Research Article

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Semilinear Robin problems resonant at both zero and infinity

DOI: 10.1515/forum-2016-0264
Received December 27, 2016; revised May 2, 2017

Abstract: We consider a semilinear elliptic problem, driven by the Laplacian with Robin boundary condition. We consider a reaction term which is resonant at $\pm\infty$ and at 0. Using variational methods and critical groups, we show that under resonance conditions at $\pm\infty$ and at zero the problem has at least two nontrivial smooth solutions.

Keywords: Resonance, critical groups, Morse index, nullity, unique continuation property

MSC 2010: 35J20, 35J60, 58E05

Communicated by: Christopher D. Sogge

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary. In this paper, we study the following semilinear Robin problem:

$$-\Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

In this problem, the reaction term $f(z, x)$ is a measurable function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$, which is continuously differentiable in the x-variable. The boundary coefficient $\beta(\cdot)$ belongs to $W^{1,\infty}(\partial\Omega)$ and satisfies $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we recover the Neumann problem. It is well known that the existence and multiplicity of nontrivial solutions depends on the interaction of the limits

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{x} \quad \text{and} \quad \lim_{x\to 0} \frac{f(z,x)}{x}$$

with the spectrum of the Robin Laplacian.

The most difficult and therefore interesting case is when the above limits hit the spectrum. Such problems are called “resonant”.

Resonant Dirichlet problems were studied by Landesman, Robinson and Rumbos [10], Liang and Su [14], Li and Willem [13], de Paiva [6], and Su [21]. Neumann problems were investigated by Gasinski and Papageorgiou [9], Li [11], Li and Li [12], Papageorgiou and Rădulescu [18], and Qian [20]. We mention also the work of Barletta and Papageorgiou [2] on periodic ordinary differential equations.

In this paper, combining variational methods with Morse theory (critical groups), we show that under conditions of resonance at both zero and infinity (double resonance), problem (1.1) can have at least two nontrivial solutions.

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In the following section, for easy reference, we recall the main mathematical tools which we will use in the sequel.

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the Cerami condition (the C-condition for short), if the following property holds:

- Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

  $$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty,$$

  admits a strongly convergent subsequence.

This compactness-type condition is central in the minimax theory of the critical values of φ.

On $\partial\Omega$ we employ the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. From the general theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the trace map, such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

We know that γ_0 is compact from $H^1(\Omega)$ into $L^\tau(\partial\Omega)$ with $1 \leqslant \tau < \frac{2(N-1)}{N-2}$ if $N > 2$, and with $1 \leqslant \tau < \infty$ if $N = 1, 2$. We have

$$\operatorname{im} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker \gamma_0 = H_0^1(\Omega).$$

In the sequel, for notational simplicity, we drop the use of the map γ_0. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Throughout this work, the standing hypothesis on $\beta(\cdot)$ is the following:

(H(β)) $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

In what follows, $\vartheta : H^1(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\vartheta(u) = \|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2 \, d\sigma \quad \text{for all } u \in H^1(\Omega).$$

Also by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$, that is,

$$\|u\| = [\|u\|_2^2 + \|Du\|_2^2]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

Let $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant 0$ for almost all $z \in \Omega$, $\eta \neq 0$. We consider the following weighted linear eigenvalue problem:

$$-\Delta u(z) = \hat{\lambda}\eta(z)u(z) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue if problem (2.1) has a nontrivial solution $\hat{u} \in H^1(\Omega)$ which is known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. Using the spectral theorem for compact self-adjoint operators, we can show that problem (2.1) has a sequence $\{\hat{\lambda}_k(\eta)\}_{k\geqslant 1} \subseteq [0, +\infty)$ of eigenvalues such that

$$\hat{\lambda}_k(\eta) \to +\infty \quad \text{as } k \to +\infty.$$

If $\eta \equiv 1$, then we write $\hat{\lambda}_k(1) = \hat{\lambda}_k$ for all $k \in \mathbb{N}$ (see Papageorgiou and Rădulescu [17, 18]).

By $E(\hat{\lambda}_k(\eta))$ (for $k \in \mathbb{N}$) we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(\eta)$. The regularity theory implies that

$$E(\hat{\lambda}_k(\eta)) \subseteq C^1(\overline{\Omega}) \quad \text{for all } k \in \mathbb{N}.$$

Moreover, each eigenspace $E(\hat{\lambda}_k(\eta))$ has the so-called "unique continuation property" (UCP for short), which asserts that if $u \in E(\hat{\lambda}_k(\eta))$ vanishes on a set of positive measure, then $u \equiv 0$ (see Motreanu, Motreanu and Papageorgiou [15, p. 234] and Papageorgiou and Rădulescu [18]). We set

$$\bar{H}_k = \bigoplus_{i=1}^{k} E(\hat{\lambda}_i(\eta)) \quad \text{and} \quad \hat{H}_k = \bar{H}_k^{\perp} = \overline{\bigoplus_{i \geqslant k+1} E(\hat{\lambda}_i(\eta))}.$$

For every $k \in \mathbb{N}$, $\bar{H}_k$ is finite-dimensional and we have the orthogonal direct sum decomposition

$$H^1(\Omega) = \bar{H}_k \oplus \hat{H}_k \quad \text{for all } k \in \mathbb{N}.$$

We have the following variational characterizations of the eigenvalues:

$$\hat{\lambda}_1(\eta) = \inf\Big[\frac{\vartheta(u)}{\int_\Omega \eta u^2\,dz} : u \in H^1(\Omega), u \neq 0\Big], \tag{2.2}$$

and for $k \geqslant 2$ we have

$$\begin{aligned}\hat{\lambda}_k(\eta) &= \inf\Big[\frac{\vartheta(u)}{\int_\Omega \eta u^2\,dz} : u \in \hat{H}_{k-1}, u \neq 0\Big] \\ &= \sup\Big[\frac{\vartheta(u)}{\int_\Omega \eta u^2\,dz} : u \in \bar{H}_k, u \neq 0\Big].\end{aligned} \tag{2.3}$$

In both (2.2) and (2.3), the infimum and the supremum are realized on the corresponding eigenspace $E(\hat{\lambda}_k(\eta))$, $k \in \mathbb{N}$. We know that $\hat{\lambda}_1(\eta)$ is simple (that is, $\dim E(\hat{\lambda}_1(\eta)) = 1$) and from (2.2) it is clear that the elements of $E(\hat{\lambda}_1(\eta))$ do not change sign. If by $\hat{u}_1$ we denote the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(\eta) \geqslant 0$, then $\hat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$.

