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DOUBLE-PHASE PROBLEMS AND A DISCONTINUITY PROPERTY OF THE SPECTRUM

NIKOLAOS S. PAPAGEORGIOU, VICENŢIU D. RĂDULESCU, AND DUŠAN D. REPOVŠ

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ABSTRACT. We consider a nonlinear eigenvalue problem driven by the sum of p and q-Laplacians. We show that the problem has a continuous spectrum. Our result reveals a discontinuity property for the spectrum of a parametric (p, q)-differential operator as the parameter β goes to 1^-.

1. Introduction

This paper was motivated by several recent contributions to the qualitative analysis of nonlinear problems with unbalanced growth. First, we refer to the pioneering contributions of Marcellini [17, 18], who studied lower semicontinuity and regularity properties of minimizers of certain quasiconvex integrals. Problems of this type arise in nonlinear elasticity and they are connected with deformations of an elastic body; cf. Ball [1].

In order to recall the roots of the double-phase problems, let us assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary. If $u : \Omega \to \mathbb{R}^N$ is the displacement and Du is the $N \times N$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$I(u) = \int_\Omega F(z, Du(z))dz, \tag{1}$$

where the energy function $F = F(z, \xi) : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}$ is quasiconvex with respect to ξ. One of the simplest examples considered by Ball is given by functions F of the type

$$F(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions which satisfy the growth conditions

$$g(\xi) \geqslant c_1 |\xi|^p; \quad \lim_{t\to+\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < N$. The condition $p \leqslant N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the

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integral (1) that are discontinuous at one point where a cavity appears; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus u is a continuous function if $p > N$.

In accordance with these problems arising in nonlinear elasticity, Marcellini [17, 18] considered continuous functions $F = F(z, u)$ with *unbalanced growth* that satisfy

$$c_1 |u|^p \leqslant |F(z,u)| \leqslant c_2 (1 + |u|^q) \quad \text{for all } (z,u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 < p < q$. Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions were studied in [18].

The study of nonautonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point was continued in a series of remarkable papers by Mingione et al. [2,3,6,8]. These contributions are related to the work of Zhikov [28], and they describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenization. We also point out that Zhikov's functionals turned out to be important in the study of duality theory and in the context of the Lavrentiev phenomenon [29]. One of the problems considered by Zhikov was the *double-phase* functional

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|Du|^p + a(z)|Du|^q)dz, \quad 0 \leqslant a(z) \leqslant L, \ 1 < p < q,$$

where the modulating coefficient $a(z) \geqslant 0$ dictates the geometry of the composite made by two different materials. More precisely, considering two different materials with power hardening exponents p and q, respectively, the variable coefficient $a(z)$ dictates the geometry of a composite of the materials. In the region where a is positive, the q-material is present; otherwise the p-material is the only one making the composite.

Another significant model example of functional with unbalanced growth studied by Mingione et al. [2,3,6] is given by

$$\mathcal{E}(u) := \int_\Omega |Du|^p \log(1 + |Du|)dz, \quad p \geqslant 1.$$

General models with (p,q)-growth in the context of geometrically constrained problems have been recently studied by De Filippis [7]. A key role is played by the method developed by Esposito, Leonetti, and Mingione [8] in order to prove the equivalence between the absence of Lavrentiev phenomenon and the extra-regularity of the minimizers for unconstrained, nonautonomous variational problems.

Motivated by these results, we study in this paper a problem with the (p,q)-growth. More precisely, we consider the following nonlinear, nonhomogeneous parametric Dirichlet problem:

$$(P_\lambda) \qquad \left\{ \begin{array}{l} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda |u(z)|^{q-2} u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \alpha > 0, \ \beta > 0, \ \lambda > 0, \ 1 < p, q < \infty, \ p \neq q \end{array} \right\},$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$.

