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Positive solutions of singular elliptic systems with multiple parameters and Caffarelli–Kohn–Nirenberg exponents

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Abstract This paper is concerned with the existence of positive solutions for a class of quasilinear singular elliptic systems with Dirichlet boundary condition. By studying the competition between the Caffarelli–Kohn–Nirenberg exponents, the sign-changing potentials and the nonlinear terms, we establish an interval on the range of multiple parameters over which solutions exist in an appropriate weighted Sobolev space. The arguments rely on the method of weak sub- and super-solutions.

Keywords Caffarelli–Kohn–Nirenberg exponents · Elliptic system · Multiple parameters · Sub-supersolution method

Mathematics Subject Classification 35J55 · 35J65

1 Introduction

The study of positive solutions of singular partial differential equations or systems has been an extremely active research topic during the past few years. Such singular nonlinear prob-

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lems arise naturally and they occupy a central role in the interdisciplinary research between analysis, geometry, biology, elasticity, mathematical physics, etc.

In this paper, we are concerned with the existence of positive solutions to the boundary value problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c_1}(\lambda_1 A(x)f(v) + \mu_1 C(x)h(u)) & \text{in } \Omega, \\ -\mathrm{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = |x|^{-(b+1)q+c_2}(\lambda_2 B(x)g(u) + \mu_2 D(x)\tau(v)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $c_1, c_2, \lambda_1, \lambda_2, \mu_1, \mu_2$ are positive parameters. We assume that the nonlinear terms $f, g, h, \tau : [0,\infty) \longrightarrow [0,\infty)$ are nondecreasing continuous functions. We also suppose that the potentials A, B, C, and D are C^1 sign-changing functions, that may be negative only near the boundary.

Nonlinear problems like (1.1) are introduced in relationship with models for physical phenomena related to the equilibrium of anisotropic media that possibly are somewhere "perfect" insulators or "perfect" conductors, see Dautray and Lions [14, p. 79]. The qualitative analysis of these problems has been much developed after the pioneering paper by Murthy and Stampacchia [23]. A crucial milestone in the understanding of the elliptic problems involving the singular quasilinear elliptic operator $-\mathrm{div}\,(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$ is the paper by Caffarelli, Kohn and Nirenberg [9] (see also [11]).

The study of this type of problems is motivated by its various applications, for example, in fluid mechanics, population genetics, Newtonian fluids, flow through porous media, and glaciology, see [2,7,13,18]. On the other hand, quasilinear elliptic systems have an extensive practical background. They are used to describe the multiplicate chemical reaction catalyzed by the catalyst grains under constant or variant temperature, in the theory of quasi-regular and quasi-conformal mappings in Riemannian manifolds with boundary, or in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [16,28]. We refer to [1,8,17,20] for additional results on elliptic problems. For the regular case, that is, when $a = b = 0$, $c_1 = p$ and $c_2 = q$ the quasilinear elliptic equation has been studied by several authors (see, e.g., [3]). We refer to [4,12], where the authors discussed the system (1.1) when $p = q = 2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2 = 0$, f, g are increasing, and $f, g \geq 0$. In [19], the authors extended the study of [12] to the case when no sign conditions on $f(0)$ or $g(0)$ were required and in [21] they extend this study to the case when $p = q > 1$.

In the present paper we focus on further extending the study in [3] for the quasilinear elliptic systems involving singularity. Due to this singularity in the weights, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [10,15]. Several methods have been used to treat quasilinear equations and systems. In the scalar case, weak solutions can be obtained through variational methods which provide critical points of the corresponding energy functional, an approach which is also fruitful in the case of potential systems, that is, the nonlinearities on the right hand side are the gradient of a C^1-functional [5]. However, due to the loss of the variational structure, the treatment of nonvariational systems like (1.1) is more complicated and is based mostly on topological methods [6]. Here we focus on further extending the study in [26] to the system (1.1), which features multiple parameters, weight functions and stronger coupling. More precisely, under suitable conditions on f, g, h, τ we show that the singular problem (1.1) has a positive solution for λ_i, μ_i, $(i = 1, 2)$ sufficiently large.

2 Auxiliary results and technical assumptions

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $0 \in \Omega$. Let $W_0^{1,p}(\Omega, |x|^{-ap})$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |x|^{-ap}|\nabla u|^p dx)^{1/p}$.

