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Set-valued equilibrium problems with applications to Browder variational inclusions and to fixed point theory

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ABSTRACT

In this paper, we deal with set-valued equilibrium problems under mild conditions of continuity and convexity on subsets recently introduced in the literature. We obtain that neither semicontinuity nor convexity are needed on the whole domain when solving set-valued and single-valued equilibrium problems. As applications, we derive some existence results for Browder variational inclusions, and we extend the well-known Berge maximum theorem in order to obtain two versions of Kakutani and Schauder fixed point theorems.

1. Introduction

On the trail of Browder's study of variational inclusions [1], many authors have been interested in inclusions involving set-valued mappings, see for instance [2–6]. In recent years, the notion of set-valued equilibrium problem has been employed in [3,5] in connection with the so-called equilibrium problem or inequality of Ky Fan-type (see [7–9]) which has produced an abundance of results in various areas of mathematics.

Let C be a nonempty subset of a (suitable) Hausdorff topological space and $\Phi : C \times C \rightrightarrows \mathbb{R}$ a set-valued mapping. Following [3,5], a *set-valued equilibrium problem* is a problem of the form

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \subset \mathbb{R}_+ \quad \forall y \in C. \tag{SVEP}$$

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We will also consider in the paper the following weaker set-valued equilibrium problem

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \cap \mathbb{R}_+ \neq \emptyset \quad \forall y \in C. \tag{SVEP(W)}$$

Recall that the so-called equilibrium problem is a problem of the form

$$\text{find } x^* \in C \text{ such that } \varphi(x^*, y) \geq 0 \quad \forall y \in C \tag{EP}$$

where $\varphi : C \times C \to \mathbb{R}$ is a bifunction.

It is well known that several problems arising in nonlinear analysis such as variational inequality problems, optimization problems, inverse optimization problems, mathematical programming, complementarity problems, fixed point problems and Nash equilibrium problems are special cases of equilibrium problems, see [10–12,2,13,14,3–6] and the references therein. As already mentioned in the literature, it is worth recalling that one of the interest of the equilibrium problem is that it unifies, at least, all the above mentioned problems in a common formulation and many techniques and methods established in order to solve one of them may be extended, with suitable adaptations, to equilibrium problems.

In the investigation about solving equilibrium problems, it has been considered recently in [15,16,10–12] the notion of hemicontinuity and semicontinuity on a subset. Various results on the existence of solutions of equilibrium problems have been obtained without the hemicontinuity and the semicontinuity of the bifunction on the whole domain, but just on the set of coerciveness.

On the other hand, a notion of self-segment-dense subset has been introduced in [5] allowing the authors to obtain some generalizations of the results of [3] on set-valued equilibrium problems.

In this paper, we deal mainly with set-valued equilibrium problems. We introduce and develop some notions of semicontinuity of set-valued mappings on a subset where the beginning notions have been first considered in [15,16,10–12]. We give some characterizations of lower and upper semicontinuity of set-valued mappings on a subset by means of lower and upper inverse sets in order to establish our results rather than using generalized sequences (nets).

In Section 2, we present the notions of semicontinuity of extended real valued functions and the semicontinuity of set-valued mappings, and give some preliminary results we need in the sequel. We also recall the necessary background on the subject such as the Ky Fan lemma and the notion of KKM mapping. The notion of self-segment-dense subset and its related results are also given.

In Section 3, we deal with single-valued and set-valued equilibrium problems and obtain, under these mild conditions of semicontinuity and convexity, existence results for the above three equilibrium problems considered in the paper.

In Section 4, we apply our results to Browder variational inclusions and obtain as a corollary an existence result for the well-known Browder–Hartman–Stampacchia variational inequality problems. We also give a generalization to the Berge maximum theorem and apply it to carry out two versions of Kakutani and Schauder fixed point theorems.

2. Notations and preliminary results

In this section we give the necessary background related to continuity and convexity of functions and set-valued mappings we need in the paper. We also establish some characterizations and preliminary results which will play a key role in the sequel.

In all the paper, $\mathbb{R} =]-\infty, +\infty[$ denotes the set of real numbers and $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$. We also make use of the following notation: $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\mathbb{R}_-^* = -\mathbb{R}_+^*$.

Let X be a Hausdorff topological space. Recall that an extended real valued function $f : X \to \overline{\mathbb{R}}$ is said to be *lower semicontinuous* at $x_0 \in X$ if for every $\epsilon > 0$, there exists an open neighborhood U of x_0 such that

$$f(x) \geq f(x_0) - \epsilon \quad \forall x \in U.$$

A function $f : X \to \overline{\mathbb{R}}$ is said to be *upper semicontinuous* at x_0 if $-f$ is lower semicontinuous at x_0.

We have considered extended real valued functions in the above definitions because such functions are more general and convenient in our study. In particular, they are crucial in the last section of this paper. As mentioned in [17], considering such definitions for extended real valued functions is also convenient in many purposes of variational analysis studies.

A function $f : X \to \overline{\mathbb{R}}$ is said to be lower (*resp.* upper) semicontinuous on a subset S of X if it is lower (*resp.* upper) semicontinuous at every point of S. Obviously, if f is lower (*resp.* upper) semicontinuous on a subset S of X, then the restriction $f_{|S} : S \to \overline{\mathbb{R}}$ of f on S is lower (*resp.* upper) semicontinuous on S. The converse does not hold true in general.

The following results are more general and make more precise those in [12] obtained for real valued functions, see also [15,16,10,11] for older versions.

Proposition 2.1. *Let X be Hausdorff topological space, $f : X \to \overline{\mathbb{R}}$ a function and let S be a subset of X. If the restriction $f_{|U}$ of f on an open subset U containing S is upper (resp. lower) semicontinuous on S, then any extension of $f_{|U}$ to the whole space X is upper (resp. lower) semicontinuous on S.*

Proposition 2.2. *Let X be Hausdorff topological space, $f : X \to \overline{\mathbb{R}}$ a function and S a subset of X. Then, the following statements hold.*

1. *The following conditions are equivalent*
   (1) *f is lower semicontinuous on S;*
   (2) *for every $a \in \mathbb{R}$,*
   $$cl(\{x \in X \mid f(x) \leq a\}) \cap S = \{x \in S \mid f(x) \leq a\};$$
   (3) *for every $a \in \mathbb{R}$,*
   $$int(\{x \in X \mid f(x) > a\}) \cap S = \{x \in S \mid f(x) > a\};$$
   In particular, if f is lower semicontinuous on S, then the trace on S of any lower level set of f is closed in S and the trace on S of any strict upper level set of f is open in S.
2. *The following conditions are equivalent*
   (a) *f is upper semicontinuous on S;*
   (b) *for every $a \in \mathbb{R}$,*
   $$cl(\{x \in X \mid f(x) \geq a\}) \cap S = \{x \in S \mid f(x) \geq a\};$$
   (c) *for every $a \in \mathbb{R}$,*
   $$int(\{x \in X \mid f(x) < a\}) \cap S = \{x \in S \mid f(x) < a\}.$$
   In particular, if f is upper semicontinuous on S, then the trace on S of any upper level set of f is closed in S and the trace on S of any strict lower level set of f is open in S.

Let X and Y be two Hausdorff topological spaces. We also say that a function $f : X \to Y$ is continuous on a subset S of X if it is continuous at every point of S. Recall that f is said to be continuous at $x_0 \in X$ if for every neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0.

In the sequel, we denote by $F : X \rightrightarrows Y$ a set-valued mapping from X to Y. The graph of F is the set

$$\operatorname{grph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

For a subset B of Y, we define

$$F^{-}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$$

the lower inverse set of B by F. We also define

$$F^{+}(B) = \{x \in X \mid F(x) \subset B\}$$

the upper inverse set of B by F. The upper inverse set of B by F is called sometimes the core of B, see [18]. It is easily seen that for every subset B of Y, we have

$$F^{+}(B) = X \setminus F^{-}(Y \setminus B).$$

This characterization provides an important relation between lower and upper inverse sets.

