PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 11, November 2016, Pages 4913–4928
http://dx.doi.org/10.1090/proc/13107
Article electronically published on April 20, 2016

POSITIVE SOLUTIONS OF NONLINEAR ROBIN EIGENVALUE PROBLEMS

NIKOLAOS S. PAPAGEORGIOU AND VICENŢIU D. RĂDULESCU

(Communicated by Catherine Sulem)

Abstract. We consider a nonlinear eigenvalue problem driven by the p-Laplacian with Robin boundary condition. Using variational methods and truncation techniques, we prove a bifurcation-type result describing the set of positive solutions as the positive parameter λ varies. We also produce extremal positive solutions and study their properties.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the nonlinear parametric Robin problem

$$(P_\lambda) \qquad \left\{ \begin{array}{ll} -\Delta_p u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z) u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0, \lambda > 0, \ 1 < p < \infty. & \end{array} \right\}$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \text{ for all } u \in W^{1,p}(\Omega).$$

Also, $\frac{\partial u}{\partial n_p}$ denotes the generalized normal derivative defined by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable, and for almost all $z \in \Omega$, the function $x \mapsto f(z, x)$ is continuous) which satisfies certain asymptotic conditions as $x \to +\infty$ and as $x \to 0^+$. These conditions incorporate as a special case the p-logistic equation of superdiffusive type.

Received by the editors May 26, 2015 and, in revised form, January 9, 2016 and January 21, 2016.

2010 *Mathematics Subject Classification.* Primary 35J66, 35J92.

Key words and phrases. Superdiffusive reaction, bifurcation type theorem, extremal positive solutions, p-Laplacian, nonlinear regularity.

Our aim in this paper is to produce a precise description of the dependence of positive solutions on the parameter $\lambda > 0$. When $f(z,x) = x^{\tau-1} - x^{r-1}$ $(x \geqslant 0)$, we have the classical p-logistic equation, which is important in problems of population dynamics (mathematical biology) and in the study of reaction-diffusion processes. Depending on the relation of the exponents $1 < \tau$, p, $r < \infty$, we distinguish different types of logistic equations. More precisely we have:

(i) *subdiffusive equations* when $\tau < p < r$;
(ii) *equidiffusive equations* when $\tau = p < r$;
(iii) *superdiffusive equations* when $p < \tau < r$.

Our hypotheses on the reaction term $f(z,x)$ incorporate as a very special case the superdiffusive p-logistic equation, and we show that it exhibits bifurcation phenomena for large values of the parameter $\lambda > 0$. So, we prove that there exists a critical parameter $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, problem (P_λ) admits at least two positive solutions: for $\lambda = \lambda^*$ there is at least one positive solution, and for $\lambda \in (0, \lambda^*)$ problem (P_λ) has no positive solutions. We also show that for every admissible parameter $\lambda \geqslant \lambda^*$, problem (P_λ) has a smallest positive solution $\underline{u}_\lambda$ and establish the continuity and monotonicity properties of the map $\lambda \mapsto \underline{u}_\lambda$. The two main results of this paper read as follows. Hypothesis $H(\beta)$ is given in the following section, while hypotheses H and H' are given in sections 3 and 4 respectively.

Theorem A. *Assume that hypotheses $H(\beta)$ and H hold. Then there exists $\lambda^* > 0$ such that*

(i) *for all $\lambda > \lambda^*$, problem (P_λ) admits at least two positive solutions,*

$$u_0, \hat{u} \in \operatorname{int} C_+, \; u_0 \neq \hat{u};$$

(ii) *for $\lambda = \lambda^*$, problem (P_{λ_*}) has at least one positive solution, $u_* \in \operatorname{int} C_+$;*
(iii) *for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has no positive solution.*

Theorem B. (a) *If hypotheses $H(\beta)$ and H hold and $\lambda \in \mathcal{L} = [\lambda^*, +\infty)$, then problem (P_λ) has a smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$.*
(b) *If hypotheses $H(\beta)$ and H' hold and $\lambda \in \mathcal{L} = [\lambda^*, +\infty)$, then problem (P_λ) has a biggest positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$ and the map $\lambda \mapsto \overline{u}_\lambda$ from $\mathcal{L} = [\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$ is nondecreasing and right continuous.*

Recently, Papageorgiou and Rădulescu [12] considered a more general class of nonlinear parametric Robin problems and proved a bifurcation result near the origin. In [12], the differential operator is in general nonhomogeneous (the p-Laplacian is included as a special case) and the parameter $\lambda > 0$ enters into the reaction in general in a nonlinear fashion. Our work here partially complements that paper. For Dirichlet p-logistic equations of superdiffusive type, we mention the works of Dong [3] and Takeuchi [13], [14]. Extensions to equations with more general reactions of superdiffusive type can be found in Gasiński and Papageorgiou [5] (Dirichlet problems) and Cardinali, Papageorgiou and Rubbioni [2], and Papageorgiou and Rădulescu [10] (Neumann problems).

Our approach uses variational methods based on critical point theory, coupled with suitable truncation techniques. In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the sequel.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Palais-Smale condition" (the "PS-condition" for short) if the following holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ and leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ.

Throughout this work, the hypotheses on the boundary coefficient $\beta(\cdot)$ are the following:

$H(\beta)$: $\beta \in C^{1,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$, $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$, $\beta \neq 0$.

