Volume 3, Number Proof, 2018, 1–

MULTIVALUED PERIODIC SYSTEMS WITH MAXIMAL MONOTONE TERMS

NIKOLAOS S. PAPAGEORGIOU AND VICENŢIU D. RĂDULESCU

Abstract. We consider a first order nonlinear periodic system with a maximal monotone term whose domain is not all of $\mathbb{R}^N$ and a multivalued perturbation $F(t,x)$. First we prove an existence theorem for the convex problem (that is, F is convex-valued). Then by strengthening the continuity on $F(t,\cdot)$ we show the existence of extremal periodic solutions (that is, solutions passing from the extreme points of $F(t,x)$) and we prove a strong relaxation theorem (approximating the solutions of the convex problem, by certain extremal trajectories).

1. Introduction

In this paper we study the existence of solutions for the following multivalued first order periodic system

$$\left\{ \begin{array}{l} -u'(t) \in A(u(t)) + \operatorname{ext} F(t,u(t)) \quad \text{for a.a. } t \in T = [0,b], \\ u(0) = u(b). \end{array} \right\} \tag{1.1}$$

In this problem, $A : D \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map with $0 \in A(0)$ while $F : T \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a multifunction with compact convex values and ext $F(t,x)$ denotes the extreme points of the set $F(t,x)$. We stress that in our formulation the set $D = \{x \in \mathbb{R}^N : A(x) \neq \emptyset\}$ (the domain of A) need not be all of $\mathbb{R}^N$. So, we incorporate in our framework differential variational inequalities (see Aubin & Cellina [1] and Hu & Papageorgiou [12]). Also, we mention that although the multifunction $F(\cdot,\cdot)$ is compact and convex valued, the set $\operatorname{ext}F(t,x)$ (for $(t,x) \in T \times \mathbb{R}^N$) is not convex and need not be closed. Moreover, even if $F(t,\cdot)$ exhibits nice continuity properties, these are not transferred to the multifunction $x \mapsto \operatorname{ext} F(t,x)$. So, the solvability of problem (1.1) is a highly nontrivial problem.

Multivalued periodic systems were studied primarily when $A \equiv 0$ and F is convex valued. We refer to the books of Aubin & Cellina [1] an Hu & Papageorgiou [12] and the references therein. Periodic differential inclusions with $A \equiv 0$ and a nonconvex valued oriented field F, were studied by Hu & Papageorgiou [13] and Hu, Kandilakis & Papageorgiou [14]. Later, Bader & Papageorgiou [3] examined periodic problems for evolution inclusion in a Hilbert space H, driven by a subdifferential operator (that is, $A = \partial\varphi$ with $\varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ a proper, convex, lower semicontinuous function and $\partial\varphi(\cdot)$ being the subdifferential of φ in the sense of convex analysis). They proved existence theorems for both the convex and nonconvex problems. More recently, in a nice paper Frigon [8] studied differential inclusions in $\mathbb{R}^N$ with $A \neq 0$ and proved existence theorems for convex and

---

2010 *Mathematics Subject Classification.* 34A34, 34B15, 47N20.

Key words and phrases. Extreme points, extremal trajectories, maximal monotone map, fixed point, strong relaxation, Hartman condition.

nonconvex problems under general conditions. For related recent results concerning multi-valued problems we refer to Papageorgiou, Rădulescu and Repovš [15, 16]. However, none of the aforementioned works deals with the problem of existence of extremal trajectories and how these trajectories are related to the conclusions of the convexified problem (strong relaxation).

In this work, we establish the existence of extremal solutions and then show that every solution of the convexified problem (that is, $\operatorname{ext} F(t,x)$ is replaced by $F(t,x) = \overline{\operatorname{conv}} \operatorname{ext} F(t,x)$) can be approximated by certain extremal trajectories (strong relaxation). Our approach uses tools from multivalued analysis (see Hu & Papageorgiou [11]) and the theory of nonlinear operators of monotone type (see Barbu [5]).

2. Mathematical background

Suppose that (Ω, Σ) is a measurable space and X is a separable Banach space. We introduce the following families of subsets of X:

$$P_{f(c)}(X) = \{E \subseteq X : E \text{ is nonempty, closed (and convex)}\},$$
$$P_{(w)k(c)}(X) = \{E \subseteq X : E \text{ is nomempty, (weakly-) compact (and convex)}\}.$$

Consider a multifunction $F : \Omega \to 2^X \backslash \{\emptyset\}$. The "graph of F" is the set

$$\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}.$$

We say that $F(\cdot)$ is "graph measurable", if

$$\operatorname{Gr} F \in \Sigma \times B(X)$$

with $B(X)$ being the Borel σ-field of X. If $\mu(\cdot)$ is a σ-finite measure defined on Σ and $F : \Omega \to 2^X \backslash \{\emptyset\}$ is a graph measurable multifunction, then the Yankov-von Neumann-Aumann selection theorem implies that there exists a Σ-measurable map $f : \Omega \to X$ such that

$$f(\omega) \in F(\omega) \text{ for } \mu-\text{a.a. } \omega \in \Omega.$$

Such a map is called a "measurable selection" of F. In fact, we can find a whole sequence of measurable selections $f_n : \Omega \to X$ $n \in \mathbb{N}$ such that

$$F(\omega) \subseteq \overline{\{f_n(\omega)\}}_{n \geq 1} \text{ for } \mu-\text{a.a. } \omega \in \Omega.$$

Given a multifunction $F : \Omega \to P_f(X)$, we say that it is "measurable", if for every $u \in X$ the $\mathbb{R}_+$-valued function

$$\omega \mapsto d(u, F(\omega)) = \inf[||u - x|| : x \in F(\omega)]$$

is Σ-measurable. A multifunction $F : \Omega \to P_f(X)$ which is measurable, it is also graph measurable. The converse is true if there is a complete σ-finite measure $\mu(\cdot)$ defined on Σ.