Consider the nonlinear eigenvalue problem

$$\begin{cases} -\text{div}\,(|x|^{-sr}|\nabla\phi|^{r-2}\nabla\phi) = \lambda|x|^{-(s+1)r+t}|\phi|^{r-2}\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

For $r = p, s = a$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of problem (2.1) such that $\phi_{1,p} > 0$ in Ω, and $\|\phi_{1,p}\|_\infty = 1$. For $r = q$, $s = b$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of problem (2.1) such that $\phi_{1,q} > 0$ in Ω and $\|\phi_{1,q}\|_\infty = 1$, see [22,29].

The maximum principle implies that $\frac{\partial\phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r \in \{p, q\}$, where n is the outward normal. Thus, there are positive constants m_0, δ and σ_p, $\sigma_q \in (0, 1)$ such that

$$\begin{cases} \lambda_{1,r}|x|^{-(s+1)r+t}\phi_{1,r}^r - |x|^{-sr}|\nabla\phi_{1,r}|^r \le -m_0 & \text{in } \Omega_\delta \\ \phi_{1,r} \ge \sigma_r & \text{in } \Omega\backslash\overline{\Omega}_\delta, \end{cases} \tag{2.2}$$

with $r \in \{p, q\}$, $s \in \{a, b\}$, $t \in \{c_1, c_2\}$, and $\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$, see [22].

We also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ of the quasilinear singular system

$$\begin{cases} -\text{div}\,(|x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p) = |x|^{-(a+1)p+c_1} & \text{in } \Omega \\ -\text{div}\,(|x|^{-bq}|\nabla\zeta_q|^{q-2}\nabla\zeta_q) = |x|^{-(b+1)q+c_2} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.3}$$

Then, by [22], we have $\zeta_r > 0$ in Ω and $\frac{\partial\zeta_r}{\partial n} < 0$ on $\partial\Omega$ for $r \in \{p, q\}$.

Throughout this paper, we assume that the weight functions A, B, C, D take negative values in $\overline{\Omega}_\delta$ but require A, B, C, D to be strictly positive in $\Omega\backslash\Omega_\delta$. To be precise we assume that there exist positive constants a_0, b_0, c_0, d_0 and a_1, b_1, c_1, d_1 such that

$$A(x) \ge -a_0, \quad B(x) \ge -b_0, \quad C(x) \ge -c_0, \quad D(x) \ge -d_0 \ \text{ on } \overline{\Omega}_\delta \tag{2.4}$$

and

$$A(x) \ge a_1, \quad B(x) \ge b_1, \quad C(x) \ge c_1, \quad D(x) \ge d_1 \ \text{ on } \Omega\backslash\overline{\Omega}_\delta. \tag{2.5}$$

Let $s_0 \ge 0$ be such that

$$\lambda_1 a_1 f(s) + \mu_1 c_1 h(s) > 0 \text{ and } \lambda_2 b_1 g(s) + \mu_2 c_2 \tau(s) > 0 \ \text{ for every } s > s_0. \tag{2.6}$$

3 Main result

In this section, we establish our abstract existence result via the method of sub- and super-solutions. The concepts of sub- and super-solution were introduced by Nagumo [24] in 1937 who proved, using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm-Liouville problems. In fact, the premises of the sub- and super-solution method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub- and super-solutions in connection with monotone methods. Picard's techniques were applied later by Poincaré [25] in connection with problems arising in astrophysics. We refer to [27].

A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) are called a sub-solution and super-solution of problem (1.1) if they satisfy $(\psi_1, \psi_2) = (z_1, z_2) = (0, 0)$ on $\partial\Omega$ and

$$\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla\omega\, dx \le \int_\Omega |x|^{-(a+1)p+c_1}[\lambda_1 A(x) f(\psi_2) + \mu_1 C(x) h(\psi_1)]\omega\, dx$$
$$\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2 \cdot \nabla\omega\, dx \le \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2 B(x) g(\psi_1) + \mu_2 D(x) \tau(\psi_2)]\omega\, dx$$

and

$$\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla\omega\, dx \ge \int_\Omega |x|^{-(a+1)p+c_1}[\lambda_1 A(x) f(z_2) + \mu_1 C(x) h(z_1)]\omega dx$$
$$\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla\omega\, dx \ge \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2 B(x) g(z_1) + \mu_2 D(x) \tau(z_2)]\omega dx$$

for all $\omega \in W = \{\omega \in C_0^\infty(\Omega) : \omega \ge 0 \text{ in } \Omega\}$.