For every $r \in (1, +\infty)$ we denote by Δ_r the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Equations driven by the sum of a p-Laplacian and a q-Laplacian, known as (p,q)-equations, arise in many problems of mathematical physics such as particle physics; see e.g. Benci, D'Avenia, Fortunato and Pisani [4], and for nonlinear elasticity, see

Zhikov [28]. For instance, Zhikov [28] introduced models for strongly anisotropic materials in the context of homogenization.

Such problems with unbalanced growth were studied by Chorfi and Rădulescu [5], Gasinski and Papageorgiou [10, 12], Marano, Mosconi, and Papageorgiou [15, 16], Papageorgiou and Rădulescu [19, 20], Papageorgiou, Rădulescu, and Repovš [21–24], Rădulescu [25], Rădulescu and Repovš [26], and Yin and Yang [27], under different conditions on the data of the problem.

In the present paper, we show that problem (P_λ) has a continuous spectrum: it is the half-line $(\beta\hat{\lambda}_1(q), +\infty)$, with $\hat{\lambda}_1(q) > 0$ being the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. So, for every $\lambda \in (\beta\hat{\lambda}_1(q), +\infty)$, problem (P_λ) admits a nontrivial solution. Our result reveals an interesting fact, which is better illustrated in the special case corresponding to

$$1 < p < \infty,\ q = 2,\ p \neq q,\ \alpha = 1 - \beta,\ \beta \in (0, 1).$$

Let $L_\beta = -(1-\beta)\Delta_p u - \beta\Delta u$ and let $\hat{\sigma}(\beta)$ denote the spectrum of L_β. We have that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty) \quad \text{for } \beta \in (0, 1).$$

The set function $\beta \mapsto \hat{\sigma}(\beta)$ is h-continuous (Hausdorff continuous) on $(0, 1]$, whereas it exhibits a discontinuity at $\beta = 0$, since $L_0 = \Delta$ which has a discrete spectrum.

Our approach is based on the use of the Nehari manifold. So, we perform a minimization under constraint.

2. Mathematical background

Let $r \in (1, +\infty)$. We recall some basic facts about the spectrum of $(-\Delta_r, W_0^{1,r}(\Omega))$. So, we consider the following nonlinear eigenvalue problem:

$$-\Delta_r u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2}$$

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ if problem (2) admits a nontrivial solution $\hat{u} \in W_0^{1,r}(\Omega)$, known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. From the nonlinear regularity theory (see, for example, Gasinski and Papageorgiou [9, pp. 737-738]), we know that $\hat{u} \in C_0^1(\overline{\Omega}) = \big\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\big\}$. There is a smallest eigenvalue $\hat{\lambda}_1(r)$ which has the following properties:

- $\hat{\lambda}_1(r)$ is isolated (that is, there exists $\epsilon > 0$ such that the interval $(\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \epsilon)$ contains no eigenvalues of $(-\Delta_r, W_0^{1,r}(\Omega))$);
- $\hat{\lambda}_1(r)$ is simple (that is, if $\hat{u}, \hat{v}$ are eigenfunctions corresponding to $\hat{\lambda}_1(r)$, then $\hat{u} = \mu\hat{v}$ with $\mu \in \mathbb{R}\backslash\{0\}$);
- $\hat{\lambda}_1(r) > 0$ and admits the following variational characterization:

$$\hat{\lambda}_1(r) = \inf\left\{\frac{||Du||_r^r}{||u||_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\}. \tag{3}$$