The aforementioned properties of the eigenvalues and eigenspaces lead to the following easy but useful results (see Gasinski and Papageorgiou [8, 9]).

Proposition 2.1. *If $\eta(z) \leqslant \eta'(z)$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure, then $\hat{\lambda}_k(\eta') < \hat{\lambda}_k(\eta)$ for all $k \in \mathbb{N}$.*

Proposition 2.2. (i) *If $e \in L^\infty(\Omega)$, $e(z) \leqslant \hat{\lambda}_k$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure, then $\vartheta(u) - \int_\Omega e(z)u^2\,dz \geqslant \hat{c}\|u\|^2$ for some $\hat{c} > 0$ and all $u \in \hat{H}_{k-1}$.*
(ii) *If $e \in L^\infty(\Omega)$, $e(z) \geqslant \hat{\lambda}_k$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure, then $\vartheta(u) - \int_\Omega e(z)u^2\,dz \leqslant -\hat{c}_1\|u\|^2$ for some $\hat{c}_1 > 0$ and all $u \in \bar{H}_k$.*

Finally, we mention that all nonprincipal eigenvalues $\hat{\lambda}_k(\eta)$ (that is, $k \geqslant 2$) have nodal (that is, sign-changing) eigenfunctions.

Next we recall some basic definitions and facts for critical groups. So let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:
- $\varphi^c = \{u \in X : \varphi(u) \leqslant c\}$ (the sublevel set at $c \in \mathbb{R}$),
- $K_\varphi = \{u \in X : \varphi'(u) = 0\}$ (the critical set of φ),
- $K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}$ (the critical set of φ at the level $c \in \mathbb{R}$).

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For very $k \in \mathbb{N}_0$, let $H_k(Y_1, Y_2)$ denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Let $u \in K_\varphi$ be isolated and $c = \varphi(u)$ (that is, $u \in K_\varphi^c$). The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. From the excision property of singular homology, we see that the above definition of critical groups is independent of the choice of the neighborhood U.

Now suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [8, p. 628]) implies that the above definition is independent of the choice of the level $c \in \mathbb{R}$.

Assuming that K_φ is finite, we have

$$\operatorname{rank} C_k(\varphi, \infty) \leqslant \sum_{u \in K_\varphi} \operatorname{rank} C_k(\varphi, u) \quad \text{for all } k \in \mathbb{N} \tag{2.4}$$

(see, for example, Motreanu, Motreanu and Papageorgiou [15, p. 160]). From (2.4) we infer the following result.

Proposition 2.3. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, K_φ is finite and for some $k \in \mathbb{N}_0$ we have $C_k(\varphi, \infty) \neq 0$, then there exists $u \in K_\varphi$ such that $C_k(\varphi, u) \neq 0$.*

A related result is the following (see [15, p. 173]).

Proposition 2.4. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, K_φ is finite and for some $k \in \mathbb{N}_0$ we have $C_k(\varphi, 0) \neq 0$ and $C_k(\varphi, \infty) = 0$, then we can find $u \in K_\varphi$ such that either*

$$\varphi(u) < 0 \quad \textit{and} \quad C_{k-1}(\varphi, u) \neq 0,$$

or

$$\varphi(u) > 0 \quad \textit{and} \quad C_{k+1}(\varphi, u) \neq 0.$$

Suppose $X = H$ is a Hilbert space and $\varphi \in C^2(H, \mathbb{R})$. Let $u \in K_\varphi$. The Morse index of u, denoted by $\hat{m}(u)$, is defined to be the supremum of the dimensions of the vector subspace on which $\varphi''(u)$ is negative definite. The nullity of u, denoted by $\hat{\nu}(u)$, is defined to be the dimension of $\ker \varphi''(u)$. We say that $u \in K_\varphi$ is nondegenerate, if $\varphi''(u)$ is invertible (that is, $\hat{\nu}(u) = 0$). For a nondegenerate $u \in K_\varphi$ with Morse index $\hat{m}(u) = \hat{m}$, we have

$$C_k(\varphi, u) = \delta_{k,\hat{m}} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Hereafter by $\delta_{k,\hat{m}}$ we denote the Kronecker symbol

$$\delta_{k,\hat{m}} = \begin{cases} 1 & \text{if } k = \hat{m}, \\ 0 & \text{if } k \neq \hat{m}, \end{cases} \quad k \in \mathbb{N}_0.$$

In dealing with degenerate critical points, the main tool is the so-called “shifting theorem” (see, for example, Motreanu, Motreanu and Papageorgiou [15, p. 156]). A consequence of this theorem which we will need in the sequel, is the following proposition.