In what follows, if $f_1,\ f_2,\ g_1,\ g_2$ are real-valued functions, we write

$$(f_1, f_2) \le (g_1, g_2) \text{ if and only if } f_1 \le g_1 \text{ and } g_1 \le g_2.$$

In such a case, we write

$$(u, v) \in [(f_1, f_2), (g_1, g_2)] \text{ if and only if } f_1 \le u \le g_1 \text{ and } g_1 \le v \le g_2.$$

A key role in our arguments will be played by the following auxiliary result, which is due to Miyagaki and Rodrigues, see [22].

Lemma 3.1 *Suppose there exist sub- and super-solutions* (ψ_1, ψ_2) *and* (z_1, z_2)*, respectively of problem* (1.1)*, such that* $(\psi_1, \psi_2) \le (z_1, z_2)$*. Then problem* (1.1) *has at least one positive solution* (u, v) *such that* $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

In order to introduce our main result, we introduce the following hypotheses:

(H1): $f, g : [0, +\infty) \to \mathbb{R}$ are C^1 increasing functions such that

$$\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} g(t) = +\infty;$$

(H2): We have

$$\lim_{t\to+\infty} \frac{f(Kg(t)^{\frac{1}{q-1}})}{t^{p-1}} = 0,$$

for every constant $K > 0$;

(H3): $h, \tau \in C^1[0, \infty)$ are nonnegative and nondecreasing functions such that

$$\lim_{u\to+\infty} \frac{h(u)}{u^{p-1}} = 0, \quad \lim_{u\to+\infty} \frac{\tau(u)}{u^{q-1}} = 0, \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \tau(u) = +\infty;$$

(H4): If $\alpha_p = \frac{p-1}{p}\sigma_p^{\frac{p}{p-1}}$, $\alpha_q = \frac{q-1}{q}\sigma_q^{\frac{q}{q-1}}$, and $\overline{\alpha} = \min\{\alpha_p, \alpha_q\}$ then there exists $\gamma > \frac{s_0}{\overline{\alpha}}$ such that

$$\max\left\{\frac{\gamma\lambda_{1,q}}{b_1 g(\gamma^{\frac{1}{p-1}}\alpha_p) + d_1\tau(\gamma^{\frac{1}{q-1}}\alpha_q)}, \frac{\gamma\lambda_{1,p}}{a_1 f(\gamma^{\frac{1}{q-1}}\alpha_q) + c_1 h(\gamma^{\frac{1}{p-1}}\alpha_p)}\right\} <$$
$$\min\left\{\frac{m_0\gamma}{b_0 g(\gamma^{\frac{1}{p-1}}) + d_0\tau(\gamma^{\frac{1}{q-1}})}, \frac{m_0\gamma}{a_0 f(\gamma^{\frac{1}{q-1}}) + c_0 h(\gamma^{\frac{1}{p-1}})}\right\}.$$

We recall that m_0, σ_p, σ_q are introduced in relation (2.2) while s_0 is defined in (2.6).

We now state our main result for the problem (1.1).

Theorem 3.2 *Assume that the conditions* $(H1)$–$(H4)$ *are fulfilled. Then there exists a closed interval* $[\alpha, \beta]$ *such that problem* (1.1) *has a positive solution* (u, v) *for every* $\lambda_i, \mu_i \in [\alpha, \beta]$, $i = 1, 2$.

Proof Choose $r > 0$ such that

$$r \leq \min\left\{|x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}\right\} \quad \text{in } \overline{\Omega}_\delta.$$

Pick $\gamma > \frac{s_0}{\alpha}$ as in hypothesis **(H4)**. Define

$$\alpha := \max\left\{\frac{\gamma\lambda_{1,q}}{b_1 g\left(\gamma^{\frac{1}{p-1}}\frac{p-1}{p}\alpha_p^{\frac{p}{p-1}}\right) + d_1\tau\left(\gamma^{\frac{1}{q-1}}\frac{q-1}{q}\alpha_q^{\frac{q}{q-1}}\right)},\right.$$
$$\left.\frac{\gamma\lambda_{1,p}}{a_1 f\left(\gamma^{\frac{1}{q-1}}\frac{q-1}{q}\alpha_q^{\frac{q}{q-1}}\right) + c_1 h\left(\gamma^{\frac{1}{p-1}}\frac{p-1}{p}\alpha_p^{\frac{p}{p-1}}\right)}\right\}$$

and

$$\beta := \min\left\{\frac{m_0\gamma}{b_0 g(\gamma^{\frac{1}{p-1}}) + d_0\tau(\gamma^{\frac{1}{q-1}})}, \frac{m_0\gamma}{a_0 f(\gamma^{\frac{1}{q-1}}) + c_0 h(\gamma^{\frac{1}{p-1}})}\right\}.$$

