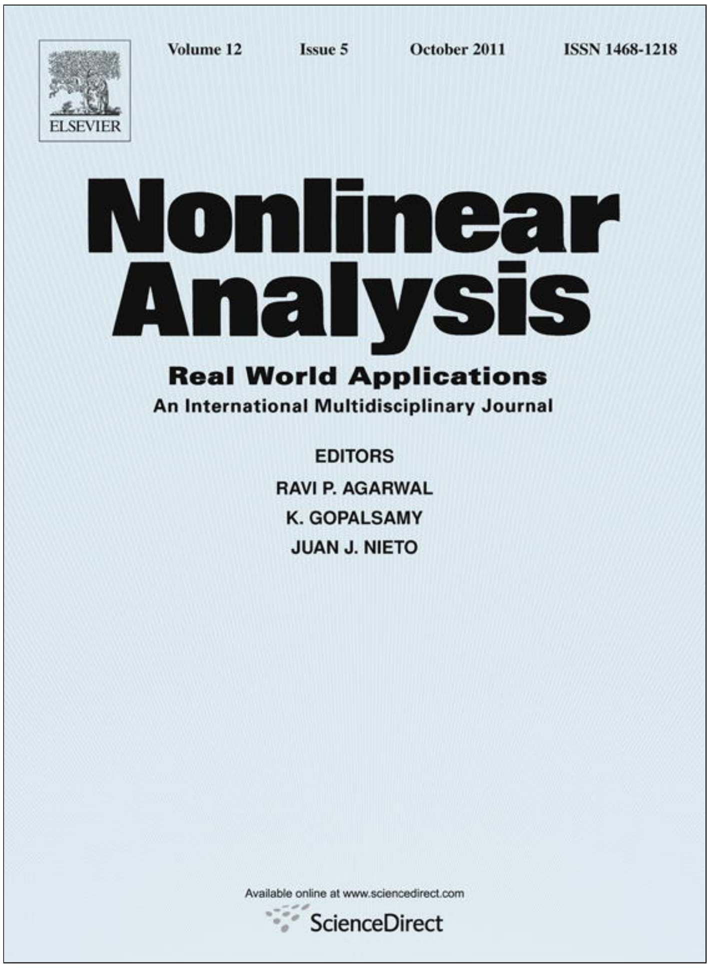

Volume 12 Issue 5 October 2011 ISSN 1468-1218

ELSEVIER

Nonlinear Analysis

Real World Applications

An International Multidisciplinary Journal

Nonlinear Analysis: Real World Applications 12 (2011) 2656–2665

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Multiple solutions of generalized Yamabe equations on Riemannian manifolds and applications to Emden–Fowler problems

Gabriele Bonanno [a], Giovanni Molica Bisci [b], Vicenţiu Rădulescu [c,d,*]

[a] Department of Science for Engineering and Architecture (Mathematics Section), Engineering Faculty, University of Messina, 98166 - Messina, Italy
[b] Dipartimento P.A.U., Università degli Studi Mediterranea di Reggio Calabria, Salita Melissari - Feo di Vito, 89100 Reggio Calabria, Italy
[c] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania
[d] Department of Mathematics, University of Craiova, 200585 Craiova, Romania

ARTICLE INFO

Article history:
Received 14 January 2011
Accepted 16 March 2011

The paper is devoted to the memory of Vicenţiu's beloved mother, Ana Rădulescu (1923–2011).

Keywords:
Yamabe problem
Emden–Fowler equation
Sublinear eigenvalue problem
Multiple solutions

ABSTRACT

The existence of three nontrivial solutions for a nonlinear problem on compact d-dimensional ($d \geq 3$) Riemannian manifolds without boundary, is established. This multiplicity result is then applied to solve Emden–Fowler equations that involve sublinear terms at infinity. Two concrete examples are also provided in the present paper. Our results apply to problems arising in conformal Riemannian geometry, astrophysics, and in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion.