Now suppose that (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $F : \Omega \to 2^X \backslash \{\emptyset\}$ is a graph measurable multifunction. For $1 \leq p \leq \infty$, we introduce the set

$$S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \mu-\text{a.e.}\}.$$

An easy application of the Yankov-von Neumann-Aumann selection theorem reveals that

$$S_F^p \neq \emptyset \text{ if and only if } \omega \mapsto \inf[||x|| : x \in F(\omega)] \text{ belongs in } L^p(\Omega).$$

The set S_F^p is "decomposable" in the sense that

$$(E, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p \Rightarrow \chi_E f_1 + \chi_{\Omega \setminus E} f_2 \in S_F^p,$$

with χ_E being the characteristic function of $E \in \Sigma$.

For every $C \subseteq X$ and $x^* \in X^*$, we define

$$|C| = \sup[||x|| : x \in C],$$
$$\sigma(x^*, C) = \sup[\langle x^*, x\rangle : x \in C]$$

with $\langle \cdot, \cdot \rangle$ being the duality brackets for the pair (X^*, X). The function $x^* \mapsto \sigma(X^*, C)$ is known as the "support function" of the set C.

Suppose V, Y are Hausdorff topological spaces and $G : V \to 2^Y \setminus \{\emptyset\}$. We say that $G(\cdot)$ is "upper semicontinuous" (usc for short), if for all $U \subseteq Y$ open, the set

$$G^+(U) = \{v \in V : G(v) \subseteq U\} \text{ is open.}$$

We say that $G(\cdot)$ is "lower semicontinuous" (lsc for short), if for all $U \subseteq Y$ open, the set

$$G^-(U) = \{v \in V : G(v) \cap U \neq 0\} \text{ is open.}$$

For any Banach space Z, on $P_f(Z)$ we can define a generalized metric, be setting

$$h(C, M) = \sup[|d(z, C) - d(z, M)| : z \in Z]$$
$$= \max\left[\max_{c \in C} d(c, M), \sup_{m \in M} d(m, C)\right] \text{ for all } C, M \in P_f(Z).$$

This is known as the *Hausdorff metric* on $P_f(Z)$.

We know that $(P_f(Z), h)$ is a complete metric space. If Y is a Hausdorff topological space, then a multifunction $G : Y \to P_f(Z)$ is said to be "h-continuous", if it is continuous from Y into the metric space $(P_f(Z), h)$.

Let X be a reflexive Banach space and X^* its topological dual. As before, by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). A multivalued map $A : D \subseteq X \to 2^{X^*}$ is said to be "monotone", if

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } (x, x^*), (y, y^*) \in \operatorname{Gr} A.$$

We say that A is "strictly monotone", if

$$\langle x^* - y^*, x - y\rangle = 0 \Rightarrow x = y.$$

The map $A(\cdot)$ is "maximal monotone", if

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } (x, x^*) \in \operatorname{Gr} A \Rightarrow (y, y^*) \in \operatorname{Gr} A.$$

This means that $\operatorname{Gr} A$ is maximal with respect to inclusion among the graphs of all monotone maps. If $A(\cdot)$ is maximal monotone, then $\operatorname{Gr} A$ is sequentially closed in $X_w \times X^*$ and in $X \times X_w^*$ (here by X_w (respectively X_w^*) we denote the Banach space X (respectively X^*) endowed with the weak topology). If A is maximal monotone, then for all $x \in D$, $A(x) \in P_{f_c}(X^*)$.

Finally by $L^1_w(T,\mathbb{R}^N)$ we denote the Lebesgue space $L^1(T,\mathbb{R}^N)$ equipped with the weak norm $||\cdot||_w$ defined by

$$||u||_w = \sup\left[||\int_s^t u(\tau)d\tau|| : 0\le s\le t\le b\right],\ u\in L^1(T,\mathbb{R}^N).$$

Equivalently, we can define the weak norm as

$$||u||_w = \sup\left[||\int_0^t u(\tau)d\tau : 0\le t\le b\right].$$

3. The convex problem

In this section we consider the convexification of problem (1.1), namely the following periodic system

$$\left\{\begin{array}{ll} -u'(t)\in A(u(t))+F(t,u(t)) & \text{for a.a. } t\in T, \\ u(0)=u(b). & \end{array}\right\} \tag{3.1}$$

We prove an existence theorem for this system, complementing Theorem 3.6 of Frigon [8].

The hypotheses on the data of (3.1) are the following.

$H(A):\ A: D\subseteq\mathbb{R}^N\to 2^{\mathbb{R}^N}$ is a maximal monotone map with $0\in A(0)$.

$H(F):\ F: T\times\mathbb{R}^N\to P_{k_c}(\mathbb{R}^N)$ is a multifunction such that

(i) for every $x\in\mathbb{R}^N,\ t\mapsto F(t,x)$ is graph measurable;
(ii) for a.a. $t\in T,\ \operatorname{Gr} F(t,\cdot)$ is closed in $\mathbb{R}^N\times\mathbb{R}^N$;
(iii) for every $r>0$, there exists $a_r\in L^1(T)$ such that

$$|F(t,x)|\le a_r(t) \text{ for a.a. } t\in T,\ \text{all } |x|\le r;$$

(iv) there exists $M>0$ such that for a.a. $t\in T$, all $x\in\mathbb{R}^N$ with $|x|=M$ and all $v\in F(t,x)$ we have

$$(v,x)_{\mathbb{R}^N}\ge 0.$$

Remark 3.1. Hypothesis $H(F)(iv)$ is a multivalued version of the so-called "Hartman condition". It was first used by Hartman [10] in the context of single-valued Dirichlet systems. Hypothesis $H(F)(iv)$ is more restrictive than condition $(ST-2)$ of Frigon [8] (the L^2-tube condition). On the other hand hypothesis $H(F)(iii)$ is more general than hypothesis $(F2-2)$ of Frigon [8]. In addition we do not need condition (A3) of Frigon [8].