Recall that a set-valued mapping $F : X \rightrightarrows Y$ is said to be lower semicontinuous at a point $x_0 \in X$ if whenever V is an open subset of Y such that $F(x_0) \cap V \neq \emptyset$, the lower inverse set $F^{-}(V)$ of V by F is a neighborhood of x_0. It turns out that F is lower semicontinuous at $x_0 \in X$ if and only if F is continuous at $x_0 \in X$ as a function from X to the set of subsets of Y endowed with the lower Vietoris topology.

By analogy, a set-valued mapping $F : X \rightrightarrows Y$ is said to be upper semicontinuous at a point $x_0 \in X$ if it is continuous at $x_0 \in X$ as a function from X to the set of subsets of Y endowed with the upper Vietoris topology. That is, F is upper semicontinuous at $x_0 \in X$ if whenever V is an open subset of Y such that $F(x_0) \subset V$, the upper inverse set $F^{+}(V)$ of V by F is a neighborhood of x_0.

A set-valued mapping $F : X \rightrightarrows Y$ is said to be continuous at a point $x_0 \in X$ if it is lower and upper semicontinuous at $x_0 \in X$.

The set-valued mapping F is said to be lower semicontinuous on X if it is lower semicontinuous at every point of X. The continuity and upper semicontinuity on the space X are defined in the same manner. Clearly, F is lower (*resp.* upper) semicontinuous on X if and only if the lower (*resp.* upper) inverse set of any open subset V of Y is open.

We say that a set-valued mapping $F : X \rightrightarrows Y$ is lower semicontinuous (*resp.* upper semicontinuous, *resp.* continuous) on a subset S of X if it is lower semicontinuous (*resp.* upper semicontinuous, *resp.* continuous) at every point of S.

The following result shows how easy is to construct lower (*resp.* upper) semicontinuous set-valued mappings on a subset without being lower semicontinuous on the whole space. It is easy to prove.

Proposition 2.3. *Let X and Y be two topological spaces, $F : X \rightrightarrows Y$ a set-valued mapping and let S be a subset of X. If the restriction $F_{|U} : U \rightrightarrows Y$ of F on an open subset U containing S is lower (resp. upper) semicontinuous, then any extension of $F_{|U}$ to the whole space X is lower (resp. upper) semicontinuous on S.*

For a subset S of X, we denote respectively by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, the closure and the interior of S with respect to X.

The following lemma provides us with a characterization of lower and upper semicontinuity of set-valued mappings on a subset.

Proposition 2.4. *Let X and Y be two topological spaces, $F : X \rightrightarrows Y$ a set-valued mapping and let S be a subset of X. Then, the following statements hold.*

(1) *The following conditions are equivalent:*
   (a) *F is lower semicontinuous on S;*
   (b) *for every open subset V of Y, we have*
   $$F^{-}(V) \cap S = \mathit{int}\left(F^{-}(V)\right) \cap S;$$
   (c) *for every closed subset B of Y, we have*
   $$F^{+}(B) \cap S = \mathit{cl}\left(F^{+}(B)\right) \cap S.$$
   In particular, if F is lower semicontinuous on S, then $F^{-}(V) \cap S$ is open in S for every open subset V of Y, and $F^{+}(B) \cap S$ is closed in S for every closed subset B of Y.

(2) *The following conditions are equivalent:*
   (a) *F is upper semicontinuous on S;*
   (b) *for every open subset V of Y, we have*
$$F^{+}(V) \cap S = int\left(F^{+}(V)\right) \cap S;$$
   (c) *for every closed subset B of Y, we have*
$$F^{-}(B) \cap S = cl\left(F^{-}(B)\right) \cap S.$$
   In particular, if F is upper semicontinuous on S, then $F^{+}(V) \cap S$ is open in S for every open subset V of Y, and $F^{-}(B) \cap S$ is closed in S for every closed subset B of Y.

Proof. Since the second statement is similar to the first one, we state only the case of a lower semicontinuous set-valued mapping.

Assume first that F is lower semicontinuous on S and let V be an open subset of Y. Then, for every $x \in F^{-}(V) \cap S$, $F^{-}(V)$ is a neighborhood of x which implies that $x \in \operatorname{int}(F^{-}(V))$. Thus, (1a) $\Longrightarrow$ (1b).

To prove (1b) $\Longrightarrow$ (1c), let B be a closed subset of Y and put $V = Y \setminus B$ which is open. By the properties of lower and upper inverse sets, we have
$$\begin{aligned}
\operatorname{cl}\left(F^{+}(B)\right) \cap S &= \left(X \setminus \left(\operatorname{int}\left(X \setminus F^{+}(B)\right)\right)\right) \cap S \\
&= \left(X \setminus \left(\operatorname{int}\left(F^{-}(V)\right)\right)\right) \cap S \\
&= S \setminus \left(\operatorname{int}\left(F^{-}(V)\right) \cap S\right) \\
&= S \setminus \left(F^{-}(V) \cap S\right) \\
&= \left(X \setminus \left(F^{-}(V)\right)\right) \cap S \\
&= F^{+}(B) \cap S.
\end{aligned}$$

To prove (1c) $\Longrightarrow$ (1a), let $x \in S$ and V be an open subset of Y such that $F(x) \cap V \neq \emptyset$. It follows that $x \in F^{-}(V)$ and then, $x \notin F^{+}(B)$ where $B = Y \setminus V$. Since $F^{+}(B) \cap S = \operatorname{cl}(F^{+}(B)) \cap S$, it follows that $x \notin \operatorname{cl}(F^{+}(B))$ which implies that $x \in \operatorname{int}(X \setminus F^{+}(B)) = \operatorname{int}(F^{-}(V))$. Therefore $F^{-}(V)$ is a neighborhood of x. □

It is worthwhile noticing that based on the notion of lower and upper limit of nets of subsets in the sense of Kuratowski–Painlevé convergence, the lower and upper semicontinuity of set-valued mappings can be also characterized by means of nets (see [18–20]). Although, these characterizations are important in many studies, we will not follow this approach in our proofs, but make use of the techniques developed in Proposition 2.4 which are based only on lower and upper inverse sets.

For our purpose, we also need the following notions of convexity of functions and set-valued mappings defined on real topological Hausdorff vector spaces.

Let X be a real topological Hausdorff vector space and D a (non necessarily convex) subset of X. The following definitions have been introduced in [5].

(1) A function $f : D \longrightarrow \mathbb{R}$ is said to be
   (a) convex on D if for every finite subset $\{x_1, \dots, x_n\} \subset D$ and $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\sum_{i=1}^{n} \lambda_i x_i \in D$, then
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i);$$
   (b) concave on D if $-f$ is convex on D.
(2) A set-valued mapping $F : D \rightrightarrows \mathbb{R}$ is said to be
   (a) convex on D if for every finite subset $\{x_1, \dots, x_n\} \subset D$ and $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\sum_{i=1}^{n} \lambda_i x_i \in D$, then

$$F\left(\sum_{i=1}^{n}\lambda_i x_i\right)\supset\sum_{i=1}^{n}\lambda_i F(x_i)$$

where the sum denotes here the usual Minkowski sum of sets;

(b) concave on D if instead of the last inclusion, the following holds

$$F\left(\sum_{i=1}^{n}\lambda_i x_i\right)\subset\sum_{i=1}^{n}\lambda_i F(x_i).$$

Note that the notion of convex set-valued mappings on the whole space X has been already considered in the literature. One can easily verify that a set-valued mapping $F : X \rightrightarrows \mathbb{R}$ is convex on X if and only if its graph is convex.

We say that a function $f : D \to \mathbb{R}$ is quasiconvex on D if for every finite subset $\{x_1, \ldots, x_n\} \subset D$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n}\lambda_i = 1$ and $\sum_{i=1}^{n}\lambda_i x_i \in D$, then

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right)\le\max_{i=1,\ldots,n} f(x_i).$$

In the quest of weakening conditions when solving equilibrium problems, a suitable notion of denseness has been recently introduced in [5]. Let X be a real topological Hausdorff vector space. For $x, y \in X$, we denote the closed line segment in X with the endpoints x and y by

$$[x, y] = \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}.$$

Let V be a convex subset of X. Following [5], a subset U of V is said to be self-segment-dense set in V if

(1) $V \subset \mathrm{cl}(U)$;
(2) for every $x, y \in U$, $[x, y] \subset \mathrm{cl}([x, y] \cap U)$.