Proposition 2.5. *If H is a Hilbert space, $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$ has finite Morse index $\hat{m}$ and nullity $\hat{\nu}$, then one of the following statements holds:*
(i) *$C_k(\varphi, u) = 0$ for all $k \leqslant \hat{m}$ and all $k \geqslant \hat{m} + \hat{\nu}$;*
(ii) *$C_k(\varphi, u) = \delta_{k,\hat{m}} \mathbb{Z}$ for all $k \in \mathbb{N}_0$;*
(iii) *$C_k(\varphi, u) = \delta_{k,\hat{m}+\hat{\nu}} \mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Finally, we fix our notation. Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega)$$

(the Nemitski or superposition operator corresponding to the function h). By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and, as we already mentioned earlier in this section, by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$, that is,

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

By $\tilde{H}_k(X)$ (for $k \in \mathbb{N}_0$), we denote the reduced homology groups, that is, $\tilde{H}_k(X) = H_k(X, *)$ for all $k \in \mathbb{N}_0$, with $* \in X$. By $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the operator defined by

$$\langle A(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \, dz \quad \text{for all } u, h \in H^1(\Omega).$$

3 Pairs of nontrivial solutions

In this section, we establish the existence of two nontrivial solutions for problem (1.1). The hypothesis on the boundary term $\beta(\cdot)$ is (H(β)) from the previous section. Note that the case $\beta \equiv 0$ corresponds to the Neumann problem. So, our result here subsumes the works on resonant Neumann problems.

The hypotheses on the reaction term $f(z,x)$ are the following:

(H) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$ we have $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and the following hold:

(i) $|f'_x(z,x)| \leqslant a(z)(1+|x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)$ and $2 \leqslant r < 2^*$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 3, \\ +\infty & \text{if } N = 1,2 \end{cases}$$

(the critical Sobolev exponent);

(ii) there exist $m \geqslant 3$ and $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z) \leqslant \hat{\lambda}_{m+1} \quad \text{for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_{m+1},$$
$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \eta(z) \quad \text{uniformly for almost all } z \in \Omega,$$

and if $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\lim_{x\to\pm\infty} [f(z,x)x - 2F(z,x)] = -\infty \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) there exist $l \in \mathbb{N}$ and $\delta > 0$ such that

$$l \leqslant m-2,$$
$$\hat{\lambda}_l x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}x^2 \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \delta;$$

(iv) $f(z,x)x \leqslant f'_x(z,x)x^2$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, $f'_x(z,x) \leqslant \hat{\lambda}_{m+1}$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, and for every $\rho > 0$, there exists $\Omega_\rho \subseteq \Omega$ with $|\Omega_\rho|_N > 0$ such that

$$f'_x(z,x) < \hat{\lambda}_{m+1} \quad \text{for almost all } z \in \Omega_\rho \text{ and all } |x| \leqslant \rho.$$

The following function satisfies hypotheses (H):

$$f(x) = \begin{cases} \hat{\lambda}_m x - (\hat{\lambda}_l - \hat{\lambda}_m)\Big(\dfrac{1}{2}\ln|x| + \sqrt{|x|}\Big) & \text{if } x < -1, \\ \hat{\lambda}_l x & \text{if } -1 \leqslant x \leqslant 1, \\ \hat{\lambda}_m x + (\hat{\lambda}_l - \hat{\lambda}_m)\Big(\dfrac{1}{2}\ln|x| + \sqrt{|x|}\Big) & \text{if } x > 1. \end{cases}$$

Remark 3.1. Hypotheses (H(ii)), (H(iii)) imply that we can have resonance at both $\pm\infty$ and zero (double resonance). Hypothesis (H(iv)) implies that for almost all $z \in \Omega$,

$$x \mapsto \frac{f(z,x)}{x} = \begin{cases} \text{nondecreasing on } (0,+\infty), \\ \text{nonincreasing on } (-\infty,0). \end{cases}$$

To see this, note that

$$\Big(\frac{f(z,x)}{x}\Big)' = \frac{f'_x(z,x)x - f(z,x)}{x^2} \begin{cases} \geqslant 0 & \text{if } x > 0, \\ \leqslant 0 & \text{if } x < 0. \end{cases}$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}\vartheta(u) - \int_\Omega F(z,u(z))\,dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently, $\varphi \in C^2(H^1(\Omega))$.

Proposition 3.2. *If hypotheses* (H(β)), (H) *hold, then the functional* φ *satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \in \mathbb{N}, \tag{3.1}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$|\langle \varphi'(u_n), h\rangle| \leqslant \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H^1(\Omega), \text{ with } \epsilon_n \to 0^+$$
$$\Longrightarrow \left|\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n h \, d\sigma - \int_\Omega f(z, u_n) h \, dz\right| \leqslant \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

In (3.3) we choose $h = u_n \in H^1(\Omega)$. Then

$$-\epsilon_n \leqslant -\vartheta(u_n) + \int_\Omega f(z, u_n) u_n dz \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

On the other hand, from (3.1) we have

$$-2M_1 \leqslant \vartheta(u_n) - \int_\Omega 2F(z, u_n)\, dz \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$-M_2 \leqslant \int_\Omega [f(z, u_n) u_n - 2F(z, u_n)]\, dz \quad \text{for some } M_2 > 0 \text{ and all } n \in \mathbb{N}. \tag{3.6}$$

Suppose that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is unbounded. By passing to a subsequence if necessary, we may assume that

$$\|u_n\| \to \infty.$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.7}$$

From (3.3) we have for all $n \in \mathbb{N}$,

$$\left|\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h \, d\sigma - \int_\Omega \frac{N_f(u_n)}{\|u_n\|} h \, dz\right| \leqslant \frac{\epsilon_n \|h\|}{(1 + \|u_n\|)\|u_n\|}. \tag{3.8}$$

Hypotheses (H(i)), (H(ii)) imply that

$$|f(z, x)| \leqslant c_1(1 + |x|) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}, \text{ with } c_1 > 0$$
$$\Longrightarrow \left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \quad \text{is bounded.}$$

We may assume that

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \xi \quad \text{in } L^2(\Omega) \text{ as } n \to \infty. \tag{3.9}$$

Moreover, hypothesis (H(ii)) implies that

$$\xi(z) = \hat{\eta}(z) y(z) \quad \text{for almost all } z \in \Omega \text{ with } \hat{\lambda}_m \leqslant \hat{\eta}(z) \leqslant \eta(z) \text{ for almost all } z \in \Omega \tag{3.10}$$

(see [1]).