In the analysis of problem (P_λ), in addition to the Sobolev space $W^{1,p}(\Omega)$ we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

By $||\cdot||$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, given by

$$||u|| = \left[||u||_p^p + ||Du||_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff measure $\sigma(\cdot)$ (the surface measure on $\partial\Omega$). Using this measure on $\partial\Omega$, we can define the Lebesgue spaces $L^q(\partial\Omega)$, $1 \leqslant q \leqslant \infty$. We know that there exists a unique continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the trace map, such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. This map is compact (more precisely, $\gamma_0 : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is compact for any $1 \leqslant q < \frac{Np-p}{N-p}$). Moreover, we have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ and } \ker \gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the explicit use of the trace map. All restrictions of Sobolev functions from $W^{1,p}(\Omega)$ on $\partial\Omega$ are defined in the sense of traces.

We say that a map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is of type $(S)_+$ if the following property holds:

$$\text{"}u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0 \Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega)\text{."}$$

In what follows, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $x \in \mathbb{R}$, then $x^\pm = \max\{\pm x, 0\}$. Given $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \; u = u^+ - u^-, \; |u| = u^+ + u^-.$$

3. Bifurcation-type theorem

In this section, we prove a bifurcation-type theorem describing in a precise way the dependence of positive solutions of (P_λ) on the parameter $\lambda > 0$.

We introduce the following conditions on the reaction $f(z,x)$:

H : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;
(ii) $\limsup\limits_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leqslant 0$ uniformly for almost all $z \in \Omega$;
(iii) $-\eta_0 \leqslant \liminf\limits_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} \leqslant \limsup\limits_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} \leqslant 0$ uniformly for almost all $z \in \Omega$, with $\eta_0 > 0$;
(iv) if $F(z,x) = \int_0^x f(z,s)ds$, then there exists $\tilde{u} \in L^r(\Omega)$ such that $\int_\Omega F(z,\tilde{u})dz > 0$;
(v) there exists $\mu > p$ such that $x \mapsto \frac{f(z,x)}{x^{\mu-1}}$ is strictly decreasing uniformly for almost all $z \in \Omega$; that is, if $x - y \geqslant \eta > 0$, there exists $c_\eta > 0$ such that $c_\eta \leqslant \frac{f(z,y)}{y^{\mu-1}} - \frac{f(z,x)}{x^{\mu-1}}$ for almost all $z \in \Omega$;
(vi) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for almost all $z \in \Omega$, the mapping $x \mapsto f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 1. Since we are looking for positive solutions and the above hypotheses concern the positive semi-axis, without any loss of generality we may assume that $f(z,x) = 0$ for almost all $z \in \Omega$ and all $x \leqslant 0$.

Example 1. The following function satisfies hypotheses H above. For the sake of simplicity, we drop the z-dependence:

$$f(x) = x^{\tau-1} - x^{r-1} \text{ for all } x \geqslant 0 \text{ and with } p < \tau < r < p^*.$$

This function is the classical superdiffusive reaction for the p-logistic equation and was used by Dong [3] and Takeuchi [13, 14].

The reader interested in applications can use the above function $f(x)$ as a reaction term in what follows.

We introduce the following two sets:

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\},$$
$$S(\lambda) \text{ is the set of positive solutions for problem } (P_\lambda).$$

Let $\lambda^* = \inf \mathcal{L} \geqslant 0$.

Proposition 1. *If hypotheses $H(\beta)$ and H hold, then for every $\lambda > 0$ we have $S(\lambda) \subseteq \operatorname{int} C_+$ and $\lambda^* > 0$.*

Proof. Let $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S(\lambda)$ such that

$$-\Delta_p u_\lambda(z) = \lambda f(z, u_\lambda(z)) \text{ for almost all } z \in \Omega, \ \frac{\partial u_\lambda}{\partial n_p} + \beta(z)u_\lambda(z)^{p-1} = 0 \text{ on } \partial\Omega$$

(see Papageorgiou and Rădulescu [9]). From Winkert [15], we know that $u_\lambda \in L^\infty(\Omega)$. So, we apply Theorem 2 of Lieberman [7] and we have $u_\lambda \in C_+\backslash\{0\}$. Let

$\rho = ||u_\lambda||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(vi)$. We have

$$
\begin{aligned}
& -\Delta_p u_\lambda(z) + \lambda \xi_\rho u_\lambda(z)^{p-1} = \lambda f(z, u_\lambda(z)) + \lambda \xi_\rho u_\lambda(z)^{p-1} \geqslant 0 \\
& \text{for almost all } z \in \Omega, \\
& \Rightarrow \Delta_p u_\lambda(z) \leqslant \lambda \xi_\rho u_\lambda(z)^{p-1} \text{ for almost all } z \in \Omega, \\
& \Rightarrow u_\lambda \in \operatorname{int} C_+ \text{ (by the nonlinear maximum principle)} \\
& \Rightarrow S(\lambda) \subseteq \operatorname{int} C_+.
\end{aligned}
$$

Hypotheses $H(i), (ii), (iii)$ imply that we can find $c_1 > 0$ such that

$$f(z,x) \leqslant c_1 x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0. \tag{1}$$

Let $\hat{\lambda}_1$ denote the principal eigenvalue of $-\Delta_p$ with Robin boundary condition (see Papageorgiou and Rădulescu [9]). We know that $\hat{\lambda}_1 > 0$. We choose $\lambda \in \left(0, \frac{\hat{\lambda}_1}{c_1}\right)$. Then we have

$$\lambda f(z,x) \leqslant \lambda c_1 x^{p-1} < \hat{\lambda}_1 x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x > 0 \text{ (see (1))}. \tag{2}$$

Suppose that $\lambda \in \mathcal{L}$. Then there exists $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ such that

$$\langle A(u_\lambda, h)\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma = \lambda \int_\Omega f(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{3}$$