The infimum in (3) is realized on the corresponding one-dimensional eigenspace. The above properties imply that the elements of this eigenspace lie in $C_0^1(\overline{\Omega})$ and do not change sign. By $\hat{u}_1(r)$ we denote the positive, L^r-normalized (that is, $||\hat{u}_1(r)||_r = 1$) eigenfunction corresponding to $\hat{\lambda}_1(r) > 0$. We have

$$\hat{u}_1(r) \in C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

In fact, the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [12, p. 738]) implies that

$$\hat{u}_1(r) \in \operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0 \right\},$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ being the outward normal derivative of u. Note that if $\hat{u}$ is an eigenfunction corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(r)$, then $\hat{u}$ is nodal (that is, sign-changing). The Ljusternik-Schnirelmann minimax scheme gives, in addition, to $\hat{\lambda}_1(r)$ a whole strictly increasing sequence $\{\hat{\lambda}_k(r)\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\hat{\lambda}_k(r) \to +\infty$. These eigenvalues are called "variational eigenvalues", and we do not know if they exhaust the entire spectrum of $(-\Delta_r, W_0^{1,r}(\Omega))$. However, if $r = 2$ (linear eigenvalue problem), then the spectrum is the sequence $\{\hat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ of variational eigenvalues.

Let $r = \max\{p, q\}$ and let $\lambda > 0$. The energy (Euler) functional for problem (P_λ) is defined by

$$\varphi_\lambda(u) = \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{\lambda}{q}||u||_q^q \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(W_0^{1,r}(\Omega), \mathbb{R})$.

The Nehari manifold for the functional φ_λ is the set

$$N_\lambda = \{u \in W_0^{1,r}(\Omega) : \langle \varphi_\lambda'(u), u\rangle = 0,\ u \neq 0\}.$$

In what follows, we denote by $\hat{\sigma}(\alpha, \beta)$ the spectrum of the differential operator

$$u \mapsto -\alpha\Delta_p u - \beta\Delta_q u \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

So, $\lambda \in \hat{\sigma}(\alpha, \beta)$ if and only if problem (P_λ) admits a nontrivial solution $\hat{u} \in C_0^1(\overline{\Omega})$. This solution is an eigenvector for the eigenvalue λ.

In what follows, for every $\tau \in (1, +\infty)$ we denote by $||\cdot||_{1,\tau}$ the norm of $W_0^{1,\tau}(\Omega)$. On account of the Poincaré inequality, we have

$$||u||_{1,\tau} = ||Du||_\tau \quad \text{for all } u \in W_0^{1,\tau}(\Omega).$$

Also, we denote by $A_\tau : W_0^{1,\tau}(\Omega) \to W_0^{-1,\tau'}(\Omega) = W_0^{1,\tau}(\Omega)^* \left(\frac{1}{\tau} + \frac{1}{\tau'} = 1\right)$ the nonlinear operator defined by

$$\langle A_\tau(u), h\rangle = \int_\Omega |Du|^{\tau-2}(Du, Dh)_{\mathbb{R}^N}\,dt \quad \text{for all } u, h \in W_0^{1,\tau}(\Omega).$$

This operator is bounded (that is, it maps bounded sets to bounded sets) and continuous, strictly monotone (hence maximal monotone, too).

3. The spectrum of (P_λ)

We first deal with the easier case when $1 < q < p$. As we will see in what follows, $\varphi_\lambda(\cdot)$ is coercive in this case and so we can use the direct method of the calculus of variations.

Proposition 1. *If $1 < q < p$, then $\hat{\sigma}(\alpha, \beta) = (\beta\hat{\lambda}_1(q), +\infty)$ and the eigenvectors belong to $C_0^1(\overline{\Omega})$.*

Proof. We have $r = \max\{p, q\} = p$. Evidently, if $\lambda \leqslant \beta\hat{\lambda}_1(q)$, then $\lambda \notin \hat{\sigma}(\alpha, \beta)$ or otherwise we violate (3).

