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Qualitative Phenomena for Some Classes of Quasilinear Elliptic Equations with Multiple Resonance

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Abstract We consider nonlinear nonhomogeneous Dirichlet problems driven by the sum of a p-Laplacian and a Laplacian. The hypotheses on the reaction term incorporate problems resonant at both $\pm\infty$ and zero. We consider both cases $p>2$ and $1<p<2$ (singular case) and we prove four multiplicity theorems producing three or four nontrivial solutions. For the case $p>2$ we provide precise sign information for all the solutions. Our approach uses critical point theory, truncation and comparison techniques, Morse theory and the Lyapunoff-Schmidt reduction method.

Keywords Strong comparison principle · Nonlinear maximum principle · Critical group · Nodal and constant sign solutions · Resonant equations · Lyapunoff-Schmidt reduction method

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = f\big(z, u(z)\big) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{1}$$

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Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\big(\|Du\|^{p-2} Du\big) \quad \text{for all } u \in W_0^{1,p}(\Omega)\ (1 < p < \infty). \tag{2}$$

We consider separately the cases $p > 2$ and $1 < p < 2$ and prove multiplicity theorems for both of them. A similar study was conducted recently by Sun [42] but under stronger conditions on the reaction $f(z, x)$ and the results proved there are weaker than ours. We stress that in (1) the differential operator is nonhomogeneous and this is the source of difficulties which require new techniques. Such nonhomogeneous elliptic equation were investigated recently by Cingolani & Degiovanni [14], Cingolani & Vannella [16], and He & Li [26], who proved existence theorems. Multiplicity theorems can be found in Papageorgiou & Smyrlis [39]. We mention that equations like (1) (we call them $(p, 2)$-equations for short) are important in quantum physics in the search for solitons, see Benci, D'Avenia, Fortunato, & Pisani [7].

Compared with [39] our setting here is different. In [39] the authors deal only with the case $2 < p < \infty$ and assume that the reaction $f(z, \cdot)$ exhibits a kind of oscillatory behavior near zero and so the geometry is different and leads to the existence of more nontrivial solutions of constant sign.

Compared with [42], our results here are considerably stronger. In [42], for the case $p > 2$, the reaction $f(z, x)$ is a C^1-function on $\overline{\Omega} \times \mathbb{R}$, which satisfies stronger asymptotic conditions (see (f_2)). The author proves a multiplicity theorem (Theorem 1.1) producing three solutions. However, no nodal solution is obtained. For the case $1 < p < 2$ only an existence theorem is proved (Theorem 1.2) under the hypothesis that $f \in C(\overline{\Omega} \times \mathbb{R})$.

Our approach uses critical point theory, combined with suitable truncation and comparison techniques, Morse theory and in the case where $1 < p < 2$, we also employ the so-called Lyapunoff-Schmidt reduction technique. In the next section for the convenience of the reader we recall some of the main mathematical tools which we will use in this paper.

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\big\{\varphi(x_n)\big\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$\big(1 + \|x_n\|\big)\varphi'(x_n) \to 0 \in X^* \text{ as } n \to \infty,$$

then $\{x_n\}_{n\geq 1}$ admits a strongly convergent subsequence".

This compactness-type condition is in general weaker than the more common Palais-Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of φ (see, for example Gasinski & Papageorgiou [24], Kristaly, V. Rădulescu & Varga [28] and Rădulescu [41]). In particular, we have the following result known in the literature as the "mountain pass theorem".

Theorem 2.1 *Assume* $\varphi \in C^1(X)$ *satisfies the* C*-condition*, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \rho$

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x) : \|x - x_0\| = \rho] = \eta_\rho,$$

and $C = \inf_{\gamma \in \Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))$, *where* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$.

Then $C \geq \eta_\rho$ *and* C *is a critical value of* φ.

In the analysis of problem (1), in addition to the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$, we will also use the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$.

This is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u \geq 0 \text{ in } \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where n stands for the outward unit normal on $\partial\Omega$.

Suppose $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the map $z \longmapsto f_0(z,x)$ is measurable and for a.a. $z \in \Omega$, the map $x \longmapsto f_0(z,x)$ is continuous) with subcritical growth in $x \in \mathbb{R}$, that is,

$$|f_0(z,x)| \leq a_0(z)\big(1 + |x|^{r-1}\big) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

$$\text{with } a_0 \in L^\infty(\Omega)_+ \text{ and } 1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leq p. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\Psi_0 : X \to \mathbb{R}$ defined by

$$\Psi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0\big(z, u(z)\big)\,dz \quad \text{for all } u \in X,$$

where $X = W_0^{1,p}(\Omega)$ if $2 < p < \infty$ and $X = H_0^1(\Omega)$ if $p \in (1,2)$.

The next result is a special case of Proposition 2 of Aizicovici, Papageorgiou & Staicu [2] and essentially it is a consequence of the nonlinear regularity results of Ladyzhenskaya & Uraltseva [29] (p. 286) and Lieberman [30, p. 320].

Proposition 2.1 *Assume that* $u_0 \in W_0^{1,p}(\Omega)$ *is a local* $C_0^1(\overline{\Omega})$*-minimizer of* Ψ_0, *that is, there exists* $\rho_0 > 0$ *such that* $\Psi_0(u_0) \leq \Psi_0(u_0 + h)$ *for all* $h \in C_0^1(\overline{\Omega})$ *with* $\|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0$.

Then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ *with* $\beta \in (0,1)$ *and* u_0 *is a local* X*-minimizer of* Ψ_0, *that is, there exists* $\rho_1 > 0$ *such that* $\Psi_0(u_0) \leq \Psi_0(u_0 + h)$ *for all* $h \in X$ *with* $\|h\|_X \leq \rho_1$.

Remark We should mention that first such a result was proved by Brezis & Nirenberg [10] for semilinear problems (that is, $p = 2$).

Next let $h, g \in L^{\infty}(\Omega)$. We write $h \prec g$, if for every compact $K \subseteq \Omega$ we can find $\epsilon = \epsilon(K) > 0$ such that

$$h(z) + \epsilon \leq g(z) \quad \text{for a.a. } z \in K.$$

Note that if $h, g \in C(\Omega)$ and $h(z) < g(z)$ for all $z \in \Omega$, then $h \prec g$.

The next proposition is essentially due to Arcoya & Ruiz [4, Proposition 2.6]. The presence of the extra linear term $-\Delta u$ does not affect their proof.

Proposition 2.2 *Assume that* $\xi \geq 0, h, g \in L^{\infty}(\Omega), h \prec g$. *Let* $u, v \in C_0^1(\overline{\Omega})$ *with* $v \in \operatorname{int} C_+$ *be solutions of*

$$-\Delta_p u(z) - \Delta u(z) + \xi \left|u(z)\right|^{p-2} u(z) = h(z) \quad \text{in } \Omega$$
$$-\Delta_p v(z) - \Delta v(z) + \xi v(z)^{p-1} = g(z) \quad \text{in } \Omega.$$

Then $v - u \in \operatorname{int} C_+$.

We will also need some basic facts concerning the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ for $1 < p < \infty$. So, we consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \hat{\lambda}\left|u(z)\right|^{p-2} u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0.$$

A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ if the above problem admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$ which is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. We know that there exists a smallest eigenvalue $\hat{\lambda}_1(p)$ with the following properties: (i) $\hat{\lambda}_1(p) > 0$; (ii) $\hat{\lambda}_1(p)$ is isolated, that is, there exists $\epsilon > 0$ such that the interval $[\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ contains no other eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$; (iii) $\hat{\lambda}_1(p)$ is simple, that is, if u, v are eigenfunctions corresponding to the eigenvalue $\hat{\lambda}_1(p)$, then $u = \xi v$ for some $\xi \in \mathbb{R}$; (iv) the eigenvalue $\hat{\lambda}_1(p) > 0$ admits the following variational characterization:

$$\hat{\lambda}_1(p) = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}. \tag{3}$$

Moreover, in relation (3) the infimum is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(p)$. From (3) it is clear that the elements of this eigenspace do not change sign. By $\hat{u}_{1,p}$ we denote the L^p-normalized (that is, $\|\hat{u}_{1,p}\|_p = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(p)$. In fact $\hat{\lambda}_1(p) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (that is, sign-changing) eigenfunctions. The nonlinear regularity theory (see [29, 30]) implies that $\hat{u}_{1,p} \in C_+ \setminus \{0\}$ and the nonlinear maximum principle of Vazquez [44] says that $\hat{u}_{1,p} \in \operatorname{int} C_+$.

When $p = 2$ (linear eigenvalue problem), then the spectrum of $(-\Delta, H_0^1(\Omega))$ is a sequence $\{\hat{\lambda}_k(2)\}_{k \geq 1} \subseteq (0, +\infty)$ of eigenvalues such that $\hat{\lambda}_k(2) \to +\infty$ as $k \to +\infty$. By $E(\hat{\lambda}_k(2))$ we denote the eigenspace corresponding to $\hat{\lambda}_k(2)$, We have the following orthogonal direct sum decomposition $H_0^1(\Omega) = \overline{\bigoplus_{k\geq 1} E(\hat{\lambda}_k(2))}$. These eigenspaces have the so-called "Unique Continuation Property" (UCP for short), namely if $u \in E(\hat{\lambda}_k(2))$ and u vanishes on a set of positive Lebesgue measure, then $u = 0$. The eigenvalues $\{\hat{\lambda}_k(2)\}_{k\geq 1}$ have the following variational characterizations:

$$\hat{\lambda}_1(2) = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right\} \tag{4}$$

$$\begin{aligned}\hat{\lambda}_n(2) &= \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \overline{\bigoplus_{k\geq 1} E\big(\hat{\lambda}_k(2)\big)}, u \neq 0\right\} \\ &= \sup\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bigoplus_{k=1}^{n} E\big(\hat{\lambda}_k(2)\big), u \neq 0\right\} \quad \text{for } n \geq 2.\end{aligned} \tag{5}$$

The infimum in (4) and both the infimum and the supremum in (5), are realized on the corresponding eigenspaces $E(\hat{\lambda}_n(2))$. Similar remarks can be made for a weighted version of the linear eigenvalue problem. So, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following weighted linear eigenvalue problem

$$-\Delta u(z) = \widetilde{\lambda} m(z) u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{6}$$

Problem (6) has a sequence $\{\widetilde{\lambda}_k(2, m)\}_{k\geq 1}$ of eigenvalues such that $\widetilde{\lambda}_k(2, m) \to +\infty$ as $k \to +\infty$. These eigenvalues still have the unique continuation property and have variational characterizations analogous to (4) and (5), in terms of the Rayleigh quotient $\frac{\|Du\|_2^2}{\int_\Omega mu^2\,dz}$. As a consequence of these variational characterizations and of the UCP, we have the following monotonicity of the eigenvalues $\widetilde{\lambda}_k(2, m)$ with respect to the weight function $m \in L^\infty(\Omega)_+$.

Proposition 2.3 *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ a.e. in Ω, $m_1 \neq 0$, $m_1 \neq m_2$, then $\widetilde{\lambda}_k(2, m_2) < \widetilde{\lambda}_k(2, m_1)$, for all $k \geq 1$.*

Next, let us recall some basic definitions and facts from Morse theory especially concerning critical groups. So, let $H_k(Y_1, Y_2)$ denote the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2). For all integers $k < 0$, we have $H_k(Y_1, Y_2) = 0$.

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \big\{x \in X : \varphi(x) \leq c\big\}, \qquad K_\varphi = \big\{x \in X : \varphi'(x) = 0\big\},$$
$$K_\varphi^c = \big\{x \in K_\varphi : \varphi(x) = c\big\}.$$

The critical groups of φ at an isolated critical point $x \in X$ of φ with $\varphi(x) = c$ (that is, $x \in K_\varphi^c$), are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \setminus \{x\}\big) \quad \text{for all } k \geq 0,$$

with $\mathcal{U}$ being a neighborhood of x such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood $\mathcal{U}$.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k\big(X, \varphi^c\big) \quad \text{for all } k \geq 0 \quad \text{(see Bartsch \& Li [6]).}$$

The second deformation theorem (see, for example, Gasinski & Papageorgiou [24, p. 628]), implies that the above definition of critical groups at infinity, is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$M(t, x) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \text{for all } t \in \mathbb{R} \text{ and for all } x \in K_\varphi$$

$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{7}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients (see Chang [13, p. 337] and Mawhin & Willem [36, p. 184]).

For $r \in (1, +\infty)$, let $A_r : W_0^{1,r}(\Omega) \to W_0^{1,r}(\Omega)^* = W^{-1,r'}(\Omega)$ ($\frac{1}{r} + \frac{1}{r'} = 1$) be the nonlinear map defined by

$$\big\langle A_r(u), y \big\rangle = \int_\Omega \|Du\|^{r-2} (Du, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } u, y \in W_0^{1,r}(\Omega). \tag{8}$$

If $r = 2$, then $A_2 = A \in \mathcal{L}\ (H_0^1(\Omega), H^{-1}(\Omega))$. Concerning the map A_r we can state the following well-known result (see Aizicovici, Papageorgiou & Staicu [1]).

Proposition 2.4 *Let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ be the nonlinear map defined by* (8). *Then A_r is continuous*, *strictly monotone* (*hence*, *maximal monotone*) *and of type* $(S)_+$.