1. Introduction

Analysis of Riemannian manifolds is a field currently undergoing great development. Moreover, analysis proves to be a very powerful tool for solving geometrical problems. Conversely, geometry may help us to solve certain problems in analysis, as pointed out in Aubin [1].

Let $(\mathcal{M}, g)$ be a compact d-dimensional Riemannian manifold without boundary, where $d \geq 3$. Let Δ_g denote the Laplace–Beltrami operator on $(\mathcal{M}, g)$ and assume that the functions $\alpha, K \in C^\infty(\mathcal{M})$ are positive. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function with *sublinear* growth and λ is a positive real parameter. In this paper, we are interested in the existence of (classical) solutions to the following eigenvalue problem:

$$-\Delta_g w + \alpha(\sigma) w = \lambda K(\sigma) f(w), \quad \sigma \in \mathcal{M}, \ w \in H_1^2(\mathcal{M}). \qquad (P_\lambda)$$

This problem generalizes the celebrated Yamabe equation (see [2, p. 126])

$$4\,\frac{d-1}{d-2}\,\Delta_g \varphi + R\varphi = \mu\, \varphi^{q-1} \quad \text{in } \mathcal{M},$$

where $2 < q < 2d/(d-2)$ and R denotes the scalar curvature of $\mathcal{M}$. According to Berger [3], curvature is "the No. 1 Riemannian invariant and the most natural. Gauss and then Riemann saw it instantly". The main question in the fundamental Yamabe's paper [4] was whether there are any restrictions needed to have a metric of constant scalar curvature. Yamabe

* Corresponding author at: Department of Mathematics, University of Craiova, 200585 Craiova, Romania. Tel.: +40 251412615.
E-mail addresses: bonanno@unime.it (G. Bonanno), gmolica@unirc.it (G. Molica Bisci), vicentiu.radulescu@imar.ro, vicentiu.radulescu@gmail.com (V. Rădulescu).

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2011.03.012

proved that for compact manifolds there always exists such a metric conformally equivalent to any given metric on the manifold. This proof turned out to be incomplete and Aubin [5] proved the theorem for all manifolds of dimension $d \geq 6$ that were not conformally flat. To settle the problem completely, a new global type of argument was needed, and that was provided in 1984 by Schoen [6].

By using variational methods (see Theorem 2.1 below), we find a well determined open interval of values of the parameter λ for which problem (P_λ) admits at least three nontrivial solutions. It is worth noticing that, to the best of our knowledge, this is the first result in which all the three solutions are nontrivial.

A remarkable case of problem (P_λ) is

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma)f(w), \quad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d), \tag{S_λ}$$

where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$, h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, s is a constant such that $1-d<s<0$, and Δ_h denotes the Laplace–Beltrami operator on $(\mathbb{S}^d, h)$.

Indeed, existence results for problem (S_λ) yield, by using an appropriate change of coordinates, the existence of solutions to the following parameterized Emden–Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s}u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\}; \tag{$\mathfrak{F}_\lambda$}$$

see Remark 4.2 and Corollary 4.2.

Moreover, we observe that the existence of a smooth positive solution for problem (S_λ), when $s=-d/2$ or $s=-d/2+1$, and $f(t)=|t|^{\frac{4}{d-2}}t$, can be viewed as an affirmative answer to the famous Yamabe problem [4] on $\mathbb{S}^d$ (see also the Nirenberg problem [7]); for these topics we refer to Aubin [1], Cotsiolis and Iliopoulos [8,9], Hebey [10], Kazdan and Warner [11], Vázquez and Véron [12], and to the excellent survey by Lee and Parker [13]. In these cases, the right-hand side of problem (S_λ) involves the critical Sobolev exponent.

Cotsiolis and Iliopoulos [9] and Vázquez and Véron [12] studied problem $(\mathfrak{F}_\lambda)$, by applying either minimization or minimax methods, provided that $f(t)=|t|^{p-1}t$, with $p>1$. Successively, in Kristály and Rădulescu [14], the authors are interested in the existence of multiple solutions of problem (P_λ) in order to obtain solutions for the parameterized Emden–Fowler equation $(\mathfrak{F}_\lambda)$ considering nonlinear terms of sublinear type at infinity. In particular, in [14, Theorem 1.1], for λ sufficiently large, the existence of two nontrivial solutions for problem (P_λ) has been successfully obtained through a careful analysis of the standard mountain pass geometry.