Let $\lambda > \beta\hat{\lambda}_1(q)$ and $u \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
\varphi_\lambda(u) &\geqslant \frac{\alpha}{p}||Du||_p^p - \frac{\lambda}{\hat{\lambda}_1(q)q}||Du||_q^q \quad \text{(see (3))}\\
&\geqslant \frac{\alpha}{p}||Du||_p^p - c_1||Du||_q^q \quad \text{for some } c_1 > 0 \text{ (since } q < p),\\
\Rightarrow \quad \varphi_\lambda(u) &\geqslant c_2||u||^p - c_3||u||^q \quad \text{for some } c_2, c_3 > 0,\\
\Rightarrow \quad \varphi_\lambda(\cdot) &\quad \text{is coercive (since } q < p).
\end{aligned}$$

Also, by the Sobolev embedding theorem, $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(\hat{u}_\lambda) = \inf\{\hat{\varphi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{4}$$

For $t > 0$ we have

$$\begin{aligned}
\varphi_\lambda(t\hat{u}_1(q)) &= \frac{t^p\alpha}{p}||D\hat{u}_1(q)||_p^p + \frac{t^q}{q}\left[\beta\hat{\lambda}_1(q) - \lambda\right] \quad \text{(recall that } ||\hat{u}_1(q)||_q = 1)\\
&= c_4t^p - c_5t^q \quad \text{for some } c_4, c_5 > 0 \quad \text{(recall that } \lambda > \beta\hat{\lambda}_1(q)).
\end{aligned}$$

Since $q < p$, choosing small $t \in (0, 1)$ we have

$$\begin{aligned}
&\varphi_\lambda(t\hat{u}_1(q)) < 0,\\
\Rightarrow \quad &\varphi_\lambda(\hat{u}_\lambda) < 0 = \varphi_\lambda(0) \quad \text{(see (4))},\\
\Rightarrow \quad &\hat{u}_\lambda \neq 0.
\end{aligned}$$

From (4) we have

$$\begin{aligned}
&\varphi'_\lambda(\hat{u}_\lambda) = 0,\\
\Rightarrow \quad &\langle \alpha A_p(\hat{u}_\lambda), h\rangle + \langle \beta A_q(\hat{u}_\lambda), h\rangle = \lambda\int_\Omega |\hat{u}_\lambda|^{q-2}\hat{u}_\lambda h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),\\
\Rightarrow \quad &-\alpha\Delta_p\hat{u}_\lambda(z) - \beta\Delta_q\hat{u}_\lambda(z) = \lambda|\hat{u}_\lambda(z)|^{q-2}\hat{u}_\lambda(z) \quad \text{for almost all } z \in \Omega,\ \hat{u}_\lambda|_{\partial\Omega} = 0,\\
\Rightarrow \quad &\hat{u}_\lambda \in C_0^1(\overline{\Omega}) \quad \text{(by the nonlinear regularity theory; see Lieberman [14]).}
\end{aligned}$$

The proof is now complete. □

When $1 < p < q$, the energy functional is no longer coercive. So, the direct method of the calculus of the variations fails, and we have to use a different approach. Instead, we will minimize φ_λ on the Nehari manifold N_λ.

First, we show that $N_\lambda \neq \emptyset$.

Proposition 2. *$\lambda > \beta\hat{\lambda}_1(q)$ if and only if $N_\lambda \neq \emptyset$.*

Proof. As before (see the proof of Proposition 1), using (3) we see that

$$N_\lambda \neq \emptyset \Rightarrow \lambda > \beta\hat{\lambda}_1(q).$$

Now, suppose that $\lambda > \beta\hat{\lambda}_1(q)$. Then on account of (3), we can find $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ such that

$$||Du||_q^q < \frac{\lambda}{\beta}||u||_q^q. \tag{5}$$

Consider the function $\xi_\lambda : (0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned} \xi_\lambda(t) &= \langle \varphi'_\lambda(tu), tu \rangle \\ &= \alpha t^p ||Du||_p^p + \beta t^q ||Du||_q^q - \lambda t^q ||u||_q^q \\ &= t^p \alpha ||Du||_p^p + t^q (\beta ||Du||_q^q - \alpha ||u||_q^q) \\ &= c_6 t^p - c_7 t^q \quad \text{for some } c_6, c_7 > 0 \text{ (see (5))}. \end{aligned} \tag{6}$$