Therefore, if in (3.8) we pass to the limit as $n \to \infty$ and use (3.7), (3.9) and (3.10), we obtain

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h \, d\sigma = \int_{\Omega} \hat{\eta}(z) y h \, dz \quad \text{for all } h \in H^1(\Omega)$$

$$\implies -\Delta y(z) = \hat{\eta}(z) y(z) \quad \text{for almost all } z \in \Omega, \qquad \frac{\partial y}{\partial n} + \beta(z) y = 0 \quad \text{on } \partial\Omega \tag{3.11}$$

(see Papageorgiou and Rădulescu [17]).

If $\hat{\eta} \not\equiv \hat{\lambda}_m$ (see (3.10)), then using Proposition 2.1, we have

$$\hat{\lambda}_m(\hat{\eta}) < \hat{\lambda}_m(\hat{\lambda}_m) = 1 \quad \text{and} \quad 1 = \hat{\lambda}_{m+1}(\hat{\lambda}_{m+1}) < \hat{\lambda}_{m+1}(\hat{\eta})$$
$$\implies y = 0 \quad \text{(see (3.11))}.$$

From (3.8) with $h = y_n \in H^1(\Omega)$, we see that

$$\|Dy_n\|_2 \to 0$$
$$\implies y_n \to 0 \quad \text{in } H^1(\Omega) \text{ (see (3.7))},$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

If $\hat{\eta}(z) = \hat{\lambda}_m$ for almost all $z \in \Omega$ (see (3.10)), then from (3.11) we have

$$y \in E(\hat{\lambda}_m).$$

Also, if in (3.8) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.7) and (3.9), then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0$$
$$\implies \|Dy_n\|_2 \to \|Dy\|_2$$
$$\implies y_n \to y \quad \text{(by the Kadec–Klee property of Hilbert spaces; see [8, p. 911])}$$
$$\implies \|y\| = 1 \quad \text{and so } y \in E(\hat{\lambda}_m) \setminus \{0\}.$$

The UCP of the eigenspaces implies that

$$y(z) \neq 0 \quad \text{for almost all } z \in \Omega$$
$$\implies |u_n(z)| \to +\infty \quad \text{for almost all } z \in \Omega.$$

Then hypothesis (H(ii)) implies that

$$f(z, u_n(z))u_n(z) - 2F(z, u_n(z)) \to -\infty \quad \text{for almost all } z \in \Omega. \tag{3.12}$$

Using (3.12), hypothesis (H(ii)) and Fatou's lemma, we have

$$\int_{\Omega} [f(z, u_n)u_n - 2F(z, u_n)] \, dz \to -\infty \quad \text{as } n \to \infty. \tag{3.13}$$

Comparing (3.6) and (3.13), we reach a contradiction.

So, we have proved that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.14}$$

In (3.3) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.14). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0$$
$$\implies \|Du_n\|_2 \to \|Du\|_2$$
$$\implies u_n \to u \quad \text{in } H^1(\Omega) \quad \text{(again by the Kadec–Klee property; see [8, p. 911])}$$
$$\implies \varphi \text{ satisfies the C-condition,}$$

as desired. □

Let

$$\bar{H}_l = \bigoplus_{i=1}^{l} E(\hat{\lambda}_i) \quad \text{and} \quad \hat{H}_l = \bar{H}_l^{\perp} = \overline{\bigoplus_{i \geqslant l+1} E(\hat{\lambda}_1)},$$

where l comes from hypothesis (H(iii)). We have the following orthogonal direct sum decomposition:

$$H^1(\Omega) = \bar{H}_l \oplus \hat{H}_l. \tag{3.15}$$

So, every $u \in H^1(\Omega)$ admits a unique decomposition

$$u = \bar{u} + \hat{u} \tag{3.16}$$

with $\bar{u} \in \bar{H}_l$ and $\hat{u} \in \hat{H}_l$.

Proposition 3.3. *If hypotheses* (H(β)), (H) *hold, then* $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$ *with* $d_l = \dim \bar{H}_l < +\infty$.

Proof. We assume that $0 \in K_\varphi$ is isolated. Otherwise, we already have a sequence of distinct nontrivial solutions of (1.1) converging to zero, and so we are done.

Let $\eta_0 \in (\hat{\lambda}_l, \hat{\lambda}_{l+1})$ and consider the functional $\psi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\vartheta(u) - \frac{\eta_0}{2}\|u\|_2^2 \quad \text{for all } u \in H^1(\Omega).$$

Evidently, $\psi \in C^2(H^1(\Omega))$ and using (2.2) and (2.3), we see that

$$\psi|_{\hat{H}_l} \geqslant 0 \quad \text{and} \quad \psi|_{\bar{H}_l} \leqslant 0.$$

From Motreanu, Motreanu and Papageorgiou [15, Proposition 6.84 (ii) and Theorem 6.87], we have

$$C_{d_l}(\psi, 0) \neq 0 \quad \text{with } d_l = \dim \bar{H}_l < \infty.$$

Since $\psi \in C^2(H^1(\Omega))$, from Proposition 2.5 we infer that

$$C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\varphi(u) + t\psi(u) \quad \text{for all } t \in [0, 1] \text{ and all } u \in H^1(\Omega).$$

For $t \in (0, 1]$ and $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leqslant \delta$ ($\delta > 0$ is as in hypothesis (H(iii))), we have

$$\langle h_u'(t, u), v\rangle = (1 - t)\langle \varphi'(u), v\rangle + t\langle \psi'(u), v\rangle \quad \text{for all } v \in H^1(\Omega). \tag{3.17}$$