Let $\epsilon>0$ and $g\in L^1(T,\mathbb{R}^N)$ be given and consider the following periodic system

$$\left\{\begin{array}{ll} -u'(t)\in (A+\epsilon I)(u(t))+g(t) & \text{for a.a. } t\in T, \\ u(0)=u(b). & \end{array}\right\} \tag{3.2}$$

Let $\hat{A}:\hat{D}\subseteq\mathbb{R}^N\to 2^{\mathbb{R}^N}$ be a map such that for some $\vartheta>0$, $x\mapsto(\hat{A}-\vartheta I)(x)$ is maximal monotone. Given $(u_0,g)\in\overline{\hat{D}}\times L^1(T,\mathbb{R}^N)$, we consider the Cauchy problem

$$-u'(t)\in\hat{A}(u(t))+g(t) \text{ for a.a. } t\in T,\ u(0)=u_0.$$

From Barbu [6, p. 128] we know that this Cauchy problem has a unique solution $u(u_0, g) \in W^{1,1}((0,b), \mathbb{R}^N)$. Moreover, if $(u_0, g), (\hat{u}_0, \hat{g}) \in \hat{\overline{D}} \times L^1(T, \mathbb{R}^N)$ and $u = u(u_0, g)$, $\hat{u} = u(\hat{u}_0, \hat{g})$ are the corresponding solutions, then

$$|u(t) - \hat{u}(t)| \le e^{-\vartheta t}|u_0 - \hat{u}_0| + \int_0^t e^{-\vartheta(t-s)}|g(s) - \hat{g}(s)|ds \text{ for all } t \in T.$$

Therefore, if $g = \hat{g}$, then from the above inequality we see that the Poincaré map

$$u_0 \mapsto u(u_0, g)(b)$$

is a contraction and so by the Banach fixed point theorem we infer that it has a unique fixed point.

Using these observations on problem (3.2), we can say that it has a unique solution $\xi_\epsilon(g) \in W^{1,1}((0,b), \mathbb{R}^N) \subseteq C(T, \mathbb{R}^N)$. The next proposition examines the solution map $\xi_\epsilon : L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$.

Proposition 3.2. *If hypotheses $H(A), H(F)$ hold, then the solution map $\xi_\epsilon : L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ is completely continuous (that is, if $g_n \xrightarrow{w} g$ in $L^1(T, \mathbb{R}^N)$, then $\xi_\epsilon(g_n) \to \xi_\epsilon(g)$ in $C(T, \mathbb{R}^N)$).*

Proof. Suppose that $g_n \xrightarrow{w} g$ in $L^1(T, \mathbb{R}^N)$ and let $u_n = \xi_\epsilon(g_n)$ and $u = \xi_\epsilon(g)$. Exploiting the monotonicity of A and the fact that $0 \in A(0)$, we have

$$|u_n(t)|^2 \le |u_n(s)|^2 + \int_{s\wedge t}^{s\vee t} 2|g_n(\tau)||u_n(\tau)|d\tau \text{ for all } 0 \le s, t \le b$$

with $s \vee t = \max\{s, t\}$, $s \wedge t = \min\{s, t\}$. From Brezis [7, p. 157] it follows that

$$|u_n(t)| \le |u_n(s)| + \int_{s\wedge t}^{s\vee t} 2|g_n(\tau)|d\tau \text{ for all } 0 \le s, t \le b, \text{ all } n \in \mathbb{N}. \tag{3.3}$$

Let $m_n = \min_{t\in T} |u_n(t)|$ and $\hat{m}_n = ||u_n||_\infty$ for all $n \in \mathbb{N}$. From (3.3) we have

$$\hat{m}_n \le M_n + c_1 \text{ with } c_1 = 2 \sup_{n\in\mathbb{N}} ||g_n||_1 < \infty\,. \tag{3.4}$$

On the other hand, from (3.2) we have

$$(u_n'(t), u_n(t))_{\mathbb{R}^N} + \epsilon|u_n(t)|^2 \le -(g_n(t), u_n(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T$$

(see hypothesis H(A)),

$$\Rightarrow \quad |u_n(b)|^2 + 2\epsilon \int_0^b |u_n(\tau)|^2 d\tau \le |u_n(0)|^2 - 2\int_0^b (g_n(\tau), u_n(\tau))_{\mathbb{R}^N} d\tau,$$

$$\Rightarrow \quad \epsilon \int_0^b |u_n(\tau)|^2 d\tau \le \int_0^b |g_n(\tau)||u_n(\tau)|d\tau \text{ for all } n \in \mathbb{N}$$

(recall that $u_n(0) = u_n(b)$ for all $n \in \mathbb{N}$);

$$\Rightarrow \quad \epsilon m_n^2 b \le \hat{m}_n \frac{c_1}{2},$$

$$\Rightarrow \quad \epsilon m_n^2 b \le \frac{c_1}{2}(m_n + c_1) \text{ for all } n \in \mathbb{N} \text{ (see (3.4))},$$

$$\Rightarrow \quad \{m_n\}_{n\ge1} \subseteq \mathbb{R}_+ \text{ is bounded},$$

$$\Rightarrow \quad \{u_n\}_{n\ge1} \subseteq C(T, \mathbb{R}^N) \text{ is bounded (see (3.4))}.$$

Since we are in a finite dimensional Banach space, the nonlinear contraction semigroup generated by A is compact (see Barbu [5]). So, from Baras [4] (see also Vrabie [17]), it follows that

$$\begin{aligned}
&\{u_n(b)\}_{n\geq 1} \subseteq \mathbb{R}^N \text{ is relatively compact,}\\
\Rightarrow\ &\{u_n(0)\}_{n\geq 1} \subseteq \mathbb{R}^N \text{ is relatively compact,}\\
\Rightarrow\ &\{u_n\}_{n\geq 1} \subseteq C(T,\mathbb{R}^N) \text{ is relatively compact (see Vrabie [17]).}
\end{aligned}$$