Since $q > p$, we see from (6) that

$$\xi_\lambda(t) \to -\infty \quad \text{as } t \to +\infty.$$

On the other hand, for small $t \in (0, 1)$ we have

$$\xi_\lambda(t) > 0 \quad \text{(see (6))}.$$

Therefore, by Bolzano's theorem, we can find $t_0 > 0$ such that

$$\begin{aligned} & \xi_\lambda(t_0) = 0, \\ \Rightarrow\ & \langle \varphi'_\lambda(t_0 u), t_0 u \rangle = 0 \quad \text{with } t_0 u \neq 0, \\ \Rightarrow\ & t_0 u \in N_\lambda \quad \text{and so } N_\lambda \neq \emptyset. \end{aligned}$$

This completes the proof. ☐

We define

$$m_\lambda = \inf\{\varphi_\lambda(u) : u \in N_\lambda\}. \tag{7}$$

For $u \in N_\lambda$, we have

$$\alpha ||Du||_p^p + \beta ||Du||_q^q = \lambda ||u||_q^q. \tag{8}$$

Therefore

$$\begin{aligned} \varphi_\lambda(u) &= \frac{\alpha}{p} ||Du||_p^p + \frac{\beta}{q} ||Du||_q^q - \frac{1}{q} \left(\alpha ||Du||_p^p + \beta ||Du||_q^q \right) \quad \text{(see (8))} \\ &= \alpha \left(\frac{1}{p} - \frac{1}{q} \right) ||Du||_p^p, \end{aligned} \tag{9}$$

$$\Rightarrow \quad m_\lambda \geqslant 0 \quad \text{(see (7))}.$$

From (9) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

Proposition 3. *If $\lambda > \beta \hat{\lambda}_1(q)$, then every minimizing sequence of (7) is bounded in $W_0^{1,q}(\Omega)$.*

Proof. We argue by contradiction. So, suppose that $\{u_n\}_{n \geqslant 1} \subseteq W_0^{1,q}(\Omega)$ is a minimizing sequence of (7) such that

$$||u_n||_{1,q} \to +\infty.$$

We have

$$\alpha ||Du_n||_p^p + \beta ||Du_n||_q^q = \lambda ||u_n||_q^q \quad \text{for all } n \in \mathbb{N}, \tag{10}$$

$$\Rightarrow \quad \beta ||Du_n||_q^q = \beta ||u_n||_{1,q}^q \leqslant \lambda ||u_n||_q^q \quad \text{for all } n \in \mathbb{N}, \tag{11}$$

$$\Rightarrow \quad ||u_n||_q \to +\infty \quad \text{as } n \to \infty.$$

We set $y_n = \frac{u_n}{||u_n||_q}$ for all $n \in \mathbb{N}$; hence $||y_n||_q = 1$. Also, from (11) we have

$$||Dy_n||_q^q \leqslant \frac{\lambda}{\beta}||y_n||_q^q = \frac{\lambda}{\beta} \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \quad \{y_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,q}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^q(\Omega). \tag{12}$$

We multiply (10) with $\frac{1}{||u_n||_q^q}$. We obtain

$$\alpha||Dy_n||_p^p = \frac{\lambda||u_n||_q^q - \beta||Du_n||_q^q}{||u_n||_q^p}, \; n \in \mathbb{N}. \tag{13}$$

Recall that $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ is a minimizing sequence for (7). So, we have

$$\left(\frac{1}{p} - \frac{1}{q}\right)\left(\lambda||u_n||_q^q - \beta||Du_n||_q^q\right) \to m_\lambda \quad \text{as } n \to \infty \text{ (see (8), (9)).} \tag{14}$$

Using (14) in (13), we can infer that

$$y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad y_n \to 0 \quad \text{in } L^q(\Omega) \text{ (see (12))},$$

a contradiction, since $||y_n||_q = 1$ for all $n \in \mathbb{N}$.