The importance of the notion of self-segment-dense set has been highlighted in [5] and especially for dimensions greater than one. The following result (see [5, Lemma 3.1]) has been also obtained and it is important in the sequel. It is valid in the settings of Hausdorff locally convex topological vector spaces since the origin has a local base of convex, balanced and absorbent sets.

Lemma 2.5. *Let X be a Hausdorff locally convex topological vector space, V a convex set of X and let $U \subset V$ a self-segment-dense set in V. Then, for all finite subset $\{x_1, \ldots, x_n\} \subset U$, we have*

$$cl\,(conv\,\{x_1, \ldots, x_n\} \cap U) = conv\,\{x_1, \ldots, x_n\}.$$

We also need in the sequel the notion of KKM mappings and the well-known intersection lemma due to Ky Fan, see [21].

Let X be a real topological Hausdorff vector space and M a subset of X. Recall that a set-valued mapping $F : M \rightrightarrows X$ is said to be a KKM mapping if for every finite subset $\{x_1, \ldots, x_n\}$ of M, we have

$$\mathrm{conv}\,\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

It is well known by Ky Fan's lemma [21] that if

(1) F is a KKM mapping,
(2) $F(x)$ is closed for every $x \in M$ and
(3) there exists $x_0 \in M$ such that $F(x_0)$ is compact,

then $\bigcap_{x\in M} F(x) \neq \emptyset$.

3. Existence of solutions of set-valued equilibrium problems

As well known, the compactness of the domain C in the existence of solutions of equilibrium problems is a rather restrictive condition. We will consider here a condition involving a set of coerciveness. Note that due to the restriction imposed in Lemma 2.5, all the results obtained in this section remain true on real topological Hausdorff vector spaces if we omit the condition involving a self-segment-dense set. The following results extend the corresponding ones obtained in [5].

Theorem 3.1. *Let X be a Hausdorff locally convex topological vector space, C a nonempty, closed and convex subset of X and $D \subset C$ a self-segment-dense set in C. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

(1) $\forall x \in D$, $\Phi(x,x) \subset \mathbb{R}_+$;
(2) $\forall x \in D$, $y \mapsto \Phi(x,y)$ *is convex on* D;
(3) $\forall x \in C$, $y \mapsto \Phi(x,y)$ *is lower semicontinuous on* $C \setminus D$;
(4) *there exist a compact set* K *of* C *and* $y_0 \in D$ *such that* $\Phi(x,y_0) \cap \mathbb{R}_-^* \neq \emptyset$, $\forall x \in C \setminus K$;
(5) $\forall y \in D$, $x \mapsto \Phi(x,y)$ *is lower semicontinuous on* K.

Then, the set-valued equilibrium problem (SVEP) has a solution.

Proof. We define the set-valued mapping $\Phi^+ : C \rightrightarrows C$ by

$$\Phi^+(y) = \{x \in C \mid \Phi(x,y) \subset \mathbb{R}_+\} \quad \forall y \in C.$$

Clearly, $x_0 \in C$ is a solution of the set-valued equilibrium problem (SVEP) if and only if $x_0 \in \bigcap_{y \in C} \Phi^+(y)$.

Assumption (1) yields $\Phi^+(y)$ is nonempty, for every $y \in D$. Now, consider the set-valued mapping $\mathrm{cl}(\Phi^+) : D \rightrightarrows \mathbb{R}$ defined by

$$\mathrm{cl}\left(\Phi^+\right)(y) = \mathrm{cl}\left(\Phi^+(y)\right) \quad \forall y \in D.$$

Clearly, $\mathrm{cl}(\Phi^+)(y)$ is closed for every $y \in D$, and $\mathrm{cl}(\Phi^+)(y_0)$ is compact since it lies in K by assumption (4).

Now we prove that, the set-valued mapping $\mathrm{cl}(\Phi^+)$ is a KKM mapping. Let $\{y_1,\ldots,y_n\} \subset D$ be a finite subset and $\{\lambda_1,\ldots,\lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$. First we assume that $\sum_{i=1}^n \lambda_i y_i \in D$. Then, by assumption (1) and assumption (2), we have

$$\sum_{i=1}^n \lambda_i \Phi\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) \subset \Phi\left(\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i y_i\right) \subset \mathbb{R}_+.$$

The convexity of $\mathbb{R}_-^*$ yields that there exists $i_0 \in \{1,\ldots,n\}$ such that $\Phi\left(\sum_{i=1}^n \lambda_i y_i, y_{i_0}\right) \subset \mathbb{R}_+$ which implies that

$$\sum_{i=1}^n \lambda_i y_i \in \Phi^+(y_{i_0}) \subset \bigcup_{i=1}^n \Phi^+(y_i).$$

Consequently, for every finite subset $\{y_1,\ldots,y_n\} \subset D$ and $\{\lambda_1,\ldots,\lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\mathrm{conv}\{y_1,\ldots,y_n\} \cap D \subset \bigcup_{i=1}^n \Phi^+(y_i)$$

and then,

$$\mathrm{cl}\left(\mathrm{conv}\left\{y_1,\ldots,y_n\right\}\cap D\right)\subset \mathrm{cl}\left(\bigcup_{i=1}^{n}\Phi^{+}\left(y_i\right)\right)=\bigcup_{i=1}^{n}\mathrm{cl}\left(\Phi^{+}\left(y_i\right)\right).$$

By applying Lemma 2.5, we obtain that

$$\mathrm{conv}\left\{y_1,\ldots,y_n\right\}\subset\bigcup_{i=1}^{n}\mathrm{cl}\left(\Phi^{+}\left(y_i\right)\right)$$

which proves that the set-valued mapping $\mathrm{cl}\left(\Phi^{+}\right)$ is a KKM mapping.

Now, by applying Ky Fan's lemma, we have

$$\bigcap_{y\in D}\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)\neq\emptyset.$$

Since $y_0\in D$ and $\mathrm{cl}\left(\Phi^{+}\left(y_0\right)\right)$ is contained in K, then we have

$$\bigcap_{y\in D}\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)=\left(\bigcap_{y\in D}\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)\right)\cap K=\bigcap_{y\in D}\left(\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)\cap K\right).$$

According to our notation, we remark that for every $y\in D$, $\Phi^{+}\left(y\right)$ is the upper inverse set $\Phi^{+}\left(\mathbb{R}_{+},y\right)$ of $\mathbb{R}_{+}$ by the set-valued mapping $\Phi\left(.,y\right)$ which is lower semicontinuous on K by assumption (5). Then, by applying Proposition 2.4, we obtain that for every $y\in D$,

$$\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)\cap K=\Phi^{+}\left(y\right)\cap K.$$

Since $y_0\in D$ and $\Phi^{+}\left(y_0\right)$ is contained in K, then we have

$$\bigcap_{y\in D}\left(\mathrm{cl}\left(\Phi^{+}\left(y\right)\right)\cap K\right)=\bigcap_{y\in D}\left(\Phi^{+}\left(y\right)\cap K\right)=\bigcap_{y\in D}\Phi^{+}\left(y\right).$$

It results that $\bigcap_{y\in D}\Phi^{+}\left(y\right)\neq\emptyset$ which means that there exists $x_0\in C$ such that $\Phi\left(x_0,y\right)\subset\mathbb{R}_{+}$, for every $y\in D$.

It remains now to extend the above statement to the whole C in order to state that x_0 is a solution of the set-valued equilibrium problem (SVEP). Let $y\in C\setminus D$. Since $D\subset\Phi^{+}\left(x_0,\mathbb{R}_{+}\right)=\left\{y'\in C\mid\Phi\left(x_0,y'\right)\subset\mathbb{R}_{+}\right\}$ and D is dense in C, then $y\in\mathrm{cl}\left(\Phi^{+}\left(x_0,\mathbb{R}_{+}\right)\right)\cap\left(C\setminus D\right)$. According to Proposition 2.4 again, assumption (3) yields that

$$\mathrm{cl}\left(\Phi^{+}\left(x_0,\mathbb{R}_{+}\right)\right)\cap\left(C\setminus D\right)=\Phi^{+}\left(x_0,\mathbb{R}_{+}\right)\cap\left(C\setminus D\right).$$

It results that $y\in\Phi^{+}\left(x_0,\mathbb{R}_{+}\right)$ which means that $\Phi^{+}\left(x_0,y\right)\subset\mathbb{R}_{+}$ and completes the proof. □

Remark 1. We note that Φ is convex (*resp.* lower semicontinuous) if and only if $-\Phi$ is convex (*resp.* lower semicontinuous). Therefore, if we replace Φ by $-\Phi$ in the above theorem, we obtain the inclusion in $\mathbb{R}_{-}$.

