), the hypothesis on $\eta(\cdot)$ and (23), it follows that

$$\rho_\varphi(Dy) = \hat{\lambda}_1 \rho_\varphi(y).$$

By the Lagrange multiplier rule, this means that $y \in W_0^{1,\varphi}(\Omega)$ is an eigenfunction corresponding to $\hat{\lambda}_1$. From Colasuonno & Squassina [12, p. 1933], we have that $y \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$, while from Proposition 2, we have that $|y(z)| > 0$ for a.a. $z \in \Omega$. Then (34) and the hypothesis on $\eta(\cdot)$, imply

$$\rho_\varphi(Dy) < \hat{\lambda}_1 \rho_\varphi(y),$$

which contradicts (23).

This proof is now complete. □

Using these results, we can now treat resonant double phase problems.

5. A multiplicity theorem

In this section we prove a multiplicity theorem for problem (1), when the reaction $f(z,\cdot)$ is resonant with respect to $\hat{\lambda}_1 > 0$ as $x \to \pm\infty$. Our method of proof combines variational techniques and Morse theoretic (critical groups) arguments.

The hypotheses on the reaction $f(z,\cdot)$ are the following:

H_2 : $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)\left(1 + |x|^{r-2}\right)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and

$$p \leqslant r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geqslant 3 \\ +\infty & \text{if } N = 2 \end{cases};$$

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\limsup_{x\to\infty} \frac{pF(z,x)}{a(z)|x|^p} \leqslant \hat{\lambda}_1$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\beta_0 > 0$ such that

$$-\beta_0 \leqslant f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iv) there exists $m \in \mathbb{N}_0$ such that

$$\begin{aligned} & f'_x(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega, \\ & f'_x(z,0) \in \left[\widetilde{\lambda}_m(2), \widetilde{\lambda}_{m+1}(2)\right] \text{ for a.a. } z \in \Omega, \\ & f'_x(\cdot,0) \not\equiv \widetilde{\lambda}_m(2),\ f'_x(\cdot,0) \not\equiv \widetilde{\lambda}_{m+1}(2). \end{aligned}$$

Remark 2. Hypothesis H_2(ii) implies that the problem can be resonant with respect to $\hat{\lambda}_1 > 0$ as $x \to \pm\infty$. As we will show in the process of the proof, hypothesis H_2(iii) implies that the resonance occurs from the left of $\hat{\lambda}_1 > 0$, making the problem coercive.

Example 1. The following function satisfies hypotheses H_2 (for the sake of simplicity we drop the z-dependence)

$$f(x) = \begin{cases} \hat{\lambda}_1 |x|^{p-2}x - c\ln|x| & \text{if } x < -1, \\ \theta x + (\hat{\lambda}_1 - \theta)|x|^{r-2}x & \text{if } |x| \leqslant 1, \\ \hat{\lambda}_1 x^{p-1} + c\ln x & \text{if } 1 < x, \end{cases}$$

with $r > 1$ and $\theta \in \left(\widetilde{\lambda}_m(2), \widetilde{\lambda}_{m+1}(2)\right)$, where $m \in \mathbb{N}_0$ is large enough so that

$$c = \hat{\lambda}_1(r-p) - \theta(r-2) < 0.$$

Let $\psi : W_0^{1,\theta}(\Omega) \mapsto \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\psi(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p dz + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,\theta}(\Omega).$$

Evidently, $\psi \in C^2(W_0^{1,\theta}(\Omega))$.

Proposition 6. *If hypotheses* H_0, H_1, H_2 *hold, then the functional* $\psi(\cdot)$ *is coercive.*

Proof. For $x \neq 0$, we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{F(z,x)}{|x|^p}\right) &= \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{|x|^p x} \\ &\begin{cases} \geqslant \frac{\beta_0}{x^{p+1}} & \text{if } x > 0 \\ \leqslant -\frac{\beta_0}{|x|^p x} & \text{if } x < 0 \end{cases} \quad \text{for a.a. } z \in \Omega \text{ (see hypothesis } \mathrm{H}_2\text{(iii))}, \\ \Rightarrow \frac{F(z,v)}{|v|^p} - \frac{F(z,x)}{|x|^p} &\geqslant \frac{\beta_0}{p}\left(\frac{1}{|v|^p} - \frac{1}{|x|^p}\right) \text{ for a.a. } z \in \Omega, \text{ all } 0 < |x| < |v|.\end{aligned}$$