Let $v = \hat{u} - \bar{u} \in H^1(\Omega)$ (see (3.16)). Exploiting the orthogonality of the component subspaces in (3.15), we have

$$\langle \varphi'(u), \hat{u} - \bar{u}\rangle = \vartheta(\hat{u}) - \vartheta(\bar{u}) - \int_\Omega f(z, u)(\hat{u} - \bar{u})\, dz. \tag{3.18}$$

Recall that $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leqslant \delta$. So, using hypothesis (H(iii)), we have

$$f(z, u(z))(\hat{u} - \bar{u})(z) \leqslant \hat{\lambda}_{l+1}\hat{u}(z)^2 - \hat{\lambda}_l\bar{u}(z)^2 \quad \text{for almost all } z \in \Omega. \tag{3.19}$$

Using (3.19) in (3.18), we have

$$\langle \varphi'(u), \hat{u} - \bar{u}\rangle \geqslant \gamma(\hat{u}) - \hat{\lambda}_{l+1}\|\hat{u}\|_2^2 - [\gamma(\bar{u}) - \hat{\lambda}_l\|u\|_2^2] \geqslant 0 \text{ (see (2.2), (2.3))}. \tag{3.20}$$

Also, we have

$$\langle \psi'(u), \hat{u} - \bar{u}\rangle = \gamma(\hat{u}) - \eta_0\|\hat{u}\|_2^2 - [\gamma(\bar{u}) - \eta_0\|\bar{u}\|_2^2] \geqslant c_2\|u\|^2 \quad \text{for some } c_2 > 0 \tag{3.21}$$

(see Proposition 2.2 and recall that $\eta_0 \in (\hat{\lambda}_l, \hat{\lambda}_{l+1})$).

So, for $t \in (0,1]$, from (3.20) and (3.21) we have

$$\langle h'_u(t,u), \hat{u} - \bar{u} \rangle \geqslant tc_2 \|u\|^2.$$

On the other hand, $h(0,\cdot) = \varphi(\cdot)$ and $0 \in K_\varphi$ is isolated. Therefore, the homotopy

$$h|_{[0,1]\times C^1(\overline{\Omega})}$$

preserves the isolation of the critical point $u = 0$. From Corvellec and Hantoute [4, Theorem 5.2] we have

$$\begin{aligned} C_k(\varphi|_{C^1(\overline{\Omega})}, 0) = C_k(\psi|_{C^1(\overline{\Omega})}, 0) \quad & \text{for all } k \in \mathbb{N}_0 \\ \Longrightarrow C_k(\varphi, 0) = C_k(\psi, 0) \quad & \text{for all } k \in \mathbb{N}_0 \text{ (since } C^1(\overline{\Omega}) \text{ is dense in } H^1(\Omega);\text{ see Palais [16])} \\ \Longrightarrow C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad & \text{for all } k \in \mathbb{N}_0 \text{ (see (3.17))}, \end{aligned}$$

as desired. □

Next we compute the critical groups of φ at infinity.

For the same integer m as in hypothesis (H(ii)), we consider the following orthogonal direct sum decomposition:

$$H^1(\Omega) = \bar{H}_m \oplus \hat{H}_m, \tag{3.22}$$

where

$$\bar{H}_m = \bigoplus_{i=1}^{m} E(\hat{\lambda}_i) \quad \text{and} \quad \hat{H}_m = \bar{H}_m^\perp = \overline{\bigoplus_{i \geqslant m+1} E(\hat{\lambda}_i)}.$$

Proposition 3.4. *If hypotheses* (H(β)), (H) *hold, then* $C_{d_m}(\varphi, \infty) \neq 0$ *with* $d_m = \dim \bar{H}_m$ *(see* (3.22)*).*

Proof. Hypothesis (H(ii)) implies that given any $\xi > 0$, we can find $M_3 = M_3(\xi) > 0$ such that

$$f(z,x)x - 2F(z,x) \leqslant -\xi \quad \text{for almost all } z \in \Omega \text{ and all } |x| \geqslant M_3. \tag{3.23}$$

For almost all $z \in \Omega$ and all $s \neq 0$, we have

$$\begin{aligned} \frac{d}{ds}\frac{F(z,s)}{s^2} &= \frac{f(z,s)s^2 - 2F(z,s)s}{s^4} \\ &= \frac{f(z,s)s - 2F(z,s)}{|s|^2 s}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d}{ds}\frac{F(z,s)}{s^2} = \frac{f(z,s)s - 2F(z,s)}{s^3} \leqslant -\frac{\xi}{s^3} \quad & \text{for almost all } z \in \Omega \text{ and all } s \geqslant M_3 \text{ (see (3.23))} \\ \Longrightarrow \frac{F(z,v)}{v^2} - \frac{F(z,s)}{x^2} \leqslant \frac{\xi}{2}\Big[\frac{1}{v^2} - \frac{1}{x^2}\Big] \quad & \text{for almost all } z \in \Omega \text{ and all } v \geqslant x \geqslant M_3. \end{aligned} \tag{3.24}$$

Note that hypothesis (H(ii)) implies that

$$\hat{\lambda}_m \leqslant \liminf_{u \to \pm\infty} \frac{2F(z,u)}{u^2} \leqslant \limsup_{u \to \pm\infty} \frac{2F(z,u)}{u^2} \leqslant \eta(z) \tag{3.25}$$

uniformly for almost all $z \in \Omega$.

So, if in (3.24) we pass to the limit as $v \to +\infty$ and use (3.25), then

$$\begin{aligned} \frac{\hat{\lambda}_m}{2} - \frac{F(z,x)}{x^2} \leqslant -\frac{\xi}{2}\frac{1}{x^2} \quad & \text{for almost all } z \in \Omega \text{ and all } x \geqslant M_3 \\ \Longrightarrow \hat{\lambda}_m x^2 - 2F(z,x) \leqslant -\xi \quad & \text{for almost all } z \in \Omega \text{ and all } x \geqslant M_3 \\ \Longrightarrow \hat{\lambda}_m x^2 - 2F(z,x) \to -\infty \quad & \text{as } x \to -\infty, \text{ uniformly for almost all } z \in \Omega \end{aligned} \tag{3.26}$$

(recall that $\xi > 0$ is arbitrary).