So, we may assume that

$$u_n \to \hat{u} \text{ in } C(T,\mathbb{R}^N). \tag{3.5}$$

Hence $\hat{u}(0)=\hat{u}(b)$. Also, on account of the monotonicity of A, we have

$$\begin{aligned}
&(u_n'(t)-u'(t),u_n(t)-u(t))_{\mathbb{R}^N}+\epsilon|u_n(t)-u(t)|^2\leq\\
&-(g_n(t)-g(t),u_n(t)-u(t))_{\mathbb{R}^N}\\
&\text{for a.a. } t\in T,\\
\Rightarrow\ &\frac{1}{2}|u_n(b)-u(b)|^2+\epsilon||u_n-u||_2^2\leq\frac{1}{2}|u_n(0)-u(0)|^2\\
&-\int_0^b(g_n(t)-g(t),u_n(t)-u(t))_{\mathbb{R}^N}dt,\\
\Rightarrow\ &|\hat{u}(b)-u(b)|^2+2\epsilon||\hat{u}-u||_2^2\leq|\hat{u}(0)-u(0)|^2,\\
\Rightarrow\ &||\hat{u}-u||_2^2\leq 0 \text{ (exploiting the periodic boundary condition),}\\
\Rightarrow\ &\hat{u}=u.
\end{aligned}$$

Therefore the Urysohn criterion for the convergence of sequences, implies that for the original sequence we have

$$\begin{aligned}
&u_n\to u \text{ in } C(T,\mathbb{R}^N) \text{ (see (3.5)),}\\
\Rightarrow\ &\xi_\epsilon(g_n)\to\xi_\epsilon(g) \text{ in } C(T,\mathbb{R}^N),\\
\Rightarrow\ &\xi_\epsilon(\cdot) \text{ is completely continuous.}
\end{aligned}$$

$\square$

Let $p_M:\mathbb{R}^N\to\mathbb{R}^N$ be the M-radial retraction defined by

$$p_M(x)=\begin{cases} x & \text{if } |x|\leq M\\ M\frac{x}{|x|} & \text{if } M<|x|\end{cases} \quad \text{for all } x\in\mathbb{R}^N.$$

We introduce the following modification of the multivalued perturbation $F(t,x)$:

$$\hat{F}(t,x)=\begin{cases} F(t,x) & \text{if } |x|\leq M\\ F(t,p_M(x)) & \text{if } M<|x|.\end{cases} \tag{3.6}$$

Using $\hat{F}(t,x)$, we consider the following periodic system:

$$\left\{\begin{array}{l} -u'(t)\in(A+\epsilon I)(u(t))+\hat{F}(t,u(t)) \text{ for a.a. } t\in T,\\ u(0)=u(b).\end{array}\right\} \tag{3.7}$$

Proposition 3.3. *Assume that hypotheses $H(A),H(F)(iv)$ hold and $u\in W^{1,1}((0,b),\mathbb{R}^N)$ is a solution of (3.7). Then $|u(t)|\leq M$ for all $t\in T$.*

Proof. We argue by contradiction. So, suppose that the conclusion of the proposition is not true. Then one of the following situations can occur:

(a) There exist $0 \leq t_1 < t_2 \leq b$ such that

$$|u(t_1)| = M < |u(t)| \text{ for all } t \in (t_1, t_2]$$

or

(b) $|u(t)| > M$ for all $t \in T$.

First suppose that (a) holds. Then

$$
\begin{aligned}
& (u'(t), u(t))_{\mathbb{R}^N} + ((A + \epsilon I)(u(t)), u(t))_{\mathbb{R}^N} = -(\hat{f}(t), u(t))_{\mathbb{R}^N} \\
& \text{for a.a. } t \in T \text{ with } \hat{f} \in S^1_{\hat{F}(\cdot, u(\cdot))}, \\
\Rightarrow \quad & \frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq -(\hat{f}(t), u(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T \text{ (see hypothesis H(A))} \\
\Rightarrow \quad & |u(t)|^2 - |u(t_1)|^2 \leq -2\int_{t_1}^{t} (\hat{f}(\tau), u(\tau))_{\mathbb{R}^N} d\tau \text{ for all } t \in (t_1, t_2]\,. \qquad (3.8)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& (\hat{f}(\tau), u(\tau))_{\mathbb{R}^N} = (\hat{f}(\tau), p_M(u(\tau)))_{\mathbb{R}^N} \frac{|u(\tau)|}{M} \text{ for a.a. } \tau \in (t_1, t_2] \\
\Rightarrow \quad & (\hat{f}(\tau), u(\tau))_{\mathbb{R}^N} \geq 0 \text{ for a.a. } \tau \in (t_1, t_2] \text{ (see (3.6) and hypothesis H(F)(iv))}, \\
\Rightarrow \quad & 0 < |u(t)|^2 - |u(t_1)|^2 \leq 0 \text{ for all } t \in (t_1, t_2] \text{ (see (3.8))},
\end{aligned}
$$

a contradiction. Hence (a) cannot happen.

Next suppose that (b) holds. We have

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \epsilon|u(t)|^2 \leq -(\hat{f}(t), u(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T \\
& \text{(see hypothesis H(A))}, \\
\Rightarrow \quad & \epsilon||u||_2^2 \leq -\int_0^b (\hat{f}(t), p_M(u(t)))_{\mathbb{R}^N} \frac{|u(t)|}{M} dt \leq 0 \\
& \text{(see (3.6) and hypothesis } H(F)(iv)) \\
\Rightarrow \quad & u = 0,
\end{aligned}
$$

again a contradiction. □

Now we produce a solution for problem (3.7).

Proposition 3.4. *If hypotheses $H(A), H(F)$ hold, then problem (3.7) has a solution $u_0 \in W^{1,1}((0,b), \mathbb{R}^N)$.*

Proof. Consider the multifunction $\hat{N} : C(T, \mathbb{R}^N) \to 2^{L^1(T, \mathbb{R}^N)}$ defined by

$$\hat{N}(u) = S^1_{\hat{F}(\cdot, u(\cdot))} \text{ for all } u \in C(T, \mathbb{R}^N).$$

Hypotheses $H(F)(i), (ii), (iii)$ imply that

$$\hat{N}(u) \in P_{wkc}(L^1(T, \mathbb{R}^N)) \text{ for all } u \in C(T, \mathbb{R}^N)$$

(see Hu & Papageorgiou [11, p. 21] and use the Dunford-Pettis theorem).