Therefore we can conclude that every minimizing sequence of (7) is bounded in $W_0^{1,q}(\Omega)$. □

We have already seen that $m_\lambda \geqslant 0$. We can now say more.

Proposition 4. *If $\lambda > \beta\hat{\lambda}_1(q)$, then $m_\lambda > 0$.*

Proof. Arguing by contradiction, suppose that $m_\lambda = 0$. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ such that $\varphi_{\lambda_-}(u_n) \to 0^+$. From (9) we have

$$\alpha\left(\frac{1}{p} - \frac{1}{q}\right)||Du_n||_p^p \to 0,$$
$$\Rightarrow \quad u_n \to 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{15}$$

Then by (15) and Proposition 3, we infer that

$$u_n \xrightarrow{w} 0 \quad \text{in } W_0^{1,q}(\Omega) \quad \text{and} \quad u_n \to 0 \quad \text{in } L^q(\Omega). \tag{16}$$

It follows from (15), (16), and (8) that

$$||Du_n||_q \to 0,$$
$$\Rightarrow \quad u_n \to 0 \quad \text{in } W_0^{1,q}(\Omega). \tag{17}$$

Let $v_n = \frac{u_n}{||u_n||_q}$ for all $n \in \mathbb{N}$; hence $||v_n||_q = 1$. We have

$$\lambda||v_n||_q^q - \beta||Dv||_q^q = \frac{\alpha}{||u_n||_q^{q-p}}||Dv_n||_p^p > 0 \quad \text{for all } n \in \mathbb{N}, \tag{18}$$
$$\Rightarrow \quad ||Dv_n||_q^q \leqslant \frac{\lambda}{\beta} \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \quad \{v_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega) \quad \text{is bounded.} \tag{19}$$

Then it follows from (18) and (19) that

$$\frac{\alpha}{||u_n||_q^{q-p}}||Dv_n||_p^p \leqslant c_8 \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad ||Dv_n||_p \to 0 \quad (\text{see (17) and recall that } p < q),$$
$$\Rightarrow \quad v_n \to 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{20}$$

From (19) and (20), we can infer that

$$v_n \to 0 \quad \text{in } L^q(\Omega),$$

a contradiction, since $||v_n||_q = 1$ for all $n \in \mathbb{N}$. From this we can conclude that $m_\lambda > 0$. ☐

Proposition 5. *If $\lambda > \beta\hat{\lambda}_1(q)$, then there exists $\hat{u}_\lambda \in N_\lambda$ such that $m_\lambda = \varphi_\lambda(\hat{u}_\lambda)$.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ be such that $\varphi_\lambda(u_n) \to m_\lambda$. According to Proposition 3, $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u}_\lambda \quad \text{in } W_0^{1,q}(\Omega) \quad \text{and} \quad u_n \to \hat{u}_\lambda \quad \text{in } L^q(\Omega). \tag{21}$$

Since $u_n \in N_\lambda$ for all $n \in \mathbb{N}$, we have

$$\alpha||Du_n||_p^p + \beta||Du_n||_q^q = \lambda||u_n||_q^q \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

Passing to the limit as $n \to \infty$ and using (21) and the weak lower semicontinuity of the norm functional in a Banach space, we obtain

$$\alpha||D\hat{u}_\lambda||_p^p \leqslant \lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q. \tag{23}$$

Note that $\lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q \neq 0$, or otherwise from (19), we have

$$||Du_n||_p \to 0,$$
$$\Rightarrow \quad u_n \to 0 \quad \text{in } W_0^{1,p}(\Omega).$$

Recall that

$$\varphi_\lambda(u_n) = \alpha\left(\frac{1}{p} - \frac{1}{q}\right)||Du_n||_p^p \quad \text{for all } n \in \mathbb{N} \text{ (see (9))}.$$