Set

$$\psi_1 = (\gamma r)^{\frac{1}{p-1}}\frac{p-1}{p}\phi_{1,p}^{\frac{p}{p-1}} \quad \text{and} \quad \psi_2 = (\gamma r)^{\frac{1}{q-1}}\frac{q-1}{q}\phi_{1,q}^{\frac{q}{q-1}}.$$

We shall verify that (ψ_1, ψ_2) is a sub-solution of problem (1.1) for $\lambda_i, \mu_i \in [\alpha, \beta]$, $i = 1, 2$ Indeed, let $\omega \in W$ with $\omega \geq 0$ in Ω. Then a simple calculation shows that

$$\begin{aligned}
&\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\omega\,dx \\
&= \gamma r\int_\Omega |x|^{-ap}\phi_{1,p}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla\omega\,dx \\
&= \gamma r\left\{\int_\Omega |x|^{-ap}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot[\nabla(\phi_{1,p}\omega) - \nabla\phi_{1,p}\omega]dx\right\} \\
&= \gamma r\int_\Omega\left[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right]\omega\,dx. \qquad (3.1)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\omega\,dx \\
&= \gamma r\int_\Omega\left[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}^q - |x|^{-bq}|\nabla\phi_{1,q}|^q\right]\omega\,dx. \qquad (3.2)
\end{aligned}$$

By relation (2.2) we have $\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p \le -m_0$ on $\overline{\Omega}_\delta$. Using hypothesis (2.4) in combination with the fact that f is nondecreasing we have for all $x \in \overline{\Omega}_\delta$

$$A(x)f(\psi_2) \ge -a_0 f(\psi_2) \ge f(\gamma^{\frac{1}{q-1}}). \tag{3.3}$$

In the last inequality we have also used

$$\gamma^{\frac{1}{q-1}} \ge (\gamma r)^{\frac{1}{q-1}} \frac{q-1}{q} \phi_{1,q}^{\frac{q}{q-1}} =: \psi_2.$$

A similar argument shows that

$$C(x)h(\psi_1) \ge -c_0 h(\gamma^{\frac{1}{p-1}}) \quad \text{in } \overline{\Omega}_\delta. \tag{3.4}$$

Combining relations (3.3), (3.4) and the definition of β we find

$$\begin{aligned}
\gamma\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right) &\le -\gamma m_0 \\
&\le -\lambda_1 a_0 f\left(\gamma^{\frac{1}{q-1}}\right) - \mu_1 c_0 h\left(\gamma^{\frac{1}{p-1}}\right) \\
&\le \lambda_1 A(x) f(\psi_2) + \mu_1 C(x) h(\psi_1). \quad (3.5)
\end{aligned}$$

Thus, for all $\lambda_1, \mu_1 \in [\alpha, \beta]$

$$\begin{aligned}
&\gamma r \int_{\overline{\Omega}_\delta} \left[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right]\omega\, dx \\
&\le \int_{\overline{\Omega}_\delta} |x|^{-(a+1)p+c_1}\Big[\lambda_1 A(x)f(\psi_2) + \mu_1 C(x)h(\psi_1)\Big]\omega dx. \quad (3.6)
\end{aligned}$$

We recall that $\phi_{1,p} \ge \sigma_p, \phi_{1,q} \ge \sigma_q$ on $\Omega\backslash\overline{\Omega}_\delta$, for some $\sigma_p, \sigma_q \in (0, 1)$. Thus, by the condition $(H1)$ and the definition of ψ_1, ψ_2, it follows that

$$\begin{aligned}
&\gamma\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right) \le \gamma\lambda_{1,p} \\
&\le \lambda_1 a_1 f\left(\gamma^{\frac{1}{q-1}}\frac{q-1}{q}\alpha_q^{\frac{q}{q-1}}\right) + \mu_1 c_1 h\left(\gamma^{\frac{1}{p-1}}\frac{p-1}{p}\alpha_p^{\frac{p}{p-1}}\right) \\
&\le \lambda_1 A(x)f(\psi_2) + \mu_1 C(x)h(\psi_1), \quad (3.7)
\end{aligned}$$

for all $\lambda_1, \mu_1 \in [\alpha, \beta]$.