In (3) we choose $h = u_\lambda \in W^{1,p}(\Omega)$ and use (2). Then

$$||Du_\lambda||_p^p + \int_{\partial\Omega} \beta(z) u_\lambda^p d\sigma < \hat{\lambda}_1 ||u_\lambda||_p^p,$$

which contradicts the variational characterization of $\hat{\lambda}_1 > 0$ (see Papageorgiou and Rădulescu [9]). Therefore $\lambda \notin \mathcal{L}$ and so $\lambda^* \geqslant \frac{\hat{\lambda}_1}{c_1} > 0$. $\square$

Proposition 2. *If hypotheses $H(\beta)$ and H hold, then $\mathcal{L} \neq \varnothing$ and $(\lambda^*, +\infty) \subseteq \mathcal{L}$.*

Proof. We introduce the Carathéodory function

$$\hat{f}_\lambda(z,x) = \lambda f(z,x) + (x^+)^{p-1} \text{ for all } (z,x) \in \Omega \times \mathbb{R}, \text{ all } \lambda > 0. \tag{4}$$

We set $\hat{F}_\lambda(z,x) = \int_0^x \hat{f}_\lambda(z,s) ds$, and for every $\lambda > 0$, we consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}_\lambda(z,u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses $H(i), (ii)$ imply that given $\epsilon \in \left(0, \frac{\hat{\lambda}_1}{\lambda}\right)$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p} x^p + c_2 \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0. \tag{5}$$

Then for all $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\hat{\varphi}_\lambda(u) &\geqslant \frac{1}{p}||Du||_p^p + \frac{1}{p}||u^-||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \frac{\lambda\epsilon}{p}||u^+||_p^p - \lambda c_2|\Omega|_N \\ &\qquad\qquad \text{(see (4), (5))} \\ &= \frac{1}{p}\left[||Du^+||_p^p + \int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \lambda\epsilon||u^+||_p^p\right] + \frac{1}{p}||u^-||^p - \lambda c_2|\Omega|_N \\ &\geqslant \frac{c_3}{p}||u^+||^p + \frac{1}{p}||u^-||^p - \lambda c_2|\Omega|_N \text{ for some } c_3 > 0 \\ &\qquad\qquad \text{(see Proposition 1 of Papageorgiou and Rădulescu [9])} \\ &\geqslant c_4||u||^p - \lambda c_2|\Omega|_N \text{ for some } c_4 > 0, \\ \Rightarrow \quad & \hat{\varphi}_\lambda \text{ is coercive.}\end{aligned}$$

Also, using the Sobolev embedding theorem and the trace theorem, we see that $\hat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(u_0) = \inf\left[\hat{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{6}$$

Consider the integral functional $I_F : L^r(\Omega) \to \mathbb{R}$,

$$I_F(u) = \int_\Omega F(z, u(z))dz \text{ for all } u \in L^r(\Omega).$$

From hypothesis $H(i)$ and the dominated convergence theorem, we see that I_F is continuous. By hypothesis $H(iv)$, we have

$$I_F(\tilde{u}) > 0.$$

Exploiting the density of $W^{1,p}(\Omega)$ in $L^r(\Omega)$, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$I_F(\tilde{u}) > 0. \tag{7}$$

So, because of (7) and by choosing $\lambda > 0$ appropriately big, we have

$$\begin{aligned}& \hat{\varphi}_\lambda(\bar{u}) < 0, \\ \Rightarrow \quad & \hat{\varphi}_\lambda(u_0) < 0 = \hat{\varphi}_\lambda(0) \text{ (see (6)); hence } u_0 \neq 0.\end{aligned}$$

From (6), we have

$$\begin{aligned}& \hat{\varphi}'_\lambda(u_0) = 0, \\ \Rightarrow \quad & \langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h dz + \int_{\partial\Omega}\beta(z)(u_0^+)^{p-1}h d\sigma = \int_\Omega \hat{f}_\lambda(z, u_0)h dz \\ & \qquad\qquad \text{for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{8}$$

In (8) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}& ||Du_0^-||_p^p + ||u_0^-||_p^p = 0 \text{ (see (4))}, \\ \Rightarrow \quad & ||u_0^-||^p = 0, \text{ and so } u_0 \geqslant 0,\ u_0 \neq 0.\end{aligned}$$

Therefore (8) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega}\beta(z)u_0^{p-1}h d\sigma = \int_\Omega f(z, u_0)h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (4))}. \tag{9}$$

From (9), reasoning as in Papageorgiou and Rădulescu [9], using the nonlinear Green's identity, we infer that $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ for $\lambda > 0$ big; hence $\mathcal{L} \neq \varnothing$.

Next, let $\lambda \in \mathcal{L}$ and let $\vartheta > \lambda$. Consider $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and let $\xi \in (0,1)$ such that

$$\lambda = \xi^{\mu-p}\vartheta \text{ (recall that } \mu > p;\text{ see hypothesis } H(v)). \tag{10}$$

We set $\underline{u} = \xi u_\lambda \in \operatorname{int} C_+$. We have

$$\begin{aligned} -\Delta_p \underline{u}(z) = \xi^{p-1}(-\Delta_p u_\lambda(z)) &= \xi^{p^+}\lambda f(z, u_\lambda(z)) \text{ (since } u_\lambda \in S(\lambda)) \\ &= \xi^{\mu-1}\vartheta f(z, u_\lambda(z)) \text{ for almost all } z \in \Omega \text{ (see (10))} \end{aligned} \tag{11}$$

$$\frac{\partial \underline{u}}{\partial n_p} + \beta(z)\underline{u}^{p-1} = 0 \text{ on } \partial\Omega \tag{12'}$$