Here we give an example of a set-valued mapping verifying all the condition of Theorem 3.1 without being lower semicontinuous in its first variable on the whole space.

Example 1. Let $X=D=\mathbb{R}$, $K=\left[-1,+1\right]$ and $y_0=0$. Define the set-valued mapping $\Phi:\mathbb{R}\times\mathbb{R}\rightrightarrows\mathbb{R}$ by

$$\Phi\left(x,y\right)=\begin{cases}\left[\dfrac{y^2-x^2}{2},+\infty\right[& \text{if } x=2,\\ \left[y^2-x^2,+\infty\right[& \text{otherwise.}\end{cases}$$

It is easily verified that Φ is convex in its second variable and that $\Phi(x,x) = \mathbb{R}_+$, for every $x \in X$. Also, for every $x \notin K$, we have $\Phi(x,0) = -x^2 < 0$ if $x \neq 2$ and $\Phi(x,0) = -2 < 0$ if $x = 2$.

To show that Φ is lower semicontinuous in its first variable on K, fix $y \in \mathbb{R}$ and let V be an open subset of $\mathbb{R}$ such that $\Phi(\overline{x}, y) \cap V \neq \emptyset$ where $\overline{x} \in K$. Then, $2 - \overline{x} \geq 1$ and furthermore $\Phi(\overline{x}, y) = \left[y^2 - \overline{x}^2, +\infty\right[$. Let $z \in \left[y^2 - \overline{x}^2, +\infty\right[\cap V$ and $\varepsilon > 0$ be such that $]z - \varepsilon, z + \varepsilon[\subset V$. Choose $0 < \delta < 1$ such that $\left|x^2 - \overline{x}^2\right| < \varepsilon$ whenever $|x - \overline{x}| < \delta$. Note that $\left|x^2 - \overline{x}^2\right| = \left|\left(y^2 - x^2\right) - \left(y^2 - \overline{x}^2\right)\right|$ and then, $(y^2 - x^2) < y^2 - \overline{x}^2 + \varepsilon \leq z + \varepsilon$ whenever $|x - \overline{x}| < \delta$. Thus, $\Phi(x, y) \cap V \neq \emptyset$, for every $x \in]\overline{x} - \delta, \overline{x} + \delta[$.

Now, let us show that Φ is not lower semicontinuous in its first variable on the whole space $\mathbb{R}$. Take for example $y = 3$ and show that the set-valued mapping $\Phi(\cdot, 3)$ is not lower semicontinuous at the point 2. To do this, consider the open interval $V =]2,3[$, and since $\Phi(2,3) = \left[\frac{5}{2}, +\infty\right[$, then $\Phi(2,3) \cap V \neq \emptyset$. However, if U is an open neighborhood of 2, then $\left]2, \sqrt{5}\right[\cap U \neq \emptyset$ and for $z \in \left]2, \sqrt{5}\right[\cap U$, we have $\Phi(z,3) = \left[9 - z^2, +\infty\right[$. Since $9 - z^2 > 4$, then $\Phi(z,3) \cap V = \emptyset$.

Of course, the set-valued equilibrium problem (SVEP(W)) is also solvable under the conditions of Theorem 3.1 since it is a particular case and a weaker version of the set-valued equilibrium problem (SVEP). However, we provide here some other conditions involving concavity and upper semicontinuity to obtain an existence result for the set-valued equilibrium problem (SVEP(W)).

Theorem 3.2. *Let X be a Hausdorff locally convex topological vector space, C a nonempty, closed and convex subset of X and $D \subset C$ a self-segment-dense set in C. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

(1) *$\forall x \in D$, $\Phi(x,x) \cap \mathbb{R}_+ \neq \emptyset$;*
(2) *$\forall x \in D$, $y \mapsto \Phi(x,y)$ is concave on D;*
(3) *$\forall x \in C$, $y \mapsto \Phi(x,y)$ is upper semicontinuous on $C \setminus D$;*
(4) *there exist a compact set K of C and $y_0 \in D$ such that $\Phi(x, y_0) \subset \mathbb{R}_-^*$, $\forall x \in C \setminus K$;*
(5) *$\forall y \in D$, $x \mapsto \Phi(x,y)$ is upper semicontinuous on K.*

Then, the set-valued equilibrium problem (SVEP(W)) has a solution.

Proof. We define the following set-valued mapping $\Phi^- : C \rightrightarrows C$ by

$$\Phi^-(y) = \{x \in C \mid \Phi(x,y) \cap \mathbb{R}_+ \neq \emptyset\} \quad \forall y \in C.$$

Clearly, $x_0 \in C$ is a solution of the set-valued equilibrium problem (SVEP(W)) if and only if $x_0 \in \bigcap_{y \in C} \Phi^-(y)$.

Now, consider the set-valued mapping $\mathrm{cl}(\Phi^-) : D \rightrightarrows \mathbb{R}$ defined by

$$\mathrm{cl}\left(\Phi^-\right)(y) = \mathrm{cl}\left(\Phi^-(y)\right) \quad \forall y \in D.$$

As above, $\Phi^-(y)$ is nonempty for every $y \in D$. Also, $\mathrm{cl}(\Phi^-)(y)$ is closed for every $y \in D$, and $\mathrm{cl}(\Phi^-)(y_0)$ is compact since it lies in K.

To prove that the set-valued mapping $\mathrm{cl}(\Phi^-)$ is a KKM mapping, let $\{y_1, \ldots, y_n\} \subset D$ be a finite subset and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i y_i \in D$. Then, by assumption (1) and assumption (2), we obtain

$$\sum_{i=1}^n \lambda_i \Phi\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) \supset \Phi\left(\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i y_i\right) \cap \mathbb{R}_+ \neq \emptyset.$$

The convexity of $\mathbb{R}_-^*$ yields that there exists $i_0 \in \{1,\ldots,n\}$ such that $\Phi\left(\sum_{i=1}^n \lambda_i y_i, y_{i_0}\right) \cap \mathbb{R}_+ \neq \emptyset$ which implies that

$$\sum_{i=1}^n \lambda_i y_i \in \Phi^-\left(y_{i_0}\right) \subset \bigcup_{i=1}^n \Phi^-\left(y_i\right).$$

Consequently, for every finite subset $\{y_1,\ldots,y_n\} \subset D$ and $\{\lambda_1,\ldots,\lambda_n\} \subset \mathbb{R}_-$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\operatorname{conv}\{y_1,\ldots,y_n\} \cap D \subset \bigcup_{i=1}^n \Phi^-\left(y_i\right)$$

and then,

$$\operatorname{cl}\left(\operatorname{conv}\{y_1,\ldots,y_n\} \cap D\right) \subset \operatorname{cl}\left(\bigcup_{i=1}^n \Phi^-\left(y_i\right)\right) = \bigcup_{i=1}^n \operatorname{cl}\left(\Phi^-\left(y_i\right)\right).$$

By applying Lemma 2.5, we have

$$\operatorname{conv}\{y_1,\ldots,y_n\} \subset \bigcup_{i=1}^n \operatorname{cl}\left(\Phi^-\left(y_i\right)\right)$$

which proves that the set-valued mapping $\operatorname{cl}\left(\Phi^-\right)$ is a KKM mapping.