Further, in Kristály et al. [15, Theorem 9.4, p. 222], the existence of an open interval of positive parameters for which problem (P_λ) admits two distinct nontrivial solutions is established by using an abstract three critical points theorem contained in Bonanno [16].

In the present paper we use a new approach to attach sublinear problems at infinity, previously developed in Bonanno and Molica Bisci [17]. We obtain the existence of a well localized open interval of positive parameters for which problem (P_λ) admits at least three nontrivial solutions; see Theorem 3.1 and Remark 3.2.

The present paper is organized as follows. In Section 2 we recall some basic definitions and preliminary facts on the Sobolev spaces defined on compact Riemannian manifolds, while Section 3 is devoted to the existence of at least three solutions for the eigenvalue problem (P_λ). In Section 4 we give some consequences of the main results, as well as the existence of three nontrivial solutions for Emden–Fowler equations. A concrete example of application of our main theorems is then presented in the last section. We cite the very recent monograph by Kristály et al. [15] as general reference on this subject.

2. Preliminaries

We start this section with a short list of notions in Riemannian geometry. We refer to Aubin [1] and Hebey [10] for detailed derivations of the geometric quantities, their motivation and further applications.

Let $(\mathcal{M}, g)$ be a smooth compact d-dimensional ($d \geq 3$) Riemannian manifold without boundary and let g_{ij} be the components of the metric g. As usual, we denote by $C^\infty(\mathcal{M})$ the space of smooth functions defined on $\mathcal{M}$. Let $\alpha \in C^\infty(\mathcal{M})$ be a positive function and put $\|\alpha\|_\infty := \max_{\sigma\in\mathcal{M}} \alpha(\sigma)$. For every $w \in C^\infty(\mathcal{M})$, set

$$\|w\|^2_{H^2_\alpha} := \int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2 d\sigma_g,$$

where ∇w is the covariant derivative of w, and $d\sigma_g$ is the Riemannian measure. In local coordinates $(x^1,\ldots,x^d)$, the components of ∇w are given by

$$(\nabla^2 w)_{ij} = \frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial w}{\partial x^k},$$

where

$$\Gamma^k_{ij} := \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) g^{lk},$$

are the usual Christoffel symbols and g^{lk} are the elements of the inverse matrix of g.

Here, and in what follows, the Einstein summation convention is adopted. Moreover, the measure element $d\sigma_g$ assumes the form $d\sigma_g = \sqrt{\det g}\, dx$, where dx stands for the Lebesgue volume element of $\mathbb{R}^d$. Hence, let

$$\mathrm{Vol}_g(\mathcal{M}) := \int_{\mathcal{M}} d\sigma_g.$$

In particular, if $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$ and h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, we set

$$\omega_d := \mathrm{Vol}_h(\mathbb{S}^d) := \int_{\mathbb{S}^d} d\sigma_h.$$

The Sobolev space $H^2_\alpha(\mathcal{M})$ is defined as the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Then $H^2_\alpha(\mathcal{M})$ is a Hilbert space endowed with the inner product

$$\langle w_1, w_2\rangle_{H^2_\alpha} = \int_{\mathcal{M}} \langle \nabla w_1, \nabla w_2\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w_1, w_2\rangle_g d\sigma_g, \quad w_1, w_2 \in H^2_\alpha(\mathcal{M}),$$

where $\langle\cdot,\cdot\rangle_g$ is the inner product on covariant tensor fields associated to g.

Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the standard norm

$$\|w\|_{H^2_1} := \left(\int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} |w(\sigma)|^2 d\sigma_g\right)^{1/2}.$$

Moreover, if $w \in H^2_\alpha(\mathcal{M})$, the following inequalities hold

$$\min\{1, \min_{\sigma\in\mathcal{M}} \alpha(\sigma)^{1/2}\}\|w\|_{H^2_1} \le \|w\|_{H^2_\alpha} \le \max\{1, \|\alpha\|_\infty^{1/2}\}\|w\|_{H^2_1}. \tag{1}$$

From the Rellich–Kondrachov theorem for compact manifolds without boundary one has

$$H^2_\alpha(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}),$$

for every $q \in [1, 2d/(d-2)]$. In particular, the embedding is compact whenever $q \in [1, 2d/(d-2))$.
Hence, there exists a positive constant S_q such that

$$\|w\|_q \le S_q\|w\|_{H^2_\alpha}, \quad \text{for all } w \in H^2_\alpha(\mathcal{M}). \tag{2}$$

From now on, we assume that the nonlinearity f satisfies the following structural condition

$f : \mathbb{R} \to \mathbb{R}$ *is a locally Hölder continuous function sublinear at infinity, that is,*

$$\lim_{|t|\to\infty} \frac{f(t)}{t} = 0. \tag{h_∞}$$

Let $K \in C^\infty(\mathcal{M})$ be a positive function.

We recall that a function $w \in H^2_1(\mathcal{M})$ is a *weak solution* of problem (P_λ) if

$$\int_{\mathcal{M}} \langle \nabla w, \nabla v\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w, v\rangle_g d\sigma_g - \lambda \int_{\mathcal{M}} K(\sigma)f(w(\sigma))v(\sigma)d\sigma_g = 0,$$

for every $v \in H^2_1(\mathcal{M})$.

Further, due to the regularity assumption on f, the weak solutions are classical.

Hence, the main purpose is to study the following problem.

Find $\lambda > 0$ *and* $w \in H^2_1(\mathcal{M})$ *such that*

$$-\Delta_g w + \alpha(\sigma)w = \lambda K(\sigma)f(w), \quad \text{for all } \sigma \in \mathcal{M},\ w \in H^2_1(\mathcal{M}). \tag{P_λ}$$

Here, Δ_g represents the Laplace–Beltrami operator that, applied to a function $w \in H^2_1(\mathcal{M})$, is given (locally) by the following expression

$$\Delta_g w = g^{ij}\left(\frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial w}{\partial x^k}\right).$$

Remark 2.1. For a fixed $\lambda > 0$, the function $w_\lambda(\sigma) = c \in \mathbb{R}\setminus\{0\}$, is a solution of (P_λ) if and only if the function $\sigma \mapsto \lambda K(\sigma)/\alpha(\sigma)$ is constant. In this case, nontrivial constant solutions of (P_λ), appear as fixed points of the function $t \mapsto k_\lambda f(t)$, where κ_λ denotes the constant value $\lambda K(\sigma)/\alpha(\sigma)$.

In order to obtain multiple solutions of (P_λ) not only in the case of constant solutions but also other solutions, we use variational methods. The main tool is a critical point theorem that we recall here in a convenient form. This result has been obtained in Bonanno and Marano [18] and it is a more precise version of Theorem 3.2 of Bonanno and Candito [19].

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(a_1) $\frac{\sup_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}\right[$ *the functional* $J_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional J_λ has at least three distinct critical points in X.

We recall that the derivative of Φ admits a continuous inverse on X^* when there exists a continuous operator $T : X^* \to X$ such that $T(\Phi'(x)) = x$ for all $x \in X$.

3. Main results

We set

$$\kappa_\alpha := \left(\frac{2}{\|\alpha\|_{L^1(\mathcal{M})}}\right)^{1/2},$$

and

$$K_1 := \frac{S_1}{\sqrt{2}}\|\alpha\|_{L^1(\mathcal{M})}, \qquad K_2 := \frac{S_q^q}{2^{\frac{2-q}{2}}q}\|\alpha\|_{L^1(\mathcal{M})}.$$

Further, let

$$F(\xi) := \int_0^\xi f(t)\,dt,$$

for every $\xi \in \mathbb{R}$.

