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A CHARACTERIZATION FOR ELLIPTIC PROBLEMS ON FRACTAL SETS

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Abstract. In this paper we prove a characterization theorem on the existence of one non-zero strong solution for elliptic equations defined on the Sierpiński gasket. More generally, the validity of our result can be checked studying elliptic equations defined on self-similar fractal domains whose spectral dimension $\nu \in (0, 2)$. Our theorem can be viewed as an elliptic version on fractal domains of a recent contribution obtained in a recent work of Ricceri for a two-point boundary value problem.

1. Introduction

In recent years a great deal of effort has gone into investigating PDEs on fractals (see, for instance, [6, 10, 12, 21] and the excellent monograph [7]). A major difficulty is how to define differential operators on non-smooth sets. Analysis on fractal sets has been made possible by the definition of operators that play the role of the Laplacian.

Originally defined as a by-product of the construction of the analog of Brownian motion [1], these Laplace-type operators have been shown by direct limit-of-difference-quotient definitions in the papers by Kigami [13–16], for a class of self-similar fractals that includes the *Sierpiński gasket*. In this way, elliptic equations have been studied by using a suitable energy functional defined on an appropriate Hilbert space (see [2, 3, 8, 11]).

Motivated by this large interest in the current literature, the purpose of this paper is to prove a characterization result on the existence of non-negative and non-zero strong solutions for the following Dirichlet problem:

$$(S^f_{\lambda,\alpha}) \qquad \begin{cases} \Delta u(x) = \lambda \alpha(x) f(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

where V stands for the Sierpiński gasket in $(\mathbb{R}^{N-1}, |\cdot|)$, $N \geq 2$, V_0 is its intrinsic boundary (consisting of its N corners), Δ denotesthe weak Laplacian on V and λ

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is a positive real parameter. We assume that

(h_f) $f:[0,+\infty)\to[0,+\infty)$ is a continuous function with $f(0)=0$ and such that, for some $a>0$, the map $h_F:(0,+\infty)\to[0,+\infty)$ defined by

$$h_F(\xi):=\frac{F(\xi)}{\xi^2}$$

is non-increasing in the real interval $(0,a]$, where

$$F(\xi):=\int_0^\xi f(t)dt$$

for each $\xi\in[0,+\infty)$.

We assume that the variable potential $\alpha : V\to\mathbb{R}$ satisfies the following hypothesis:

(h_α) $\alpha\in C(V)$ with $\alpha(x)<0$, for every $x\in V$.

The main result of this paper is the following.

Theorem 1.1. *Assume that hypotheses* (h_f) *and* (h_α) *are fulfilled. Then, the following properties are equivalent*:

(i_1) *h_F is not constant in $(0,b]$ for each $b>0$;*

(i_2) *for each $r>0$ there exists an open interval $I\subseteq(0,+\infty)$ such that, for every $\lambda\in I$, problem $(S^f_{\lambda,\alpha})$ has a strong non-negative and non-zero solution, whose norm in $H^1_0(V)$ is less than r.*

The above characterization can be regarded as an elliptic version, for some classes of fractal sets, of a very recent result obtained by Ricceri for a two-point boundary value problem (see [19, Theorem 1]).

The extension of the cited result to $(S^f_{\lambda,\alpha})$ is not trivial and is required to overcome some difficulties which arise in this new geometrical context. In particular, some analytical properties on the Hilbert space $H^1_0(V)$ and the distribution of the spectrum of the corresponding linear problem defined on fractal sets, need special care. More precisely, in our setting, a key ingredient is the validity of the following Morrey-type inequality:

$$\sup_{x,y\in V_*}\frac{|u(x)-u(y)|}{|x-y|^\sigma}\leq(2N+3)\sqrt{W(u)}, \tag{1}$$

where

$$\sigma:=\frac{\log((N+2)/N)}{2\log 2},$$

and V_* is defined as in the next section (see [8, Lemma 2.4] for details). Inequality (1) and the Arzela-Ascoli theorem yield that the embedding

$$H^1_0(V)\hookrightarrow C_0(V) \tag{2}$$

is compact (see [9]). This fact will be crucial in our approach.