We recall (see Brezis [9]) that if X is a real Banach space and $A : X \to X^*$ is a nonlinear operator, then A is said to be of type $(S)_+$ if for any sequence $\{u_n\}$ converging weakly to u in X and $\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$.

We will also make use of some results of Cingolani & Vannella [15, 16], which we recall here for easy reference. In those works, the authors showed that $(p, 2)$-equations can be embedded in a Hilbert space setting and this makes possible the use of Morse theoretic methods. So, let $f(z, x)$ be a measurable function such that $f(z, \cdot) \in C^1(\mathbb{R})$ for a.a. $z \in \Omega$ and assume that

$$\big|f_x'(z, x)\big| \leq a(z)\big(1 + |x|^{r-1}\big) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$ and $2 < p \le r < p^*$. We set $F(z,x) = \int_0^x f(z,s)\,ds$ and consider the C^2-functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F\big(z, u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have that for all $u, y, v \in W_0^{1,p}(\Omega)$

$$\begin{aligned}\big\langle \varphi''(u)y, v\big\rangle = {} & \int_\Omega \big(1 + \|Du\|^{p-2}\big)(Dy, Dv)_{\mathbb{R}^N}\,dz \\ & + (p-2)\int_\Omega \|Du\|^{p-4}(Du, Dy)_{\mathbb{R}^N}(Du, Dv)_{\mathbb{R}^N}\,dz \\ & - \int_\Omega f_x'(z,u)yv\,dz.\end{aligned}$$

The nonlinear regularity theory implies that if $u_0 \in K_\varphi$, then $u_0 \in C_0^1(\overline{\Omega})$ (see [26, 27]). Therefore

$$b = \|Du_0\|^{(p-4)/2} Du_0 \in L^\infty\big(\Omega, \mathbb{R}^N\big).$$

Let H_b be the closure of $C_c^\infty(\Omega)$ under the inner product

$$(y, v)_b = \int_\Omega \big[\big(1 + \|b\|^2\big)(Dy, Dv)_{\mathbb{R}^N} + (p-2)(b, Dy)_{\mathbb{R}^N}(b, Dv)_{\mathbb{R}^N}\big]\,dz.$$

Let $\|\cdot\|_b$ be the corresponding Hilbert norm, which is evidently equivalent to the Sobolev norm $\|\cdot\|_{H_0^1(\Omega)}$. Therefore, we have that $W_0^{1,p}(\Omega)$ is embedded continuously into H_b. Defining $L_b \in \mathcal{L}(H_b, H_b^*)$ by

$$\big\langle L_b(u), v\big\rangle = (u, v)_b - \int_\Omega f_x'(z, u_0)uv\,dz \quad \text{for all } u, v \in H_b,$$

we note that L_b is a Fredholm operator of index zero and in fact is the extension of $\varphi''(u_0)$ on H_b. We consider the orthogonal direct sum decomposition

$$H_b = H^- \oplus H^0 \oplus H^+,$$

where H^-, H^0, H^+ are respectively the negative, null and positive subspaces according to the spectral decomposition of L_b. The spaces H^- and H^0 are finite dimensional. Since $u_0 \in C_0^1(\overline{\Omega})$, from standard regularity theory, we have

$$H^- \oplus H^0 \subseteq W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Set $V = H^- \oplus H^0$ and $W = H^+ \cap W_0^{1,p}(\Omega)$. Then

$$W_0^{1,p}(\Omega) = V \oplus W \quad \text{and} \quad \big\langle \varphi''(u_0)y, y\big\rangle \ge \beta\|y\|_b^2 \quad \text{for all } y \in W,$$

for some $\beta > 0$ (see Cingolani & Vannella [15]).

Hereafter, by $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$, where $1 < p < \infty$. By virtue of the Poincaré inequality, we have $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$.

By $\|\cdot\|$ we also denote the norm of $\mathbb{R}^N$. No confusion is possible, since it will always be clear from the context which norm is used.

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ $(1 < p < \infty)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

Finally, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we define

$$N_h(u)(\cdot) = h\big(\cdot, u(\cdot)\big) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We denote by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$.

3 The Case $2 < p < \infty$

Throughout this section we assume that $2 < p < \infty$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$. We impose the following conditions on the reaction $f(z, x)$:

H_1: (i) $|f(z, x)| \le a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$ and for all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$ and $p \le r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then $\limsup_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \le \hat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$ and there exists $\xi > 0$ such that $f(z, x)x - pF(z, x) \ge -\xi$ for a.a. $z \in \Omega$ and for all $x \in \mathbb{R}$;

(iii) there are an integer $m \ge 2$, $\delta_0 > 0$ and $\eta \in L^\infty(\Omega)$ with $\eta(z) \ge \hat{\lambda}_m(2)$ a.e. in Ω, $\eta \ne \hat{\lambda}_m(2)$ such that $\eta(z)x^2 \le f(z, x)x \le \hat{\lambda}_{m+1}(2)x^2$ for a.a. $z \in \Omega$ and for $|x| \le \delta_0$;

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, $x \longmapsto f(z, x) + \xi_\rho |x|^{p-2}x$ is nondecreasing in $[-\rho, \rho]$.

Remarks Hypothesis H_1(ii) allows for resonance to occur at $\pm\infty$ with respect to the principal eigenvalue $\hat{\lambda}_1(p)$. Such resonant p-Laplacian equations were studied by Jiu & Su [27], Liu & Liu [33], Liu & Su [34] and Zhang, Li, Liu & Feng [45], using the additional condition that

$$\lim_{x\to\pm\infty} \big[f(z, x)x - pF(z, x)\big] = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Evidently our hypothesis H_1(ii) is weaker. Hypothesis H_1(iii) implies that we can have resonance at zero with respect to $\hat{\lambda}_{m+1}(2)$. So, we have a "double resonance" situation. Clearly, hypothesis H_1(iv) is much weaker than assuming the monotonicity of $f(z, \cdot)$.

We consider the positive and negative truncations of $f(z, \cdot)$, namely

$$f_\pm(z, x) = f\big(z, \pm x^\pm\big).$$

Both are Carathéodory functions. We set $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$ and consider the C^1-functionals $\varphi, \varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F\big(z,u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

$$\varphi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_\pm\big(z,u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.1 *If hypotheses H_1 hold, then the functionals φ and $\varphi_\pm$ are coercive.*

Proof We do the proof for φ_+, the proofs for φ_- and φ being similar.

Note that for a.a. $z \in \Omega$ and all $x > 0$, we have

$$\begin{aligned} \frac{d}{dx}\frac{F_+(z,x)}{x^p} &= \frac{f_+(z,x)x^p - px^{p-1}F_+(z,x)}{x^{2p}} \\ &= \frac{f_+(z,x)x - pF_+(z,x)}{x^{p+1}} \\ &\geq -\frac{\xi}{x^{p+1}} \quad \big(\text{see hypothesis } H_1(\text{ii})\big) \\ \Rightarrow \quad \frac{F_+(z,x)}{x^p} - \frac{F_+(z,u)}{u^p} &\geq \frac{\xi}{p}\left[\frac{1}{x^p} - \frac{1}{u^p}\right] \\ &\text{for a.a. } z \in \Omega \text{ and fo all } x \geq u \geq 0. \end{aligned}$$

Let $x \to +\infty$. Then by virtue of hypothesis H_1(ii), we have

$$\begin{aligned} &\frac{\hat{\lambda}_1(p)}{p} - \frac{F_+(z,u)}{u^p} \geq -\frac{\xi}{p}\frac{1}{u^p} \quad \text{for a.a. } z \in \Omega \text{ and for all } u > 0 \\ &\Rightarrow \quad pF_+(z,u) - \hat{\lambda}(p)\big(u^+\big)^p \leq \xi \quad \text{for all } u \in \mathbb{R}. \end{aligned} \tag{9}$$

Arguing by contradiction, suppose that φ_+ is not coercive. Then we can find $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ and $M_1 > 0$ such that

$$\|u_n\| \to +\infty \quad \text{and} \quad \varphi_+(u_n) \leq M_1 \quad \text{for all } n \geq 1. \tag{10}$$

From relations (10), (9) and (3), we deduce that

$$\big\{u_n^-\big\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \text{ is bounded.} \tag{11}$$

So, from (10) it follows that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^r(\Omega). \tag{12}$$

We have

$$\frac{1}{p}\|Dy_n\|_p^p + \frac{1}{2\|u_n^+\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \frac{F_+(z,u_n^+)}{\|u_n^+\|^p}\,dz \le \frac{M_2}{\|u_n^+\|^p}$$

for some $M_2 > M_1 > 0$ (see (11))

$$\begin{aligned}
\Rightarrow\quad & \frac{1}{p}\|Dy_n\|_p^p + \frac{1}{2\|u_n^+\|^{p-2}}\|Dy_n\|_2^2 - \frac{\hat{\lambda}_1(p)}{p}\|y_n\|_p^p - \frac{\xi|\Omega|_N}{p\|u_n^+\|^p} \le \frac{M_2}{\|u_n^+\|^p} \\
& \quad \text{for all } n \ge 1 \ \big(\text{see } (9)\big) \\
\Rightarrow\quad & \|Dy\|_p^p \le \hat{\lambda}_1(p)\|y\|_p^p \quad \big(\text{see } (12) \text{ and recall } 2 < p\big) \\
\Rightarrow\quad & y = \mu\hat{u}_{1,p} \quad \text{with } \mu \ge 0 \quad (\text{recall } y \ge 0).
\end{aligned}$$

If $\mu = 0$, then $y = 0$ and so we have $y_n \to 0$ in $W_0^{1,p}(\Omega)$ (since $\|Dy_n\|_p \to 0$), a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$. So, $\mu > 0$ and we have $y(z) > 0$ for all $z \in \Omega$ (recall $\hat{u}_{1,p} \in \operatorname{int} C_+$). This implies that $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$ as $n \to \infty$. Recall that

$$\begin{aligned}
& \frac{1}{p}\big\|Du_n^+\big\|_p^p + \frac{1}{2}\big\|Du_n^+\big\|_2^2 - \int_\Omega F\big(z,u_n^+\big)\,dz \le M_1 \quad \text{for all } n \ge 1 \quad \big(\text{see } (10)\big) \\
\Rightarrow\quad & \int_\Omega \left[\frac{\hat{\lambda}_1(p)}{p}\big(u_n^+\big)^p - F\big(z,u_n^+\big)\right]dz + \int_\Omega \frac{\hat{\lambda}_1(2)}{2}\big(u_n^+\big)^2\,dz \le M_1 \quad \big(\text{see } (3)\big) \\
\Rightarrow\quad & \frac{\hat{\lambda}_1(2)}{2}\int_\Omega \big(u_n^+\big)^2\,dz \le M_1 + \xi|\Omega|_N \quad \big(\text{see } (9)\big).
\end{aligned} \tag{13}$$

But $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$ and so, by Fatou's lemma, $\int_\Omega (u_n^+)^2\,dz \to +\infty$ as $n \to \infty$, which contradicts (13). This proves the coercivity of φ_+. Similarly we show the coercivity of the functionals φ_- and φ. □

Using this proposition and the direct method, we can produce two nontrivial solutions of constant sign.

Proposition 3.2 *Assume that hypotheses H_1 hold. Then problem* (1) *has at least two nontrivial solutions of constant sign $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and both are local minimizers of the functional φ.*

Proof From Proposition 3.1, we know that the functional φ_+ is coercive. Also, using the Sobolev embedding theorem, we can easily check that φ_+ is sequentially weakly lower semi-continuous. So, by the Weierstrass theorem we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \inf\big\{\varphi_+(u) : u \in W_0^{1,p}(\Omega)\big\}. \tag{14}$$

Let $\delta_0 > 0$ be as in hypotheses H_1(iii) and let $t \in (0,1)$ be small such that $t\hat{u}_{1,2}(z) \in [0,\delta_0]$ for all $z \in \bar{\Omega}$ (recall that $\hat{u}_{1,2} \in \operatorname{int} C_+$). Using H_1(iii) and $\|\hat{u}_{1,2}\|_2 =$

1 we obtain

$$\begin{aligned}\varphi_+(t\hat{u}_{1,2}) &= \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}\|D\hat{u}_{1,2}\|_2^2 - \int_\Omega F_+(z, t\hat{u}_{1,2})\,dz \\ &\le \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}\big[\hat{\lambda}_1(2) - \hat{\lambda}_m(2)\big]. \end{aligned} \tag{15}$$

We have $\hat{\lambda}_1(2) < \hat{\lambda}_m(2)$ (recall $m \ge 2$). Since $p > 2$, by choosing $t \in (0, 1)$ even smaller if necessary, from (15) we have

$$\varphi_+(t\hat{u}_{1,2}) < 0 \quad \Rightarrow \quad \varphi_+(u_0) < 0 = \varphi_+(0) \quad \big(\text{see (14)}\big),$$

hence $u_0 \neq 0$. From (14) we have

$$\varphi_+'(u_0) = 0 \quad \Rightarrow \quad A_p(u_0) + A(u_0) = N_{f_+}(u_0). \tag{16}$$

On (16) we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then $\|Du_0^-\|_p^p + \|Du_0^-\|_2^2 = 0$, hence $u_0 \ge 0$, $u_0 \neq 0$. Then (16) becomes

$$\begin{aligned}&A_p(u_0) + A(u_0) = N_f(u_0) \\ &\quad\Rightarrow \quad -\Delta_p u_0(z) - \Delta u_0(z) = f\big(z, u_0(z)\big) \quad \text{a.e. in } \Omega, u_0|_{\partial\Omega} = 0. \end{aligned} \tag{17}$$

From Ladyzhenskaya & Uraltseva [29, p. 286] and Fan & Zhao [20, Theorem 4.1], we have $u_0 \in L^\infty(\Omega)$. Then we can apply the regularity result of Lieberman [30, p. 320] and infer that $u_0 \in C_+\backslash\{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis H_1(iv). Then from (17) we have

$$\Delta_p u_0(z) + \Delta u_0(z) \le \xi_\rho u_0(z)^{p-1} \quad \text{a.e. in } \Omega.$$

From the strong maximum principle of Pucci & Serrin [40, p. 111], we have $u_0(z) > 0$ for all $z \in \Omega$. Then we can apply the boundary point theorem of Pucci & Serrin [40, p. 120] and conclude that $u_0 \in \operatorname{int} C_+$. Note that $\varphi_{+|C_+} = \varphi_{|C_+}$. So, $u_0 \in \operatorname{int} C_+$ is a local $C_0^1(\bar{\Omega})$-minimizer of φ, hence by virtue of Proposition 2.1 it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ. Similarly, working with the functional φ_-, we produce a second nontrivial constant sign solution $v_0 \in -\operatorname{int} C_+$, which is also a local minimizer of φ. □

In fact, we can show that problem (1) has extremal constant sign solutions, that is, a smallest nontrivial positive solution and a biggest nontrivial negative solution. The existence of such extremal constant sign solutions will lead to nodal (sign–changing) solutions.