The main abstract theorem in this paper is the following multiplicity result.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that* (h_∞) *holds and assume that*

(h_1) *There exist two nonnegative constants a_1, a_2 such that*

$$|f(t)| \le a_1 + a_2|t|^{q-1}, \quad \text{for all } t \in \mathbb{R},$$

where $q \in]1, 2d/(d-2)[$.

(h_2) *There exist two positive constants γ and δ, with $\delta > \gamma\kappa_\alpha$, such that*

$$\frac{F(\delta)}{\delta^2} > \frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}}\left(a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2}\right).$$

Then, for each parameter λ belonging to

$$\Lambda_{(\gamma,\delta)} := \left]\frac{\delta^2\|\alpha\|_{L^1(\mathcal{M})}}{2F(\delta)\|K\|_{L^1(\mathcal{M})}}, \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty\left(a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2}\right)}\right[,$$

the problem (P_λ) possesses at least three distinct solutions in $H_1^2(\mathcal{M})$.

Proof. Our aim is to apply Theorem 2.1. Hence, let $X := H_1^2(\mathcal{M})$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(w) := \frac{\|w\|^2_{H^2_\alpha}}{2}, \qquad \Psi(w) := \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g, \quad \text{for all } w \in X.$$

Clearly $\Phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*. On the other hand, Ψ is well defined, continuously Gâteaux differentiable and with compact derivative. Moreover, one has

$$\Phi'(w)(v) = \int_{\mathcal{M}}\langle\nabla w, \nabla v\rangle_g d\sigma_g + \int_{\mathcal{M}}\alpha(\sigma)\langle w(\sigma), v(\sigma)\rangle_g d\sigma_g,$$

and

$$\Psi'(w)(v) = \int_{\mathcal{M}} K(\sigma)f(w(\sigma))v(\sigma)d\sigma_g,$$

for every $w, v \in X$.

Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $w \in X$ such that

$$\Phi'(w)(v) - \lambda\Psi'(w)(v) = 0,$$

for every $v \in X$. Hence, the critical points of the functional J_λ are weak solutions (hence classical solutions) of problem (P_λ). Now, $\Phi(0) = \Psi(0) = 0$ and since condition (h_1) holds, one has

$$F(\xi) \le a_1|\xi| + a_2\frac{|\xi|^q}{q}, \tag{3}$$

for every $\xi \in \mathbb{R}$.

Let $\varrho \in]0, +\infty[$ and consider the function

$$\chi(\varrho) := \frac{\sup_{w\in\Phi^{-1}(]-\infty,\varrho])} \Psi(w)}{\varrho}.$$

Taking into account (3), it follows that

$$\Psi(w) = \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g \le \|K\|_\infty \left(a_1\|w\|_{L^1(\mathcal{M})} + \frac{a_2}{q}\|w\|^q_{L^q(\mathcal{M})}\right).$$

Then, for every $w \in X$ such that $w \in \Phi^{-1}(]-\infty, \varrho])$, owing to (2), we get

$$\Psi(w) \le \|K\|_\infty\left((2\varrho)^{1/2}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}\varrho^{q/2}\right).$$

Hence, by using the definition of Φ, one has

$$\sup_{w\in\Phi^{-1}(]-\infty,\varrho])} \Psi(w) \le \|K\|_\infty \left((2\varrho)^{1/2}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}\varrho^{q/2}\right). \tag{4}$$

From (4), the following inequality holds

$$\chi(\varrho) \le \|K\|_\infty \left(\sqrt{\frac{2}{\varrho}}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}\varrho^{q/2-1}\right) \tag{5}$$

for every $r > 0$.

Next, put $w_\delta(\sigma) := \delta$ for every $\sigma \in M$. Clearly $w_\delta \in X$ and we have

$$\begin{aligned}\Phi(w_\delta) &= \frac{1}{2}\left(\int_{\mathcal{M}} |\nabla w_\delta(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)|w_\delta(\sigma)|^2 d\sigma_g\right)\\ &= \frac{1}{2}\int_{\mathcal{M}} \alpha(\sigma)\delta^2\, d\sigma_g = \frac{\delta^2}{2}\|\alpha\|_{L^1(\mathcal{M})}.\end{aligned} \tag{6}$$

Taking into account that $\delta > \gamma\kappa_\alpha$, by a direct computation, one has $\gamma^2 < \Phi(w_\delta)$.

