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Rodrigues-type formulae for Hermite and Laguerre polynomials

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Abstract

In this paper we give new proofs of some elementary properties of the Hermite and Laguerre orthogonal polynomials. We establish Rodrigues-type formulae and other properties of these special functions, using suitable operators defined on the Lie algebra of endomorphisms to the vector space of infinitely many differentiable functions.

1 Introduction and preliminary results

Special orthogonal polynomials began appearing in mathematics before the significance of such a concept became clear. For instance, Laplace used Hermite polynomials in his studies in probability while Legendre and Laplace utilized Legendre polynomials in celestial mechanics. We devote this paper to the study of some elementary properties of Hermite and Laguerre polynomials because these are the most extensively studied and have the longest history. We also point out that the properties we establish in the present paper can be extended to other special functions. We refer to [1, 2, 3, 4] for related properties.

The classical orthogonal polynomials of Hermite and Laguerre satisfy linear differential equations of the form

$$a(x)y'' + b(x)y' + c(x)y = 0\,. \tag{1}$$

In both cases, the factor $c(x)$ is actually independent of x (resp, $c(x) = 2n$ for Hermite's polynomials, and $c(x) = n$ for Laguerre's polynomials) and depends

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only on the integer parameter n, which turns out to be also the exact degree of the polynomial solution of (1).

In the study of special functions, solutions to differential equations in the form of Rodrigues' formulae are of considerable interest. More precisely, for each of the classical families of orthogonal polynomials (Hermite, Laguerre, and Jacobi) there is a generalized Rodrigues formula through which the nth member of the family is given (except for a normalization factor) by the relation

$$P_n(x) = \frac{1}{w}\,\left(w(x)f^n(x)\right)^{(n)}\,.$$

A particular family of polynomials is characterized by the choice of functions w and f, For the Hermite polynomials we have $w(x) = e^{-x^2}$ and $f(x) = 1$, for the Laguerre polynomials $w(x) = e^{-x}$ and $f(x) = x$; and, for the Jacobi polynomials, $w(x) = (1-x)^\alpha(1+x)^\beta$ and $f(x) = 1 - x^2$. The method we develop in this paper relies on the study of suitable linear operators defined on the Lie algebra of endomorphisms of a vector space. We start with a simple spectral property of these operators.

Let $\operatorname{End} V$ be the Lie algebra of endomorphisms of the vector space V, endowed with the Lie bracket $[\,,\,]$ defined by

$$[A,B] = AB - BA, \quad \text{for every } A, B \in \operatorname{End} V.$$

We denote by I the identity operator of V.

Theorem 1 *Let $A, B \in \operatorname{End} V$ be such that $[A,B] = I$. We define the sequence $(y_n)_n \subset V$ as follows: $Ay_0 = 0$ and $y_n = By_{n-1}$, for every $n \geq 1$. Then y_n is an eigenvector of eigenvalue n for BA, for every $n \geq 1$.*

Proof. We first show that

$$Ay_n = ny_{n-1}, \quad \text{for every } n \geq 1.$$

For $n = 1$ this equality is evident, because $(AB - BA)y_0 = y_0$, $Ay_0 = 0$ and $y_1 = By_0$.

We suppose that $Ay_n = ny_{n-1}$. We may write, equivalently:

$$[A,B]y_n = y_n$$

$$ABy_n - BAy_n = y_n$$

$$Ay_{n+1} - nBy_{n-1} = y_n$$

$$Ay_{n+1} = (n+1)y_n$$

It follows that $BAy_n = nBy_{n-1}$, that is, $BAy_n = ny_n$, which completes our proof. □

2 Hermite polynomials

Throughout this paper we assume that $V = C^\infty(\mathbb{R})$.

We define the operators $A, B \in \operatorname{End} V$ by

$$(Af)x = (1/2)f'(x),\ (Bf)x = -f'(x) + 2xf(x), \quad \text{for every } x \in \mathbb{R}.$$

We prove that these operators satisfy the commutation relation $[A, B] = I$.

Indeed,

$$A(Bf)x - B((Af)x) = -(1/2)f''(x) + xf'(x) + f(x) + (1/2)f''(x) - xf'(x) = f(x).$$

Next, we prove that $(B^n f)x = (-1)^n e^{x^2}(f(x)e^{-x^2})^{(n)}$. From the definition of B, the above equality holds for $n = 1$. Inductively, taking into account $B^{n+1}f = B(B^n f)$, it follows that

$$\begin{aligned}(B^{n+1}f)x &= -(-1)^n(e^{x^2}(f(x)e^{-x^2})^{(n)})' + 2x(-1)^n e^{x^2}(f(x)e^{-x^2})^{(n)} \\ &= (-1)^{n+1}e^{x^2}(f(x)e^{-x^2})^{(n+1)},\end{aligned}$$

which ends our proof.