By applying Ky Fan's lemma, we obtain

$$\bigcap_{y\in D} \operatorname{cl}\left(\Phi^-(y)\right) \neq \emptyset,$$

and since $y_0 \in D$ and $\operatorname{cl}\left(\Phi^-(y_0)\right) \subset K$, then

$$\bigcap_{y\in D} \operatorname{cl}\left(\Phi^-(y)\right) = \bigcap_{y\in D} \left(\operatorname{cl}\left(\Phi^-(y)\right)\right) \cap K = \bigcap_{y\in D} \left(\operatorname{cl}\left(\Phi^-(y)\right) \cap K\right).$$

According to our notation, we remark that for every $y \in D$, $\Phi^-(y)$ is the lower inverse set $\Phi^-(\mathbb{R}_+, y)$ of $\mathbb{R}_+$ by the set-valued mapping $\Phi(.,y)$ which is upper semicontinuous on K by assumption (5). Then, by applying Proposition 2.4, we obtain that for every $y \in D$,

$$\operatorname{cl}\left(\Phi^-(y)\right) \cap K = \Phi^-(y) \cap K.$$

Since $y_0 \in D$ and $\Phi^-(y_0)$ is contained in K, we have

$$\bigcap_{y\in D} \left(\operatorname{cl}\left(\Phi^-(y)\right) \cap K\right) = \bigcap_{y\in D} \left(\Phi^-(y) \cap K\right) = \bigcap_{y\in D} \Phi^-(y).$$

It follows that $\bigcap_{y\in D} \Phi^-(y) \neq \emptyset$ which means that there exists $x_0 \in C$ such that $\Phi(x_0, y) \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in D$.

It remains now to extend the above statement to the whole C in order to state that x_0 is a solution of the set-valued equilibrium problem (SVEP(W)). Let $y \in C \setminus D$. Since $D \subset \Phi^-(x_0, \mathbb{R}_+) = \{y' \in C \mid \Phi(x_0, y') \cap \mathbb{R}_+ \neq \emptyset\}$ and D is dense in C, then $y \in \operatorname{cl}\left(\Phi^-(x_0, \mathbb{R}_+)\right) \cap (C \setminus D)$. According to Proposition 2.4 again, assumption (3) yields that

$$\operatorname{cl}\left(\Phi^-(x_0, \mathbb{R}_+)\right) \cap (C \setminus D) = \Phi^-(x_0, \mathbb{R}_+) \cap (C \setminus D).$$

It results that $y \in \Phi^-(x_0, \mathbb{R}_+)$ which means that $\Phi^-(x_0, y) \cap \mathbb{R}_+ \neq \emptyset$ and completes the proof. □

Once again, remark that any solution of the set-valued equilibrium problem (SVEP) or the set-valued equilibrium problem (SVEP(W)) is a solution of the classical equilibrium problem (EP) where $\Phi : C\times C \rightrightarrows \mathbb{R}$ is defined by $\Phi(x,y) = \{\varphi(x,y)\}$ and φ is a single-valued bifunction. However, when the notions of convexity and concavity in the sense of set-valued mappings are applied to Φ, we obtain the linearity (on D) of the single-valued mapping φ which is a strong condition for solving equilibrium problems. Also, lower and upper semicontinuity in the sense of set-valued mappings applied to Φ turn out to be the continuity of φ. Here, we provide conditions weaker than linearity and continuity to obtain an existence result for the equilibrium problems (EP).

Theorem 3.3. *Let X be a Hausdorff locally convex topological vector space, C a nonempty, closed and convex subset of X and $D \subset C$ a self-segment-dense set in C. Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction, and assume that the following conditions hold:*

(1) $\forall x \in D$, $\varphi(x,x) \geq 0$;
(2) $\forall x \in D$, $y \mapsto \varphi(x,y)$ *is convex on* D;
(3) $\forall x \in C$, $y \mapsto \varphi(x,y)$ *is lower semicontinuous on* $C \setminus D$;
(4) *there exist a compact set K of C and $y_0 \in D$ such that* $\varphi(x,y_0) < 0$, $\forall x \in C \setminus K$;
(5) $\forall y \in D$, $x \mapsto \varphi(x,y)$ *is lower semicontinuous on* K.

Then, the equilibrium problem (EP) *has a solution.*

Proof. Proceed as above and define the following set-valued mapping $\varphi^+ : C \rightrightarrows C$ by

$$\varphi^+(y) = \{x \in C \mid \varphi(x,y) \geq 0\}.$$

Clearly, $x_0 \in C$ is a solution of the equilibrium problem (EP) if and only if $x_0 \in \bigcap_{y \in C} \varphi^+(y)$.

We also consider the set-valued mapping $\operatorname{cl}(\varphi^+) : D \rightrightarrows \mathbb{R}$ defined by

$$\operatorname{cl}\left(\varphi^+\right)(y) = \operatorname{cl}\left(\varphi^+(y)\right) \quad \forall y \in D.$$

We have that $\varphi^+(y)$ is nonempty, for every $y \in D$. Also, $\operatorname{cl}(\varphi^+)(y)$ is closed for every $y \in D$, and $\operatorname{cl}(\varphi^+)(y_0)$ is compact.

To prove that $\operatorname{cl}(\varphi^+)$ is a KKM mapping, let $\{y_1, \ldots, y_n\} \subset D$ be a finite subset and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i y_i \in D$. We have

$$\max_{i=1,\ldots,n} \varphi\left(\sum_{i=1}^n \lambda_i y_i, y_i\right) \geq \varphi\left(\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i y_i\right) \geq 0.$$

Then, there exists $i_0 \in \{1, \ldots, n\}$ such that $\varphi\left(\sum_{i=1}^n \lambda_i y_i, y_{i_0}\right) \geq 0$ which implies that

$$\sum_{i=1}^n \lambda_i y_i \in \varphi^+(y_{i_0}) \subset \bigcup_{i=1}^n \varphi^+(y_i).$$

Consequently, for every finite subset $\{y_1, \ldots, y_n\} \subset D$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_-$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\operatorname{conv}\{y_1, \ldots, y_n\} \cap D \subset \bigcup_{i=1}^n \varphi^+(y_i)$$

and then,

$$\operatorname{cl}(\operatorname{conv}\{y_1, \ldots, y_n\} \cap D) \subset \operatorname{cl}\left(\bigcup_{i=1}^n \varphi^+(y_i)\right) = \bigcup_{i=1}^n \operatorname{cl}\left(\varphi^+(y_i)\right).$$

By applying Lemma 2.5, we obtain that

$$\operatorname{conv}\{y_1,\dots,y_n\}\subset\bigcup_{i=1}^{n}\operatorname{cl}\left(\varphi^{+}(y_i)\right)$$

which proves that the set-valued mapping $\operatorname{cl}(\varphi^{+}) : D \rightrightarrows \mathbb{R}$ is a KKM mapping.

By applying Ky Fan's lemma, we have

$$\bigcap_{y\in D}\operatorname{cl}\left(\varphi^{+}(y)\right)\neq\emptyset.$$

Since $y_0\in D$ and $\operatorname{cl}(\varphi^{+}(y_0))\subset K$, then

$$\bigcap_{y\in D}\operatorname{cl}\left(\varphi^{+}(y)\right)=\bigcap_{y\in D}\left(\operatorname{cl}\left(\varphi^{+}(y)\right)\right)\cap K=\bigcap_{y\in D}\left(\operatorname{cl}\left(\varphi^{+}(y)\right)\cap K\right).$$

By applying Proposition 2.2, assumption (5) yields that for every $y\in D$,

$$\operatorname{cl}\left(\varphi^{+}(y)\right)\cap K=\varphi^{+}(y)\cap K$$

and then,

$$\bigcap_{y\in D}\left(\operatorname{cl}\left(\varphi^{+}(y)\right)\cap K\right)=\bigcap_{y\in D}\left(\varphi^{+}(y)\cap K\right)=\bigcap_{y\in D}\varphi^{+}(y).$$

It follows that $\bigcap_{y\in D}\varphi^{+}(y)\neq\emptyset$ which means that there exists $x_0\in C$ such that $\varphi(x_0,y)\geq 0$, for every $y\in D$.

As above, by assumption (3), we can apply Proposition 2.2 to the set $C\setminus K$ and obtain that $\varphi(x_0,y)\geq 0$, for every $y\in C$. That is, x_0 is a solution of the equilibrium problem (EP). □

Remark 2. Let us point out that the solutions sets of the equilibrium problems studied above in Theorems 3.1, 3.2 or Theorem 3.3 are always included in the set of coerciveness K.