Let $\eta = (1-\xi)\min_{\overline{\Omega}} u_\lambda > 0$ (recall $u_\lambda \in \operatorname{int} C_+,\ \xi \in (0,1)$). So, by virtue of hypothesis $H(v)$, we can find $c_\eta > 0$ such that

$$\begin{aligned} & c_\eta \leqslant \frac{f(z, \xi u_\lambda(z))}{\xi^{\mu-1}u_\lambda(z)^{\mu-1}} - \frac{f(z, u_\lambda(z))}{u_\lambda(z)^{\mu-1}} \text{ for almost all } z \in \Omega, \\ \Rightarrow\ & c_\eta \underline{u}(z)^{\mu-1} + \xi^{\mu-1} f(z, u_\lambda(z)) \leqslant f(z, \underline{u}(z)) \text{ for almost all } z \in \Omega. \end{aligned} \tag{12}$$

Using (12) in (11), we obtain

$$-\Delta_p \underline{u}(z) \leqslant \vartheta f(z, \underline{u}(z)) - c_\eta \underline{u}(z)^{\mu-1} \text{ for almost all } z \in \Omega. \tag{13}$$

We introduce the following truncations:

$$\hat{g}_\vartheta(z,x) = \begin{cases} \hat{f}_\theta(z, \underline{u}(z)) & \text{if } x \leqslant \underline{u}(z) \\ \hat{f}_\theta(z,x) & \text{if } \underline{u}(z) < x \end{cases} \text{ for all } (z,x) \in \Omega \times \mathbb{R}, \tag{14}$$

$$\hat{k}(z,x) = \begin{cases} \beta(z)\underline{u}(z)^{p-1} & \text{if } x \leqslant \underline{u}(z) \\ \beta(z)x^{p-1} & \text{if } \underline{u}(z) < x \end{cases} \text{ for all } (z,x) \in \partial\Omega \times \mathbb{R}\,. \tag{15}$$

Both are Carathéodory functions. We set

$$\hat{G}_\vartheta(z,x) = \int_0^x \hat{g}_\vartheta(z,s)ds \text{ and } \hat{K}(z,x) = \int_0^x \hat{k}(z,s)ds$$

and consider the C^1-functional $\hat{\psi}_\vartheta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\vartheta(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \int_{\partial\Omega} \hat{K}(z,u)d\sigma - \int_\Omega \hat{G}_\vartheta(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (14) and (15) it is clear that the functional $\hat{\psi}_\vartheta$ has essentially the same structure as the functional $\hat{\varphi}_\vartheta$. In particular, $\hat{\psi}_\vartheta$ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_\vartheta \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\psi}_\vartheta(u_\vartheta) = \inf[\hat{\psi}_\vartheta(u) : u \in W^{1,p}(\Omega)], \\ \Rightarrow\ & \hat{\psi}'_\vartheta(u_\vartheta) = 0, \\ \Rightarrow\ & \langle A(u_\vartheta), h\rangle + \int_\Omega |u_\vartheta|^{p-2}u_\vartheta h dz + \int_{\partial\Omega} \hat{k}(z, u_\vartheta)h d\sigma = \int_\Omega \hat{g}_\vartheta(z, u_\vartheta)h dz \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{16}$$

In (16) we choose $h = (\underline{u} - u_\vartheta)^+ \in W^{1,p}(\Omega)$ and obtain $\underline{u} \leqslant u_\vartheta$.

Using (14) and (15), we see that equation (16) becomes

$$\begin{aligned}
&\langle A(u_\vartheta), h\rangle + \int_{\partial\Omega} \beta(z) u_\vartheta^{p-1} h d\sigma - \int_\Omega \vartheta f(z, u_\vartheta) dz \text{ for all } h \in W^{1,p}(\Omega),\\
\Rightarrow\quad & u_\vartheta \in S(\vartheta) \subseteq \operatorname{int} C_+ \text{ and so } \vartheta \in \mathcal{L},\\
\Rightarrow\quad & (\lambda^*, \infty) \subseteq \mathcal{L}.
\end{aligned}$$

This completes the proof. □

Proposition 3. *If hypotheses $H(\beta)$ and H hold and $\vartheta > \lambda^*$, then problem (P_ϑ) admits at least two positive solutions,*

$$u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \neq 0.$$

Proof. Fix $\lambda \in (\lambda^*, \vartheta)$. From Proposition 2 we know that $\lambda \in \mathcal{L}$, and so there exists $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$. Let $\xi \in (0,1)$ such that $\lambda = \xi^{\mu-p}\vartheta$ (recall $\mu > p$; see hypothesis $H(v)$). We set $\underline{u} = \xi u_\lambda \in \operatorname{int} C_+$. Reasoning as in the proof of Proposition 2 and using the truncated functional $\hat{\psi}_\vartheta \in C^1(W^{1,p}(\Omega))$, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in S(\vartheta) \subseteq \operatorname{int} C_+, \ \underline{u} \leqslant u_0, \tag{17}$$

$$\hat{\psi}_\vartheta(u_0) = \inf[\hat{\psi}_\vartheta(u) : u \in W^{1,p}(\Omega)]. \tag{18}$$

Let $\rho = ||u_0||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(vi)$. For $\delta > 0$, let $\underline{u}^\delta = \underline{u} + \delta \in \operatorname{int} C_+$. We have