The Hermite equation $y'' - 2xy' + 2ny = 0$, where n is a positive integer, may be written

$$-y'' + 2xy' = 2ny,$$

or

$$By' = 2ny, \quad \text{that is, } BAy = ny.$$

By Theorem 1, it follows that y_n is a solution of the Hermite equation. Setting $y_0 = 1$, we obtain $y_n = B^n(1)$. Therefore, defining $H^n(x) = y_n$, we deduce the Rodrigues-type formula

$$H_n(x) = (-1)^n e^{x^2}(e^{-x^2})^{(n)},$$

for every n positive integer.

The Hermite polynomials H_n are orthogonal with respect to the weight function $w(x) = e^{-x^2}$, in the sense that

$$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \sqrt{\pi}\, 2^n n! & \text{if } m = n. \end{cases}$$

A straightforward computation shows that

$$H_n(x) = \sum_{j=0}^{[n/2]} (-1)^j (2x)^{n-2j} \frac{n!}{j!(n-2j)!}\,.$$

We also point out that it follows that the *functions of the parabolic cylinder* are closely related (see [5]) with the Hermite polynomials by the relation $u_n = e^{-x^2} H_n$.

We deduce in what follows additional properties of the Hermite polynomials.

Proposition 1 *The Hermite functions satisfy the following recurrence relations:*
i)$H'_n(x) = 2nH_{n-1}(x), \quad n \in \mathbb{N}$.
ii)$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \quad n \in \mathbb{N}$.

Proof. i) Since $H_n(x) = y_n = B^n(1)$, we can use the equality $Ay_n = ny_{n-1}$, which was proved in Theorem 1. From the definition of the operator A, it follows that

$$(1/2y'_n) = ny_{n-1}, \text{ that is, } H'_n(x) = 2nH_{n-1}(x).$$

ii) By the definition of B, one has

$$By_n + y'_n - 2xy_n = 0.$$

But $y'_n = 2ny_{n-1}$. Thus,

$$By_n - 2xy_n + 2ny_{n-1} = 0,$$

or

$$y_{n+1} - 2xy_n + 2ny_{n-1} = 0,$$

that is,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0.$$

□

Theorem 2 *Let $A, B \in \operatorname{End} V$ be such that $[A, B] = I$. Define $S = (A-I)BA$, $T_n = (A-I)^n B^n$, for every $n \geq 1$. Then:*
i) $[(A-I)^n, B] = n(A-I)^{n-1}$.
ii) $T_{n+1} = (T_1 + nI)T_n$.
iii) $S(T_1 + nI) = (T_1 + nI)S + (S + nI) - (T_1 + nI)$.
iv) If $y_0 \in V$ is an eigenvector of S with eigenvalue 0, then y_n is an eigenvector of eigenvalue $-n$ for S, where $y_n = T_n y_0$, for every $n \geq 1$.
v) If $y_n \in V$ is an eigenvector of eigenvalue $-n$ for S, then $w_n = (T_1 + nI)y_n$ is an eigenvector for S, with eigenvalue $-(n+1)$.

Proof. i) For $n = 1$, the equality follows from the commutation relation $[A, B] = I$. Inductively, let's suppose that $[(A - I)^n B] = n(A - I)^{n-1}$. It follows that

$$\begin{aligned} B(A-I)^{n+1} &= A(-I)^n B(A-I) - n(A-I)^n \\ &= (A-I)^n(BA - B - nI) = (A-I)^n(AB - I - B - nI) \\ &= (A-I)^{n+1}B - (n+1)(A-I)^n, \end{aligned}$$

that is,

$$[(A-I)^{n+1}B_0 = (n+1)(A-I)^n.$$

ii) From the equality proved at i), it follows that

$$(A-I)((A-I)^n B - B(A-I)^n)B^n = n(A-I)^n B^n$$
$$(A-I)^{n+1}B^{n+1} = (A-I)B(A-I)^n B^n + n(A-I)^n B^n$$
$$T_{n+1} = (T_1 + nI)T_n.$$

iii) The equality we have to prove is equivalent to $[S, T_1] = S - T_1$. But

$$\begin{aligned} [S, T_1] &= (A-I)BA(A-I)B - (A-I)B(A-I)BA \\ &= (A-I)B(A^2B - AB - ABA + BA) \\ &= (A-I)B(A[A,B] - [A,B]) \\ &= (A-I)B(A-I) \\ &= (A-I)BA - (A-I)B \\ &= S - T_1. \end{aligned}$$

iv) We'll prove our assertion by recurrence. For $n = 0$, it is obvious. We suppose that $Sy_n = -ny_n$, that is, $ST_ny_0 = -nT_ny_0$. From ii) and iii), it follows that

$$\begin{aligned} ST_{n+1}y_0 &= (T_1 + nI)ST_ny_0 - (S + nI)T_ny_0 - (T_1 + nI)T_ny_0 \\ &= (T_1 + nI)(n - T_ny_0) - T_{n+1}y_0 \\ &= -nT_{n+1}y_0 - T_{n+1}y_0 \\ &= -(n+1)T_{n+1}y_0. \end{aligned}$$

v) From iii), it follows that

$$Sw_n = (T_1 + nI)Sy_n + (S + nI)y_n - w_n.$$

But $(S + nI)y_n = 0$. Therefore,

$$Sw_n = -n(T_1 + nI)y_n - w_n = -(n+1)w_n.$$

□

3 Laguerre polynomials

Define the operators $A, B \in \operatorname{End} V$ by

$$(Af)x = f'(x)$$

and

$$(Bf)x = x \cdot f(x), \text{ for every } x \in \mathbb{R} \text{ and } f \in C^\infty(\mathbb{R}),$$