Consider the set

$$S_\epsilon = \{u \in C(T, \mathbb{R}^N) : u \in \lambda(\xi_\epsilon \circ \hat{N})(u), \lambda \in (0,1)\}.$$

For $u \in S_\epsilon$, we have

$$-\frac{1}{\lambda}u'(t) \in A\left(\frac{1}{\lambda}u(t)\right) + \hat{f}(t) \text{ for a.a. } t \in T, u(0) = u(b)$$

with $\hat{f} \in \hat{N}(u)$. As in the proof of Proposition, via hypothesis $H(F)(iv)$, we have

$$|u(t)| \le M \text{ for all } t \in T.$$

This means that S_ϵ is bounded. So, we can apply the multivalued Leray-Schauder alternative principle due to Bader [2, Theorem 8] (see also Gasinski & Papageorgiou [9, p. 890]) and produce $\hat{u}_\epsilon \in C(T, \mathbb{R}^N)$ such that

$$\begin{aligned} & \hat{u}_\epsilon \in (\xi_\epsilon \circ \hat{N})(\hat{u}_\epsilon), \\ \Rightarrow \quad & \hat{u}_\epsilon \in W^{1,1}((0,b), \mathbb{R}^N) \text{ is a solution of problem (3.7).} \end{aligned}$$

□

Now we consider the following periodic system

$$\left\{ \begin{array}{l} -u'(t) \in (A + \epsilon I)(u(t)) + F(t, u(t)) \text{ for a.a. } t \in T, \\ u(0) = u(b). \end{array} \right\} \tag{3.9}$$

Combining Propositions 3.3 and 3.4 and using (3.6), we can state the following existence theorem for problem (3.9).

Proposition 3.5. *If hypotheses $H(A), H(F)$ hold and $\epsilon > 0$, then problem (3.9) has a solution $u_\epsilon \in W^{1,1}((0,b), \mathbb{R}^N)$.*

Next we let $\epsilon \downarrow 0$ to produce a solution of the "convex" problem (3.1).

Theorem 3.6. *If hypotheses $H(A), H(F)$ hold, then problem (3.1) admits a solution $u_0 \in W^{1,1}((0,b), \mathbb{R}^N)$.*

Proof. Let $\epsilon_n \downarrow 0$ and let $u_n = u_{\epsilon_n}$ (for $n \in \mathbb{N}$) be the solutions of the approximate problem (3.9), produced in Proposition 3.5. From Proposition 3.3 we know that

$$|u_n(t)| \le M \text{ for all } t \in T, \text{ all } n \in \mathbb{N}. \tag{3.10}$$

As in the proof of Proposition 3.2, using the result of Baras [4], we have that

$$\{u_n\}_{n \ge 1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.}$$

So, by passing to a subsequence if necessary, we may assume that

$$u_n \to u_0 \text{ in } C(T, \mathbb{R}^N). \tag{3.11}$$

From Barbu [5, p. 124] we have

$$\frac{1}{2}|u_n(t) - x|^2 \le \frac{1}{2}|u_n(s) - x|^2 + \int_s^t (f_n(\tau) - y, u_n(\tau) - x)_{\mathbb{R}^N} d\tau \tag{3.12}$$

for all $(x, y) \in \mathrm{Gr}\,(A + \epsilon_n I)$, all $0 \le s \le t \le b$ and with $f \in S^1_{F(\cdot, u_n(\cdot))}$, $n \in \mathbb{N}$.

Hypothesis $H(F)(iii)$ and (3.10) imply that

$$|f_n(t)| \le a_M(t) \text{ for a.a. } t \in T, \text{ all } n \in \mathbb{N}.$$

So, by the Dunford-Pettis theorem, we may assume that

$$f_n \xrightarrow{w} f_0 \text{ in } L^1(T, \mathbb{R}^N) \text{ as } n \to \infty\,. \tag{3.13}$$

If in (3.12) we pass to the limit as $n \to \infty$ and use (3.11) and (3.13), then

$$\frac{1}{2}|u_0(t) - x|^2 \leq \frac{1}{2}|u_0(s) - x|^2 + \int_s^t (f_0(\tau) - y, u_0(\tau) - y)_{\mathbb{R}^N} d\tau \tag{3.14}$$
$$\text{for all } (x, y) \in \operatorname{Gr} A, \text{ all } 0 \leq s \leq t \leq b.$$

From (3.13) and Proposition 3.9 of Hu & Papageorgiou [11, p. 694], we have

$$\begin{aligned}
& f_0(t) \in \overline{\operatorname{conv}} \limsup_{n\to\infty} \{f_n(t)\}_{n\geq 1} \\
& \subseteq \overline{\operatorname{conv}} \limsup_{n\to\infty} F(t, u_n(t)) \\
& \subseteq \overline{\operatorname{conv}} F(t, u_0(t)) \text{ (see (3.11) and hypothesis H(f)(ii))} \\
& = F(t, u_0(t)) \text{ for a.a. } t \in T, \\
\Rightarrow\ & f_0 \in S^1_{F(\cdot, u_0(\cdot))}\,.
\end{aligned}$$

Then from (3.14) and Propositions 3.6 and 3.8, pp. 70 and 82 of Brezis [7] (see also Theorem 2.1 of Barbu [5, p. 124]), we infer that $u_0 \in W^{1,1}((0, b), \mathbb{R}^N)$ is a solution of problem (3.1). □

4. Extremal solutions

In this section, we turn our attention to problem (1.1). To produce a solution for this problem, we need to strengthen the conditions on the maximal monotone term $A(\cdot)$ and the multivalued perturbation $F(t, x)$.