Moreover,

$$\Psi(w_\delta) = \int_{\mathcal{M}} K(\sigma)F(w_\delta(\sigma))\, d\sigma_g = F(\delta)\|K\|_{L^1(\mathcal{M})}. \tag{7}$$

Hence, from (6) and (7), one has

$$\frac{\Psi(w_\delta)}{\Phi(w_\delta)} = 2\frac{F(\delta)\|K\|_{L^1(\mathcal{M})}}{\delta^2\|\alpha\|_{L^1(\mathcal{M})}}. \tag{8}$$

In view of (h_2) and taking into account (5) and (8), we get

$$\chi(\gamma^2) = \frac{\sup_{w\in\Phi^{-1}(]-\infty,\gamma^2])} \Psi(w)}{\gamma^2} \le \|K\|_\infty\left(\frac{\sqrt{2}}{\gamma}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}\gamma^{q-2}\right)$$

$$= \frac{2\|K\|_\infty}{\|\alpha\|_{L^1(\mathcal{M})}}\left(a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2}\right)$$
$$< 2\frac{F(\delta)\|K\|_{L^1(\mathcal{M})}}{\delta^2\|\alpha\|_{L^1(\mathcal{M})}}$$
$$= \frac{\Psi(w_\delta)}{\Phi(w_\delta)}.$$

Therefore, the assumption (a_1) of Theorem 2.1 is satisfied taking $\bar{x} := w_\delta$ and by choosing $r := \gamma^2$. Moreover, owing to (h_∞), for every $\varepsilon > 0$ sufficiently small there is $c(\varepsilon) > 0$ such that $|f(t)| \leq \varepsilon|t| + c(\varepsilon)$ for every $t \in \mathbb{R}$. Consequently, for every $w \in H_1^2(\mathcal{M})$, we have

$$J_\lambda(w) \geq \frac{1}{2}(1 - \lambda\varepsilon\|K\|_\infty S_2^2)\|w\|_{H_\alpha^2}^2 - c(\varepsilon)\lambda\|K\|_\infty S_1\|w\|_{H_\alpha^2}.$$

Therefore, the functional J_λ is coercive for every positive parameter, in particular, for every

$$\lambda \in \Lambda_{(\gamma,\delta)} \subseteq \left]\frac{\Phi(w_\delta)}{\Psi(w_\delta)}, \frac{\gamma^2}{\sup\limits_{w\in\Phi^{-1}(]-\infty,\gamma^2])}\Psi(w)}\right[.$$

So, condition (a_2) holds and hence, all the assumptions of Theorem 2.1 are satisfied. Then, for each $\lambda \in \Lambda_{(\gamma,\delta)}$, the functional J_λ has at least three distinct critical points that are classical solutions of problem (P_λ). □

Remark 3.1. Hypothesis (h_∞) can be substituted by the following growth condition.

(h'_∞) *There exist two positive constants b and $s < 2$ such that*

$$F(\xi) \leq b(1 + |\xi|^s),$$

for every $\xi \in \mathbb{R}$.

Indeed, owing to (h_1), problem (P_λ) is well defined. Therefore, the functional J_λ is coercive for every $\lambda \in (0, \infty)$. Indeed, fixing $\lambda > 0$, since $s < 2$, from the Hölder inequality we have

$$\int_{\mathcal{M}} |w(\sigma)|^s d\sigma_g \leq \|w\|_{L^2(\mathcal{M})}^s \mathrm{Vol}_g(\mathcal{M})^{\frac{2-s}{2}}, \quad \text{for all } w \in H_1^2(\mathcal{M}).$$