From hypotheses H_1(i), (iii) we see that we can find $c_1 > \hat{\lambda}_1(2)$ and $c_2 > 0$ such that

$$f(z, x)x \ge c_1 x^2 - c_2|x|^r \quad \text{for a.a. } z \in \Omega \text{ and for all } x \in \mathbb{R}. \tag{18}$$

This unilateral growth estimate leads to the following auxiliary problem

$$-\Delta_p u(z) - \Delta u(z) = c_1 u(z) - c_2\big|u(z)\big|^{r-2}u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{19}$$

Proposition 3.3 *Problem* (19) *has a unique nontrivial positive solution* $u_* \in \operatorname{int} C_+$ *and because the problem is odd we have that* $v_* = -u_* \in -\operatorname{int} C_+$ *is the unique nontrivial negative solution of* (19).

Proof We start by proving the existence of a nontrivial positive solution for problem (19).

Let $\Psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\Psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{c_1}{2}\|u^+\|_2^2 + \frac{c_2}{r}\|u^+\|_r^r \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $p > 2$, then Ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$\Psi_+(u_*) = \inf\{\Psi_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{20}$$

Since $c_1 > \hat{\lambda}_1(2)$ and $2 < p \le r$, as before (see the proof of Proposition 3.2), for $t \in (0,1)$ small, we have $\Psi_+(t\hat{u}_{1,2}) < 0$. Thus, by (20), $\Psi_+(u_*) < 0 = \Psi_+(0)$, hence $u_* \neq 0$.

Relation (20) yields

$$\Psi_+'(u_*) = 0 \quad \Rightarrow \quad A_p(u_*) + A(u_*) = c_1 u_*^+ - c_2 (u_*^+)^{r-1}. \tag{21}$$

On (21) we act with $-u_*^- \in W_0^{1,p}(\Omega)$ and obtain $u_* \ge 0$ and $u_* \neq 0$. Then

$$\begin{aligned} &A_p(u_*) + A(u_*) = c_1 u_* - c_2 u_*^{r-1} \\ &\quad \Rightarrow \quad -\Delta_p u_*(z) - \Delta u_*(z) = c_1 u_*(z) - c_2 u_*(z)^{r-1} \quad \text{a.e. in } \Omega,\ u_*|_{\partial\Omega} = 0. \end{aligned}$$

Hence u_* is a nontrivial positive solution of the auxiliary problem (19). As in the proof of Proposition 3.2, using the nonlinear regularity theory (see [20, 29, 30]) and the results of Pucci & Serrin [40, pp. 111 and 120], we show that $u_* \in \operatorname{int} C_+$.

Next, we show the uniqueness of this nontrivial positive solution. To this end, let $G_0(t) = \frac{t^p}{p} + \frac{t^2}{2}$ for all $t \ge 0$ and set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. Then $G \in C^1(\mathbb{R}^N)$ and $\nabla G(y) = a(y) = \|y\|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. The mapping $G_0(\cdot)$ is increasing on $\mathbb{R}_+$ and $t \longmapsto G_0(t^{1/2})$ is convex and we have

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the integral functional $\mu_+ : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\mu_+(u) = \begin{cases} \int_\Omega G(Du^{1/2})\,dz & \text{if } u \ge 0 \text{ and } u^{1/2} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom}\mu_+$ and let $y = (tu_1 + (1-t)u_2)^{1/2} \in W_0^{1,p}(\Omega)$ for $t \in [0,1]$. From Benguria, Brezis & Lieb [8, Lemma 4] (see also Diaz & Saa [17, Lemma 1])

we have

$$
\begin{aligned}
&\|Dy(z)\| \le \big(t\|Du_1(z)^{1/2}\|^2 + (1-t)\|Du_2(z)^{1/2}\|^2\big)^{1/2} \text{ a.e. in } \Omega \\
\Rightarrow\quad & G_0\big(\|Dy(z)\|\big) \le G_0\big(\big(t\|Du_1(z)^{1/2}\|^2 + (1-t)\|Du_2(z)^{1/2}\|^2\big)^{1/2}\big) \\
&\quad \text{(since } G_0 \text{ is increasing)} \\
&\qquad\qquad \le tG_0\big(\|Du_1(z)^{1/2}\|\big) + (1-t)G_0\big(\|Du_2(z)^{1/2}\|\big) \\
&\quad \big(\text{since } t \longmapsto G_0\big(t^{1/2}\big) \text{ is convex}\big) \\
\Rightarrow\quad & G\big(Dy(z)\big) \le tG\big(Du_1(z)^{1/2}\big) + (1-t)G\big(Du_2(z)^{1/2}\big) \\
\Rightarrow\quad & \mu_+ \text{ is convex.}
\end{aligned}
$$

Suppose $u, y \in W_0^{1,p}(\Omega)$ are two nontrivial positive solutions of (19). From the first part of the proof, we have $u, y \in \operatorname{int} C_+$ and so $u^2, y^2 \in \operatorname{dom}\mu_+$. Let $h \in C_0^1(\bar{\Omega})$. For $t \in [-1, 1]$ small in absolute value, we have $u^2 + th, y^2 + th \in \operatorname{dom}\mu_+$. The Gâteaux derivative of μ_+ at u^2, y^2 in the direction h exists and by the chain rule and the density of $C_0^1(\bar{\Omega})$ in $W_0^{1,p}(\Omega)$, we have for all $h \in W_0^{1,p}(\Omega)$

$$\mu_+'\big(u^2\big)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p u - \Delta u}{u} h\,dz \tag{22}$$

$$\mu_+'\big(y^2\big)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p y - \Delta y}{y} h\,dz. \tag{23}$$

Since μ_+ is convex, μ_+' is monotone, we have

$$
\begin{aligned}
0 &\le \big\langle \mu_+'\big(u^2\big) - \mu_+'\big(y^2\big), u^2 - y^2\big\rangle_{L^1(\Omega)} \\
&= \frac{1}{2}\int_\Omega \bigg(\frac{-\Delta_p u - \Delta u}{u} - \frac{-\Delta_p y - \Delta y}{y}\bigg)\big(u^2 - y^2\big)\,dz \\
&= \frac{c_2}{2}\int_\Omega \big(y^{r-2} - u^{r-2}\big)\big(u^2 - y^2\big)\,dz \le 0 \\
\Rightarrow\quad & u = y.
\end{aligned}
$$

This proves the uniqueness of u_*.

Since the auxiliary problem (19) is odd, then $v_* = -u_* \in -\operatorname{int} C_+$ is the unique nontrivial negative solution of (19). □

Using $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$, we can produce extremal constant sign solutions for problem (1).

Proposition 3.4 *Assume that hypotheses H_1 hold. Then problem* (1) *has a smallest nontrivial positive solution $u_+ \in \operatorname{int} C_+$ and a biggest nontrivial negative solution $v_- \in -\operatorname{int} C_+$.*

Proof Let S_+ be the set of nontrivial positive solutions for problem (1). From Proposition 3.2 and its proof, we have $S_+ \neq \emptyset$ and $S_+ \subseteq \operatorname{int} C_+$.

Claim: If $\tilde{u} \in S_+$, then $u_* \leq \tilde{u}$.

We consider the following Carathéodory function

$$h(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_1 x - c_2 x^{r-1} & \text{if } 0 \leq x \leq \tilde{u}(z) \\ c_1 \tilde{u}(z) - c_2 \tilde{u}(z)^{r-1} & \text{if } \tilde{u}(z) < x. \end{cases} \tag{24}$$

We set $H(z,x) = \int_0^x h(z,s)\,ds$ and consider the C^1-functional $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega H\big(z, u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (24) that Ψ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\Psi(\tilde{u}_*) = \inf\big\{\Psi(u) : u \in W_0^{1,p}(\Omega)\big\}. \tag{25}$$

Recall that $\tilde{u} \in \operatorname{int} C_+$. So we can find $t \in (0,1)$ small such that $t\tilde{u}_{1,2} \leq \tilde{u}$. Then we have

$$\begin{aligned} \Psi(t\hat{u}_{1,2}) &= \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}\|D\hat{u}_{1,2}\|_2^2 - \int_\Omega H(z, t\hat{u}_{1,2})\,dz \\ &\leq \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}\big[\hat{\lambda}_1(2) - c_1\big] + \frac{t^r c_2}{r}\|\hat{u}_{1,2}\|_r^r \\ &\qquad \big(\text{see (24) and recall } \|\hat{u}_{1,2}\|_2 = 1\big). \end{aligned}$$

Since $c_1 > \hat{\lambda}_1(2)$ and $2 < p \leq r$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\Psi(t\hat{u}_{1,2}) < 0 \\ &\quad \Rightarrow \quad \Psi(\tilde{u}_*) < 0 = \Psi(0) \quad \big(\text{see (25)}\big), \text{ hence } \tilde{u}_* \neq 0. \end{aligned}$$

From (25) we have

$$\Psi'(\tilde{u}_*) = 0 \quad \Rightarrow \quad A_p(\tilde{u}_*) + A(\tilde{u}_*) = N_h(\tilde{u}_*). \tag{26}$$

On (26) we act with $-\tilde{u}_*^- \in W_0^{1,p}(\Omega)$ and obtain $\tilde{u}_* \geq 0, \tilde{u}_* \neq 0$. Also, we act with $(\tilde{u}_* - \tilde{u})^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\big\langle A_p(\tilde{u}_*), (\tilde{u}_* - \tilde{u})^+\big\rangle + \big\langle A(\tilde{u}_*), (\tilde{u}_* - \tilde{u})^+\big\rangle \\ &\quad = \int_\Omega h(z, \tilde{u}_*)(\tilde{u}_* - \tilde{u})^+\,dz \end{aligned}$$

$$= \int_\Omega \left(c_1\tilde{u} - c_2\tilde{u}^{r-1}\right)(\tilde{u}_* - \tilde{u})^+ \, dz \quad \left(\text{see (24)}\right)$$

$$\le \int_\Omega h(z, \tilde{u})(\tilde{u}_* - \tilde{u})^+ \, dz \quad \left(\text{see (18)}\right)$$

$$= \left\langle A_p(\tilde{u}), (\tilde{u}_* - \tilde{u})^+\right\rangle + \left\langle A(\tilde{u}), (\tilde{u}_* - \tilde{u})^+\right\rangle \quad (\text{since } \tilde{u} \in S_+)$$

$$\Rightarrow \quad \left\| D(\tilde{u}_* - \tilde{u})^+ \right\|_2^2 \le 0 \quad (\text{see Proposition 2.4}),$$

hence $\tilde{u}_* \le \tilde{u}$.

So, we have proved that

$$\tilde{u}_* \in [0, \tilde{u}] = \left\{u \in W_0^{1,p}(\Omega) : 0 \le u(z) \le \tilde{u}(z) \text{a.e. in } \Omega\right\}, \tilde{u}_* \ne 0.$$

Then (26) becomes

$$A_p(\tilde{u}_*) + A(\tilde{u}_*) = c_1\tilde{u}_* - c_2\tilde{u}_*^{r-1} \quad \left(\text{see (24)}\right)$$

$$\Rightarrow \quad -\Delta_p\tilde{u}_*(z) - \Delta\tilde{u}_*(z) = c_1\tilde{u}_*(z) - c_2\tilde{u}_*(z)^{r-1} \quad \text{a.e. in } \Omega, \tilde{u}_*|_{\partial\Omega} = 0$$

$$\Rightarrow \quad \tilde{u}_* = u_* \quad (\text{see Proposition 3.3})$$

$$\Rightarrow \quad u_* \le \tilde{u}.$$

This proves the Claim.