The new conditions are the following:

$H(A)'$: $A : D \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$ and

$$c_0|x|^2 \leq (h, x)_{\mathbb{R}^N} \text{ for all } (x, h) \in \operatorname{Gr} A, \text{ some } c_0 > 0.$$

$H(F)'$: $F : T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t, x)$ is graph measurable;
(ii) for a.a. $t \in T$, $x \mapsto F(t, x)$ is h-continuous;
(iii) for every $r > 0$, there exists $a_r \in L^1(T)_+$ such that

$$|F(t, x)| \leq a_r(t) \text{ for a.a. } t \in T, \text{ all } |x| \leq r;$$

(iv) there exists $M > 0$ such that for a.a. $t \in T$, all $x \in \mathbb{R}^N$ with $|x| = M$ and all $v \in F(t, x)$ we have

$$(v, x)_{\mathbb{R}^N} \geq 0.$$

Remark 4.1. Hypotheses $H(F)'(i)$, (ii) imply that $(t, u) \mapsto F(t, u)$ is graph measurable and in particular for every $u : T \to \mathbb{R}^N$ measurable, the multifunction $t \mapsto F(t, u(t))$ is graph measurable (hence measurable, see Hu & Papageorgiou [11]).

Theorem 4.2. *If hypotheses $H(A)', H(F)'$ hold, then problem (1.1) admits a solution $\hat{u} \in W^{1,1}((0, b), \mathbb{R}^N)$.*

Proof. Hypothesis $H(A)'$ implies that Proposition 3.3 is true for the solution $u(\cdot)$ of (1.1). So, we may replace $F(t,x)$ by $\hat{F}(t,x)$ (see (3.6)). We have

$$|\hat{F}(t,x)| \le a_M(t) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N. \tag{4.1}$$

Let $E = \{g \in L^1(T,\mathbb{R}^N) : |g(t)| \le a_M(t) \text{ for a.a. } t \in T\}$. For every $g \in E$, let $\xi_0(g) \in W^{1,1}((0,b),\mathbb{R}^N)$ be the unique solution of

$$-u'(t) \in A(u(t)) + g(t) \text{ for a.a. } t \in T, \ u(0) = u(b).$$

On account of hypothesis $H(A)'$ and Proposition 3.2 we have that the solution map $\xi_0 : L^1(T,\mathbb{R}^N) \to C(T,\mathbb{R}^N)$ is completely continuous, in particular then compact. So, $\xi_0(E)$ is relatively compact in $C(T,\mathbb{R}^N)$. We set

$$K = \overline{\text{conv}}\, \xi_0(E) \in P_{kc}(C(T,\mathbb{R}^N)).$$

Invoking Theorem 8.31 of Hu & Papageorgiou [11, p. 260], we obtain a continuous map $\gamma : K \to L^1_w(T,\mathbb{R}^N)$ such that

$$\gamma(u) \in \text{ext}\, S^1_{\hat{F}(\cdot,u(\cdot))} = S^1_{\text{ext}\,\hat{F}(\cdot,u(\cdot))} \text{ for all } u \in K. \tag{4.2}$$

Consider the map $\eta : K \to C(T,\mathbb{R}^N)$ defined by

$$\eta(u) = (\xi_0 \circ \gamma)(u) \text{ for all } u \in K.$$

Using Lemma 2.8 of Hu & Papageorgiou [11, p. 24] together with the complete continuity of ξ_0, we infer that η is continuous. Also, by virtue of (4.1), η maps K into itself. Therefore we can apply the Schauder fixed point theorem and find $u_0 \in K$ such that

$$u_0 = \eta(u_0),$$

$$\Rightarrow \ -u_0'(t) \in A(u_0(t)) + \text{ext}\,\hat{F}(t,u_0(t)) \text{ for a.a. } t \in T, u_0(0) = u_0(b) \text{ (see (4.2))}. \tag{4.3}$$

From Proposition 3.3, we have

$$\begin{aligned} & |u_0(t)| \le M \text{ for all } t \in T, \\ \Rightarrow \ & F(t,u_0(t)) = \hat{F}(t,u_0(t)) \text{ for a.a. } t \in T \text{ (see (3.6))}, \\ \Rightarrow \ & u_0 \in W^{1,1}((0,b),\mathbb{R}^N) \text{ is a solution of (1.1) (see (4.3), (3.6))}. \end{aligned}$$

□

5. Strong relaxation

Let S_c be the solution set of the convexified problem (3.1). In this section we prove a strong relaxation result approximating the elements of S_c with certain extremal trajectories. To do this we need to strengthen further the conditions on the multivalued perturbation $F(t,x)$.

The new stronger conditions on $F(t,x)$, are the following.

$H(F)''$: $F : T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is graph measurable;
(ii) $h(F(t,x),F(t,y)) \le k(t)\ |x-y|$ for a.a. $t \in T$, all $x,y \in \mathbb{R}^N$, with $k \in L^1(T)_+$;

(iii) there exists $a \in L^2(T)_+$ such that

$$|F(t,x)| \leq a(t)(1+|x|) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N;$$

(iv) there exists $M > 0$ such that for a.a. $t \in T$, all $x \in \mathbb{R}^N$ with $|x| = M$ and all $v \in F(t,x)$, we have

$$(v,x)_{\mathbb{R}^N} \geq 0.$$

Remark 5.1. Now the continuity and growth conditions on $F(t,\cdot)$ are more restrictive.