4. Applications

In this section, we give two applications of our results developed above. The first application is about existence of solutions of Browder variational inclusions, and the second is to establish two versions of Kakutani and Schauder fixed point theorems. These results extend those obtained in [3].

4.1. Browder variational inclusions

Browder variational inclusion problems have been considered in the literature as a generalization of Browder–Hartman–Stampacchia variational inequality problems. These problems are also presented in the literature as a weak type of multivalued variational inequalities, see [22,2,4].

In the sequel, for a real normed vector space X, we denote by X^* the dual space of X and by $\langle .,. \rangle$ the duality pairing between X^* and X. For $x\in X$ and a subset A of X^*, we put $\langle A,x\rangle=\{\langle x^*,x\rangle \mid x^*\in A\}$.

Theorem 4.1. *Let X be a real normed vector space, C a nonempty, closed and convex subset of X. Suppose that $F : C \rightrightarrows X^*$ has the following conditions:*

(1) *there exist a compact subset K of C and $y_0\in C$ such that $\langle x^*, y-x\rangle<0$, for every $x\in C\setminus K$ and every $x^*\in F(x)$;*

(2) *F is upper semicontinuous on K;*
(3) *F has weak * compact values on K.*

Then, there exists $x_0 \in K$ such that $\langle F(x_0), y - x_0\rangle \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$.

Proof. Define the set-valued mapping $\Phi : C \times C \rightrightarrows \mathbb{R}$ by

$$\Phi(x, y) = \langle F(x), y - x\rangle.$$

We will show that all the conditions of Theorem 3.2 are satisfied with $D = C$.

Condition (1) and Condition (3) are obviously satisfied. Condition (4) holds easily from our assumption on the subset K.

To prove Condition (2), let $x \in C$, $\{y_1, \ldots, y_n\} \subset C$ a finite subset and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$. Take $x^* \in F(x)$, and by linearity, we have

$$\left\langle x*, \sum_{i=1}^n \lambda_i y_i - x \right\rangle = \sum_{i=1}^n \lambda_i \langle x*, y_i - x\rangle \in \sum_{i=1}^n \lambda_i \langle F(x), y_i - x\rangle$$

which implies that $\langle F(x), \sum_{i=1}^n \lambda_i y_i - x\rangle \subset \sum_{i=1}^n \lambda_i \langle F(x), y_i - x\rangle$. Thus, $\Phi(x, \sum_{i=1}^n \lambda_i y_i) \subset \sum_{i=1}^n \lambda_i \Phi(x, y_i)$.

To prove Condition (5), fix $y \in D$, V an open subset of $\mathbb{R}$ and let $x \in \Phi^+(V, y) \cap (C \setminus K)$ where $\Phi^+(V, y) = \{x' \in C \mid \langle F(x'), y - x'\rangle \subset V\}$. First, we claim that there exists $\delta > 0$ such that

$$B_{\mathbb{R}}(\langle x^*, y - x\rangle, \delta) \subset V \quad \forall x^* \in F(x)$$

where $B_{\mathbb{R}}(\langle x^*, y - x\rangle, \delta) = \{t \in \mathbb{R} \mid |t - \langle x^*, y - x\rangle| < \delta\}$. Indeed; for every $x^* \in F(x)$, let $\varepsilon_{x^*} > 0$ such that $B_{\mathbb{R}}(\langle x^*, y - x\rangle, 2\varepsilon_{x^*}) \subset V$ and put $U_{x^*} = \{z^* \in X^* \mid \langle z^*, y - x\rangle \in B_{\mathbb{R}}(\langle x^*, y - x\rangle, \varepsilon_{x^*})\}$. The family $\{U_{x^*} \mid x^* \in F(x)\}$ being a weak * open cover of $F(x)$ which is weak * compact, let $\{x_1^*, \ldots, x_n^*\} \subset F(x)$ be such that $F(x) \subset \bigcup_{i=1}^n U_{x_i^*}$. Put

$$\delta = \min_{i=1,\ldots,n} \varepsilon_{x_i^*}.$$

If $t \in B_{\mathbb{R}}(\langle x^*, y - x\rangle, \delta)$ for some $x^* \in F(x)$, then $x^* \in U_{x_i^*}$, for some $i = 1, \ldots, n$. Since

$$\begin{aligned} |t - \langle x_i^*, y - x\rangle| &\leq |t - \langle x^*, y - x\rangle| + |\langle x^*, y - x\rangle - \langle x_i^*, y - x\rangle| \\ &< \delta + \varepsilon_{x_i^*} \leq 2\varepsilon_{x_i^*}, \end{aligned}$$

then, $t \in B_{\mathbb{R}}\left(\langle x_i^*, y - x\rangle, 2\varepsilon_{x_i^*}\right) \subset V$.

Now, put

$$\delta_1 = \min\left\{\frac{\delta}{3(\|x\| + 1)}, \frac{\delta}{3(\|y\| + 1)}\right\}$$

and $O = \bigcup_{x^* \in F(x)} B_{X^*}(x^*, \delta_1)$ where $B_{X^*}(x^*, \delta_1) = \{z \in X^* \mid \|z - x^*\|_* < \delta_1\}$, and $\|.\|$ and $\|.\|_*$ denote respectively the norm of X and X^*. Clearly O is an open set containing $F(x)$, and by the upper semicontinuity of F on K, let $\eta > 0$ be such that $F(w) \subset O$ for every $w \in B_X(x, \eta) \cap C$, where $B_X(x, \eta) = \{w \in X \mid \|w - x\| < \eta\}$. Put

$$\eta_1 = \min\left\{\frac{\delta}{3(\|F(x)\|_* + 1)}, \eta, 1\right\}$$

where $\|F(x)\|_* = \max\{\|x^*\|_* \mid x^* \in F(x)\}$. Put $U = B_X(x, \eta_1) \cap C$ which is an open subset of C containing x.

We will show that $\Phi(z,y) \subset V$, for every $z \in U$. To do this, let $z \in U$ and $z^* \in F(z)$. Let $x_0^* \in F(x)$ be such that $F(z^*) \subset B_{X^*}(x_0^*, \delta_1)$. We have

$$\begin{aligned}
|\langle z^*, y-z\rangle - \langle x_0^*, y-x\rangle| &= |\langle x_0^* - z^*, z\rangle + \langle x_0^*, x-z\rangle - \langle x_0^* - z^*, y\rangle| \\
&\leq \|x_0^* - z^*\|_* \|z\| + \|x_0^*\|_* \|x-z\| + \|x_0^* - z^*\|_* \|y\| \\
&< \frac{\delta(\|x\| + \eta_1)}{3(\|x\|+1)} + \frac{\delta\|x_0^*\|_*}{3(\|F(x)\|_* + 1)} + \frac{\delta\|y\|}{3(\|y\|+1)} \\
&< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned}$$

It follows that $\langle z^*, y-z\rangle \in B_{\mathbb{R}}(\langle x_0^*, y-x\rangle, \delta) \subset V$. Since z is arbitrary in U and z^* is arbitrary in $F(z)$, then $\Phi(z,y) \subset V$, for every $z \in U$.

We conclude, by applying Theorem 3.2, that there exists $x_0 \in C$ such that $\Phi(x_0, y) \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$. $\square$

When F is a single-valued mapping, we obtain a solution to the well-known Browder–Hartman–Stampacchia variational inequality problems.

Corollary 4.2. *Let X be a real normed vector space, C a nonempty, closed and convex subset of X. Suppose that $f : C \to X^*$ has the following conditions:*

(1) *there exist a compact subset K of C and $y_0 \in C$ such that $\langle f(x), y-x\rangle < 0$, for every $x \in C \setminus K$;*
(2) *F is continuous on K.*

Then, there exists $x_0 \in K$ such that $\langle f(x_0), y-x_0\rangle \geq 0$, for every $y \in C$.

4.2. Fixed point theory

In order to obtain a version of Kakutani fixed point theorem, we need to develop some results on the continuity of the distance function and the marginal function generalizing some older results in the literature.

Recall that if X is a real normed vector space, $x \in X$ and A is a nonempty subset of X, then

$$\operatorname{dist}(x, A) = \inf_{z \in A} \|x - z\|$$

is called the distance between x and A, where $\|.\|$ is the norm of X. Obviously, the (real valued) distance function $x \mapsto \operatorname{dist}(x, A)$ is nonexpansive, and therefore continuous. It is also convex whenever A is convex, see for example, [18,23,24,20].