$$\begin{aligned}
& -\Delta_p \underline{u}^\delta + \vartheta\xi_\rho(\underline{u}^\delta)^{p-1}\\
\leqslant\ & -\Delta_p \underline{u} + \vartheta\xi_\rho \underline{u}^{p-1} + \gamma(\delta) \text{ with } \gamma(\delta) \to 0^+ \text{ as } \delta \to 0^+\\
\leqslant\ & \vartheta f(z, \underline{u}) + \vartheta\xi_\rho \underline{u}^{p-1} - c_\eta \underline{u}^{\mu-1} + \gamma(\delta) \text{ (see (13))}\\
\leqslant\ & \vartheta f(z, u_0) + \vartheta\xi_\rho u_0^{p-1} - c_\eta \overline{m}^{\mu-1} + \gamma(\delta) \text{ with } \overline{m} = \min_{\overline{\Omega}} \underline{u} > 0\\
& \qquad\qquad \text{(see (17), hypothesis } H(vi) \text{ and recall } \underline{u} \in \operatorname{int} C_+)\\
<\ & \vartheta f(z, u_0) + \vartheta\xi_\rho u_0^{p-1} \text{ for } \delta > 0 \text{ small (recall } \gamma(\delta) \to 0^+)\\
=\ & -\Delta_p u_0 + \vartheta\xi_\rho u_0^{p-1} \text{ for almost all } z \in \Omega, \text{ all } \delta > 0 \text{ small,}\\
\Rightarrow\quad & \underline{u}^\delta \leqslant u_0 \text{ for } \delta > 0 \text{ small,}\\
\Rightarrow\quad & u_0 - \underline{u} \in \operatorname{int} C_+.
\end{aligned}\tag{19}$$

Let

$$[\underline{u}) = \{u \in W^{1,p}(\Omega) : \underline{u}(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}.$$

From (14) and (15), we see that

$$\hat{\psi}_\vartheta|_{[\underline{u})} = \hat{\varphi}_\vartheta|_{[\underline{u})} + \xi_\vartheta^* \text{ with } \xi_\vartheta^* \in \mathbb{R}. \tag{20}$$

From (18), (19) and (20), it follows that

$$\begin{aligned}
& u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \hat{\varphi}_\vartheta,\\
\Rightarrow\quad & u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\vartheta \text{ (see [9]).}
\end{aligned}\tag{21}$$

Hypothesis $H(iii)$ implies that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}x^p \text{ for almost all } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{22}$$

Then for all $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta$, we have

$$\begin{aligned}
(23)\ \hat{\varphi}_\vartheta(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{p}||u^-||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega F(z,u^+)dz \text{ (see (4))} \\
&\geqslant \frac{1}{p}\left[||Du^+||_p^p + \int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \epsilon||u^+||_p^p\right] + \frac{1}{p}||u^-||^p \text{ (see (22))} \\
&\geqslant c_5||u||^p \text{ for some } c_5 > 0 \text{ (see [11, Proposition 4])}, \\
\Rightarrow\quad & u = 0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \hat{\varphi}_\vartheta, \\
\Rightarrow\quad & u = 0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\vartheta \text{ (see [9])}.
\end{aligned}$$

We may assume that $\hat{\varphi}_\vartheta(0) = 0 \leqslant \hat{\varphi}_\vartheta(u_0)$ (the reasoning is similar if the opposite inequality holds). Since $K_{\hat{\varphi}_\vartheta}\backslash\{0\} \subseteq \operatorname{int} C_+$, we may assume that $K_{\hat{\varphi}_\vartheta}$ is finite; otherwise we already have an infinity of distinct positive solutions for problem (P_λ). Because of (21) (if the opposite inequality holds, we use (23)), we can find $\rho \in (0,1)$, small such that

$$\hat{\varphi}_\vartheta(0) = 0 \leqslant \varphi_\vartheta(u_0) < \inf[\hat{\varphi}_\vartheta(u) : ||u-u_0|| = \rho] = \hat{m}_\rho \ ||u_0|| > \rho \tag{24}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). The functional $\hat{\varphi}_\vartheta$ being coercive, it satisfies the PS-condition. This fact and (24) permit the use of the mountain pass-theorem; see [4, p. 648]. So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\varphi}_\vartheta} \text{ and } \hat{m}_\rho \leqslant \hat{\varphi}_\vartheta(\hat{u}). \tag{25}$$

From (24) and (25) it follows that $\hat{u} \in S(\vartheta) \subseteq \operatorname{int} C_+,\ \hat{u} \neq u_0$. □

Next we examine what happens at the critical parameter $\lambda = \lambda^*$.

Proposition 4. *If hypotheses $H(\beta)$ and H hold, then $\lambda^* \in \mathcal{L}$, and so $\mathcal{L} = [\lambda^*, +\infty)$.*

Proof. Let $\lambda_n > \lambda^*$ for all $n \geqslant 1$ and assume that $\lambda_n \downarrow \lambda^*$ as $n \to \infty$. From Proposition 2, we can find $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ for all $n \geqslant 1$. We have

(26)
$$\langle A(u_n), h\rangle + \int_{\partial\Omega}\beta(z)u_n^{p-1}h d\sigma = \lambda_n\int_\Omega f(z,u_n)hdz \text{ for all } n \geqslant 1, \text{ all } h \in W^{1,p}(\Omega).$$

In (26) we choose $h = u_n \in \operatorname{int} C_+$ and have

$$||Du_n||_p^p + \int_{\partial\Omega}\beta(z)u_n^p d\sigma = \lambda_n\int_\Omega f(z,u_n)u_n dz \text{ for all } n \geqslant 1. \tag{27}$$

Hypotheses $H(i),(ii)$ imply that given $\epsilon > 0$, we can find $c_6 = c_6(\epsilon) > 0$ such that

$$f(z,x)x \leqslant \epsilon x^p + c_6 x \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0. \tag{28}$$

We use (28) in (27) and obtain

$$||Du_n||_p^p + \int_{\partial\Omega}\beta(z)u_n^p d\sigma \leqslant \lambda_1\epsilon||u_n||_p^p + \lambda_1 c_6||u_n||_1 \text{ for all } n \geqslant 1.$$

Choosing $\epsilon \in \left(0, \frac{\hat{\lambda}_1}{\lambda_1}\right)$ and using Proposition 4 of Papageorgiou and Rădulescu [11], we have

$$
\begin{aligned}
& ||u_n||^p \leqslant c_7||u_n|| \text{ for some } c_7 > 0, \text{ all } n \geqslant 1, \\
\Rightarrow \quad & \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\end{aligned}
$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$
u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{29}
$$