So, it follows that

$$\varphi_\lambda(u_n) \to 0 \quad \text{as } n \to \infty,$$
$$\Rightarrow \quad m_\lambda = 0,$$

which contradicts Proposition 4. Therefore

$$\lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q \neq 0,$$
$$\Rightarrow \quad \hat{u}_\lambda \neq 0.$$

Also, exploiting the sequential weak lower semicontinuity of $\varphi_\lambda(\cdot)$, we have

$$\varphi_\lambda(\hat{u}_\lambda) \leqslant \lim_{n\to\infty} \varphi_\lambda(u_n) = m_\lambda \quad (\text{see (21)}).$$

If we show that $\hat{u}_\lambda \in N_\lambda$, then $\varphi_\lambda(\hat{u}_\lambda) = m_\lambda$, and this will conclude the proof.

To this end, let

$$\hat{\xi}_\lambda(t) = \langle\varphi_\lambda'(t\hat{u}_\lambda, t\hat{u}_\lambda)\rangle \quad \text{for all } t \in [0, 1].$$

Evidently, $\hat{\xi}_\lambda(\cdot)$ is a continuous function. Arguing by contradiction, suppose that $\hat{u}_\lambda \notin N_\lambda$. Then since $u_n \in N_\lambda$ for all $n \in \mathbb{N}$, we can infer from (21) that

$$\hat{\xi}_\lambda(1) < 0. \tag{24}$$

On the other hand, note that since $\lambda > \beta\hat{\lambda}_1(q)$, we have

$$\begin{aligned} & \hat{\xi}_\lambda(t) \geqslant c_9 t^p - c_{10} t^q \quad \text{for some } c_9, c_{10} > 0, \\ (25) \quad \Rightarrow \quad & \hat{\xi}_\lambda(t) > 0 \quad \text{for all } t \in (0,\epsilon) \text{ with small } \epsilon \in (0,1) \text{ (recall that } p < q). \end{aligned}$$

By (24), (25), and Bolzano's theorem, we see that there exists $t^* \in (0,1)$ such that

$$\begin{aligned} & \hat{\xi}_\lambda(t^*\hat{u}_\lambda) = 0, \\ \Rightarrow \quad & t^*\hat{u}_\lambda \in N_\lambda. \end{aligned}$$

Then using (9), we have

$$\begin{aligned} m_\lambda \leqslant \varphi_\lambda(t^*\hat{u}_\lambda) &= \alpha\left(\frac{1}{p}-\frac{1}{q}\right)(t^*)^p||D\hat{u}_\lambda||_p^p \\ &< \alpha\left(\frac{1}{p}-\frac{1}{q}\right)||D\hat{u}_\lambda||_p^p \quad (\text{since } t^* \in (0,1)) \\ &\leqslant \alpha\left(\frac{1}{p}-\frac{1}{q}\right)\liminf_{n\to\infty}||Du_n||_p^p \quad (\text{see } (21)) \\ &= m_\lambda, \end{aligned}$$

a contradiction. Therefore $\hat{u}_\lambda \in N_\lambda$, and this finishes the proof. $\square$

So, we can state the following theorem concerning problem (P_λ). This property establishes the existence of a continuous spectrum that concentrates at infinity.

Theorem 6. *If $\lambda > \beta\hat{\lambda}_1(q)$, then λ is an eigenvalue of problem (P_λ) with eigenfunction $\hat{u}_\lambda \in C_0^1(\overline{\Omega})$.*

Proof. For $1 < q < p$, this follows from Proposition 1.