From Filippakis, Kristaly & Papageorgiou [21] (Proposition 4.2 and Lemma 4.3) we have that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \le u_1, u \le u_2$). So, without any loss of generality we may assume that there exists $M_3 > 0$ such that $u(z) \le M_3$ for all $z \in \overline{\Omega}$ and all $u \in S_+$. Let $C \subseteq S_+$ be a chain (that is, a totally ordered subset of S_+). From Dunford & Schwartz [19], we know that we can find $\{u_n\}_{n\ge1} \subseteq C$ such that $\inf C = \inf_{n\ge1} u_n$.

We have

$$A_p(u_n) + A(u_n) = N_f(u_n), u_* \le u_n \le M_3$$
$$\text{for all } n \ge 1 \text{ (see the Claim)}$$
$$\Rightarrow \quad \{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{27}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega). \tag{28}$$

On (27) we act with $u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (28). Then

$$\lim_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \left\langle A(u_n), u_n - u\right\rangle\right] = 0$$

$$\Rightarrow \quad \limsup_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \left\langle A(u), u_n - u\right\rangle\right] \le 0$$
$$\text{(due to the monotonicity of } A)$$

$$\Rightarrow \limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \le 0 \quad \big(\text{see (28)}\big)$$

$$\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.4) and } u_* \le u \ \big(\text{see (27)}\big). \tag{29}$$

So, if in (27) we pass to the limit as $n \to \infty$ and use (29), then

$$A_p(u) + A(u) = N_f(u), \quad u_* \le u$$
$$\Rightarrow \quad u \in S_+ \text{ and } u = \inf C.$$

Since C is an arbitrary chain in S_+, from the Kuratowski–Zorn lemma we know that we can find a minimal element $u_+ \in S_+ \subseteq \operatorname{int} C_+$. Since S_+ is downward directed, if $u \in S_+$ we can find $\hat{u} \in S_+$ such that $\hat{u} \le u_+$ and $\hat{u} \le u$. The minimality of u_+ implies that $\hat{u} = u_+$ and so $u_+ \le u$ for all $u \in S_+$.

Let S_- be the set of nontrivial negative solutions of (1). We have

$$S_- \ne \emptyset \text{ and } S_- \subset -\operatorname{int} C_+ \quad \text{(see Proposition 3.2).}$$

The set S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then we can find such that $v_1 \le v, v_2 \le v$; see [21]). Reasoning as above, via the Kuratowski-Zorn lemma, we produce $v_- \in -\operatorname{int} C_+$ the biggest nontrivial negative solution of (1). □

Using these extremal constant sign solutions, we can produce nodal solutions. To do this, we need to strengthen the conditions on the reaction $f(z,x)$. The new hypotheses on $f(z,x)$, are the following:

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z,x)| \le a(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p \le r < p^*$;

(ii) $\limsup_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \le \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$ and there exists $\xi > 0$ such that

$$f(z,x)x - pF(z,x) \ge -\xi \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iii) there are $\delta_0 > 0$, an integer $m \ge 2$ and $\eta \in L^\infty(\Omega)$, such that $\eta(z) \ge \hat{\lambda}_m(2)$ a.e. in Ω, $\eta \ne \hat{\lambda}_m(2)$ and

$$\eta(z)x^2 \le f(z,x)x \le \hat{\lambda}_{m+1}(2)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta_0;$$

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the map $x \longmapsto f(z,x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

Remark In this setting hypothesis H_2(iv) is satisfied, if for example there exists $\delta_0 > 0$ such that $f_x'(z,x) \ge 0$ for a.a. $z \in \Omega$, all $|x| \le \delta_0$ (that is, $f(z,\cdot)$ is increasing near zero).

Proposition 3.5 *Assume that hypotheses H_2 are fulfilled. Then problem* (1) *has a nodal solution $y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_-, u_+]$, that is, $y_0 \in C_0^1(\bar{\Omega})$ and $u_+ - y_0, y_0 - v_- \in \operatorname{int} C_+$.*

Proof Let $u_+ \in \operatorname{int} C_+$ and $v_- \in \operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 3.4. We introduce the following Carathéodory function

$$\hat{g}(z,x) = \begin{cases} f(z, v_-(z)) & \text{if } x < v_-(z) \\ f(z,x) & \text{if } v_-(z) \le x \le u_+(z) \\ f(z, u_+(z)) & \text{if } u_+(z) < x. \end{cases} \tag{30}$$

We set $\hat{G}(z,x) = \int_0^x \hat{g}(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{G}\big(z, u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also let $\hat{g}_\pm(z,x) = \hat{g}(z, \pm x^\pm)$, $\hat{G}_\pm(z,x) = \int_0^x \hat{g}_\pm(z,s)\,ds$ and consider the C^1-functionals $\hat{\varphi}_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{G}_\pm\big(z, u(z)\big)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: $K_{\hat{\varphi}} \subseteq [v_-, u_+] = \{u \in W_0^{1,p}(\Omega) : v_-(z) \le u(z) \le u_+(z) \text{ a.e. in } \Omega\}$, $K_{\hat{\varphi}_+} = \{0, u_+\}$, $K_{\hat{\varphi}_-} = \{0, v_-\}$.

Let $u \in K_{\hat{\varphi}}$. Then we have

$$A_p(u) + A(u) = N_{\hat{g}}(u). \tag{31}$$

On (31) we act with $(u - u_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\big\langle A_p(u), (u-u_+)^+\big\rangle + \big\langle A(u), (u-u_+)^+\big\rangle \\ &\quad = \int_\Omega \hat{g}(z,u)(u-u_+)^+\,dz \\ &\quad = \int_\Omega f(z,u_+)(u-u_+)^+\,dz \quad \big(\text{see (30)}\big) \\ &\quad = \big\langle A_p(u_+), (u-u_+)^+\big\rangle + \big\langle A(u_+), (u-u_+)^+\big\rangle \\ &\qquad \Rightarrow \quad \big\|D(u-u_+)^+\big\|_2^2 \le 0 \quad (\text{since } A_p \text{ is monotone, see Proposition 2.4}) \\ &\qquad \Rightarrow \quad u \le u_+. \end{aligned}$$

Similarly, acting on (31) with $(v_- - u)^+ \in W_0^{1,p}(\Omega)$, we show that $v_- \le u$. It follows that $K_{\hat{\varphi}} \subseteq [v_-, u_+]$.

Reasoning in a similar way, we show that

$$\begin{aligned} K_{\hat{\varphi}_+} &\subset [0, u_+] = \big\{u \in W_0^{1,p}(\Omega) : 0 \le u(z) \le u_+(z) \text{ a.e. in } \Omega\big\} \\ K_{\hat{\varphi}_-} &\subset [v_-, 0] = \big\{u \in W_0^{1,p}(\Omega) : v_-(z) \le u(z) \le 0 \text{ a.e. in } \Omega\big\}. \end{aligned}$$

The extremality of $v_- \in -\operatorname{int} C_+$ and $u_+ \in \operatorname{int} C_+$ (see Proposition 3.4), implies that

$$K_{\hat{\varphi}_+} = \{0, u_+\} \quad \text{and} \quad K_{\hat{\varphi}_-} = \{0, v_-\}.$$

This proves Claim 1.

Claim 2: $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ are local minimizers of the functional $\hat{\varphi}$.

From (30) it is clear that $\hat{\varphi}$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(\tilde{u}) = \inf\bigl\{\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\bigr\}. \tag{32}$$

Integrating hypothesis in H_2(iii), we have

$$\frac{\eta(z)}{2}x^2 \leq F(z,x) \leq \frac{\hat{\lambda}_{m+1}(2)}{2}x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.$$

As before (see the proof of Proposition 3.4), for $t \in (0,1)$ small we have by (32)

$$\hat{\varphi}_+(t\hat{u}_{1,2}) < 0 \quad \Rightarrow \quad \hat{\varphi}_+(\tilde{u}) < 0 = \hat{\varphi}_+(0),$$

hence $\tilde{u} \neq 0$.

From (32) we have

$$\tilde{u} \in K_{\hat{\varphi}_+}, \tilde{u} \neq 0 \quad \Rightarrow \quad \tilde{u} = u_+ \quad \text{(see Claim 1)}.$$

Note that $\hat{\varphi}_+|_{C_+} = \hat{\varphi}|_{C_+}$. So, u_+ is a local $C_0^1(\bar{\Omega})$-minimizer of $\hat{\varphi}$. Hence by virtue of Proposition 2.1, it is also a local $W_0^{1,p}(\Omega)$-minimizer of $\hat{\varphi}$.

Similarly for $v_- \in -\operatorname{int} C_+$ using this time the functional $\hat{\varphi}_-$. This proves Claim 2.

Without any loss of generality we may assume that $\hat{\varphi}(v_-) \leq \hat{\varphi}(u_+)$ (the analysis is similar if the opposite inequality is true). From Claim 2 we know that $u_+ \in \operatorname{int} C_+$ is a local minimizer of the functional $\hat{\varphi}$. So, as in Filippakis, Kristaly & Papageorgiou [21] (proof of Proposition 3.2) or from de Figueiredo [23, Theorem 5.10], we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}(v_-) \leq \hat{\varphi}(u_+) < \inf\bigl[\hat{\varphi}(u) : \|u - u_+\| = \rho\bigr] = \hat{\eta}_\rho^+, \quad \|v_- - u_+\| > \rho. \tag{33}$$

Since $\hat{\varphi}$ is coercive (see (30)), it satisfies the C-condition. This fact and (33) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\hat{\varphi}} \subset [v_-, u_+] \quad \text{(see Claim 1)} \quad \text{and} \quad \hat{\eta}_\rho^+ \leq \hat{\varphi}(y_0). \tag{34}$$

From (33) and (34) it follows that

$$y_0 \notin \{v_-, u_+\}. \tag{35}$$

We have

$$A_p(y_0) + A(y_0) = N_f(y_0) \quad \big(\text{see (30)}\big)$$
$$\Rightarrow -\Delta_p y_0(z) - \Delta y_0(z) = f\big(z, y_0(z)\big) \quad \text{a.e. in } \Omega,\ y_0|_{\partial\Omega} = 0.$$

As before the nonlinear regularity theory implies that $y_0 \in C_0^1(\bar{\Omega})$. Set

$$\rho = \max\big\{\|u_+\|_\infty, \|v_-\|_\infty\big\}$$

and let $\xi_\rho > 0$ be such that for a.a. $z \in \Omega$, the mapping $x \to f(z,x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$. Then

$$\begin{aligned}
&-\Delta_p y_0(z) - \Delta y_0(z) + \xi_p \big|y_0(z)\big|^{p-2} y_0(z) \\
&\quad = f\big(z, y_0(z)\big) + \xi_\rho \big|y_0(z)\big|^{p-2} y_0(z) \\
&\quad \le f\big(z, u_+(z)\big) + \xi_\rho u_+(z)^{p-1} \quad (\text{since } y_0 \le u_+) \\
&\quad = -\Delta_p u_+(z) - \Delta u_+(z) + \xi_\rho u_+(z)^{p-1} \quad \text{a.e. in } \Omega.
\end{aligned} \tag{36}$$

As in the proof of Proposition 3.3, let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the C^1-map, defined by

$$a(y) = \|y\|^{p-2} y + y \quad \text{for all } y \in \mathbb{R}^N.$$

We have

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have

$$\begin{aligned}
&\nabla a(y) = \|y\|^{p-2}\left[I + (p-2)\frac{y \otimes y}{\|y\|}\right] + I \\
&\quad\Rightarrow \quad \big(\nabla a(y)\xi, \xi\big)_{\mathbb{R}^N} \ge \|\xi\|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N.
\end{aligned} \tag{37}$$

Then relation (37) permits the use of the tangency principle of Pucci & Serrin [40, p. 35] and we have

$$y_0(z) < u_+(z) \quad \text{for all } z \in \Omega.$$

Then from (36) and Proposition 2.2, we have

$$u_+ - y_0 \in \operatorname{int} C_+.$$

In a similar manner, we show that

$$y_0 - v_- \in \operatorname{int} C_+.$$

Therefore

$$y_0 \in \operatorname{int}_{C_0^1(\Omega)}[v_-, u_+].$$

So, we have

$$C_k(\hat{\varphi}|_{C_0^1(\bar{\Omega})}, y_0) = C_k(\varphi|_{C_0^1(\bar{\Omega})}, y_0)$$
$$\Rightarrow \quad C_k(\hat{\varphi}, y_0) = C_k(\varphi, y_0) \quad \text{for all } k \geq 0 \tag{38}$$

(see Bartsch [5, Proposition 2.6] and Palais [38]).

Recall that $y_0 \in C_0^1(\bar{\Omega})$ is a critical point of mountain pass type for the functional $\hat{\varphi}$. Therefore $C_1(\hat{\varphi}, y_0) \neq 0$, hence $C_1(\varphi, y_0) \neq 0$.

The next Claim can be found in [39]. For completeness and the convenience of the reader, we present the detailed proof.

Claim 3: $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.

It is known from [16, Lemma 2.2] that there are $\rho > 0$ and $\xi : V \cap \overline{B_\rho} \to \mathbb{R}$ (with V as in Sect. 2 and $\overline{B_\rho} = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho\}$) such that

$$\langle \xi''(0)v, w\rangle = \langle \varphi''(y_0)v, w\rangle \quad \text{for all } v, w \in V.$$

Moreover, $\xi''(0)$ is Fredholm and $\ker \xi''(0) = H^0$ (see Sect. 2). From [15, p. 286], we have

$$C_k(\varphi, y_0) = C_k(\xi, 0) \quad \text{for all } k \geq 0.$$

Hence it follows that

$$C_1(\xi, 0) \neq 0.$$

Therefore we have $d = \dim H^- \leq 1$ (see for example [15, Theorem 2.5]). Let $d_0 = \dim H^0$.