The situation is more complicated when A is depending on x as the image of x by a set-valued mapping.

First, we establish the following result on the distance function generalizing the second item in [20, Theorem 6.1.15].

Proposition 4.3. *Let X be a Hausdorff topological space, S a subset of X, (Y, d) a metric space and $F : X \rightrightarrows Y$ a set-valued mapping with nonempty values. If F is upper semicontinuous on S, then for every $y \in Y$, the function $x \mapsto$ dist $(y, F(x))$ is lower semicontinuous on S.*

Proof. Fix $y \in Y$ and let $a \in \mathbb{R}$. By Proposition 2.2, we have to prove that

$$\operatorname{cl}(\{x \in X \mid \operatorname{dist}(y, F(x)) \leq a\}) \cap S = \{x \in S \mid \operatorname{dist}(y, F(x)) \leq a\}.$$

Let $\overline{x} \in \mathrm{cl}\left(\{x \in X \mid \mathrm{dist}\left(y, F\left(x\right)\right) \leq a\}\right) \cap S$ and take a net $\left(x_\alpha\right)_{\alpha \in \Lambda}$ in the set $\{x \in X \mid \mathrm{dist}\left(y, F\left(x\right)\right) \leq a\}$ converging to $\overline{x}$. Let $\varepsilon > 0$ be arbitrary, and put

$$V = \{y' \in Y \mid \mathrm{dist}\left(y', F\left(\overline{x}\right)\right) < \varepsilon\}$$

which is an open subset containing $F\left(\overline{x}\right)$. By the upper semicontinuity of F at $\overline{x}$, there exists $\alpha_0 \in \Lambda$ such that $F\left(x_\alpha\right) \subset V$, for every $\alpha \geq \alpha_0$. For every $\alpha \geq \alpha_0$ and every $z \in F\left(x_\alpha\right)$, we remark that

$$\mathrm{dist}\left(y, F\left(\overline{x}\right)\right) \leq d\left(y, z\right) + \mathrm{dist}\left(z, F\left(\overline{x}\right)\right) < d\left(y, z\right) + \varepsilon$$

and then, $\mathrm{dist}\left(y, F\left(\overline{x}\right)\right) \leq \mathrm{dist}\left(y, F\left(x_\alpha\right)\right) + \varepsilon \leq a + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then $\mathrm{dist}\left(y, F\left(\overline{x}\right)\right) \leq a$. $\square$

In the sequel, we need to establish the following generalization of the well-known Berge maximum theorem which is useful in many applications, see [20, Theorem 6.1.18].

Let X and Y be two Hausdorff topological spaces, $F : X \rightrightarrows Y$ a set-valued mapping and $\psi : X \times Y \to \overline{\mathbb{R}}$ a function. The marginal (or value) extended real valued function $g : X \to \overline{\mathbb{R}}$ is defined by

$$g\left(x\right) = \sup_{y \in F(x)} \psi\left(x, y\right).$$

Theorem 4.4. *Let X and Y be two Hausdorff topological spaces, S a nonempty subset of X, $F : X \rightrightarrows Y$ a set-valued mapping and $\psi : X \times Y \to \overline{\mathbb{R}}$ a function.*

(1) *If ψ is lower semicontinuous on $S \times Y$ and F is lower semicontinuous on S, then the marginal function $g : X \to \overline{\mathbb{R}}$ is lower semicontinuous on S.*
(2) *If ψ is upper semicontinuous on $S \times Y$ and there exists an open subset U containing S such that the function $y \mapsto \psi\left(x, y\right)$ is upper semicontinuous on Y for every $x \in U$, and F is upper semicontinuous on S and has nonempty compact values on U, then the marginal function $g : X \to \overline{\mathbb{R}}$ is upper semicontinuous on S.*

Proof. *Proof of the first statement.* Let $a \in \mathbb{R}$. By Proposition 2.2, we have to prove that

$$\mathrm{int}\left(\{x \in X \mid g\left(x\right) > a\}\right) \cap S = \{x \in S \mid g\left(x\right) > a\}.$$

Let $\overline{x} \in \{x \in S \mid g\left(x\right) > a\}$. Then by the definition of the marginal function g, let $\overline{y} \in F\left(\overline{x}\right)$ such that $\psi\left(\overline{x}, \overline{y}\right) > a$. The function ψ being lower semicontinuous on $S \times Y$, then by Proposition 2.2, let $W_1 \times V$ be an open neighborhood of $\left(\overline{x}, \overline{y}\right)$ such that

$$\psi\left(x, y\right) > a \quad \forall x \in W_1, \forall y \in V.$$

Since F is lower semicontinuous on S and $\overline{x} \in F^-\left(V\right) \cap S$, then by Proposition 2.4, let W_2 be an open neighborhood of $\overline{x}$ such that $W_2 \subset F^-\left(V\right)$. Taking $W = W_1 \cap W_2$, we have $F\left(x\right) \cap V \neq \emptyset$, for every $x \in W$. Fix $y_x \in F\left(x\right) \cap V$, for every $x \in W$. Then, $\psi\left(x, y_x\right) > a$ which implies that $g\left(x\right) > a$. It follows that $\overline{x} \in W \subset \{x \in X \mid g\left(x\right) > a\}$.

Proof of the second statement. Let $a \in \mathbb{R}$. By Proposition 2.2, we have to prove that

$$\mathrm{cl}\left(\{x \in X \mid g\left(x\right) \geq a\}\right) \cap S = \{x \in S \mid g\left(x\right) \geq a\}.$$

Let $\overline{x} \in \mathrm{cl}\left(\{x \in X \mid g\left(x\right) \geq a\}\right) \cap S$ and take a net $\left(x_\alpha\right)_{\alpha \in \Lambda}$ in the set $\{x \in X \mid g\left(x\right) \geq a\}$ converging to $\overline{x}$. Since $\overline{x} \in S \subset U$, we may assume without loss of generality that $\left(x_\alpha\right)_{\alpha \in \Lambda}$ is in U. For every $\alpha \in \Lambda$, the function $y \mapsto \psi\left(x_\alpha, y\right)$ is upper semicontinuous on Y and therefore, by Weierstrass theorem, it attains its maximum on the compact set $F\left(x_\alpha\right)$. Let $y_\alpha \in F\left(x_\alpha\right)$ be such that $g\left(x_\alpha\right) = \psi\left(x_\alpha, y_\alpha\right)$, for every $\alpha \in \Lambda$.

The net $(y_\alpha)_{\alpha\in\Lambda}$ has a cluster point in $F(\overline{x})$. Indeed, suppose the contrary holds. Then the compactness of $F(\overline{x})$ yields the existence of an open set V containing $F(\overline{x})$ and $\alpha_0 \in \Lambda$ such that $y_\alpha \notin V$, for every $\alpha \geq \alpha_0$. The upper semicontinuity of F at $\overline{x}$ yields the existence of an open neighborhood W of $\overline{x}$ such that $F(x) \subset V$, for every $x \in W$. Since $(x_\alpha)_{\alpha\in\Lambda}$ is converging to $\overline{x}$, let $\alpha_1 \in \Lambda$ be such that $x_\alpha \in W$, for every $\alpha \geq \alpha_1$. Then, $y_\alpha \in V$, for every $\alpha \geq \max\{\alpha_0, \alpha_1\}$. A contradiction.

Now, take $\overline{y} \in F(\overline{x})$ and $(y_\alpha)_{\alpha\in\Gamma}$ a subnet of $(y_\alpha)_{\alpha\in\Lambda}$ converging to $\overline{y}$. The subnet $(x_\alpha, y_\alpha)_{\alpha\in\Gamma}$ converges to $(\overline{x}, \overline{y}) \in S \times Y$ and satisfies $\psi(x_\alpha, y_\alpha) \geq a$, for every $\alpha \in \Gamma$. Since ψ is upper semicontinuous on $S \times Y$, then, we conclude by Proposition 2.2 that $\psi(\overline{x}, \overline{y}) \geq a$. It follows that $g(\overline{x}) \geq \psi(\overline{x}, \overline{y}) \geq a$ which completes the proof. $\square$

Now, we formulate the following version of Kakutani fixed point theorem.