Taking into account the results contained in [6], our method adopted here can be useful for studying the existence of weak solutions for elliptic equations defined on self-similar sets, whose spectral dimension $\nu\in(0,2)$. In such a case, the Laplacian may be defined via a suitable Dirichlet form, following the variational fractal approach developed by Mosco in [17]. An open and more delicate problem is to attack the case $\nu\geq 2$ in which the compact embedding (2) is false.

We emphasize that, as suggested by Ricceri, a possible extension of his result to the elliptic case requires a more sophisticated and delicate analysis also for equations involving the classical Laplacian and defined on bounded Euclidean domains.

This paper is organized as follows. In Section 2 we recall the geometrical construction of the Sierpiński gasket and our variational framework. Successively, Section 3 is devoted to the proof of the main theorem.

We refer to the recent book by Ciarlet [4] as a general reference for the basic notions used in the present paper.

2. Preliminaries and abstract result

Let $N \geq 2$ be a natural number and let $p_1, \ldots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \ldots, N\}$, the map $S_i \colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}\, x + \frac{1}{2}\, p_i\,.$$

Let $\mathcal{S} := \{S_1, \ldots, S_N\}$ and denote by $L \colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$L(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor* of the family $\mathcal{S}$, such that $L(V) = V$; see, Theorem 9.1 in Falconer [7]. The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$ of *intrinsic boundary* $V_0 := \{p_1, \ldots, p_N\}$.

Consider H to be the convex hull of the set V_0 and observe that S satisfies the *open set condition* (see [7, p. 129]) taking $\operatorname{int}(H)$ the interior of H, which is a non-empty bounded open set such that

$$\bigcup_{i=1}^{N} S_i(\operatorname{int}(H)) \subset \operatorname{int}(H).$$

Since the above condition holds and V is the attractor of $\mathcal{S}$, applying [7, Theorem 9.3], we deduce that V has Hausdorff and box dimensions equal to the value of d satisfying

$$\sum_{i=1}^{N} \frac{1}{2^d} = 1, \tag{3}$$

and we also have $\mathcal{H}^d(V) \in (0, +\infty)$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. By relation (3), we immediately get that $d = \log N / \log 2$. Let μ be the normalized restriction of the Hausdorff measure $\mathcal{H}^d$ on $\mathbb{R}^{N-1}$ to the subsets of V, so $\mu(V) = 1$.

We also recall, for completeness, that if $0 \leq d < \infty$, $0 < \delta < \infty$ and $A \subset \mathbb{R}^k$, then

$$\mathcal{H}^d(A) = \lim_{\delta \to 0+} \mathcal{H}^d_\delta(A) = \sup_{\delta > 0} \mathcal{H}^d_\delta(A),$$

where

$$\mathcal{H}^d_\delta(A) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \inf\left\{ \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam} C_j}{2}\right)^d ;\ A \subset \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta \right\}$$

and

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

is Euler's Gamma function (see Evans and Gariepy [5, p. 60] for details).

Further, the following property of μ will be useful in the sequel:

$$\mu(B) > 0, \text{ for every non-empty open subset } B \text{ of } V. \tag{4}$$

In other words, the support of μ coincides with V; see, for instance, Breckner, Rădulescu and Varga [2] for more details.

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_\infty$. For a function $u\colon V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y| = 2^{-m}}} (u(x) - u(y))^2, \tag{5}$$

where $V_m := L(V_{m-1})$, for $m \geq 1$ and $V_* := \bigcup_{m \geq 0} V_m$.

We have $W_m(u) \leq W_{m+1}(u)$ for very natural m, so we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{6}$$

Now define

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y| = 2^{-m}}} (u(x) - u(y))(v(x) - v(y)).$$

Set

$$\mathcal{W}(u,v) = \lim_{m \to \infty} \mathcal{W}_m(u,v).$$

Then $\mathcal{W}(u,v) \in \mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, becomes a real Hilbert space.