On the negative semiaxis, we have

$$\frac{d}{ds}\frac{F(z,s)}{|s|^2} = \frac{f(z,s)s - 2F(z,s)}{|s|^2 s} \geqslant -\frac{\xi}{|s|^2 s} \quad \text{for almost all } z \in \Omega \text{ and all } s \leqslant -M_3$$

(see (3.23) and recall that $s < 0$), which implies

$$\frac{F(z,v)}{|v|^2} - \frac{F(z,x)}{|x|^2} \geqslant -\frac{\xi}{2}\Big[\frac{1}{|x|^2} - \frac{1}{|v|^2}\Big] \quad \text{for almost all } z \in \Omega \text{ and all } x \leqslant v \leqslant -M_3.$$

Letting $x \to -\infty$ and using (3.25), we obtain

$$\begin{aligned}
&\frac{F(z,v)}{|v|^2} - \frac{\hat{\lambda}_m}{2} \geqslant \frac{\xi}{2}\frac{1}{|v|^2} \quad \text{for almost all } z \in \Omega \text{ and all } v \leqslant -M_3\\
&\implies 2F(z,v) - \hat{\lambda}_m|v|^2 \geqslant \xi \quad \text{for almost all } z \in \Omega \text{ and all } v \leqslant -M_3\\
&\implies \hat{\lambda}_m|v|^2 - 2F(z,v) \to -\infty \quad \text{as } v \to -\infty, \text{ uniformly for almost all } z \in \Omega
\end{aligned} \tag{3.27}$$

(recall that $\xi > 0$ is arbitrary).

From (3.26) and (3.27) it follows that

$$\hat{\lambda}_m|x|^2 - 2F(z,x) \to -\infty \quad \text{as } x \to \pm\infty, \text{ uniformly for almost all } z \in \Omega. \tag{3.28}$$

We introduce the two sets

$$\begin{aligned}
S_r &= \{u \in H^1(\Omega) : \|u\| = r,\ \vartheta(u) \leqslant \hat{\lambda}_m\|u\|_2^2\} \quad (r > 0),\\
A &= \{u \in H^1(\Omega) : \vartheta(u) \geqslant \hat{\lambda}_{m+1}\|u\|_2^2\}.
\end{aligned}$$

The set S_r is a C^1-Hilbert manifold with boundary and from Degiovanni and Lancelotti [5, Theorem 3.2]. We have

$$\operatorname{ind} S_r = \operatorname{ind}(H^1(\Omega) \setminus A) = d_m,$$

where "ind" denotes the Fadell–Rabinowitz index (see [7]). From Cingolani and Degiovanni [3, Theorem 3.6], we know that the sets S_r and A homologically link in dimension d_m.

Hypotheses (H(i)), (H(ii)) imply that given $\epsilon > 0$, we can find $c_3 = c_3(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\eta(z) + \epsilon)x^2 + c_3 \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.29}$$

Then for all $u \in A$, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\vartheta(u) - \int_\Omega F(z,u)\,dz\\
&\geqslant \frac{1}{2}\Big[\vartheta(u) - \int_\Omega \eta(z)u^2\,dz - \epsilon\|u\|^2\Big] - c_3|\Omega|_N \quad \text{(see (3.29))}\\
&\geqslant \frac{\hat{c} - \epsilon}{2}\|u\|^2 - c_3|\Omega|_N \quad \text{for some } \hat{c} > 0 \text{ (see Proposition 2.2).}
\end{aligned}$$

Choosing $\epsilon \in (0, \hat{c})$, we see that

$$\inf_A \varphi > -\infty.$$

Next we show that

$$\varphi(u) \to -\infty \quad \text{as } u \in S_r \text{ and } r \to +\infty.$$

Arguing by contradiction, we suppose that one can find $r_n \to +\infty$, $u_n \in S_{r_n}$ and $M_4 > 0$ such that

$$-M_4 \leqslant \varphi(u_n) \quad \text{for all } n \in \mathbb{N} \text{ and } \|u_n\| \to \infty. \tag{3.30}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.31}$$

We have

$$\vartheta(y_n) \leqslant \hat{\lambda}_m \|y_n\|_2^2 \quad \text{for all } n \in \mathbb{N}. \tag{3.32}$$

If $y = 0$, from (3.31), (3.32) and hypothesis (H(β)) we have

$$\begin{aligned} Dy_n \to 0 \qquad\qquad & \text{in } L^2(\Omega, \mathbb{R}^N) \\ \implies y_n \to 0 \quad & \text{in } H^1(\Omega) \text{ (see (3.31))}, \end{aligned}$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Therefore $y \neq 0$, and so

$$|u_n(z)| \to +\infty \quad \text{for almost all } z \in \Omega_0 = \{y \neq 0\},\ |\Omega_0|_N > 0.$$

Then from (3.28) and Fatou's lemma we have

$$\int_\Omega [\hat{\lambda}_m u_n^2 - 2F(z, u_n)]\, dz \to -\infty \quad \text{as } n \to \infty. \tag{3.33}$$

On the other hand, from (3.30) and since $u_n \in S_{r_n}$, $n \in \mathbb{N}$, we have

$$-2M_4 \leqslant 2\varphi(u_n) \leqslant \int_\Omega [\hat{\lambda}_m u_n^2 - 2F(z, u_n)]\, dz \quad \text{for all } n \in \mathbb{N}. \tag{3.34}$$

Comparing (3.33) and (3.34), we reach a contradiction. Therefore, we have

$$\varphi(u) \to -\infty \quad \text{as } u \in S_r \text{ and } r \to +\infty.$$