We let $v \to \pm\infty$ and using hypothesis H_2(ii), we obtain

$$\begin{aligned}&\frac{1}{p}\hat{\lambda}_1 a(z) - \frac{F(z,x)}{|x|^p} \geqslant -\frac{\beta_0}{|x|^p} \text{ for a.a. } z \in \Omega, \text{ all } |x| > 0, \\ &\Rightarrow \hat{\lambda}_1 a(z)|x|^p - pF(z,x) \geqslant -p\beta_0 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.\end{aligned} \tag{35}$$

Arguing by contradiction, suppose that $\psi(\cdot)$ is not coercive. Then we can find $\{u_n\}_{n\in\mathbb{N}_0} \subseteq W_0^{1,\theta}(\Omega)$ such that

$$\|u_n\|_{1,\theta} \to +\infty \text{ and } \varphi(u_n) \leqslant c_0 \text{ for some } c_0 > 0, \text{ all } n \in \mathbb{N}_0. \tag{36}$$

First suppose that $\|u_n\|_{1,\varphi} \to +\infty$ as $n \to \infty$. Then let $y_n = \frac{u_n}{\|u_n\|_{1,\varphi}}$, $n \in \mathbb{N}_0$. We have $\|y_n\|_{1,\varphi} = 1$ for all $n \in \mathbb{N}_0$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\varphi}(\Omega) \text{ and } y_n \to y \text{ in } L^\varphi(\Omega) \text{ as } n \to \infty. \tag{37}$$

From (36) we have

$$\begin{aligned}&\frac{1}{p}\int_\Omega a(z)|Du_n|^p dz - \int_\Omega F(z,u_n)dz \leqslant c_0 \text{ for all } n \in \mathbb{N}_0, \\ &\Rightarrow \frac{1}{p}\left(\rho_\varphi(Dy_n) - \int_\Omega \frac{pF(z,u_n)}{\|u_n\|_{1,\varphi}^p}dz\right) \leqslant \frac{c_0}{\|u_n\|_{1,\varphi}^p} \text{ for all } n \in \mathbb{N}_0.\end{aligned} \tag{38}$$

On account of hypothesis H_2(ii), we have

$$\limsup_{n\to\infty} \frac{pF(z,u_n(z))}{\|u_n\|_{1,\varphi}^p} \leqslant \eta(z)|y|^p \text{ for a.a. } z \in \Omega, \tag{39}$$

with $\eta \in L^\infty(\Omega)_+$, $\eta(z) \leqslant \hat{\lambda}_1 a(z)$ for a.a. $z \in \Omega$ (see Aizicovici, Papageorgiou & Staicu [30], proof of Proposition 16). If in (38) we pass to the limit as $n \to \infty$ and use (37), (39) and use the sequential weak lower semicontinuity of the modular function $\rho_\varphi(\cdot)$, we obtain

$$\rho_\varphi(Dy) \leqslant \int_\Omega \eta(z)|y|^p dz. \tag{40}$$

If $\eta \neq \hat{\lambda}_1 a$, then from (40) and Proposition 5, we have

$$\begin{aligned} & c^* \|y\|_{1,\varphi}^p \leqslant 0, \\ \Rightarrow\ & y = 0. \end{aligned}$$

From (39) it follows that

$$\begin{aligned} & \rho_\varphi(Dy_n) \to 0, \\ \Rightarrow\ & \|y_n\|_{1,\varphi} \to 0 \text{ (see Proposition (1))}, \end{aligned}$$

a contradiction since $\|y_n\|_{1,\varphi} = 1$ for all $n \in \mathbb{N}_0$.

If $\eta = \hat{\lambda}_1 a$, then from (40) we have

$$\begin{aligned} & \rho_\varphi(Dy) \leqslant \int_\Omega \hat{\lambda}_1 a(z) |y|^p dz, \\ \Rightarrow\ & \rho_\varphi(Dy) = \hat{\lambda}_1 \rho_\varphi(y) \text{ (see (23))}. \end{aligned} \tag{41}$$

From (41) we see that $y = 0$ or $y \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$ is an eigenfunction corresponding to $\hat{\lambda}_1 > 0$. If $y = 0$, then the previous argument leads to a contradiction of the fact that $\|y_n\|_{1,\varphi} = 1$ for all $n \in \mathbb{N}_0$. So, y is an eigenfunction corresponding to $\hat{\lambda}_1 > 0$. By Proposition 3, $|y(z)| > 0$ for a.a. $z \in \Omega$ and so

$$|u_n(z)| \to +\infty \text{ for a.a. } z \in \Omega, \text{ as } n \to \infty. \tag{42}$$