In (26) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (29). Then we have

$$
\begin{aligned}
& \lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0, \\
\Rightarrow \quad & u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see [8, p. 314]).}
\end{aligned} \tag{30}
$$

So, if in (26) we pass to the limit as $n \to \infty$ and use (30), then

$$
\begin{aligned}
& \langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma = \lambda^* \int_\Omega f(z, u_*) h dz \text{ for all } h \in W^{1,p}(\Omega), \\
\Rightarrow \quad & u_* \geqslant 0 \text{ solves problem } (P_{\lambda_*}).
\end{aligned}
$$

If we can show that $u_* \neq 0$, then $u_* \in S(\lambda^*) \subseteq \operatorname{int} C_+$, and so $\lambda^* \in \mathcal{L}$.

We argue indirectly. So, suppose that $u_* = 0$. We set $y_n = \frac{u_n}{||u_n||}$, $n \geqslant 1$. Then $||y_n|| = 1$, $y_n \geqslant 0$, $n \geqslant 1$. By passing to a subsequence if necessary, we may assume that

$$
y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{31}
$$

Hypotheses $H(i), (iii)$ imply that

$$
\begin{aligned}
& |f(z,x)| \leqslant c_8(|x|^{p-1} + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_8 > 0, \\
\Rightarrow \quad & \left\{ \frac{f(\cdot, u_n(\cdot))}{||u_n||^{p-1}} \right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded } \left(\frac{1}{p} + \frac{1}{p'} = 1\right).
\end{aligned} \tag{32}
$$

Thus, by passing to a subsequence if necessary and using hypothesis $H(iii)$ we obtain

$$
\frac{f(\cdot, u_n(\cdot))}{||u_n||^{p-1}} \xrightarrow{w} g y^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } -\eta_0 \leqslant g(z) \leqslant 0 \text{ for almost all } z \in \Omega \tag{33}
$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 14).

From (26) we have

$$
\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n^{p-1} h d\sigma = \lambda_n \int_\Omega \frac{f(z, u_n)}{||u_n||^{p-1}} h dz \text{ for all } n \geqslant 1. \tag{34}
$$

Choosing $h = y_n - y \in W^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (31) and (33), we obtain

$$
\begin{aligned}
& \lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\
\Rightarrow \quad & y_n \to y \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty, \ ||y|| = 1, \ y \geqslant 0.
\end{aligned} \tag{35}
$$

So, if in (34) we pass to the limit as $n \to \infty$ and use (35), we obtain

$$
\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y^{p-1} h d\sigma = \lambda^* \int_\Omega g y^{p-1} h dz \text{ for all } h \in W^{1,p}(\Omega).
$$

We choose $h = y \in W^{1,p}(\Omega)$. Then

$$0 < \hat{\lambda}_1 \leqslant ||Dy||_p^p + \int_{\partial\Omega} \beta(z) y^p d\sigma = \lambda_* \int_\Omega g y^p dz \leqslant 0 \text{ (see [9] and (33), (35))},$$

a contradiction.

Therefore $u_* \neq 0$, and so $u_* \in S(\lambda^*) \subseteq \operatorname{int} C_+$; hence $\lambda^* \in \mathcal{L}$, and so $\mathcal{L} = [\lambda^*, +\infty)$. ∎

The proof of Theorem A is now complete.

4. Extremal positive solutions

In this section we show that for all $\lambda \geqslant \lambda^*$, problem (P_λ) has extremal positive solutions, that is, a smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$ and a biggest positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$. We also investigate the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

To produce the smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$, we do not need to introduce any additional conditions on the reaction $f(z,x)$.

Proposition 5. *If hypotheses $H(\beta)$ and H hold and $\lambda \in \mathcal{L} = [\lambda^*, \infty)$, then problem (P_λ) admits a smallest positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$.*

Proof. The set $S(\lambda)$ is downward directed; that is, if $u, \hat{u} \in S(\lambda)$, then there exists $\tilde{u} \in S(\lambda)$ such that $\tilde{u} \leqslant u,\ \tilde{u} \leqslant \hat{u}$.

Since we are looking for the smallest positive solution, thanks to the above property of $S(\lambda)$, without any loss of generality, we may assume that there exists $c_9 > 0$ such that

$$||u||_\infty \leqslant c_9 \text{ for all } u \in S(\lambda). \tag{36}$$

From Hu and Papageorgiou [6, p. 178], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S(\lambda)$ such that

$$\inf S(\lambda) = \inf_{n\geqslant 1} u_n.$$

Then we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \lambda \int_\Omega f(z, u_n) h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{37}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (37), we obtain

$$\hat{\lambda}_1 ||u_n||_p^p \leqslant ||Du_n||_p^p + \int_{\partial\Omega} \beta(z) u_n^p d\sigma = \lambda \int_\Omega f(z, u_n) u_n dz \text{ for all } n \geqslant 1,$$

$$\Rightarrow \quad \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (36) and hypothesis } H(i)).$$

So, we may assume that

$$u_n \xrightarrow{w} \underline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to \underline{u}_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{38}$$

In (37), we choose $h = u_n - \underline{u}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (38). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - \underline{u}_\lambda \rangle = 0,$$

$$\Rightarrow \quad u_n \to \underline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 4).}$$

Arguing as in the proof of Proposition 4, via a contradiction argument and using hypothesis $H(iii)$, we show that $\underline{u}_\lambda \neq 0$. Hence $\underline{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and $\underline{u}_\lambda = \inf S(\lambda)$. ∎

To produce the biggest positive solution for (P_λ), we need an extra condition on the reaction $f(z,\cdot)$. So, we assume the following on the function $f(z,x)$:

H' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses $H'(i) \to (vi)$ are the same as the corresponding hypotheses $H(i) \to (vi)$ and

(vii) there exists $w \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $w(z) \geqslant 0$, for all $z \in \overline{\Omega}$, $w \neq 0$ and

$$\begin{aligned} &A(w) \geqslant 0 \text{ in } W^{1,p}(\Omega)^*, \\ &f(z,x) \geqslant 0 \text{ for almost all } z \in \Omega, \text{ all } x \in [0, w(z)], \\ &f(z,x) \leqslant 0 \text{ for almost all } z \in \Omega, \text{ all } x \geqslant w(z). \end{aligned}$$

Remark 2. Evidently, $f(z, w(z)) = 0$ for almost all $z \in \Omega$. If there exists $\xi > 0$ such that

$$\begin{aligned} &f(z,x) \geqslant 0 \text{ for almost all } z \in \Omega, \text{ all } x \in [0,\xi] \\ &\text{and } f(z,x) \leqslant 0 \text{ for almost all } z \in \Omega, \text{ all } x \geqslant \xi, \end{aligned}$$

then hypothesis $H(vii)$ is satisfied. In particular, the classical superdiffusive reaction

$$f(x) = x^{\tau-1} - x^{r-1} \text{ for all } x \geqslant 0 \text{ with } 1 < p < \tau < r < p^*$$

satisfies this new condition.

Proposition 6. *If hypotheses $H(\beta)$ and H' hold and $\lambda \in \mathcal{L}$, then $u_\lambda(z) \leqslant w(z)$ for all $z \in \overline{\Omega}$ and all $u_\lambda \in S(\lambda)$.*

Proof. Let $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 1). We have

$$\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma = \lambda \int_\Omega f(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega).$$

Let $h = (u_\lambda - w)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\left\langle A(u_\lambda), (u_\lambda - w)^+\right\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - w)^+ d\sigma \\ &= \lambda \int_\Omega f(z, u_\lambda)(u_\lambda - w)^+ dz \\ &\leqslant \left\langle A(w), (u_\lambda - w)^+\right\rangle + \int_{\partial\Omega} \beta(z) w^{p-1}(u_\lambda - w)^+ d\sigma \text{ (see } H'(vii) \text{ and } H(\beta)), \\ \Rightarrow\ &\int_{\{u_\lambda, w\}} (|Du_\lambda|^{p-2} Du_\lambda - |Dw|^{p-2} Dw, Du_\lambda - Dw)_{\mathbb{R}^N} dz \\ &\qquad + \int_{\partial\Omega} \beta(z)(u_\lambda^{p-1} - w^{p-1})(u_\lambda - w)^+ d\sigma \leqslant 0, \\ \Rightarrow\ &|\{u_\lambda > w\}|_N = 0, \text{ hence } u_\lambda(z) \leqslant w(z) \text{ for all } z \in \overline{\Omega}. \end{aligned}$$

This completes the proof. ∎

Remark 3. This proposition implies that $w(z) > 0$ for all $z \in \overline{\Omega}$ and hypothesis $H'(iii)$ becomes

$$\lim_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} = 0 \text{ uniformly for almost all } z \in \Omega.$$

Using Proposition 6 and reasoning similarly as in the proof of Proposition 5, we obtain:

Proposition 7. *If hypotheses $H(\beta)$ and H' hold and $\lambda \in \mathcal{L}$, then problem (P_λ) admits a biggest positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$.*

Now we investigate the properties of the map $\lambda \mapsto \overline{u}_\lambda$ from $(0, +\infty)$ into $\operatorname{int} C_+$. First we show the monotonicity of this map.

Proposition 8. *If hypotheses $H(\beta)$ and H' hold and $\lambda, \vartheta \in \mathcal{L}$, then $\overline{u}_\lambda \leqslant \overline{u}_\vartheta$.*

Proof. We have

(39)
$$\begin{aligned}\langle A(\overline{u}_\lambda, h)\rangle + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1} h d\sigma &= \lambda \int_\Omega f(z, \overline{u}_\lambda) h dz \\ &\leqslant \vartheta \int_\Omega f(z, \overline{u}_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ with } h \geqslant 0 \\ &\text{(see hypothesis } H'(vii) \text{ and Proposition 6).}\end{aligned}$$

We introduce the following truncations-perturbations of the reaction and of the boundary term:

$$\hat{g}_\vartheta(z,x) = \begin{cases} \vartheta f(z, \overline{u}_\lambda(z)) + \overline{u}_\lambda(z)^{p-1} & \text{if } x \leqslant \overline{u}_\lambda(z) \\ \vartheta f(z,x) + x^{p-1} & \text{if } \overline{u}_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{40}$$

$$\hat{k}(z,x) = \begin{cases} \beta(z)\overline{u}_\lambda(z)^{p-1} & \text{if } x \leqslant \overline{u}_\lambda(z) \\ \beta(z)x^{p-1} & \text{if } \overline{u}_\lambda(z) \leqslant x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{41}$$

Both are Carathéodory functions. We set $\hat{G}_\vartheta(z,x) = \int_0^x \hat{g}_\vartheta(z,s)ds$ and $\hat{K}(z,x) = \int_0^x \hat{k}(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\vartheta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\vartheta(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \int_{\partial\Omega} \hat{K}(z, u(z))d\sigma - \int_\Omega \hat{G}_\vartheta(z, u(z))dz \text{ for all } u \in W^{1,p}(\Omega).$$