First we assume that $d_0 = 0$. In this case the origin is a nondegenerate critical point of ξ with Morse index d. Hence

$$C_k(\xi, 0) = \delta_{k,d}\mathbb{Z} \quad \text{for all } k \geq 0.$$

It follows that $d = 1$ and so we have

$$C_k(\xi, 0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Next assume that $d_0 > 0$. In this case the origin is a degenerate critical point of ξ. Invoking the Shifting Theorem (see Chang [12]) we have

$$C_k(\xi, 0) = C_{k-d}(\hat{\xi}, 0) \quad \text{for all } k \geq 0,$$

where $\hat{\xi} = \xi|_{H^0}$.

Assume $d = 1$. Then we have

$$C_0(\hat{\xi}, 0) \neq 0$$

and so from Chang [12, Theorem 5.1.20], we have

$$C_k(\xi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Next we assume that $d=0$. We have

$$C_k(\xi,0)=C_k(\hat{\xi},0)\quad \text{for all } k\geq 0,$$

hence $C_1(\hat{\xi},0)\neq 0$.

We show that "if $\sigma(\hat{\xi}''(0))\subseteq[0,\infty)$ (the spectrum of $\hat{\xi}''(0)$ is in $\mathbb{R}_+$), then $\dim\ker\xi''(0)\leq 1$".

Under the hypothesis on the spectrum of $\hat{\xi}''(0)$, for $\ker\xi''(0)$ to be nontrivial it amounts to saying that 1 is the first eigenvalue of the weighted linear eigenvalue problem (see Sect. 2)

$$-\operatorname{div}\big(\big(1+\|b\|^2\big)Du+(p-2)(b,Du)_{\mathbb{R}^N}\big)=\lambda f_x'(z,y_0)u\quad \text{in } \Omega,\quad u|_{\partial\Omega}=0.$$

But as it is well known (see, for example, de Figueiredo [22] and Gasinski & Papageorgiou [24, Sect. 6.1]), this first eigenvalue is simple. So, we can apply Theorem 5.1.20 of Chang [12] and have

$$C_k(\varphi,y_0)=\delta_{k,1}\mathbb{Z}\quad \text{for all } k\geq 0.$$

This proves Claim 4.

From Claim 4 we have

$$C_k(\hat{\varphi},y_0)=\delta_{k,1}\mathbb{Z}\quad \text{for all } k\geq 0.$$

Claim 4: $C_{d_m}(\hat{\varphi},0)\neq 0$ where $d_m=\dim\bigoplus_{i=1}^m E(\hat{\lambda}_i(2))\geq 2$.

Set

$$Y=W_0^{1,p}(\Omega)\cap\left[\bigoplus_{i=1}^m E\big(\hat{\lambda}_i(2)\big)\right]\quad\text{and}\quad V=W_0^{1,p}(\Omega)\cap\overline{\left[\bigoplus_{i\geq m+1} E\big(\hat{\lambda}_i(2)\big)\right]}.$$

Then $W_0^{1,p}(\Omega)=Y\oplus V$.

Note that Y is finite dimensional and $Y\subseteq C_0^1(\bar{\Omega})$. So, we can find $\rho_0\in(0,1)$ such that

$$\|y\|\leq\rho_0\quad\Rightarrow\quad\|y\|_{C_0^1(\bar{\Omega})}\leq\delta_0\quad \text{for all } y\in Y.$$

Here $\delta_0>0$ is as postulated by hypothesis H_2(iii). Then for $y\in Y$ with $\|y\|\leq\rho_0$ we have

$$\begin{aligned}\varphi(y)&=\frac{1}{p}\|Dy\|_p^p+\frac{1}{2}\|Dy\|_2^2-\int_\Omega F\big(z,y(z)\big)\,dz\\&\leq\frac{1}{p}\|Dy\|_p^p+\frac{1}{2}\|Dy\|_2^2-\frac{1}{2}\int_\Omega\eta y^2\,dz\quad\big(\text{see hypothesis } H_2(\text{iii})\big)\\&\leq\frac{1}{p}\|y\|^p-\frac{c_3}{2}\|y\|^2_{H_0^1(\Omega)}\quad \text{for some } c_3>0\ \big(\text{see (5)}\big)\end{aligned}$$

$$\le \frac{1}{p}\|y\|^p - c_4\|y\|^2 \quad \text{for some } c_4 > 0$$

$(Y$ is finite dimensional so all norms are equivalent)

$$\Rightarrow \quad \varphi(y) \le 0 \quad \text{for all } \|y\| \le \hat{\rho}_0 \text{ with } \hat{\rho}_0 \le \rho_0 \text{ (since } p > 2). \tag{39}$$

Next, let $v \in V$. From hypotheses H_2(i), (iii) we have

$$F(z,x) \le \frac{\hat{\lambda}_{m+1}(2)}{2}x^2 + c_5|x|^q$$
$$\text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_5 > 0,\ p < q. \tag{40}$$

Then for $v \in V$, we have

$$\varphi(v) = \frac{1}{p}\|Dv\|_p^p + \frac{1}{2}\|Dv\|_2^2 - \int_\Omega F(z,v)\,dz$$
$$\ge \frac{1}{p}\|v\|^p - c_6\|v\|^q \quad \text{for some } c_6 > 0 \ \big(\text{see (40) and (5)}\big).$$

Because $q > p$, we can find $\tilde{\rho}_0 \in (0, \hat{\rho}_0]$ such that

$$\varphi(v) > 0 \quad \text{for all } v \in V \text{ with } 0 < \|v\| \le \tilde{\rho}_0. \tag{41}$$

From (39) and (41) we see that φ has local linking at the origin and we can apply Proposition 2.2 of Bartsch & Li [6] and infer that

$$C_{d_m}(\varphi, 0) \ne 0 \quad \Rightarrow \quad C_{d_m}(\hat{\varphi}, 0) \ne 0 \quad \big(\text{see (38)}\big).$$

This proves Claim 3.

From Claim 3 and since $d_m \ge 2$, we have that $y_0 \ne 0$. Since $y_0 \in [v_-, u_+]$, $y_0 \notin \{v_-, u_+\}$, we see that y_0 is a solution of (1) (see (30)) and the extremality of v_- and u_+ implies that y_0 is nodal. Finally from the nonlinear regularity theory (see [29, 30]) we deduce that $y_0 \in C_0^1(\bar{\Omega})$. □

Now, we can state our first multiplicity theorem for problem (1). Our theorem improves Theorem 1.1 of Sun [42], where the hypotheses on the reaction $f(z,x)$ are more restrictive, no sign information is given for the third solution, no regularity properties are established for the solutions and, finally, no location information is given for them.

Theorem 3.1 *Assume H_2 and $2 < p < \infty$. Then problem* (1) *has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_0, u_0]$ nodal.*

In fact, by strengthening the condition on $f(z,\cdot)$ near the origin (see hypothesis H_2(iii)), we can improve the conclusion of the above multiplicity theorem and produce a second nodal solution for a total of four nontrivial solutions, two of constant sign and two nodal (sign changing).

The new hypotheses on the reaction $f(z,x)$ are the following:

H_3: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that for a.a. $z\in\Omega$, $f(z,0)=0$, $f(z,\cdot)\in C^1(\mathbb{R})$, hypotheses H_3(i), (ii) are the same as the corresponding hypotheses H_2(i), (ii), and (iii) there exists an integer $m\geq 2$ such that

$$f'_x(z,0)\in\big[\hat{\lambda}_m(2),\hat{\lambda}_{m+1}(2)\big]\quad \text{a.e. in } \Omega,\ f'_x(\cdot,0)\neq\hat{\lambda}_m(2),\ f'_x(\cdot,0)\neq\hat{\lambda}_{m+1}(2)$$

and

$$f'_x(z,0)=\lim_{x\to 0}\frac{f(z,x)}{x}\quad \text{uniformly for a.a. } z\in\Omega.$$

Remark Now we do not allow for resonance to occur at the origin. Instead we have nonuniform non-resonance in the spectral interval $[\hat{\lambda}_m(2),\hat{\lambda}_{m+1}(2)]$.

Theorem 3.2 *Assume H_3 and $2<p<\infty$. Then problem* (1) *has at least four nontrivial solutions $u_0\in\operatorname{int}C_+$, $v_0\in-\operatorname{int}C_+$ and $y_0,\hat{y}\in\operatorname{int}_{C_0^1(\bar{\Omega})}[v_0,u_0]$ nodal.*

Proof From Theorem 3.1 we already have three nontrivial solutions

$$u_0\in\operatorname{int}C_+,\qquad v_0\in-\operatorname{int}C_+,\qquad \text{and}\qquad y_0\in\operatorname{int}_{C_0^1(\bar{\Omega})}[v_0,u_0]\ \text{nodal}.$$

By virtue of Proposition 3.4, we can always assume that $u_0\in\operatorname{int}C_+$ and $v_0\in-\operatorname{int}C_+$ are the extremal nontrivial constant sign solutions of (1) (that is, $u_0=u_+$, $v_0=v_-$). From Claim 2 in the proof of Proposition 3.5, we know that u_0 and v_0 are local minimizers of the functional $\hat{\varphi}$, hence

$$C_k(\hat{\varphi},u_0)=C_k(\hat{\varphi},v_0)=\delta_{k,0}\mathbb{Z}\quad \text{for all } k\geq 0. \tag{42}$$

We have

$$C_k(\hat{\varphi},y_0)=\delta_{k,1}\mathbb{Z}\quad \text{for all } k\geq 0. \tag{43}$$

Using the new stronger condition near the origin (see H_3(iii)), we can improve Claim 4 in the proof Proposition 3.5.

Claim: $C_k(\hat{\varphi},0)=\delta_{k,d_m}\mathbb{Z}$ for all $k\geq 0$, with $d_m=\dim\bigoplus_{i=1}^m E(\hat{\lambda}_i(2))\geq 2$.

Let $\epsilon\in(0,\hat{\lambda}_m(2))$. By virtue of hypothesis H_3(iii) we can find $\delta=\delta(\epsilon)>0$ such that

$$\frac{1}{2}\big[f'_x(z,0)-\epsilon\big]x^2\leq F(z,x)\leq\frac{1}{2}\big[f'_x(z,0)+\epsilon\big]x^2$$
$$\text{for a.a. } z\in\Omega,\ \text{all } |x|\leq\delta. \tag{44}$$

Let $\Psi:W_0^{1,p}(\Omega)\to\mathbb{R}$ be the C^2-functional defined by

$$\Psi(u)=\frac{1}{p}\|Du\|_p^p+\frac{1}{2}\|Du\|_2^2-\frac{1}{2}\int_\Omega f'_x(z,0)u^2\,dz\quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Evidently Ψ is coercive (recall $p>2$). Also from (44) and Chang [12, p. 336] we have

$$C_k(\varphi,0)=C_k(\Psi,0)\quad \text{for all } k\geq 0. \tag{45}$$

From H_3(iii) and Cingolani & Vannella [15] (see Theorem 1.1), we have

$$\begin{aligned} &C_k(\Psi,0)=\delta_{k,d_m}\mathbb{Z}\quad \text{for all } k\geq 0\\ &\quad\Rightarrow\quad C_k(\varphi,0)=\delta_{k,d_m}\mathbb{Z}\quad \text{for all } k\geq 0 \ \big(\text{see (45)}\big)\\ &\quad\Rightarrow\quad C_k(\hat{\varphi},0)=\delta_{k,d_m}\mathbb{Z}\quad \text{for all } k\geq 0\\ &\quad(\text{recall } \varphi|_{[v_0,u_0]}=\hat{\varphi}|_{[v_0,u_0]} \text{ and } v_0\in -\operatorname{int} C_+, u_0\in \operatorname{int} C_+). \end{aligned}$$

This proves the Claim.

Recall that $\hat{\varphi}$ is coercive (see (30)). Hence

$$C_k(\hat{\varphi},\infty)=\delta_{k,0}\mathbb{Z}\quad \text{for all } k\geq 0. \tag{46}$$

Suppose $K_{\hat{\varphi}}=\{0,u_0,v_0,y_0\}$. From (42), (43), (46), the Claim and the Morse relation (see (7)) with $t=-1$, we have $2(-1)^0+(-1)^1+(-1)^{d_m}=(-1)^0$, a contradiction. So, we can find $\hat{y}\in K_{\hat{\varphi}}\subseteq[v_0,u_0]\cap C_0^1(\bar{\Omega})$ such that $\hat{y}\notin\{0,v_0,u_0,y_0\}$. The extremality of u_0, v_0 implies that $\hat{y}\in C_0^1(\bar{\Omega})$ is the second nodal solution of (1). □

4 The Case $1<p<2$

In this section we deal with the case in which $1<p<2$. Now the ambient space is the Hilbert Sobolev space $H_0^1(\Omega)$, which creates more possibilities in the analysis of problem (1) and compensates for the fact that $-\Delta_p$ is singular.