Theorem 4.5. *Let X be a real normed vector space, C a nonempty, closed and convex subset of X and $D \subset C$ a self-segment-dense set in C. Suppose that $F : C \rightrightarrows X$ has the following conditions:*

(1) *F has nonempty convex values on C;*
(2) *there exist a compact subset K of C and $y_0 \in D$ such that $\mathrm{dist}(y_0, F(x)) < \mathrm{dist}(x, F(x))$, for every $x \in C \setminus K$;*
(3) *F is continuous on K and has closed values on K;*
(4) *$F(x) \cap K \neq \emptyset$, for every $x \in K$.*

Then, F has a fixed point.

Proof. We define the set-valued mapping $\Phi : C \times C \to \mathbb{R}$ by

$$\Phi(x, y) = \mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x)) + [0, +\infty[\,.$$

Note that

$$\Phi(x, y) = [\mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x)), +\infty[\,.$$

We are now going to verify the conditions of Theorem 3.1. Condition (1) is verified since $\Phi(x, x) = [0, +\infty[= \mathbb{R}_+$. Also, the convexity of $y \mapsto \mathrm{dist}(y, F(x))$ on C yields easily the convexity of Φ in its second variable on C, for every $x \in C$. To verify Condition (3), fix $x \in C$ and $y \in C$, and let V be an open subset of $\mathbb{R}$ such that $\Phi(x, y) \subset V$. Let $\varepsilon > 0$ such that $[\mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x)) - \varepsilon, +\infty[\subset V$. By lower semicontinuity of $y \mapsto \mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x))$, let U be an open neighborhood of y such that

$$\mathrm{dist}(y', F(x)) - \mathrm{dist}(x, F(x)) \geq \mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x)) - \varepsilon \quad \forall y' \in U.$$

This means that $\Phi(x, y') \subset V$, for every $y' \in U$, and then in particular, Condition (3) is satisfied.

Condition (4) is obvious. To verify Condition (5), fix $y \in C$ and let V be an open subset of C. Put $\Phi^+(V, y) = \{x \in C \mid \Phi(x, y) \subset V\}$ and let $x \in \Phi^+(V, y) \cap K$. By Proposition 2.4, it suffices to show that $x \in \mathrm{int}(\Phi^+(V, y)) \cap K$. We have $\Phi(x, y) \subset V$. As above, let $\varepsilon > 0$ such that $[\mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x)) - \varepsilon, +\infty[\subset V$. The function $x \mapsto \mathrm{dist}(y, F(x)) - \mathrm{dist}(x, F(x))$ is lower semicontinuous on K. Indeed; by Proposition 4.3, the function $x \mapsto \mathrm{dist}(y, F(x))$ is lower semicontinuous on K. Now, taking $\psi : C \times X \to \mathbb{R}$ defined by $\psi(x, y) = -\|y - x\|$, we have

$$\begin{aligned} g(x) &= \sup_{y\in F(x)} \psi(x, y) \\ &= \sup_{y\in F(x)} (-\|y - x\|) = -\inf_{y\in F(x)} (\|y - x\|) = -\mathrm{dist}(x, F(x)). \end{aligned}$$

It follows by Theorem 4.4 that the function $x \mapsto -\mathrm{dist}(x, F(x))$ is lower semicontinuous on K.

Let U be an open neighborhood of x such that

$$\operatorname{dist}(y, F(x')) - \operatorname{dist}(x', F(x')) \geq \operatorname{dist}(y, F(x)) - \operatorname{dist}(x, F(x)) - \varepsilon \quad \forall x' \in U.$$

This means that $\Phi(x', y) \subset V$, for every $x' \in U$.

We conclude, by applying Theorem 3.1, that there exists $x_0 \in C$ such that $\Phi(x_0, y) \subset \mathbb{R}_+$, for every $y \in C$. Then, $\operatorname{dist}(y, F(x_0)) - \operatorname{dist}(x_0, F(x_0)) \geq 0$, for every $y \in C$. Note that $x_0 \in K$ and by taking $y \in F(x_0) \cap K$, we have $\operatorname{dist}(x_0, F(x_0)) \leq 0$ which provide necessarily that $x_0 \in F(x_0)$. $\square$

Here, by applying Theorem 3.2, we derive the following version of Schauder fixed point theorem, and in particular, the Brouwer fixed point theorem.

Theorem 4.6. *Let X be a real normed vector space, C a nonempty, closed and convex subset of X and $D \subset C$ a self-segment-dense set in C. Suppose that $f : C \to C$ has the following conditions:*

(1) *there exist a compact subset K of C and $y_0 \in D$ such that $\|y_0 - f(x)\| < \|x - f(x)\|$, for every $x \in C \setminus K$;*
(2) *f is continuous on K.*

Then, f has a fixed point.

Proof. Consider the bifunction $\Phi : C \times C \to \mathbb{R}$ defined by

$$\Phi(x, y) = \|y - f(x)\| - \|x - f(x)\| +]-\infty, 0].$$

Note that

$$\Phi(x, y) =]-\infty, \|y - f(x)\| - \|x - f(x)\|].$$

We are now going to verify the conditions of Theorem 3.2. Condition (1) is verified since $\Phi(x, x) =]-\infty, 0]$ and then, $0 \in \Phi(x, x) \cap \mathbb{R}_+$, for every $x \in C$. Also, the convexity of $y \mapsto \|y - f(x)\|$ on C yields easily the concavity of Φ in its second variable on C, for every $x \in C$. Now, fix $x \in C$ and $y \in C$, and let V be an open subset of $\mathbb{R}$ such that $\Phi(x, y) \subset V$. Let $\varepsilon > 0$ such that $]-\infty, \|y - f(x)\| - \|x - f(x)\| + \varepsilon] \subset V$. By continuity of $y \mapsto \|y - f(x)\| - \|x - f(x)\|$, let U be an open neighborhood of y such that

$$\|y' - f(x)\| - \|x - f(x)\| \leq \|y - f(x)\| - \|x - f(x)\| + \varepsilon \quad \forall y' \in U.$$

This means that $\Phi(x, y') \subset V$, for every $y' \in U$, and then in particular, Condition (3) is satisfied.

Condition (4) is obvious. To verify Condition (5), fix $y \in C$ and let V be an open subset of C. Put $\Phi^+(V, y) = \{x \in C \mid \Phi(x, y) \subset V\}$ and let $x \in \Phi^+(V, y) \cap K$. By Proposition 2.4, it suffices to show that $x \in \operatorname{int}(\Phi^+(V, y)) \cap K$. We have $\Phi(x, y) \subset V$. As above, let $\varepsilon > 0$ such that $]-\infty, \|y - f(x)\| - \|x - f(x)\| + \varepsilon] \subset V$ and by continuity of $x \mapsto \|y - f(x)\| - \|x - f(x)\|$, let U be an open neighborhood of x such that

$$\|y - f(x')\| - \|x' - f(x')\| \leq \|y - f(x)\| - \|x - f(x)\| + \varepsilon \quad \forall x' \in U.$$

This means that $\Phi(x', y) \subset V$, for every $x' \in U$.

We conclude, by applying Theorem 3.2, that there exists $x_0 \in C$ such that $\Phi(x_0, y) \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$. Taking $y = f(x_0)$, we have $]-\infty, -\|x_0 - f(x_0)\|] \cap \mathbb{R}_+ \neq \emptyset$ which provide necessarily that $\|x_0 - f(x_0)\| \leq 0$ and then, $f(x_0) = x_0$. $\square$

Remark 3. In our applications, we have focused our attention on weakening only semicontinuity. We remark that in both Theorems 4.5 and 4.6, the self-segment-dense set D does not play any role in the proofs and can be replaced merely by C. One can consult [5] to see some applications with the weakened condition of self-segment-dense subsets to a generalized Debreu–Gale–Nikaïdo-type theorem and to a Nash equilibrium of noncooperative games. It is not hard to see that by introducing a set of coerciveness, it may be possible to carry out similar applications with weakened conditions of both semicontinuity and convexity.

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