We now state a useful property of the space $H_0^1(V)$ which shows, together with the facts that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space and $H_0^1(V)$ is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$.

Lemma 2.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \geq 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L\|u\|$.*

Following Falconer and Hu [8] we can define in a standard way a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, where Z is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), \|\cdot\|)$), such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \text{ for every } (u,v) \in Z \times H_0^1(V).$$

The operator Δ is called the (*weak*) *Laplacian* on V.

Precisely, let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h\in H_0^1(V)\\ \|h\|=1}} |\langle u,h\rangle|,$$

where

$$\langle v,h\rangle = \int_V v(x)h(x)d\mu, \qquad v\in L^2(V,\mu),\ h\in H_0^1(V).$$

Then $H^{-1}(V)$ is a Hilbert space. Then, the relation

$$-\mathcal{W}(u,v) = \langle \Delta u, v\rangle, \quad \forall v\in H_0^1(V),$$

uniquely defines a function $\Delta u\in H^{-1}(V)$ for every $u\in H_0^1(V)$.

Finally, fix $\lambda>0$. Let $\alpha\colon V\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be as in the Introduction. We say that a function $u\in H_0^1(V)$ is a *weak solution* of problem $(S^f_{\lambda,\alpha})$ if

$$\mathcal{W}(u,v) = -\lambda\int_V \alpha(x)f(u(x))v(x)d\mu,$$

for every $v\in H_0^1(V)$.

While we mainly work with the weak Laplacian, there is also a directly defined version. We say that Δ_s is the *standard Laplacian* of u if $\Delta_s u: V\to\mathbb{R}$ is continuous and

$$\lim_{m\to\infty}\sup_{x\in V\setminus V_0} |(N+2)^m(H_m u)(x) - \Delta_s u(x)| = 0,$$

where

$$(H_m u)(x) := \sum_{\substack{y\in V_m\\ |x-y|=2^{-m}}} (u(y)-u(x)),$$

for $x\in V_m$. We say that $u\in C_0(V)$ is a *strong solution* of problem $(S^f_{\lambda,\alpha})$ if $\Delta_s u$ exists and is continuous for all $x\in V\setminus V_0$, and

$$\Delta_s u(x) = \lambda\alpha(x)f(u(x)), \quad \forall\, x\in V\setminus V_0.$$

The existence of the standard Laplacian of a function $u\in H_0^1(V)$ implies the existence of the weak Laplacian Δ; see, for completeness, Falconer and Hu [8].

Remark 2.1. Since $f\colon\mathbb{R}\to\mathbb{R}$ is continuous and $\alpha\in C(V)$, then using Lemma 2.16 of Falconer and Hu [8], it follows that every weak solution of the problem $(S^f_{\lambda,\alpha})$ is also a strong solution.

Now, let $(X,\langle\cdot,\cdot\rangle)$ be a real Hilbert space and, for each $\gamma>0$, put

$$B_\gamma := \{u\in X : \|u\|^2\le\gamma\}.$$

Further, denote by $\operatorname{int}(B_\gamma)$ the interior of B_γ.

The proof of our main result is obtained by exploiting the following abstract theorem due to Ricceri [19] whose proof is entirely based on the results contained in [18].

Theorem 2.1. *Let $J: X\to\mathbb{R}$ be a sequentially weakly upper semicontinuous and Gâteaux differentiable functional, with $J(0_X)=0$. Assume that, for some $\gamma>0$, there exists a global maximum $\widehat{u}$ of $J|_{B_\gamma}$, such that*

$$\langle J'(\widehat{u}),\widehat{u}\rangle < 2J(\widehat{u}).$$

Then, there exists an open interval $I \subseteq (0,+\infty)$ *such that, for every* $\lambda \in I$, *the equation*

$$u = \lambda J'(u)$$

has a non-zero solution lying in $\operatorname{int}(B_\gamma)$.