We know that $\hat{\psi}_\vartheta$ is coercive and sequentially weakly lower semicontinuous (see the proof of Proposition 2). So, by the Weierstrass theorem, we can find $u_\vartheta \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\psi}_\vartheta(u_\vartheta) = \inf[\psi_\vartheta(u) : u \in W^{1,p}(\Omega)], \\ &\Rightarrow \hat{\psi}'_\vartheta(u_\vartheta) = 0, \\ &\Rightarrow \langle A(u_\vartheta), h\rangle + \int_\Omega |u_\vartheta|^{p-2} u_\vartheta h dz + \int_{\partial\Omega} \hat{k}(z, u_\vartheta) h d\sigma = \int_\Omega \hat{g}_\vartheta(z, u_\vartheta) h dz \quad (42) \\ &\qquad \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (42) we choose $h = (\overline{u}_\lambda - u_\vartheta)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A(u_\vartheta), (\overline{u}_\lambda - u_\vartheta)^+\rangle + \int_\Omega |u_\vartheta|^{p-2}u_\vartheta(\overline{u}_\lambda - u_\vartheta)^+ dz + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\vartheta)^+ d\sigma \\
&= \int_\Omega (\vartheta f(z, \overline{u}_\lambda) + \overline{u}_\lambda^{p-1})(\overline{u}_\lambda - u_\vartheta)^+ dz \text{ (see (40), (41))} \\
&\geqslant \langle A(\overline{u}_\lambda), (\overline{u}_\lambda - u_\vartheta)^+\rangle + \int_\Omega \overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\vartheta)^+ dz + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\lambda)^+ d\sigma \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(see (39))} \\
&\Rightarrow \int_{\{\overline{u}_\lambda, u_\vartheta\}} \left(|D\overline{u}_\lambda|^{p-2}D\overline{u}_\lambda - |Du_\vartheta|^{p-2}Du_\vartheta, D\overline{u}_\lambda - Du_\vartheta\right)_{\mathbb{R}^N} dz \\
&\qquad\qquad\qquad + \int_{\{\overline{u}_\lambda > u_\vartheta\}} (\overline{u}_\lambda^{p-1} - |u_\vartheta|^{p-2}u_\vartheta)(u_\lambda - u_\vartheta) dz \leqslant 0, \\
&\Rightarrow |\{\overline{u}_\lambda > u_\vartheta\}|_N = 0, \text{ hence } \overline{u}_\lambda \leqslant u_\vartheta.
\end{aligned}$$

So, (42) becomes

$$\begin{aligned}
&\langle A(u_\vartheta, h)\rangle + \int_{\partial\Omega} \beta(z)u_n^{p-1} h d\sigma = \int_\Omega \vartheta f(z, u_\vartheta) h dz \text{ for all } h \in W^{1,p}(\Omega) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(see (40), (41)),} \\
&\Rightarrow u_\vartheta \in S(\vartheta) \subseteq \operatorname{int} C_+ .
\end{aligned}$$

We conclude that

$$\overline{u}_\lambda \leqslant u_\vartheta \leqslant \overline{u}_\vartheta,$$

which completes the proof. □

Next we examine the continuity of the map $\lambda \mapsto \overline{u}_\lambda$ from $[\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$.

Proposition 9. *If hypotheses $H(\beta)$ and H' hold, then the map $\lambda \mapsto \overline{u}_\lambda$ is right continuous from $[\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L} = [\lambda^*, +\infty)$ and assume that $\lambda_n \downarrow \lambda \in \mathcal{L}$ as $n \to \infty$. Let $\overline{u}_n = \overline{u}_{\lambda_n} \in S(\lambda_n) \subseteq \operatorname{int} C_+$ be the maximal positive solution of problem (P_{λ_n}) $n \geqslant 1$ (see Proposition 7). We have

$$\begin{aligned}
&-\Delta_p u_n(z) = \lambda_n f(z, u_n(z)) \text{ for almost all } z \in \Omega, \\
&\frac{\partial u_n}{\partial n_p} + \beta(z)u_n^{p-1} = 0 \text{ on } \partial\Omega,\ n \geqslant 1
\end{aligned} \tag{43}$$

$$u_n \leqslant w \text{ for all } n \geqslant 1 \text{ (see Proposition 6).} \tag{44}$$

Then (43), (44) and Theorem 2 of Lieberman [7] imply that there exist $\alpha \in (0, 1)$ and $c_{10} > 0$ such that

$$\overline{u}_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } ||\overline{u}_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{10} \text{ for all } n \geqslant 1. \tag{45}$$

Moreover, from Proposition 8 we have that $\{\overline{u}_n\}_{n\geqslant 1}$ is decreasing. From (45) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we have

$$\overline{u}_n \downarrow \tilde{u} \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{46}$$

We show that $\tilde{u} = \overline{u}_\lambda$. Indeed, if this is not true, then we can find $z_0 \in \Omega$ such that

$$\tilde{u}(z_0) < \overline{u}_\lambda(z_0).$$

Because of (46), we can find $n_0 \in \mathbb{N}$ such that

$$\overline{u}_n(z_0) < \overline{u}_\lambda(z_0) \text{ for all } n \geqslant n_0. \tag{47}$$

On the other hand, from Proposition 8 we have

$$\overline{u}_\lambda(z_0) \leqslant \overline{u}_n(z_0) \text{ for all } n \geqslant 1. \tag{48}$$

Comparing (47) and (48) we reach a contradiction. This proves that $\tilde{u} = \overline{u}_\lambda$, and so we have established the right continuity of the map $\lambda \mapsto \overline{u}_\lambda$ from $\mathcal{L} = [\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$. $\square$

The proof of Theorem B is now complete.

Acknowledgement

The authors wish to thank a knowledgeable referee for remarks and constructive criticism.

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Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece

E-mail address: `npapg@math.ntua.gr`

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia — and — Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

E-mail address: `vicentiu.radulescu@math.cnrs.fr`