Given $x_0 \in \bar{D}$, let $S_e(x_0)$ be the solution set of the Cauchy problem

$$-u'(t) \in A(u(t)) + \operatorname{ext} F(t,u(t)) \text{ for a.a. } t \in T, \ u(0) = x_0.$$

With similar argument as in Section 4, we show that $S_e(x_0) \neq \emptyset$. In fact for this result we do not need hypothesis $H(F)(iv)$ (the Hartman condition). Indeed in this case, thanks to hypothesis $H(F)''(iii)$, we can produce an *a priori* bound for the elements of $S_e(x_0)$. To see this, let $u \in S_e(x_0) \subseteq W^{1,1}((0,b),\mathbb{R}^N)$. We have

$$\begin{aligned}
&(u'(t),u(t))_{\mathbb{R}^N} + (f(t),u(t))_{\mathbb{R}^N} \leq 0 \text{ for a.a. } t \in T, \ \text{with } f \in S^1_{\operatorname{ext} F(\cdot,u(\cdot))} \\
&\text{(see hypothesis } H(A)'\text{)} \\
\Rightarrow \ & \frac{1}{2}|u(t)|^2 \leq \frac{1}{2}|x_0|^2 + \int_0^1 |f(\tau)||u(\tau)|d\tau, \\
\Rightarrow \ & |u(t)|^2 \leq c_1 + \int_0^t 2a(t)|u(\tau)|^2 d\tau \text{ for all } t \in T, \ \text{some } c_1 > 0, \\
\Rightarrow \ & |u(t)| \leq M_0 \text{ for all } t \in T, \ \text{some } M_0 > 0 \text{ (by Growall's inequality).}
\end{aligned}$$

So the arguments in Section 4 apply with $M_0 > 0$ instead of M. In fact we have that $\overline{S_e(x_0)}^{C(T,\mathbb{R}^N)}$ is compact. Moreover, as before we may always assume that

$$|F(t,x)| \leq a_0(t) \text{ for a.a. } t \in T, \ \text{all } x \in \mathbb{R}^N \text{ with } a_0 \in L^2(T)_+,$$

since we can always replace $F(t,x)$ by

$$\hat{F}(t,x) = \begin{cases} F(t,x) & \text{if } |x| \leq M_0 \\ F(t,p_{M_0}(x)) & \text{if } M_0 < |x|. \end{cases}$$

Hypothesis $H(F)''(iv)$ is needed to guarantee that $S_c \neq \emptyset$ (see Theorem 3.6).

Given $u \in S_c$, we will approximate it in the $C(T,\mathbb{R}^N)$ norm by a sequence in $S_e(u(0))$.

Theorem 5.2. *If hypotheses $H(A)', H(F)''$ hold and $u \in S_c$, then we can find a sequence $\{u_n\}_{n\geq 1} \subseteq S_e(u(0))$ such that*

$$u_n \to u \text{ in } C(T,\mathbb{R}^N).$$

Proof. Since $u \in S_c$, we have

$$-u'(t) \in A(u(t)) + f(t) \text{ for a.a. } t \in T, \ u(0) = u(b) \tag{5.1}$$

with $f \in S^1_{F(\cdot,u(\cdot))}$. Let $K \subseteq C(T,\mathbb{R}^N)$ be the compact, convex set from the proof of Theorem 4.2 (that is, $K = \overline{\mathrm{conv}}\,\xi_0(E) \in P_{kc}(C(T,\mathbb{R}^N))$). Given $\epsilon > 0$ and $v \in K$, consider the multifunction $L^v_\epsilon : T \to 2^{\mathbb{R}^N}\setminus\{\emptyset\}$ defined by

$$L^v_\epsilon(t) = \{h \in \mathbb{R}^N : |f(t) - h| < \frac{\epsilon}{2m_0 b} + d(f(t), F(t, v(t))),\ h \in F(t, v(t))\}$$

with $m_0 = \sup[||y||_{C(T,\mathbb{R}^N)} : y \in K] < \infty$. It is clear that

$$\mathrm{Gr}\, L^v_\epsilon \in \mathcal{L}_T \times B(\mathbb{R}^N)$$

with $\mathcal{L}_T$ being the Lebesgue σ-field of T. So, we can use the Yankov-von Neumann-Aumann selection theorem and produce a measurable map $l : T \to \mathbb{R}^N$ such that $l(t) \in L^v_\epsilon(t)$ for a.a. $t \in T$.

Now we consider the multifunction $\Gamma_\epsilon : K \to 2^{L^2(T,\mathbb{R}^N)}$ defined by

$$\Gamma_\epsilon(v) = \{h \in S^2_{F(\cdot,v(\cdot))} : |f(t) - h(t)| < \frac{1}{2m_0 b} + d(f(t), F(t, v(t))) \text{ for a.a. } t \in T\}.$$

We have just seen that Γ_ϵ has nonempty values (in fact hypothesis $H(F)''(iii)$ implies that $\Gamma_\epsilon(v) \in P_{wkc}(L^2(T,\mathbb{R}^N))$ for all $v \in K$). In addition, $\Gamma_\epsilon(\cdot)$ has decomposable values and it is lsc (see Hu & Papageorgiou [11, Lemma 8.3, p. 239]). Therefore $u \mapsto \overline{\Gamma_\epsilon(u)}$ is lsc and we can apply Theorem 8.7 of Hu & Papageorgiou [11, p. 245] and produce a continuous map $\gamma_\epsilon : K \to L^2(T,\mathbb{R}^N)$ such that

$$\gamma_\epsilon(v) \in \overline{\Gamma_\epsilon(v)} \text{ for all } v \in K.$$

This means that

$$\begin{aligned} |f(t) - \gamma_\epsilon(v)(t)| &\le \frac{\epsilon}{2m_0 b} + d(f(t), F(t, v(t))) \\ &\le \frac{\epsilon}{2m_0 b} + h(F(t, u(t)), F(t, v(t))) \\ &\le \frac{\epsilon}{2m_0 b} + k(t)|u(t) - v(t)| \text{ for a.a. } t \in T \quad (5.2) \\ &\quad \text{(see hypothesis } H(F)''(ii)). \end{aligned}$$

Also Theorem 8.31 of Hu & Papageorgiou [11, p. 260], gives a continuous map $r_\epsilon : K \to L^1_w(T,\mathbb{R}^N)$ such that

(5.3) $r_\epsilon(v) \in \mathrm{ext}\, S^2_{F(\cdot,v(\cdot))} = S^2_{\mathrm{ext}\, F(\cdot,v(\cdot))}$ and $||\gamma_\epsilon(v) - r_\epsilon(v)||_w \le \epsilon$ for all $v \in K$.