It follows that for $r > 0$ big, we have

$$\sup_{S_r} \varphi < \inf_A \varphi.$$

As before, we assume that K_φ is finite (otherwise, we already have a whole sequence of distinct solutions, and so we are done). Hence we can have

$$\sup_{S_r} \varphi \leqslant a < \inf_{K_\varphi} \varphi.$$

We consider the triple of sets

$$S_r \subseteq \varphi^a \subseteq H^1(\Omega) \setminus A.$$

We have the commutative diagram

$$\begin{tikzcd} \tilde{H}_{d_m-1}(S_r) \arrow[r] \arrow[dr, "i_*"'] & \tilde{H}_{d_m-1}(\varphi^a) \arrow[d] \\ & \tilde{H}_{d_m-1}(H^1(\Omega) \setminus A). \end{tikzcd}$$

Since $i_* \neq 0$ (recall that we established earlier that S_r and A homologically link in dimension d_m; see Motreanu, Motreanu and Papageorgiou [15, Definition 6.77]), we have

$$\tilde{H}_{d_m-1}(\varphi^a) \neq 0.$$

But from Motreanu, Motreanu and Papageorgiou [15, Proposition 6.64] we have

$$C_{d_m}(\varphi, \infty) = H_{d_m}(H^1(\Omega), \varphi^a) = \tilde{H}_{d_m-1}(\varphi^a) \neq 0. \qquad \square$$

For the multiplicity theorem we need to strengthen hypothesis (H(ii)) a little. So, the new conditions on the reaction term $f(z, x)$ are the following.

(H') $f(z,x) = \hat{\lambda}_m x + f_0(z,x)$ with $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$ we have $f_0(z,0) = 0$, $f_0(z,\cdot) \in C^1(\mathbb{R})$ and the following holds:

(i) $|(f_0)'_x(z,x)| \leqslant a_0(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, with $a_0 \in L^\infty(\Omega)_+$, $2 \leqslant r < 2^*$;

(ii) there exists a function $\hat{\eta} \in L^\infty(\Omega)_+$ such that

$$0 \leqslant \hat{\eta}(z) \leqslant \hat{\lambda}_{m+1} - \hat{\lambda}_m \quad \text{for almost all } z \in \Omega,$$
$$\hat{\eta} \not\equiv \hat{\lambda}_{m+1} - \hat{\lambda}_m,$$
$$\hat{\eta}(z) = \lim_{x \to \pm\infty} \frac{f_0(z,x)}{x} \quad \text{uniformly for almost all } z \in \Omega,$$

and if $F_0(z,x) = \int_0^x f_0(z,s)\,ds$, then

$$\lim_{x \to \pm\infty} [f_0(z,x)x - 2F_0(z,x)] = -\infty \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) there exist $l \in \mathbb{N}$ and $\delta > 0$ such that

$$l \leqslant m - 2,$$
$$(\hat{\lambda}_l - \hat{\lambda}_m)x^2 \leqslant f_0(z,x) \leqslant (\hat{\lambda}_{l+1} - \hat{\lambda}_m)x^2 \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \delta;$$

(iv) $f_0(z,x)x \leqslant (f_0)'_x(z,x)x^2$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, $(f_0)'_x(z,x) \leqslant \hat{\lambda}_{m+1} - \hat{\lambda}_m$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, and for every $\rho > 0$ there exists $\Omega_\rho \subseteq \Omega$ with $|\Omega_\rho|_N > 0$ such that

$$(\hat{f}_0)'_x(z,x) < \hat{\lambda}_{m+1} \quad \text{for almost all } z \in \Omega_\rho \text{ and all } |x| \leqslant \rho.$$

Theorem 3.5. *If hypotheses* (H(β)), (H') *hold, then problem* (1.1) *admits at least two nontrivial solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.4 we know that $C_{d_m}(\varphi,\infty) \neq 0$. So, according to Proposition 2.3 we can find $u_0 \in K_\varphi$ such that

$$C_{d_m}(\varphi, u_0) \neq 0. \tag{3.35}$$

On the other hand, from Proposition 3.3 we have

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.36}$$

Since $l \leqslant m-2$ (see hypothesis (H'(iii)), we have $d_l \neq d_m$. Then from (3.35) and (3.36) we infer that $u_0 \neq 0$.

Note that hypotheses (H'(i))–(H'(iii)) imply that

$$|f(z,x)| \leqslant c_4|x| \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}, \text{ with } c_4 > 0. \tag{3.37}$$

We set

$$g(z) = \begin{cases} \dfrac{f(z,u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0, \\ f'_x(z,u_0(z)) & \text{if } u_0(z) = 0. \end{cases}$$

From (3.37) and hypothesis (H'(iv)) we see that $g \in L^\infty(\Omega)$, and we have

$$\begin{cases} -\Delta u_0(z) = g(z)u_0(z) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.38}$$

(see Papageorgiou and Rădulescu [17]).

From Wang [22, Lemma 5.1] we have $u_0 \in L^\infty(\Omega)$, and from (3.38) it follows that $\Delta u_0 \in L^\infty(\Omega)$. Using the Calderon–Zygmund estimates (see Wang [22, Lemma 5.2]), we have $u_0 \in W^{2,s}(\Omega)$ with $s > N$. Then the Sobolev embedding theorem says that

$$W^{2,s}(\Omega) \hookrightarrow C^{1+\alpha}(\overline{\Omega}) \quad \text{with } \alpha = 1 - \tfrac{N}{s} > 0$$
$$\implies u_0 \in C^1(\overline{\Omega}).$$

If $\hat{\eta} \not\equiv 0$ (nonresonant problem at $\pm\infty$), then from Papageorgiou and Rădulescu [19, Proposition 26] we have

$$C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}.$$

If $\hat{\eta}(z) = 0$ for almost all $z \in \Omega$ (resonant problem at $\pm\infty$), then by Motreanu, Motreanu and Papageorgiou [15, Theorem 6.7.1] we have