We impose the following conditions on the reaction $f(z,x)$:

H_4: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for a.a. $z\in\Omega$, $f(z,0)=0$, $|f(z,x)|\leq a(z)(1+|x|)$ for all $x\in\mathbb{R}$ with $a\in L^\infty(\Omega)_+$ and

(i) there exist an integer $m\geq 2$ and a function $\eta\in L^\infty(\Omega)$ such that

$$\begin{aligned} &\eta(z)\leq\hat{\lambda}_{m+1}(2)\quad \text{for a.a. } z\in\Omega, \eta\neq\hat{\lambda}_{m+1}(2)\\ &\big(f(z,x)-f(z,y)\big)(x-y)\leq\eta(z)(x-y)^2\quad \text{for a.a. } z\in\Omega, \text{ all } x,y\in\mathbb{R}; \end{aligned}$$

(ii) $\hat{\lambda}_m(2)\leq\liminf_{x\to\pm\infty}\frac{f(z,x)}{x}$ uniformly for a.a. $z\in\Omega$;

(iii) $\lim_{x\to\pm\infty}\frac{f(z,x)x-2F(z,x)}{|x|^p}=-\infty$ uniformly for a.a. $z\in\Omega$;

(iv) for every $\rho>0$, there exists $\xi_\rho>0$ such that

$$f(z,x)x+\xi_\rho x^2\geq 0\quad \text{for a.a. } z\in\Omega, \text{ all } |x|\leq\rho.$$

As we already mentioned in the Introduction, by employing the Lyapunoff–Schmidt reduction technique, we will prove two multiplicity theorems for problem (1) producing three and four nontrivial solutions respectively.

Let $\varphi : H_0^1(\Omega) \to \mathbb{R}$ be the C^1-energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F\big(z, u(z)\big)\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

Also as before (see Sect. 3), let $f_\pm(z,x) = f(z, \pm x^\pm)$, $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$ and consider the C^1-functionals $\varphi_\pm : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_\pm\big(z, u(z)\big)\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

Proposition 4.1 *Assume that hypotheses H_4 hold. Then the functionals $\varphi_\pm$ satisfy the C-condition.*

Proof We do the proof for the functional φ_+, the proof for φ_- being similar.

Let $\{u_n\}_{n\geq 1} \subseteq H_0^1(\Omega)$ be a sequence such that

$$\big|\varphi_+(u_n)\big| \leq M_4 \quad \text{for some } m_4 > 0, \text{ all } n \geq 1, \tag{47}$$

$$\big(1 + \|u_n\|\big)\varphi_+'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega) = H_0^1(\Omega)^* \quad \text{as } n \to \infty. \tag{48}$$

Now $\|\cdot\|$ is the norm of $H_0^1(\Omega)$ (that is, $\|u\| = \|Du\|_2$ for all $u \in H_0^1(\Omega)$). From (48) we have for all $h \in H_0^1(\Omega)$

$$\left|\big\langle A_p(u_n), h\big\rangle + \big\langle A(u_n), h\big\rangle - \int_\Omega f_+(z, u_n)h\,dz\right| \leq \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \quad \text{with } \epsilon_n \to 0^+. \tag{49}$$

In (49) we choose $h = -u_n^- \in H_0^1(\Omega)$. Then

$$\begin{aligned} &\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 \leq \epsilon_n \quad \text{for all } n \geq 1 \\ &\Rightarrow \quad u_n^- \to 0 \text{ in } H_0^1(\Omega). \end{aligned} \tag{50}$$

Using (49) and (50), we have

$$\left|\big\langle A_p\big(u_n^+\big), h\big\rangle + \big\langle A\big(u_n^+\big), h\big\rangle - \int_\Omega f\big(z, u_n^+\big)h\,dz\right| \leq \epsilon_n'\|h\| \quad \text{with } \epsilon_n' \to 0^+. \tag{51}$$

Suppose that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|} n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^2(\Omega). \tag{52}$$

From (51) we have for all $n \geq 1$

$$\left|\frac{1}{\|u_n^+\|^{2-p}}\big\langle A_p(y_n), h\big\rangle + \big\langle A(y_n), h\big\rangle - \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|}h\,dz\right| \leq \epsilon_n'\|h\|. \tag{53}$$

From the growth condition on $f(z,\cdot)$, it is clear that $\{\frac{N_f(u_n^+)}{\|u_n^+\|}\}_{n\geq 1}\subseteq L^2(\Omega)$ is bounded. So, we may assume that

$$\frac{N_f(u_n^+)}{\|u_n^+\|}\xrightarrow{w} g \quad \text{in } L^2(\Omega). \tag{54}$$

Moreover, using H_4(i), (ii), we have

$$g=\hat{\xi}y \quad \text{with} \quad \hat{\lambda}_m(2)\leq\hat{\xi}(z)\leq\eta(z) \quad \text{a.e. in } \Omega. \tag{55}$$

In (48) we choose $h=y_n-y\in H_0^1(\Omega)$, pass to the limit as $n\to\infty$ and use (52) and (54). Then

$$\lim_{n\to\infty}\langle A(y_n),y_n-y\rangle=0 \quad (\text{recall that } p<2).$$

$$\Rightarrow \quad y_n\to y \text{in } H_0^1(\Omega) \quad (\text{see Proposition 2.4}), \text{ hence } \|y\|=1. \tag{56}$$

If in (53) we pass to the limit as $n\to\infty$ and use (54), (55) and (56), then

$$\langle A(y),h\rangle=\int_\Omega\hat{\xi}yh\,dz \quad \text{for all } h\in H_0^1(\Omega) \quad \Rightarrow \quad A(y)=\hat{\xi}y$$

$$\Rightarrow \quad -\Delta y(z)=\hat{\xi}(z)y(z) \quad \text{a.e. in } \Omega,\ y|_{\partial\Omega}=0. \tag{57}$$

If $\hat{\xi}\neq\hat{\lambda}_m(2)$ (see (55)), then by Proposition 2.3, we have

$$\begin{aligned}&\tilde{\lambda}_m(2,\hat{\xi})<\tilde{\lambda}_m\big(2,\hat{\lambda}_m(2)\big)=1 \quad \text{and}\\&\tilde{\lambda}_{m+1}\big(2,\hat{\lambda}_{m+1}(2)\big)=1<\tilde{\lambda}_{m+1}(2,\eta)\leq\tilde{\lambda}_{m+1}(2,\hat{\xi}).\end{aligned} \tag{58}$$

From (57) and (58), it follows that $y=0$, which contradicts (56).

Now suppose that $\hat{\xi}(z)=\hat{\lambda}_m(2)$ a.e. in Ω. Then $y\in E(\hat{\lambda}_m(2))\backslash\{0\}$ and so $y(z)>0$ for all $z\in\Omega$ (by the UCP and since $y\geq 0$). This implies that $u_n^+(z)\to+\infty$ for a.a. $z\in\Omega$. Then hypothesis H_4(iii) and Fatou's Lemma imply that

$$\frac{1}{\|u_n^+\|^p}\int_\Omega\big[f\big(z,u_n^+\big)u_n^+-2F\big(z,u_n^+\big)\big]\,dz\to-\infty \quad \text{as } n\to\infty. \tag{59}$$

From (47) and (50), we have for some $M_5>0$ and for all $n\geq 1$

$$\frac{2}{p}\|Du_n^+\|_p^p+\|Du_n^+\|_2^2-2\int_\Omega F\big(z,u_n^+\big)\,dz\geq -M_5.$$

Also from (51) with $h=u_n^+$, we have

$$-\|Du_n^+\|_p^p-\|Du_n^+\|_2^2+\int_\Omega f\big(z,u_n^+\big)u_n^+\geq-\epsilon_n'\|u_n^+\| \tag{60}$$

Adding (59) and (60), we obtain

$$\int_\Omega\big[f\big(z,u_n^+\big)u_n^+-2F\big(z,u_n^+\big)\big]\,dz\geq-M_6\big(1+\|Du_n^+\|_p^p+\|u_n^+\|\big)$$

$$\Rightarrow \quad \frac{1}{\|u_n^+\|^p}\int_\Omega \big[f\big(z,u_n^+\big)u_n^+ - 2F\big(z,u_n^+\big)\big]\,dz$$

$$\geq -M_6\left(\frac{1}{\|u_n^+\|^p} + \|Dy_n\|_p^p + \frac{1}{\|u_n^+\|^{p-1}}\right) \quad \text{for all } n \geq 1. \tag{61}$$

Comparing (59) and (61) and since $p > 1$, we reach a contradiction. This proves that $\{u_n^+\}_{n\geq 1} \subseteq H_0^1(\Omega)$ is bounded, hence $\{u_n\}_{n\geq 1} \subseteq H_0^1(\Omega)$ is bounded (see (50)). So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H_0^1(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^2(\Omega). \tag{62}$$

If in (49) we choose $h = u_n - u \in H_0^1(\Omega)$ and pass to the limit as $n \to \infty$, then recalling that $\{N_f(u_n)\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded, we have

$$\lim_{n\to\infty}\big[\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\big] = 0$$

$$\Rightarrow \quad \limsup_{n\to\infty}\big[\langle A_p(u), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\big] \leq 0 \quad (\text{since } A_p \text{is monotone})$$

$$\Rightarrow \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$$

$$\Rightarrow \quad u_n \to u \text{ in } H_0^1(\Omega) \quad (\text{see Proposition 2.4}).$$

This proves that the functional φ_+ satisfies the C-condition. Similarly we show that the functional φ_- satisfies the C-condition. □

Straightforward changes in the above proof lead to the same result for the functional φ.

Proposition 4.2 *Assume that hypotheses H_4 hold. Then the functional φ satisfies the C-condition.*

Next we verify the mountain pass geometry for the functionals $\varphi_\pm$.

Proposition 4.3 *Assume that hypotheses H_4 hold. Then $u = 0$ is a local minimizer for the functionals $\varphi_\pm$ and φ.*

Proof We do the proof for the functional φ_+, the proofs for φ_- and φ being similar.
Hypotheses H_4(i), (iv) imply that

$$\lim_{x\to 0}\frac{f(z,x)}{|x|^{p-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega \text{ (recall } p < 2). \tag{63}$$

So, given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$\big|f(z,x)\big| \leq \epsilon|x|^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta$$

$$\Rightarrow \quad F_+(z,x) \leq \frac{\epsilon}{p}\big(x^+\big)^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{64}$$

Let $u \in C_0^1(\bar{\Omega})$ with $\|u\|_{C_0^1(\bar{\Omega})} \leq \delta$. Then

$$\begin{aligned}\varphi_+(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_+(z,u)\,dz \\ &\geq \frac{1-\epsilon}{p}\|u\|^p \quad \big(\text{see (64)}\big).\end{aligned} \tag{65}$$

Choosing $\epsilon \in (0,1)$, from (65) we see that $u=0$ is a local $C_0^1(\bar{\Omega})$-minimizer of φ_+. Invoking Proposition 2.1, we deduce that $u=0$ is a local $H_0^1(\Omega)$-minimizer of φ_+. Similarly for the functionals φ_- and φ. □

Proposition 4.4 *Assume that hypotheses H_4 hold and $u \in E(\hat{\lambda}_{m-1}(2))\backslash\{0\}$ with $\|u\|_2 = 1$. Then $\varphi_\pm(tu) \to -\infty$ as $t \to \pm\infty$.*

Proof By virtue hypothesis H_4(ii), given $\epsilon \in (0, \hat{\lambda}_m(2) - \hat{\lambda}_{m-1}(2))$, we can find $M_7 = M_7(\epsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{2}\big(\hat{\lambda}_m(2) - \epsilon\big)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_7. \tag{66}$$

For $t > 0$, we have

$$\begin{aligned}\varphi_+(tu) &= \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\|Du\|_2^2 - \int_\Omega F_+(z,tu)\,dz \\ &= \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\|Du\|_2^2 - \int_{\{|tu|\geq M_7\}} F_+(z,tu)\,dz \\ &\quad - \int_{\{0\leq|tu|<M_7\}} F_+(z,tu)\,dz \\ &\leq \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\hat{\lambda}_{m-1}(2) - \frac{t^2}{2}\big[\hat{\lambda}_m(2) - \epsilon\big] + \xi_*(t) \\ &\qquad \text{with } \xi_*(t) \text{ bounded } \big(\text{see (66)}\big) \\ &= \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\big[\hat{\lambda}_{m-1}(2) + \epsilon - \hat{\lambda}_m(2)\big] + \xi_*(t).\end{aligned} \tag{67}$$

Since $\hat{\lambda}_{m-1}(2) + \epsilon < \hat{\lambda}_m(2)$ and $p < 2$, from (67) we infer that

$$\varphi_+(tu) \to -\infty \quad \text{as } t \to +\infty.$$

Similarly for the functional φ_-. □

Now we are ready to produce two nontrivial constant sign solutions for problem (1).

Proposition 4.5 *Assume that hypotheses H_4 hold. Then problem* (1) *has at least two nontrivial constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof By virtue of Proposition 4.3, $u = 0$ is a critical point of φ_+. Then this is an isolated critical point of φ_+. Otherwise, because $K_{\varphi_+} \subseteq C_+$, we will have a whole sequence of distinct nontrivial positive solutions of (1). So, we can find $\rho \in (0, 1)$ small such that

$$\varphi_+(0) = 0 < \inf\big[\varphi_+(u) : \|u\| = \rho\big] = \eta_\rho^+. \tag{68}$$

Combing with Propositions 4.1 and 4.4, we see that we can apply Theorem 2.1 (the mountain pass theorem) and find $u_0 \in H_0^1(\Omega)$ such that

$$u_0 \in K_{\varphi_+} \quad \text{and} \quad \eta_\rho^+ \le \varphi_+(u_0). \tag{69}$$

From (68) and (69) we deduce that $u_0 \ge 0$, $u_0 \ne 0$ and $\varphi_+'(u_0) = 0$, hence u_0 is a nontrivial solution of (1) and so $u_0 \in C_+ \backslash \{0\}$ (nonlinear regularity).

Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis H_4(iv). Then

$$\begin{aligned} &-\Delta_p u_0(z) - \Delta u_0(z) + \xi_\rho u_0(z) = f\big(z, u_0(z)\big) + \xi_\rho u_0(z) \ge 0 \quad \text{a.e. in } \Omega \\ &\Rightarrow \quad \Delta_p u_0(z) + \Delta u_0(z) \le \xi_\rho u_0(z) \quad \text{a.e. in } \Omega. \end{aligned}$$

Invoking the boundary point theorem of Pucci & Serrin [40, p. 120], we deduce that $u_0 \in \operatorname{int} C_+$. Similarly, working with the functional φ_-, we produce a nontrivial negative solution $v_0 \in -\operatorname{int} C_+$ for problem (1). □

Let $Y = \bigoplus_{i=1}^m E(\hat{\lambda}_i(2))$ and $\hat{H} = Y^\perp$, hence $H_0^1(\Omega) = Y \oplus \hat{H}$. The Lyapunoff-Schmidt reduction technique will be based on this decomposition. We should mention that the reduction technique was first developed for elliptic equations with a C^2-energy functional by Amann [3], Castro & Lazer [11] and Thews [43]. The next proposition is a crucial step in the implementation of the reduction technique.

Proposition 4.6 *Assume that hypotheses H_4 hold. Then there exists a continuous map $\gamma_0 : Y \to \hat{H}$ such that*

$$\varphi\big(y + \gamma_0(y)\big) = \inf\big[\varphi(y + \hat{u}) : \hat{u} \in \hat{H}\big] \quad \textit{for all } y \in Y.$$

Proof Let $y \in Y$ and consider the C^1-functional $\varphi_y(u) : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_y(u) = \varphi(y + u) \quad \text{for all } u \in H_0^1(\Omega).$$

Let $i : \hat{H} \to H_0^1(\Omega)$ be the inclusion map. We set $\hat{\varphi}_y = \varphi_y \circ i : \hat{H} \to \mathbb{R}$. From the chain rule we have

$$\hat{\varphi}_y'(\hat{u}) = p_{\hat{H}^*} \varphi_y'(\hat{u}) \quad \text{for all } \hat{u} \in \hat{H}, \tag{70}$$

where $p_{\hat{H}^*}$ is the orthogonal projection of $H^{-1}(\Omega) = H_0^1(\Omega)^*$ onto $\hat{H}^*$. Let $\hat{u}_1, \hat{u}_2 \in \hat{H}$. We have

$$\begin{aligned}
&\langle \hat{\varphi}_y'(\hat{u}_1) - \hat{\varphi}_y'(\hat{u}_2), \hat{u}_1 - \hat{u}_2 \rangle_{\hat{H}} \\
&\quad = \langle \varphi_y'(\hat{u}_1) - \varphi_y'(\hat{u}_2), \hat{u}_1 - \hat{u}_2 \rangle \\
&\quad = \langle A_p(y+\hat{u}_1) - A_p(y+\hat{u}_2), \hat{u}_1 - \hat{u}_2 \rangle + \langle A(\hat{u}_1 - \hat{u}_2), \hat{u}_1 - \hat{u}_2 \rangle \\
&\qquad - \int_\Omega \big[f(z, y+\hat{u}_1) - f(z, y+\hat{u}_1) \big] (\hat{u}_1 - \hat{u}_2)\, dz \\
&\quad \geq \big\| D(\hat{u}_1 - \hat{u}_2) \big\|_2^2 - \int_\Omega \eta (\hat{u}_1 - \hat{u}_2)^2\, dz \\
&\qquad \big(\text{since } A_p \text{ is monotone and see } H_4(\text{i})\big) \\
&\quad \geq C_7 \|\hat{u}_1 - \hat{u}_2\|^2 \quad \text{for some } C_7 > 0 \ \big(\text{see } H_4(\text{i}) \text{ and recall } \hat{u}_1, \hat{u}_2 \in \hat{H}\big) \\
&\qquad \Rightarrow \quad \hat{\varphi}_y' \text{ is strongly monotone} \Rightarrow \hat{\varphi}_y \text{ is strictly convex.}
\end{aligned}$$

Also, note that

$$\begin{aligned}
&\langle \hat{\varphi}_y'(\hat{u}), \hat{u} \rangle = \langle \hat{\varphi}_y'(\hat{u}) - \hat{\varphi}_y'(0), \hat{u} \rangle + \langle \hat{\varphi}_y'(0), \hat{u} \rangle \geq C_7 \|\hat{u}\|^2 - C_8 \|\hat{u}\| \\
&\quad \text{for some } C_8 > 0.
\end{aligned}$$

Thus, $\hat{\varphi}_y'$ is coercive. The map $\hat{\varphi}_y'$ is continuous and strongly monotone, hence it is maximal monotone. This fact combined with the coercivity of $\hat{\varphi}_y'$ implies that the map is surjective (see, for example, Gasinski & Papageorgiou [24, p. 320]). So, we can find $\hat{u}_0 \in \hat{H}$ such that $\hat{\varphi}_y'(\hat{u}_0) = 0$. The strong monotonicity of $\hat{\varphi}_y'$ implies that this $\hat{u}_0$ is unique and in fact is the unique global minimizer of the strictly convex functional $\hat{u} \to \hat{\varphi}_y(\hat{u})$, $\hat{u} \in \hat{H}$. So, we can define the single valued map $\gamma_0 : Y \to \hat{H}$ which to each $y \in Y$ assigns this unique global minimizer of $\hat{\varphi}_y(\cdot)$. We have

$$\begin{aligned}
&0 = \hat{\varphi}_y'\big(\gamma_0(y)\big) = p_{\hat{H}^*} \varphi'\big(y + \gamma_0(y)\big) \quad \text{and} \\
&\varphi\big(y + \gamma_0(y)\big) = \inf\big[\varphi(y + \hat{u}) : \hat{u} \in \hat{H}\big].
\end{aligned} \tag{71}$$

Next we show the continuity of the map $\gamma_0 : Y \to \hat{H}$. To this end, let $y_n \to y$ in Y. The coercivity of $\hat{\varphi}_y'$ and (71) imply that

$$\big\{\gamma_0(y_n)\big\}_{n \geq 1} \subseteq \hat{H} \subseteq H_0^1(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\gamma_0(y_n) \xrightarrow{w} h \text{ in } H_0^1(\Omega) \quad \text{and} \quad h \in \hat{H}.$$

Using the Sobolev embedding theorem, we can easily check that φ is sequentially weakly lower semi-continuous. Hence

$$\varphi(y + h) \leq \liminf_{n \to \infty} \varphi\big(y_n + \gamma_0(y_n)\big). \tag{72}$$

From (71) we have

$$\begin{aligned}
&\varphi\big(y_n+\gamma_0(y_n)\big)\le\varphi(y_n+\hat{u})\quad\text{for all }\hat{u}\in\hat{H}\\
&\quad\Rightarrow\quad \limsup_{n\to\infty}\varphi\big(y_n+\gamma_0(y_n)\big)\le\varphi(y+\hat{u})\quad\text{for all }\hat{u}\text{ in }\hat{H}\ (\text{since } y_n\to y \text{in } Y)\\
&\quad\Rightarrow\quad \varphi(y+h)\le\varphi(y+\hat{u})\quad\text{for all }\hat{u}\in\hat{H}\ \big(\text{see (72)}\big)\\
&\quad\Rightarrow\quad h=\gamma_0(y)\quad\big(\text{see (71)}\big)\\
&\quad\Rightarrow\quad \gamma_0(y_n)\xrightarrow{w}\gamma_0(y)\quad\text{in } H_0^1(\Omega). \qquad (73)
\end{aligned}$$

Moreover, again from (71), we have

$$\begin{aligned}
&p_{\hat{H}^*}\varphi'\big(y_n+\gamma_0(y_n)\big)=0\quad\text{for all } n\ge 1\\
&\quad\Rightarrow\quad p_{\hat{H}^*}\big[A_p\big(y_n+\gamma_0(y_n)\big)+A\big(y_n+\gamma_0(y_n)\big)\big]\\
&\qquad\qquad =p_{\hat{H}^*}N_f\big(y_n+\gamma_0(y_n)\big)\quad\text{for all } n\ge 1.
\end{aligned}$$

Acting on this equation with $\gamma_0(y_n)-\gamma_0(y)$ and passing to the limit as $n\to\infty$, as before (see the proof of Proposition 3.4), exploiting the monotonicity of A_p, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\big\langle A\big(y_n+\gamma_0(y_n)\big),\gamma_0(y_n)-\gamma_0(y)\big\rangle\le 0\\
&\quad\Rightarrow\quad \gamma_0(y_n)\to\gamma_0(y)\text{ in } H_0^1(\Omega)\quad(\text{see Proposition 2.4})\\
&\quad\Rightarrow\quad \gamma_0(\cdot)\text{ is continuous.}
\end{aligned}$$

□

We consider the functional $\Psi:Y\to\mathbb{R}$ defined by

$$\Psi(y)=\varphi\big(y+\gamma_0(y)\big)\quad\text{for all } y\in Y.$$

The next lemma is not immediately clear, since γ_0 is only continuous.

Lemma 4.1 *Assume that hypotheses H_4 hold. Then $\Psi\in C^1(Y)$.*

Proof Let $y,v\in Y$ and $t>0$ (the analysis is similar if $t<0$). Then

$$\begin{aligned}
&\frac{\Psi(y+tv)-\Psi(y)}{t}\le\frac{\varphi(y+tv+\gamma_0(y))-\varphi(y+\gamma_0(y))}{t}\\
&\quad\Rightarrow\quad \limsup_{t\to 0}\frac{\Psi(y+tv)-\Psi(y)}{t}\le\big\langle\varphi'\big(y+\gamma_0(y)\big),v\big\rangle. \qquad (74)
\end{aligned}$$

Also, we have

$$\frac{\Psi(y+tv)-\Psi(y)}{t}\ge\frac{\varphi(y+tv+\gamma_0(y+tv))-\varphi(y+\gamma_0(y+tv))}{t}$$

$$\Rightarrow \quad \liminf_{t\to 0} \frac{\Psi(y+tv)-\Psi(y)}{t} \geq \langle \varphi'\big(y+\gamma_0(y)\big), v\rangle$$
$$\big(\text{since } \varphi \in C^1\big(H_0^1(\Omega)\big) \text{ and } \gamma_0 \text{ is continuous}\big). \tag{75}$$

From (74) and (75) it follows that Ψ is Gâteaux differentiable at $y \in Y$ in the direction $v \in Y$ and if by $i_Y : Y \to H_0^1(\Omega)$ we denote the inclusion map, then

$$\begin{aligned} &\langle \Psi_G'(y), v\rangle_Y = \langle \varphi'\big(y+\gamma_0(y)\big), i_Y(v)\rangle \quad \text{for all } v \in Y \\ &\quad \Rightarrow \quad \Psi_G'(y) = p_{Y^*}\varphi'\big(y+\gamma_0(y)\big), \end{aligned}$$

where p_{Y^*} is the orthogonal projection of $H^{-1}(\Omega) = H_0^1(\Omega)^*$ onto Y^*. Hence by virtue of the continuity of $\gamma_0(\cdot)$ (see Proposition 4.6), we see that $y \to \Psi_G'(y)$ is continuous and this proves that $\Psi \in C^1(Y)$. □

Proposition 4.7 *Assume that hypotheses H_4 hold. Then Ψ is anticoercive* (*that is*, *if* $\|y\| \to +\infty$, $y \in Y$, *then* $\Psi(y) \to -\infty$).

Proof We argue by contradiction. So, suppose we can find $\{y_n\}_{n\geq 1} \subseteq Y$ and $M_8 > 0$ such that $\|y_n\| \to \infty$ and $\Psi(y_n) \geq -M_8$ for all $n \geq 1$.

We have

$$-M_8 \leq \Psi(y_n) \leq \varphi(y_n) = \frac{1}{p}\|Dy_n\|_p^p + \frac{1}{2}\|Dy_n\|_2^2 - \int_\Omega F(z, y_n)\,dz. \tag{76}$$

Let $v_n = \frac{y_n}{\|y_n\|} n \geq 1$. Then $v_n \in Y$ and $\|v_n\| = 1$ for all $n \geq 1$. The finite dimensionality of Y implies that at least for a subsequence, we have

$$v_n \to v \text{ in } H_0^1(\Omega) \quad \text{and} \quad v \in Y, \|v\| = 1. \tag{77}$$

From (76) he have

$$-\frac{M_8}{\|y_n\|^2} \leq \frac{1}{p}\frac{1}{\|y_n\|^{2-p}}\|Dv_n\|_p^p + \frac{1}{2}\|Dv_n\|_2^2 - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^2}\,dz. \tag{78}$$

Hypothesis H_4(i) implies that

$$\left\{\frac{F(\cdot, y_n(\cdot))}{\|y_n\|^2}\right\}_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

So, by the Dunford-Pettis theorem and hypothesis H_4(i), we have

$$\frac{F(\cdot, y_n(\cdot))}{\|y_n\|^2} \xrightarrow{w} \frac{1}{2}\xi^* v^2 \text{in } L^1(\Omega) \quad \text{with} \quad \xi^*(z) \leq \eta(z) \text{ a.e. in } \Omega. \tag{79}$$

Passing to the limit as $n \to \infty$ in (78) and using (77) and (79), we obtain

$$0 \leq \frac{1}{2}\|Dv\|_2^2 - \frac{1}{2}\int_\Omega \xi^* v^2\,dz < 0 \quad \text{when } \xi^* \neq \hat{\lambda}_m(2) \text{ (recall } p < 2),$$

a contradiction. If $\xi^* = \hat{\lambda}_m(2)$, then $v \in E(\hat{\lambda}_m(2)) \setminus \{0\}$ and so the argument of Proposition 4.1 leads again to a contradiction. This proves the anti-coercivity of Ψ. □

In particular the above proposition implies that Ψ satisfies the C-condition (since Ψ is coercive). Now we can state the first multiplicity results for problem (1) when $1 < p < 2$.