Now, bearing in mind (2), we obtain

$$\int_{\mathcal{M}} |w(\sigma)|^s d\sigma_g \leq S_2^s\|w\|_{H_\alpha^2}^s \mathrm{Vol}_g(\mathcal{M})^{\frac{2-s}{2}}, \quad \text{for all } w \in H_1^2(\mathcal{M}). \tag{9}$$

So, by using (9) and from condition (h'_∞), it follows that

$$J_\lambda(w) \geq \frac{\|w\|_{H_\alpha^2}^2}{2} - \lambda b \mathrm{Vol}_g(\mathcal{M})^{\frac{2-s}{2}} S_2^s\|w\|_{H_\alpha^2}^s - \lambda b\mathrm{Vol}_g(\mathcal{M}), \quad \text{for all } w \in H_1^2(\mathcal{M}).$$

Hence, J_λ is coercive for every real positive parameter λ.

Remark 3.2. Under the additional hypothesis $f(0) \neq 0$, Theorem 3.1 ensures the existence of at least three nontrivial solutions. Indeed, in this case, zero is not a solution for problem (P_λ), as a simple computation shows. Hence, all the three solutions, attained by using our abstract framework, are nontrivial. Moreover, if f is only continuous instead of Hölder continuous, our result guarantees the existence of at least three weak (nontrivial) solutions for problem (P_λ).

Remark 3.3. The technical approach used to prove the previous result have been introduced in Bonanno and Molica Bisci [17]. In the cited work, the existence of at least three weak solutions for a Dirichlet problem is showed under suitable conditions on the potential F.

Remark 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and positive function in $]0, +\infty[$ such that

$$|f(t)| \leq a_2|t|^{q-1}, \quad \forall\, t \in \mathbb{R},$$

for some $q \in]2, 2d/(d-2)[$.

Clearly, the above growth condition is a particular case of hypothesis (h_1) and implies $f(0) = 0$. In this setting, under the additional hypothesis (h'_∞), Theorem 3.1 ensures the existence of at least three (two nontrivial) solutions for every

$$\lambda > \lambda^* := \frac{1}{2}\frac{\|\alpha\|_{L^1(\mathcal{M})}}{\|K\|_{L^1(\mathcal{M})}} \inf_{\delta>0} \frac{\delta^2}{F(\delta)}.$$

This particular result can be achieved by using [15, Theorem 9.1].

Remark 3.5. It is well known that sharp Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. There has been much work on such inequalities and their applications. In our context, a concrete upper bound for the constants S_q in Theorem 3.1 is essential for a concrete evaluation of the interval $\Lambda_{(\gamma,\delta)}$. In the particular case $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, if $q \in [1, 2d/(d-2)[$, one has

$$S_q \le \frac{\kappa_q}{\min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}}, \tag{10}$$

where, we set

$$\kappa_q := \begin{cases} \omega_d^{\frac{2-q}{2q}} & \text{if } q \in [1, 2[, \\ \max\left\{ \left(\dfrac{q-2}{d\omega_d^{\frac{q-2}{q}}}\right)^{1/2}, \dfrac{1}{\omega_d^{\frac{q-2}{2q}}} \right\} & \text{if } q \in \left[2, \dfrac{2d}{d-2}\right[. \end{cases}$$

Indeed, in Beckner [20], it is proved that for every $2 \le q < 2d/(d-2)$ and any $w \in H_1^2(\mathbb{S}^d)$, one has

$$\left(\int_{\mathbb{S}^d} |w(\sigma)|^q d\sigma_h\right)^{2/q} \le \frac{q-2}{d\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \frac{1}{\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h;$$

see also, for instance, Theorem 4.28 in Hebey [10]. Hence,

$$\|w\|_{L^q(\mathbb{S}^d)} \le \max\left\{ \left(\frac{q-2}{d\omega_d^{\frac{q-2}{q}}}\right)^{1/2}, \frac{1}{\omega_d^{\frac{q-2}{2q}}} \right\} \left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h\right)^{1/2},$$

for every $w \in H_1^2(\mathbb{S}^d)$. Owing to (1) the desiderated statement follows.
On the other hand, if $q \in [1, 2[$, as simple consequence of the Hölder inequality, it follows that

$$\|w\|_{L^q(\mathbb{S}^d)} \le \omega_d^{\frac{2-q}{2q}} \|w\|_{L^2(\mathbb{S}^d)}, \quad \text{for all } w \in L^2(\mathbb{S}^d).$$