We have

$$\begin{aligned} & \frac{1}{p}\left(\rho_\varphi(Du_n) - \int_\Omega pF(z,u_n)dz\right) + \frac{1}{2}\|Du_n\|_2^2 \leqslant c_0 \text{ for all } n \in \mathbb{N}_0 \text{ (see (36))}, \\ \Rightarrow\ & \frac{1}{p}\left(\hat{\lambda}_1 \rho_\varphi(u_n) - \int_\Omega pF(z,u_n)dz\right) + \frac{1}{2}\|Du_n\|_2^2 \leqslant c_0 \text{ for all } n \in \mathbb{N}_0 \text{ (see (23))}, \\ \Rightarrow\ & \frac{1}{p}\int_\Omega \left(\hat{\lambda}_1 a(z)|u_n|^p - pF(z,u_n)\right) dz + \frac{\widetilde{\lambda}_1(2)}{2}\|u_n\|_2^2 \leqslant c_0, \\ \Rightarrow\ & \widetilde{\lambda}_1(2)\|u_n\|_2^2 \leqslant c_1 \text{ for some } c_1 > 0, \text{ all } n \in \mathbb{N}_0 \text{ (see (35))}. \end{aligned} \tag{43}$$

From (42), (43) and Fatou's lemma, we reach a contradiction.

Therefore, by passing to a subsequence if necessary, we may assume that

$$\begin{aligned} & \|u_n\|_{1,\varphi} \leqslant c_2 \text{ for some } c_2 > 0, \text{ all } n \in \mathbb{N}_0, \\ \Rightarrow\ & \rho_\varphi(Du_n) \leqslant c_3 \text{ for some } c_3 > 0, \text{ for all } n \in \mathbb{N}_0 \text{ (see Proposition 1)}. \end{aligned} \tag{44}$$

From (36) we have that $\rho_\theta(Du_n) \to +\infty$. Note that

$$\rho_\theta(Du_n) = \rho_\varphi(Du_n) + \|Du_n\|_2^2 \text{ for all } n \in \mathbb{N}_0.$$

Then from (44), we infer that

$$\|Du_n\|_2 \to +\infty \text{ as } n \to \infty. \tag{45}$$

Also again from (44) we see that by passing to a subsequence if necessary we may assume

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\varphi}(\Omega) \text{ and } u_n \to u \text{ in } L^\varphi(\Omega). \tag{46}$$

Hypotheses H_2(i), H_2(ii) imply that we can find $c_4 > 0$ such that

$$F(z,x) \leqslant c_4 + (\hat{\lambda}_1 + 1)|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{47}$$

From (36) we

$$\begin{aligned}\|Du_n\|_2^2 &\leqslant 2c_0 + 2\int_\Omega F(z,u_n)dz \\ &\leqslant c_5 + 2(\hat{\lambda}_1+1)\rho_\varphi(u_n) \text{ for some } c_5>0, \text{ all } n\in\mathbb{N}_0 \text{ (see (47))} \\ &\leqslant c_6 \text{ for some } c_6>0, \text{ all } n\in\mathbb{N}_0 \text{ (see (43)).}\end{aligned} \tag{48}$$

Comparing (45) and (48), we have a contradiction which proves that $\psi(\cdot)$ is coercive.
This proof is now complete. □

Using Proposition 6 and the direct method of the calculus of variationals, we can produce a first nontrivial solution of problem (1).

Proposition 7. *If hypotheses* H_0, H_1, H_2 *hold, then problem* (1) *admits a nontrivial solution*

$$u_0 \in W_0^{1,\theta}(\Omega)\cap L^\infty(\Omega).$$

Proof. From Proposition 6 we know that $\psi(\cdot)$ is coercive. Also, since $r<2^*$ we have that $W_0^{1,\theta}(\Omega)\hookrightarrow L^r(\Omega)$ compactly. Hence, it follows that $\psi(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0\in W_0^{1,\theta}(\Omega)$ such that

$$\psi(u_0)=\min\left\{\psi(u): u\in W_0^{1,\theta}(\Omega)\right\}. \tag{49}$$

On account of hypothesis H_2(iv) given $\varepsilon>0$, we can find $\delta>0$ such that

$$\frac{1}{2}\left(f'_x(z,0)-\varepsilon\right)x^2 \leqslant F(z,x) \text{ for a.a. } z\in\Omega, \text{ all } |x|\leqslant\delta. \tag{50}$$