For $1 < q < p$, let $h \in W_0^{1,q}(\Omega)$. Choose $\epsilon > 0$ such that $\hat{u}_\lambda + sh \not\equiv 0$ for $s \in (-\epsilon,\epsilon)$. We set

$$t(s) = \left(\frac{\lambda||\hat{u}_\lambda + sh||_q^q - \beta||D(\hat{u}_\lambda+sh)||_q^q}{\alpha||D(\hat{u}_\lambda+sh)||_p^p}\right)^{\frac{1}{p-q}}, \; s \in (-\epsilon,\epsilon).$$

Then $s \mapsto t(s)$ is a curve in N_λ and it is differentiable. Let $\hat{\xi}_\lambda : (-\epsilon,\epsilon) \to \mathbb{R}$ be defined by

$$\hat{\xi}_\lambda(s) = \varphi_\lambda(t(s)(\hat{u}_\lambda + sh)), \; s \in (-\epsilon,\epsilon).$$

Evidently, $s = 0$ is a minimizer of $\hat{\xi}_\lambda(\cdot)$ and so

$$\begin{aligned} 0 &= \hat{\xi}_\lambda(0) \\ &= \langle \varphi'_\lambda(\hat{u}_\lambda), t'(0)\hat{u}_\lambda + h\rangle \quad (\text{by the chain rule}) \\ &= t'(0)\,\langle \varphi'_\lambda(\hat{u}_\lambda), \hat{u}_\lambda\rangle + \langle \varphi'_\lambda(\hat{u}_\lambda), h\rangle \\ &= \langle \varphi'_\lambda(\hat{u}_\lambda), h\rangle \quad (\text{since } \hat{u}_\lambda \in N_\lambda), \\ \Rightarrow \quad & \alpha\,\langle A_p(\hat{u}_\lambda), h\rangle + \beta\,\langle A_q(\hat{u}_\lambda), h\rangle = \lambda\int_\Omega |\hat{u}_\lambda|^{q-2}\hat{u}_\lambda hzd, \\ \Rightarrow \quad & -\alpha\Delta_p\hat{u}_\lambda(z) - \beta\Delta_q\hat{u}_\lambda(z) = \lambda|\hat{u}_\lambda(z)|^{q-2}\hat{u}_\lambda(z) \text{ for almost all } z \in \Omega, \; \hat{u}_\lambda|_{\partial\Omega} = 0, \\ \Rightarrow \quad & \hat{u}_\lambda \neq 0 \quad \text{is an eigenfunction with eigenvalue } \lambda > \beta\hat{\lambda}_1(q). \end{aligned}$$

Now, the nonlinear regularity theory of Lieberman [14, p. 320] implies that $\hat{u}_\lambda \in C_0^1(\overline{\Omega})$. $\square$

Remark 1. In the terminology of the critical point theory, the above proof shows that the Nehari manifold is a natural constraint for the functional φ_λ (see Gasinski and Papageorgiou [9, p. 812]).

Now suppose that $\alpha = (1-\beta)$, $\beta \in (0,1)$. Let $L_\beta = -(1-\beta)\Delta_p - \beta\Delta_q$ and let $\hat{\sigma}(\beta)$ be the spectrum of L_β. From Theorem 6, we know that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty).$$

Evidently, $\hat{\sigma}(\cdot)$ is Hausdorff and Vietoris continuous on $(0,1)$ (see Hu and Papageorgiou [11]), but it exhibits a discontinuity at $\beta = 1$, since

$$\hat{\sigma}(1) = \text{the spectrum of } (-\Delta_q, W_0^{1,q}(\Omega)),$$

and we know from Section 2 that $\hat{\lambda}_1(q) > 0$ is isolated and so $\hat{\sigma}(1) \neq (\hat{\lambda}_1(q), +\infty)$. This is more emphatically illustrated when $q = 2$. Then

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(2), +\infty) \quad \text{for all } \beta \in (0,1),$$

but at $\beta = 1$ we have

$$\hat{\sigma}(1) = \{\hat{\lambda}_k(2)\}_{k\geqslant 1} \quad \text{(the discrete spectrum)}.$$

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Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece; Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

Email address: npapg@math.ntua.gr

Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland; Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia; Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Email address: vicentiu.radulescu@imar.ro

Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana; Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

Email address: dusan.repovs@guest.arnes.si