3. Proof of the main theorem

Let us put $X := H_0^1(V)$ endowed by the inner product $\mathcal{W}$ and define

$$\widetilde{f}(t) := \begin{cases} f(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Since $f(0) = 0$ it follows that $\widetilde{f}$ is a continuous function.

We consider the truncated problem

$$(S^{\widetilde{f}}_{\lambda,\alpha}) \qquad \begin{cases} \Delta u(x) = \lambda\alpha(x)\widetilde{f}(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

and set

$$J(u) := -\int_V \alpha(x)\widetilde{F}(u(x))d\mu,$$

for every $u \in X$, where

$$\widetilde{F}(\xi) := \int_0^\xi \widetilde{f}(t)dt,$$

for every $\xi \in \mathbb{R}$.

By [3, Proposition 4.5] it follows that J is a Gâteaux differentiable and sequentially weakly continuous functional with $J(0_X) = 0$. Moreover, fixing $\lambda > 0$, the weak solutions of $(S^{\widetilde{f}}_{\lambda,\alpha})$ are exactly the solutions $u \in X$ of the following equation:

$$u = \lambda J'(u)\,;$$

see Proposition 2.19 in [8].

Further, Remark 2.1 ensures that every weak solution of problem $(S^{\widetilde{f}}_{\lambda,\alpha})$ is a strong one. Hence, exploiting the maximum principle proved by Strichartz in [20, Theorem 2.1], every solution $u \in X$ of $(S^{\widetilde{f}}_{\lambda,\alpha})$ is non-negative, so that u also solves the original problem $(S^{f}_{\lambda,\alpha})$.

$(\mathrm{i}_1) \Rightarrow (\mathrm{i}_2)$

By hypothesis (h_f), taking into account that h_F is non-increasing in $(0,a]$ and since

$$h_F'(\xi) = \frac{f(\xi)\xi - 2F(\xi)}{\xi^3}, \quad \forall\, \xi \in (0,a],$$

we obtain

$$f(\xi)\xi \leq 2F(\xi), \tag{7}$$

for every $\xi \in (0,a]$.

On the other hand, Fukushima and Shima proved in [9] that the embedding

$$(X, \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty)$$

is compact and

$$\|u\|_\infty \leq (2N+3)\|u\|, \tag{8}$$

for every $u \in X$.

Thus, let us fix

$$r \in \left(0, \frac{a^2}{(2N+3)^2}\right],$$

and denote $\gamma := r^2$.

By relations (7) and (8) it follows that

$$\widetilde{f}(u(x))u(x) \leq 2\widetilde{F}(u(x)), \tag{9}$$

for every $u \in B_\gamma$ and $x \in V$.

Let $u_0 \in X$ be a non-negative function in V with $\|u_0\| = y > 0$. Take $\varepsilon > y/\gamma$ and consider the function

$$v_\varepsilon(x) := \frac{u_0(x)}{\varepsilon}, \quad \forall\, x \in V.$$

Clearly $v_\varepsilon \in B_\gamma$ and

$$J(v_\varepsilon) = -\int_{V_a^\varepsilon} \alpha(x)F(v_\varepsilon(x))d\mu > 0, \tag{10}$$

where

$$V_a^\varepsilon := \{x \in V : 0 < v_\varepsilon(x) \leq a\},$$

with $\mu(V_a^\varepsilon) > 0$.

Now, let $\widehat{u} \in B_\gamma$ be a global maximum of J in B_γ. Then, condition (10) ensures that $J(\widehat{u}) > 0$ and consequently

$$\max_{x\in V} \widehat{u}(x) > 0.$$

At this point, note that

$$S_f := \{x \in V : f(\widehat{u}(x))\widehat{u}(x) < 2F(\widehat{u}(x))\} \neq \emptyset. \tag{11}$$

Indeed, arguing by contradiction, if $S_f = \emptyset$, bearing in mind relation (9), we would have

$$f(\widehat{u}(x))\widehat{u}(x) = 2F(\widehat{u}(x)),$$

for every $x \in V$.