Recall that $\widetilde{u}_1(2)\in \operatorname{int} C_+$ (see Section 2). So, we can find $t\in(0,1)$ small such that $0\leqslant t\widetilde{u}_1(2)(z)\leqslant\delta$ for all $z\in\overline{\Omega}$. We have

$$\begin{aligned}\psi(t\widetilde{u}_1(2)) &= \frac{t^p}{p}\rho_\varphi(D\widetilde{u}_1(2)) + \frac{t^2}{2}\widetilde{\lambda}_1(2) - \frac{1}{2}\int_\Omega\left(f'_x(z,0)-\varepsilon\right)\widetilde{u}_1(2)^2dz \\ &\quad \text{(see (50) and recall that } \|\widetilde{u}_1(2)\|_2=1) \\ &= \frac{t^p}{p}\rho_\varphi(D\widetilde{u}_1(2)) + \frac{t^2}{2}\int_\Omega\left(\widetilde{\lambda}_1(2)-f'_x(z,0)\right)\widetilde{u}_1(2)^2dz + \frac{\varepsilon}{2}.\end{aligned}$$

Note that

$$\gamma_0 = \int_\Omega\left(f'_x(z,0)-\widetilde{\lambda}_1(2)\right)\widetilde{u}_1(2)^2dz>0$$
(see hypothesis H_2(iv) and recall that $\widetilde{u}_1(2)\in\operatorname{int} C_+$).

So, choosing $\varepsilon\in(0,\gamma_0)$, we have that

$$\psi(t\widetilde{u}_1(2))\leqslant c_7t^p - c_8t^2 \text{ for some } c_7,\, c_8>0.$$

Since $2<p$, choosing $t\in(0,1)$ even smaller if necessary, we have

$$\begin{aligned}&\psi(\widetilde{u}_1(2))<0 \\ \Rightarrow\ &\psi(u_0)<0=\psi(0) \text{ (see (49))}, \\ \Rightarrow\ &u_0\neq 0.\end{aligned}$$

From (49) we have

$$\begin{aligned}
&\psi'(u_0) = 0,\\
\Rightarrow &\int_\Omega a(z)|Du_0|^{p-2}(Du_0, Dh)_{\mathbb{R}^N} dz + \int_\Omega (Du_0, Dh)_{\mathbb{R}^N} dz = \int_\Omega f(z, u_0) h dz\\
&\text{for all } h \in W_0^{1,\theta}(\Omega),\\
\Rightarrow &u_0 \in W_0^{1,\theta} \text{ is a nontrivial solution of problem (1).}
\end{aligned}$$

From Colasuonno & Squassina [12, pp. 1933–1934], we have that

$$u_0 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

This proof is now complete. □

Next, using Morse theoretic tools, we will generate a second nontrivial bounded solution for problem (1). Let $\hat{\xi} : H_0^1(\Omega) \mapsto \mathbb{R}$ be the functional defined by

$$\hat{\xi}(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u) dz \text{ for all } u \in H_0^1(\Omega).$$

We have $\hat{\xi} \in C^2\left(H_0^1(\Omega)\right)$.

In what follows by $E(\widetilde{\lambda}_k(2))$ we denote the eigenspace of $\left(-\Delta, H_0^1(\Omega)\right)$ corresponding to $\widetilde{\lambda}_k(2)$, $k \in \mathbb{N}_0$. We know that $E(\widetilde{\lambda}_k(2))$ is finite dimensional. We set $\overline{H}_m = \bigoplus_{k=1}^m E(\widetilde{\lambda}_k(2))$ and $\hat{H}_{m+1} = \overline{H}_m^\perp = \bigoplus_{k \geqslant m+1} E(\widetilde{\lambda}_k(2))$. We have $H_0^1(\Omega) = \overline{H}_m \bigoplus \hat{H}_{m+1}$.

Proposition 8. *If hypotheses* H_2(i), (iv) *hold, then* $C_k(\hat{\xi}, 0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$, *with* $d_m = \dim \overline{H}_m$.