Now let $\epsilon_n \downarrow 0$ and set $\gamma_n = \gamma_{\epsilon_n}, r_n = r_{\epsilon_n},\ x_0 = u(0) = u(b)$. We consider the following Cauchy problem

$$-u_n'(t) \in A(u_n(t)) + r_n(u_n)(t) \text{ for a.a. } t \in T,\ u_n(0) = x_0. \tag{5.4}$$

We know that problem (5.4) has a solution $u_n \in S_e(x_0)$ and $\overline{S_e(x_0)}^{C(T,\mathbb{R}^N)}$ is compact. So, we may assume that

$$u_n \to \tilde{u} \text{ in } C(T,\mathbb{R}^N). \tag{5.5}$$

From (5.1), (5.4) and hypothesis $H(A)'$ we have

$$(u_n'(t) - u'(t), u_n(t) - u(t))_{\mathbb{R}^N} + (f(t) - r_n(u_n), u_n(t) - u(t))_{\mathbb{R}^N} \le 0 \text{ for a.a. } t \in T. \tag{5.6}$$

Recall that

$$(u_n'(t) - u'(t), u_n(t) - u(t))_{\mathbb{R}^N} = \frac{1}{2}\frac{d}{dt}|u_n(t) - u(t)|^2.$$

Using this equality in (5.6) and integrating, we obtain

$$\begin{aligned} \frac{1}{2}|u_n(t) - u(t)|^2 \quad &\le \quad \int_0^t |f(\tau) - \gamma_n(u_n)(\tau)||u_n(\tau) - u(\tau)|d\tau \\ &\quad + \int_0^t (\gamma_n(u_n)(\tau) - r_n(u_n)(\tau), u_n(\tau) - u(\tau))_{\mathbb{R}^N} d\tau \\ &\quad \text{(recall that } u_n(0) = u(0) = x_0). \end{aligned} \tag{5.7}$$

From (5.3), we have

$$\begin{aligned} & \|\gamma_n(u_n) - r_n(u_n)\|_w \le \epsilon_n, \\ \Rightarrow \quad & \gamma_n(u_n) - r_n(u_n) \xrightarrow{\|\cdot\|_w} 0, \\ \Rightarrow \quad & \gamma_n(u_n) - r_n(u_n) \xrightarrow{w} 0 \text{ in } L^2(T, \mathbb{R}^N) \\ & \text{(see Hu \& Papageorgiou [12, Lemma 2.8, p. 24])} \\ \Rightarrow \quad & \int_0^t (\gamma_n(u_n) - r_n(u_n), u_n - u)_{\mathbb{R}^N} dt \to 0. \end{aligned} \tag{5.8}$$

In addition, we have

$$\begin{aligned} & \int_0^t |f(\tau) - \gamma_n(u_n)(\tau)||u_n(\tau) - u(\tau)|d\tau \\ & \le \epsilon_n + \int_0^t k(\tau)|u_n(\tau) - u(\tau)|^2 d\tau \text{ for all } n \in \mathbb{N} \text{ (see (5.2))}. \end{aligned}$$

Using (5.5) we deduce that

$$\limsup_{n\to\infty} \int_0^t |f(\tau) - \gamma_n(u_n)(\tau)||u_n(\tau) - u(\tau)|d\tau \le \int_0^t k(\tau)|\tilde{u}(\tau) - u(\tau)|^2 d\tau\,. \tag{5.9}$$

We return to (5.7), pass to the limit as $n \to \infty$ and use (5.5), (5.8), (5.9). Then

$$\begin{aligned} & |\tilde{u}(t) - u(t)|^2 \le 2\int_0^t k(\tau)|\tilde{u}(\tau) - u(\tau)|^2 d\tau \text{ for all } t \in T, \\ \Rightarrow \quad & \tilde{u} = u \text{ (by Gronwall's inequality)}. \end{aligned}$$

Then from (5.5) and Urysohn's criterion, for the original sequence we have that

$$u_n \to u \text{ in } C(T, \mathbb{R}^N) \text{ with } u_n \in S_e(u(0)) \text{ for all } n \in \mathbb{N}.$$

□

References

[1] J.-P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[2] R. Bader, *A topological fixed-point index theory for evolution inclusions*, Z. Anal. Anwend. **20** (2001), 3–15.

[3] R. Bader, N.S. Papageorgiou, *On a problem of periodic evolution inclusions of the subdifferential type*, Z. Anal. Anwend. **21** (2002), 963–984.

[4] P. Baras, *Compacité de l'opérateur $f \to u$ solution d'une équation d'évolution non homogène*, C.R. Acad. Sci. Paris, Sér. I **284** (1977), 799–802.
[5] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.
[6] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer, New York, 2010.
[7] H. Brezis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans un Espace de Hilbert*, North Holland, Amsterdam, 1973.
[8] M. Frigon, *Systems of first order differential inclusions with maximal monotone terms*, Nonlinear Anal. **66** (2007), 2064-2077.
[9] L. Gasinski and N.S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[10] P. Hartman, *On boundary value problems for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. **96** (1960), 493–509.
[11] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
[12] S. H and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Volume II: Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
[13] S. Hu and N. S. Papageorgiou, *On the existence of periodic solutions for nonconvex valued differential inclusions in $\mathbb{R}^N$*, Proc. Amer. Math. Soc. **123** (1995), 3043–3050.
[14] S. Hu, D. Kandilakis and N. S. Papageorgiou, *Periodic solutions for nonconvex differential inclusions*, Proc. Amer. Math. Soc. **127** (1999), 89–94.
[15] N.S. Papageorgiou, V. Rădulescu and D. Repovš, *Nonlinear elliptic inclusions with unilateral constraint and dependence on the gradient*, Applied Mathematics and Optimization, in press (doi:10.1007/s00245-016-9392-y).
[16] N.S. Papageorgiou, V. Rădulescu and D. Repovš, *Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions*, Advances in Nonlinear Analysis, in press (doi:10.1515/anona-2016-0096).
[17] I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. **32**, Longman Scientific and Technical, Harlow, UK, 1987.

N.S. Papageorgiou
National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr

V.D. Rădulescu
Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia & Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania
E-mail address: vicentiu.radulescu@imar.ro