Then, since $h'_F(\xi) = 0$ for every

$$\xi \in A := \left(0, \max_{x\in V} \widehat{u}(x)\right],$$

the function h_F would be constant in A against (i_1).

Finally, since α is negative in S_f, relations (4) and (11) yield

$$-\int_{S_f} \alpha(x)f(\widehat{u}(x))\widehat{u}(x)d\mu < -2\int_{S_f} \alpha(x)F(\widehat{u}(x))d\mu. \tag{12}$$

Moreover,

$$\int_{V\setminus S_f} \alpha(x)(f(\widehat{u}(x))\widehat{u}(x) - 2F(\widehat{u}(x)))d\mu = 0. \tag{13}$$

Thus, by (12) and (13), we write

$$-\int_V \alpha(x)f(\widehat{u}(x))\widehat{u}(x)d\mu < -2\int_V \alpha(x)F(\widehat{u}(x))d\mu.$$

Bearing in mind that

$$\langle J'(\widehat{u}), \widehat{u}\rangle = -\int_V \alpha(x) f(\widehat{u}(x))\widehat{u}(x)d\mu,$$

the above inequality can be rewritten as

$$\langle J'(\widehat{u}), \widehat{u}\rangle < 2J(\widehat{u}). \tag{14}$$

On the other hand, if

$$r \in \left(\frac{a}{(2N+3)}, +\infty\right),$$

by choosing $\gamma := a^2/(2N+3)^2$, and arguing as before, there exists a global maximum $\widehat{u}$ of J in B_γ such that condition (14) holds.

Hence, Theorem 2.1 ensures that there exists an open interval $I \subseteq (0, +\infty)$ such that, for every $\lambda \in I$, problem $(S^f_{\lambda,\alpha})$ has a strong non-negative and non-zero solution, whose norm in X is less than $\sqrt{\gamma}$. The conclusion follows.

$(\mathrm{i}_1) \Leftarrow (\mathrm{i}_2)$

Let us start recalling a preliminary fact on the spectrum of linear elliptic problems on the Sierpiński gasket. More precisely, let $a : V \to \mathbb{R}$ be such that

(a_1) $a(x) \geq 0$ in V and $0 < \int_V a(x)d\mu < +\infty,$

and consider the following elliptic eigenvalue problem:

$$(S_{\lambda,a}) \qquad \begin{cases} \Delta u(x) + \lambda a(x)u(x) = 0 & x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

By [11], under the structural hypothesis (a_1), it follows that problem $(S_{\lambda,a})$ possesses a sequence $\{\lambda_n\}$ of eigenvalues fulfilling

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \quad \lambda_n \to \infty \text{ as } n \to \infty; \tag{15}$$

see also [8, pp. 563–564].

From now on, we argue by contradiction. Hence, assume that there are two positive real constants b and c such that

$$\frac{F(\xi)}{\xi^2} = c, \quad \forall\, \xi \in (0, b].$$

Fixing

$$r \in \left(0, \frac{b}{(2N+3)}\right],$$

by (i_2), there exists an open interval I such that, for every $\lambda \in I$, problem $(S^f_{\lambda,\alpha})$ admits a (strong) non-negative solution $u \in C_0(V) \setminus \{0_X\}$ such that

$$\|u\| < r\,. \tag{16}$$

In view of (8) and (16), we also obtain

$$\|u\|_\infty < b,$$

and it follows that

$$f(u(x)) = 2cu(x), \quad \forall\, x \in V.$$

Then, for every $\lambda \in I$, the linear problem

$$(S_{\lambda,c\alpha}) \qquad \begin{cases} \Delta u(x) - 2\lambda c\alpha(x)u(x) = 0 & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

admits a strong non-zero solution. This fact contradicts (15) and the proof is complete. □

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