Proof. Consider the orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{H}_m \bigoplus \hat{H}_{m+1}.$$

On account of hypothesis H_2(iv), given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\frac{1}{2}\left(f_x'(z,0) - \varepsilon\right) x^2 \leqslant F(z, x) \leqslant \frac{1}{2}\left(f_x'(z,0) + \varepsilon\right) x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta. \tag{51}$$

Since $\overline{H}_m$ is finite dimensional, all norms are equivalent. So, we can find $\delta_1 > 0$ such that

$$u \in \overline{H}_m, \ \|u\|_{1,2} \leqslant \delta_1 \ \Rightarrow \ |u(z)| \leqslant \delta \text{ for a.a. } z \in \Omega. \tag{52}$$

Then for $u \in \overline{H}_m$ with $\|u\|_{1,2} \leqslant \delta_1$, we have

$$\begin{aligned}
\hat{\xi}(u) &\leqslant \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0) u^2 dz + \frac{\varepsilon}{2}\|u\|_2^2 \text{ (see (51), (52))}\\
&\leqslant \left(-c_9 + \frac{\varepsilon}{\widetilde{\lambda}_1(2)}\right)\|Du\|_2^2 \text{ for some } c_9 > 0\\
&\qquad \text{(see D'Agui, Marano \& Papageorgiou [31, Lemma 2.2]).}
\end{aligned}$$

Choosing $\varepsilon \in (0, \widetilde{\lambda}_1(2)c_9)$, we have that

$$\hat{\xi}(u) \leqslant 0 \text{ for all } u \in \overline{H}_m \text{ with } \|u\|_{1,2} \leqslant \delta_1. \tag{53}$$

On the other hand from (51) and hypothesis H_2(i), we have

$$F(z,x) \leqslant \frac{1}{2}\left(f_x'(z,0)+\varepsilon\right)x^2 + c_{10}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10}>0. \tag{54}$$

Then for $u \in \hat{H}_{m+1}$, we have

$$\begin{aligned}\hat{\xi}(u) &\geqslant \frac{1}{2}\left(\|Du\|_2^2 - \int_\Omega f_x'(z,0)u^2dz - \frac{\varepsilon}{\widetilde{\lambda}_1(2)}\|Du\|_2^2\right) - c_{11}\|Du\|_2^r \\ &\quad \text{for some } c_{11}>0 \text{ (see (54))} \\ &\geqslant \frac{1}{2}\left(c_{12} - \frac{\varepsilon}{\widetilde{\lambda}_1(2)}\right)\|Du\|_2^2 - c_{11}\|Du\|_2^r \text{ for some } c_{12}>0 \\ &\quad \text{(see D'Agui, Marano \& Papageorgiou [31]).}\end{aligned}$$

Choosing $\varepsilon \in (0, \widetilde{\lambda}_1(2)c_{12})$, we obtain

$$\hat{\xi}(u) \geqslant c_{13}\|Du\|_2^2 - c_{11}\|Du\|_2^r \text{ for some } c_{13}>0, \text{ all } u \in \hat{H}_{m+1}.$$

Since $2<p<r$, we see that we can find $\delta_2>0$ such that

$$\hat{\xi}(u) > 0 \text{ for all } u \in \hat{H}_{m+1} \text{ with } 0<\|u\|_{1,2} \leqslant \delta_2. \tag{55}$$

From (53) and (55), we infer that $\hat{\xi}(\cdot)$ has local linking at $u=0$ with respect to the decomposition $(\overline{H}_m, \hat{H}_{m+1})$.

We show that $0 \in K_{\hat{\xi}}$ is isolated. Arguing by contradiction suppose we could find $\{u_n\}_{n\in\mathbb{N}_0} \subseteq H_0^1(\Omega)$ such that

$$u_n \to 0 \text{ and } \hat{\xi}'(u_n)=0 \text{ for all } n \in \mathbb{N}_0.$$

We have

$$-\Delta u_n(z) = f(z,u_n(z)) \text{ in } \Omega,\ u_n|_{\partial\Omega}=0,\ n \in \mathbb{N}_0. \tag{56}$$

Standard semilinear regularity theory (see, for example, Gilbarg & Trudinger [32, p. 241]), implies that there exist $\alpha \in (0,1)$ and $c_{14}>0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}),\ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{14} \text{ for all } n \in \mathbb{N}_0.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, we have

$$u_n \to 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{57}$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,2}}$, $n \in \mathbb{N}_0$. Then $\|y_n\|_{1,2}=1$ for all $n \in \mathbb{N}_0$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{58}$$

From (56) we have

$$\begin{aligned}& -\Delta y_n = \frac{f(z,u_n)}{\|u_n\|_{1,2}} \text{ in } \Omega,\ y_n|_{\partial\Omega}=0,\ n \in \mathbb{N}_0, \qquad (59)\\ &\Rightarrow \int_\Omega (Dy_n, D(y_n-y))_{\mathbb{R}^N}dz = \int_\Omega \frac{f(z,u_n)}{\|u_n\|_{1,2}}(y_n-y)dz.\end{aligned}$$