A and B satisfy $[A, B] = I$. Indeed,

$$([A, B]f)x = A((Bf)x) - B((Af)x) = (x \cdot f(x))' - x'f'(x) = f(x).$$

We consider now the Laguerre equation

$$xy'' + (1 - x)y' + ny = 0, \quad n \in \mathbb{N}.$$

But $(A - I)BAy = xy'' + (1 - x)y'$. Thus, the Laguerre equation becomes

$$Sy = -ny.$$

By Theorem 2, we conclude that if $y_0 \in C^\infty(\mathbb{R})$ and $Sy_0 = 0$, then $y_n = T_n y_0$ is a solution of the Laguerre equation. We choose $y_0 = 1$. It is clear that $Sy_0 = 0$.

We prove in what follows that

$$((A - I)^n f)x = e^x (f(x)e^{-x})^{(n)}, \text{ for every } n \in \mathbb{N}. \tag{2}$$

For $n = 1$, this equality becomes $f'(x) - f(x) = e^x (f(x)e^{-x})'$, which is trivial.

Inductively, we suppose that relation (2) is true. Therefore, $((A-I)^{n+1} f)x = ((A - I)(A - I)^n f)x = e^x (((A - I)^n f)xe^{-x})' = e^x (e^x (f(x)e^{-x})^{(n)} e^{-x})' = e^x (f(x)e^{-x})^{(n+1)}$ which proves the validity of relation (2).

Since $B^n(1) = x^n$, it follows that

$$y_n = T_n y_0 = (A - I)^n B^n(1) = e^x (x^n e^{-x})^{(n)}.$$

Thus the Laguerre polynomial

$$L_n(x) = e^x (x^n e^{-x})^{(n)}$$

is a solution of the Laguerre equation.

The Laguerre polynomials L_n are orthogonal with respect to the weight function $w(x) = e^{-x}$, in the sense that

$$\int_0^{+\infty} e^{-x} L_m(x) L_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

Proposition 2 *The Laguerre functions satisfy the following recurrence relations:*
i)$L'_{n+1}(x) - (n+1)L'_n(x) + (n+1)L_n(x) = 0$
ii)$L_{n+1}(x) + (x - 2n - 1)L_n(x) + n^2 L_{n-1}(x) = 0,$ *for every* $n \in \mathbb{N}$.

Proof. i) Using the definition of A and $L_n(x) = T_n(1)$, our relation becomes

$$AT_{n+1}(1) - (n+1)AT_n(1) + (n+1)T_n(1) = 0.$$

From ii) of the Theorem 2, the above equality becomes

$$AT_1T_n(1) - AT_n(1) + (n+1)T_n(1) = 0,$$

or

$$(AT_1 - A + I)y_n = -ny_n,$$

because $T_n(1) = y_n$, by Theorem 2 iv).

Thus, by the same theorem, it suffices to prove that

$$AT_1 - A + I = S.$$

Indeed,

$$\begin{aligned} AT_1 - A + I - S &= A(A-I)B - A + I - (A-I)BA \\ &= A^2B - AB - A + I - (A-I)BA \\ &= A^2B - A - ABA \\ &= A[A,B] - A \\ &= 0. \end{aligned}$$

ii) By the definition of B, we have to prove that

$$T_{n+1}(1) + BT_n(1) - (2n+1)T_n(1) + n^2T_{n+1}(1) = 0$$

or equivalently,

$$(T_1 + nI)T_n(1) + BT_n(1) - (2n+1)T_n(1) + n^2T_{n-1}(1) = 0.$$

$$(T_1 + B - (n+1)I)T_n(1) + n^2T_{n-1}(1) = 0. \tag{3}$$

But $T_1 + B - (n+1)I = BA - nI$. Relation (3) becomes

$$(BA - nI)(T_1 + (n-1)I)y_{n-1} = -n^2y_{n-1}$$

$$(BAT_1 - nT_1 + (n-1)BA + nI)y_{n-1} = 0$$

$$B(AT_1 - A + I)y_{n-1} = -(n-1)By_{n-1}$$
$$BSy_{n-1} = -(n-1)By_{n-1},$$

which is true because y_{n-1} is an eigenvector for S of eigenvalue $-(n-1)$. $\square$

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