Theorem 4.1 *Assume H_4 and $1 < p < 2$. Then problem* (1) *has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in C_0^1(\bar{\Omega})$.*

Proof From Proposition 4.5, we already have two nontrivial solutions of constant sign, namely $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$. From the proof of Proposition 4.5 we know that $u_0 \in \operatorname{int} C_+$ is a critical point of mountain pass type for the functional φ_+, while $v_0 \in -\operatorname{int} C_+$ is a critical point of mountain pass type for the functional φ_-. We know that

$$\begin{aligned} &\varphi_{+|C_+} = \varphi_{|C_+} \quad \text{and} \quad \varphi_{-|-C_+} = \varphi_{|-C_+} \\ &\quad \Rightarrow \quad C_k(\varphi_+|_{C_0^1(\bar{\Omega})}, u_0) = C_k(\varphi|_{C_0^1(\bar{\Omega})}, u_0) \text{ and } C_k(\varphi_-|_{C_0^1(\bar{\Omega})}, v_0) \\ &\qquad\qquad = C_k(\varphi|_{C_0^1(\bar{\Omega})}, v_0) \quad \text{for all } k \geq 0. \end{aligned} \tag{80}$$

From Bartsch [5, Proposition 2.6] and Palais [38] we have for all $k \geq 0$

$$C_k(\varphi_+|_{C_0^1(\bar{\Omega})}, u_0) = C_k(\varphi_+, u_0) \quad \text{and} \quad C_k(\varphi|_{C_0^1(\bar{\Omega})}, u_0) = C_k(\varphi, u_0) \tag{81}$$

$$C_k(\varphi_-|_{C_0^1(\bar{\Omega})}, v_0) = C_k(\varphi_-, v_0) \quad \text{and} \quad C_k(\varphi|_{C_0^1(\bar{\Omega})}, v_0) = C_k(\varphi, v_0). \tag{82}$$

From (80), (81), (82) and since u_0 and v_0 are critical points of mountain pass type for φ_+ and φ_- respectively, we have

$$C_1(\varphi, u_0) \neq 0, \quad C_1(\varphi, v_0) \neq 0. \tag{83}$$

Let $\bar{u}_0 = p_Y(u_0)$ and $\bar{v}_0 = p_Y(v_0)$. From Liu & Li [32], we have

$$\begin{aligned} &C_k(\varphi, u_0) = C_k(\Psi, \bar{u}_0) \quad \text{and} \quad C_k(\varphi, v_0) = C_k(\Psi, \bar{v}_0) \quad \text{for all } k \geq 0 \\ &\quad \Rightarrow \quad C_1(\Psi, \bar{u}_0) \neq 0 \quad \text{and} \quad C_1(\Psi, \bar{v}_0) \neq 0 \quad \big(\text{see (83)}\big). \end{aligned} \tag{84}$$

From Proposition 4.7 we know that Ψ is anticoercive on Y. Hence by the Weierstrass theorem, we can find $\bar{y}_0 \in Y$ such that

$$\begin{aligned} &\Psi(\bar{y}_0) = \max\big[\Psi(y) : y \in Y\big] \\ &\quad \Rightarrow \quad C_k(\Psi, \bar{y}_0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0 \\ &\qquad\qquad \text{with } d_m = \dim \bigoplus_{i=1}^{m} E\big(\hat{\lambda}_i(2)\big) \geq 2, \end{aligned} \tag{85}$$

see Chang [12].

Finally from Proposition 4.3, we know that $u = 0$ is a local minimizer of φ, hence

$$C_k(\Psi, 0) = C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \ge 0 \text{ (see Liu \& Li [32])}. \tag{86}$$

From (84), (85) and (86) we infer that $\bar{y}_0 \notin \{0, \bar{u}_0, \bar{v}_0\}$. Therefore, if $y_0 = \bar{y}_0 + \gamma_0(\bar{y}_0)$, then y_0 is a critical point of φ distinct from $\{0, u_0, v_0\}$. This is the third nontrivial solution of (1) and the nonlinear regularity theory (see [29, 30]) implies that $y_0 \in C_0^1(\bar{\Omega})$. □

Next, by strengthening the regularity of $f(z, \cdot)$, we can improve the above multiplicity theorem and produce four nontrivial solutions.

To this end, first we compute the critical groups of Ψ at infinity. To do this we do not need the stronger conditions on $f(z, \cdot)$ and in the proof we use some ideas of Liu [31].

Proposition 4.8 *If hypotheses H_4 hold, then $C_k(\Psi, \infty) = \delta_{k,d_m}\mathbb{Z}$ for all $k \ge 0$.*

Proof Let $m_0 < \inf \Psi(K_\Psi)$. Since Ψ is anti-coercive (see Proposition 4.7), we can find $\eta < \vartheta < m_0$ and $0 < \rho < R$ such that $C_R \subseteq \Psi^\eta \subseteq C_\rho \subseteq \Psi^\vartheta$, where for every $r > 0$, $C_r = \{y \in Y : \|y\| \ge r\}$.

For the triples (C_R, C_ρ, Y) and $(\Psi^\eta, \Psi^\vartheta, Y)$ we consider the corresponding long exact sequences of homology groups. So, we have

$$\begin{array}{ccccccccc}
\cdots \to & H_k(C_\rho, C_R) & \xrightarrow{i_*} & H_k(Y, C_R) & \xrightarrow{j_*} & H_k(Y, C_\rho) & \xrightarrow{\partial_*} & H_{k-1}(C_\rho, C_R) & \to \cdots \\
& \downarrow h_{*|C_\rho} & & \downarrow h_* & & \downarrow h_* & & \downarrow h_{*|C_\rho} & \\
\cdots \to & H_k(\Psi^\vartheta, \Psi^\eta) & \xrightarrow{\hat{i}_*} & H_k(Y, \Psi^\eta) & \xrightarrow{\hat{j}_*} & H_k(Y, \Psi^\vartheta) & \xrightarrow{\hat{\partial}_*} & H_{k-1}(\Psi^\vartheta, \Psi^\eta) &
\end{array} \tag{87}$$

In (87) all the squares are commutative (see Granas & Dugundji [25] (p.377)) and the map $i_*, j_*, \hat{i}_*, \hat{j}_*, h_*$ are the group homomorphisms induced by the corresponding inclusion maps. Also, ∂_* and $\hat{\partial}_*$ are the corresponding boundary homomorphisms. Since $\eta < \vartheta < m_0 < \inf \Psi(K_\Psi)$, we have that Ψ^η is a strong deformation retract of Ψ^ϑ (by the second deformation theorem, see [24, p. 628]) and so

$$H_k\big(\Psi^\vartheta, \Psi^\eta\big) = 0 \quad \text{for all } k \ge 0. \tag{88}$$

Consider the map $\sigma : C_\rho \to C_R$ defined by

$$\sigma(u) = \begin{cases} R\frac{u}{\|u\|} & \text{if } \rho \le \|u\| \le R \\ u & \text{if } R < \|u\|. \end{cases}$$

Clearly σ is continuous and $\sigma|_{C_R} = id|_{C_R}$, So, C_R is a retract of C_ρ. Moreover, if $h : [0, 1] \times C_\rho \to Y$ is defined by

$$h(t, u) = (1 - t)u + tR\frac{u}{\|u\|} \quad \text{for all } t \in [0, 1], \text{ all } u \in C_\rho,$$

then we see that C_ρ is deformable into C_R in Y. Therefore, invoking Theorem 6.5 of Dugundji [18, p. 325], we infer that C_R is a deformation retract of C_ρ. Hence

$$H_k(C_\rho, C_R) = 0 \quad \text{for all } k \geq 0. \tag{89}$$

(see Granas & Dugundji [25] (p. 387)). From the exactness of the long homology sequences in (87), we have

$$0 = \operatorname{im} i_* = \ker j_* \text{ and } \operatorname{im} j_* = \ker \partial_* = H_k(Y, C_\rho), \quad \text{see (88)}$$
$$0 = \operatorname{im} \hat{i}_* = \ker \hat{j}_* \text{ and } \operatorname{im} \hat{j}_* = \ker \hat{\partial}_* = H_k\big(Y, \Psi^\vartheta\big), \quad \text{see (89)}.$$

If follows that both j_* and $\hat{j}_*$ are group isomorphisms. Then invoking Lemma D.1 of Granas & Dugundji [25, p. 610], we deduce that h_* is an isomorphism. So, for all $k \geq 0$

$$H_k(Y, C_\rho) = H_k\big(Y, \Psi^\vartheta\big) \quad \Rightarrow \quad H_k(Y, C_\rho) = C_k(\Psi, \infty). \tag{90}$$

As before, using the radical retraction and Theorem 6.5 of Dugundji [18, p. 325], we show that $\partial B_\rho = \{y \in Y : \|y\| = \rho\}$ is a deformation retract of C_ρ. Therefore

$$H_k(Y, C_\rho) = H_k(Y, \partial B_\rho) \quad \text{for all } k \geq 0$$
$$\Rightarrow \quad H_k(Y, C_\rho) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0 \ \big(\text{see Maunder [35, p. 121]}\big)$$
$$\Rightarrow \quad C_k(\Psi, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0 \ \big(\text{see (90)}\big). \qquad \square$$

The new stronger conditions on $f(z, x)$ which we will need in order to prove a four solutions theorem for problem (1) when $1 < p < 2$, are the following:

H_5: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) there exist an integer $m \geq 2$ and a function $\eta \in L^\infty(\Omega)$ such that

$$\eta(z) \leq \hat{\lambda}_{m+1}(2) \quad \text{a.e. in } \Omega, \eta \neq \hat{\lambda}_{m+1}(2) \quad \text{and}$$
$$\big|f'_x(z, x)\big| \leq \eta(z) \quad \text{a.e. in } \Omega, \text{ for all } x \in \mathbb{R};$$

(ii) $\hat{\lambda}_m \leq \liminf_{x \to \pm\infty} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x \to \pm\infty} \frac{f(z,x)x - 2F(z,x)}{|x|^p} = -\infty$ uniformly for a.a. $z \in \Omega$.

Remark From hypothesis H_5(i) and the mean value theorem we have

$$\left|\frac{f(z, x)}{x}\right| \leq \eta(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}\backslash\{0\}.$$

Remark Similarly, we see for all $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, $x \to f(z, x) + \xi_\rho x$ is nondecreasing on $[-\rho, \rho]$.

Then we can state the following multiplicity theorem for problem (1) (case $1 < p < 2$).

Theorem 4.2 *Assume H_5 and $1 < p < 2$. Then problem* (1) *has at least four nontrivial solutions*. $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $y_0, \hat{y} \in C_0^1(\bar{\Omega})$.

Proof From Theorem 4.1, we already have three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in C_0^1(\bar{\Omega})$.

We know that $\varphi \in C^2(H_0^1(\Omega)\backslash\{0\})$. Then as in Claim 3 in the proof of Proposition 3.5 (see also Motreanu, Motreanu & Papageorgiou [37], proof of Theorem 4.2) we can apply Proposition 2.5 of Bartsch [6] and have that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0 \tag{91}$$

(see relation (85) and recall that $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ and φ is C^2 in neighborhoods $\tilde{\mathcal{U}}$, $\tilde{\mathcal{V}}$ of u_0 and v_0 in the $C_0^1(\overline{\Omega})$ space).

From (85) and since $C_k(\varphi, y_0) = C_k(\Psi, \bar{y}_0)$ for all $k \geq 0$ (see [32]), it follows that

$$C_k(\varphi, y_0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{92}$$

From Proposition 4.8, we have for all $k \geq 0$

$$C_k(\Psi, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \Rightarrow \quad C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{(see [32])}. \tag{93}$$

Finally from Proposition 4.3, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{94}$$

Suppose $K_\varphi = \{0, u_0, v_0, y_0\}$. then from (91), (92), (93), (94) and the Morse relation (see (7)) with $t = -1$, we have $2(-1)^1 + (-1)^0 + (-1)^{d_m} = (-1)^{d_m}$, a contradiction. Thus, there exists $\hat{y} \in K_\varphi$ such that $\hat{y} \notin \{0, u_0, v_0, y_0\}$. This the fourth nontrivial solution of problem (1) and by the nonlinear regularity theory we have $\hat{y} \in C_0^1(\bar{\Omega})$. □

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