Hypotheses H_2(i), (iv) imply that $\left\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|_{1,2}}\right\}_{n\in\mathbb{N}_0} \subseteq L^2(\Omega)$ is bounded. Hence we have $\int_\Omega \frac{f(z,u_n)}{\|u_n\|_{1,2}}(y_n - y)dz \to 0$. Then

$$\lim_{n\to\infty}\int_\Omega (Dy_n, D(y_n-y))_{\mathbb{R}^N}dz = 0,$$
$$\Rightarrow \|Dy_n\|_2 \to \|Dy\|_2. \tag{60}$$

From (58), (60) and the Kadec–Klee property of Hilbert spaces we have that

$$y_n \to y \text{ in } H_0^1(\Omega), \text{ hence } \|y\|_{1,2} = 1. \tag{61}$$

Returning to (59), passing to the limit as $n\to\infty$ and using (61), (57) and hypothesis H_2(iv), we obtain

$$-\Delta y = f'_x(z,0)y^2 \text{ in } \Omega,$$
$$\Rightarrow y = 0,$$

which contradicts (61).

Therefore $0 \in K_{\hat{\xi}}$ is isolated and so we use Proposition 6.6.19 of Papageorgiou, Rădulescu & Repovš [25, p. 539], to conclude that

$$C_k(\hat{\xi},0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k\in\mathbb{N}_0.$$

This proof is now complete. □

Recall that

$$W_0^{1,\theta}(\Omega) \hookrightarrow H_0^1(\Omega) \text{ densely}.$$

Let $\xi = \hat{\xi}|_{W_0^{1,\theta}(\Omega)}$. Then Theorem 6.6.26 of Papageorgiou, Rădulescu & Repovš [25, p. 545], gives us the following result.

Proposition 9. *If hypotheses* H_2(i), (iv) *hold, then* $C_k(\xi,0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k\in\mathbb{N}_0$.

Note that

$$\psi(u) = \frac{1}{p}\rho_\varphi(Du) + \xi(u) \text{ for all } u\in W_0^{1,\theta}(\Omega).$$

Then Proposition 1, the continuous embedding of $W_0^{1,\theta}(\Omega)$ into $W_0^{1,\varphi}(\Omega)$ and the C^1-continuity property of critical groups (see Papageorgiou, Rădulescu & Repovš [25, Theorem 6.3.4, p. 503]), imply that

Proposition 10. *If hypotheses* H_0, H_1, H_2 *hold, then* $C_k(\psi,0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k\in\mathbb{N}_0$.

Now we can produce the second nontrivial solution of problem (1).

Proposition 11. *If hypotheses* H_0, H_1, H_2 *hold, then problem* (1) *admits a second nontrivial solution*

$$\hat{u}\in W_0^{1,\theta}(\Omega)\cap L^\infty(\Omega).$$

Proof. From Proposition 7 we already have a nontrivial solution $u_0 \in W_0^{1,\theta}(\Omega)\cap L^\infty(\Omega)$ which is a global minimizer of the energy functional $\psi(\cdot)$. Hence

$$C_k(\psi,u_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k\in\mathbb{N}_0. \tag{62}$$

From Proposition 10, we know that

$$C_k(\psi,0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k\in\mathbb{N}_0. \tag{63}$$

We know that $\psi(\cdot)$ is coercive (see Proposition 6). Hence $\psi(\cdot)$ is bounded below and satisfies the C-condition (see [25, Proposition 5.1.15, p. 369]). Invoking Proposition 6.2.24 of Papageorgiou, Rădulescu & Repovš [25, p. 491], we have

$$C_k(\psi, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{64}$$

Suppose that $K_\psi = \{0, u_0\}$. Then from (62), (63), (64) and the Morse relation with $t = -1$ (see Section 2), we have

$$\begin{aligned} &(-1)^{d_m} + (-1)^0 = (-1)^0, \\ \Rightarrow\ &(-1)^{d_m} = 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\hat{u} \in K_\psi$, $\hat{u} \notin \{0, u_0\}$. This is the second nontrivial solution of problem (1) and as before $\hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.

This proof is now complete. $\square$

We can state the following multiplicity theorem for problem (1).

Theorem 12. *If hypotheses* H_0, H_1, H_2 *hold, then problem* (1) *admits two distinct nontrivial solutions*

$$u_0,\ \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

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