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \notin [d_m, d_{m+1}] \tag{3.39}$$

(here $d_{m+1} = \dim \bigoplus_{i=1}^{m+1} E(\hat{\lambda}_i)$). Since $d_l < d_m$ (see hypothesis (H'(iii))), we have

$$C_{d_l}(\varphi, 0) \neq 0 \quad \text{and} \quad C_{d_l}(\varphi, \infty) = 0 \qquad \text{(see (3.36) and (3.39))}. \tag{3.40}$$

Then from (3.40) and Proposition 2.4 we know that we can find $\hat{u} \in K_\varphi$ such that

$$C_{d_l-1}(\varphi, \hat{u}) \neq 0 \quad \text{or} \quad C_{d_l+1}(\varphi, \hat{u}) \neq 0. \tag{3.41}$$

From (3.36) and (3.41) we infer that $\hat{u}$ is a nontrivial solution of (1.1) and the regularity theory implies that $\hat{u} \in C^1(\overline{\Omega})$.

We need to show that $\hat{u} \neq u_0$. According to (3.35), it suffices to show that

$$C_{d_m}(\varphi, \hat{u}) = 0. \tag{3.42}$$

Let

$$\hat{g}(z) = \begin{cases} \dfrac{f(z, \hat{u}(z))}{\hat{u}(z)} & \text{if } \hat{u}(z) \neq 0, \\ f'_x(z, \hat{u}(z)) & \text{if } \hat{u}(z) = 0. \end{cases} \tag{3.43}$$

As before we have $\hat{g} \in L^\infty(\Omega)$. We consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta v(z) = \hat{\lambda}\hat{g}(z)v(z) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} + \beta(z)v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.44}$$

Since $\hat{u} \in C^1(\overline{\Omega})$ is a solution for problem (1.1), from (3.42) it follows that $\hat{\lambda} = 1$ is an eigenvalue of problem (3.44) with $\hat{u} \in C^1(\overline{\Omega})$ as a corresponding eigenfunction.

Recall that $\hat{\lambda}_l \leqslant \hat{g}(z)$ for almost all $z \in \Omega$ (see hypothesis (H'(iii)) and Remark 3.1). Suppose that $\hat{\lambda}_l = \hat{g}(z)$ for almost all $z \in \Omega$. Then from (3.44) it follows that $\hat{u} \in C^1(\overline{\Omega})$ is an eigenfunction corresponding to $\hat{\lambda}_l$. The UCP implies that $\hat{u}(z) \neq 0$ for almost all $z \in \Omega$. Then from (3.43) and Proposition 2.5 it follows that (3.42) holds (recall $l \leqslant m-2$).

So, we may assume that the inequality $\hat{\lambda}_l \leqslant \hat{g}(z)$ for almost all $z \in \Omega$ is strict on a set of positive measure. Invoking Proposition 2.1, we have

$$\hat{\lambda}_i(\hat{g}) < \hat{\lambda}_i(\hat{\lambda}_l) \leqslant 1 \quad \text{for all } i \in \{1, \dots, l\}.$$

Since $\hat{\lambda} = 1$ is an eigenvalue of (3.44), we have

$$\hat{\lambda}_{l+1}(\hat{g}) \leqslant 1. \tag{3.45}$$

By hypothesis (H'(iv)) we have

$$\hat{\sigma}(z) = f'_x(z, \hat{u}(z)) \geqslant \hat{g}(z) \quad \text{for almost all } z \in \Omega.$$

If this last inequality is strict on a set of positive measure, then Proposition 2.1 implies that

$$\begin{aligned} &\hat{\lambda}_{l+1}(\hat{\sigma}) < \hat{\lambda}_{l+1}(\hat{g}) \\ &\qquad \Longrightarrow \hat{\lambda}_{l+1}(\hat{\sigma}) < 1 \quad \text{(see (3.45))}. \end{aligned} \tag{3.46}$$

Recall that

$$\langle \varphi''(\hat{u})y, y\rangle = \vartheta(y) - \int_\Omega f'_x(z, \hat{u}(z))y^2\,dz \quad \text{for all } y \in H^1(\Omega).$$

Then from (3.46) it follows that

$$\hat{m}(\hat{u}) > d_l.$$

From (3.41) and Proposition 2.5 we see that we must have

$$C_k(\varphi, \hat{u}) = \delta_{k,d_l+1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.47}$$

But $l \leqslant m - 2$ (see hypothesis (H'(iii))). Therefore, (3.47) implies that (3.42) holds.

Now suppose that $\hat{\sigma}(z) = f'_x(z, \hat{u}(z)) = \hat{g}(z)$ for almost all $z \in \Omega$. Hypothesis (H'(iv)) implies that

$$\hat{\sigma}(z) \leqslant \hat{\lambda}_{m+1} \quad \text{for almost all } z \in \Omega,$$

with strict inequality on a set of positive measure. Hence

$$\begin{aligned} &1 < \hat{\lambda}_{m+1}(\hat{\sigma}) \quad \text{(see Proposition 2.1)} \\ &\implies \hat{m}(\hat{u}) + \hat{v}(\hat{u}) \leqslant d_m. \end{aligned} \tag{3.48}$$

Recall that $d_l + 1 < d_m$ (see hypothesis (H'(iii))). Then from (3.41), (3.48) and Proposition 2.5, we infer that (3.42) holds again. From (3.42) and (3.35) it follows that

$$\begin{aligned} &\hat{u} \neq u_0 \\ &\implies \hat{u} \in C^1(\overline{\Omega}) \text{ is the second nontrivial smooth solution of problem (1.1).} \end{aligned}$$ □

Funding: The second author acknowledges the support through a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130.

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