Relations (3.5) and (3.7) imply that

$$\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\omega\, dx \le \int_\Omega |x|^{-(a+1)p+c_1}[\lambda_1 A(x)f(\psi_2) + \mu_1 C(x)h(\psi_1)]\omega\, dx \tag{3.8}$$

A similar argument shows that

$$\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\omega dx \le \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2 B(x)g(\psi_1) + \mu_2 D(x)\tau(\psi_2)]\omega dx \tag{3.9}$$

From (3.8) and (3.9), we deduce that (ψ_1, ψ_2) is a sub-solution of problem (1.1). Moreover, we have $\psi_1 > 0$ and $\psi_2 > 0$ in Ω.

Next, we construct a super-solution of problem (1.1). For this purpose, we prove that there exists a large enough positive constant C so that

$$(z_1, z_2) := (C\zeta_p, \left[(\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g(C\|\zeta_p\|_\infty)\right]^{\frac{1}{q-1}}\zeta_q(x))$$

is a super-solution of problem (1.1), where (ζ_p, ζ_q) is a solution of problem (2.3). We first observe that

$$\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla\omega\, dx = C^{p-1}\int_\Omega |x|^{-ap}|\nabla \zeta_p|^{p-2}\nabla \zeta_p \cdot \nabla\omega\, dx = C^{p-1}\int_\Omega |x|^{-(a+1)p+c_1}\omega\, dx \tag{3.10}$$

Using the conditions $(H2)--(H3)$, we can choose the number $C > 0$ large enough so that

$$\begin{aligned} C^{p-1} &\geq \lambda_1\|a\|_\infty f\left(\left[(\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g(C\|\zeta_p\|_\infty)\right]^{\frac{1}{q-1}}\zeta_q\right) + \mu_1\|c\|_\infty h(C\|\zeta_p\|_\infty)\\ &\geq \lambda_1\|a\|_\infty f\left(\left[(\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g(C\|\zeta_p\|_\infty)\right]^{\frac{1}{q-1}}\zeta_q\right) + \mu_1\|c\|_\infty h(C\zeta_p)\\ &\geq \lambda_1 A(x)f(z_2) + \mu_1 C(x)h(z_1). \end{aligned} \tag{3.11}$$

Therefore

$$\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla\omega\, dx \geq \int_\Omega |x|^{-(a+1)p+c_1}(\lambda_1 A(x)f(z_2) + \mu_1 C(x)h(z_1))\omega\, dx. \tag{3.12}$$

Next, from the definition of z_2, the condition $(H3)$ for $C > 0$ large enough and the fact that g is increasing

$$g\Big(C\|\zeta_p\|_\infty\Big) \geq \tau\left(\left[(\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g(C\|\zeta_p\|_\infty)\right]^{\frac{1}{q-1}}\ \|\ \zeta_q\|_\infty\right).$$

It follows that

$$\begin{aligned} &\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla\omega\, dx\\ &\geq\ (\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g(C\|\zeta_p\|_\infty)\int_\Omega |x|^{-(b+1)q+c_2}\omega dx\\ &\geq \int_\Omega |x|^{-(b+1)q+c_2}(\lambda_2\|b\|_\infty + \mu_2\|d\|_\infty)g\Big(C\|\zeta_p\|_\infty\Big)\omega dx\\ &= \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2\|b\|_\infty g\Big(C\|\zeta_p\|_\infty\Big) + \mu_2\|d\|_\infty g\Big(C\|\zeta_p\|_\infty\Big)]\omega dx\\ &\geq \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2\|b\|_\infty g(z_1) + \mu_2\|d\|_\infty \tau(z_2)]\omega dx\\ &\geq \int_\Omega |x|^{-(b+1)q+c_2}[\lambda_2\|b\|_\infty g(z_1) + \mu_2\|d\|_\infty \tau(z_2)]\omega\, dx\\ &\geq \int |x|^{-(b+1)q+c_2}[\lambda_2 B(x)g(z_1) + \mu_2 D(x)\tau(z_2)]\omega\, dx. \end{aligned} \tag{3.13}$$

Relations (3.12) and (3.13) yield that (z_1, z_2) is a super-solution of problem (1.1) with $\psi_1 \leq z_1$ and $\psi_2 \leq z_2$ for $C > 0$ large.

Thus, by Lemma 3.1 there exists a positive solution (u, v) of the system (1.1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 3.2. □

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