The thesis is achieved by taking into account that

$$\|w\|_{L^2(\mathbb{S}^d)} \le \|w\|_{H_1^2} \le \frac{\|w\|_{H_\alpha^2}}{\min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}},$$

for every $w \in H_1^2(\mathbb{S}^d)$.

4. Applications to nonlinear eigenvalue problems and Emden–Fowler equations

Let $\alpha, K \in C^\infty(\mathbb{S}^d)$ be positive and set

$$K_1^\star := \frac{\kappa_1 \|\alpha\|_{L^1(\mathbb{S}^d)}}{\sqrt{2}\min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}}. \tag{11}$$

Further, for $q \in]1, 2d/(d-2)[$, we will denote

$$K_2^\star := \frac{\kappa_q^q \|\alpha\|_{L^1(\mathbb{S}^d)}}{2^{\frac{2-q}{2}} q \min\left\{1, \min_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{q/2}\right\}}. \tag{12}$$

As a consequence of Theorem 3.1, and taking into account Remark 3.5, we get the following result on the existence of three solutions for nonlinear eigenvalues problems on the unit sphere $\mathbb{S}^d$.

Corollary 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (h_∞) and (h_1) hold. Further, assume that there exist two positive constants γ and δ, with $\delta > \gamma\kappa_\alpha$, and*

$$(\mathrm{h}_2^\star) \quad \frac{F(\delta)}{\delta^2} > \frac{\|K\|_\infty}{\|K\|_{L^1(\mathbb{S}^d)}} \left(a_1 \frac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-2} \right),$$

where $K_1^\star$ and $K_2^\star$ are given respectively by (11) and (12).
Then, for each parameter λ belonging to

$$\Lambda^\star_{(\gamma,\delta)} := \left] \frac{\delta^2 \|\alpha\|_{L^1(\mathbb{S}^d)}}{2F(\delta)\|K\|_{L^1(\mathbb{S}^d)}}, \frac{\|\alpha\|_{L^1(\mathbb{S}^d)}}{2\|K\|_\infty \left(a_1 \frac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-2} \right)} \right[,$$

the problem

$$-\Delta_h w + \alpha(\sigma) w = \lambda K(\sigma) f(\omega), \quad \sigma \in \mathbb{S}^d, \; w \in H_1^2(\mathbb{S}^d), \qquad (S_\lambda^\alpha)$$

possesses at least three distinct solutions.

Remark 4.1. Other relevant contributions on the existence of multiple solutions for elliptic problems on the sphere are contained in Kristály [21]; see also the related paper Kristály and Marzantowicz [22].

Next, we consider the following parameterized Emden–Fowler problem that arises in astrophysics, conformal Riemannian geometry, and in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion:

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\}. \qquad (\mathfrak{F}_\lambda)$$

The equation $(\mathfrak{F}_\lambda)$ has been studied when f has the form $f(t) = |t|^{p-1}t, p > 1$; see Cotsiolis–Iliopoulos [9], Vázquez–Véron [12]. In these papers, the authors obtained the existence and multiplicity results for $(\mathfrak{F}_\lambda)$, applying either minimization or minimax methods.

Remark 4.2. The solutions of $(\mathfrak{F}_\lambda)$ are being sought in the particular form

$$u(x) = r^s w(\sigma), \qquad (13)$$

where, $(r, \sigma) := (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$ and w be a smooth function defined on $\mathbb{S}^d$. This type of transformation is also used by Bidaut-Véron and Véron [23], where the asymptotic of a special form of $(\mathfrak{F}_\lambda)$ has been studied. Throughout (13), taking into account that

$$\Delta u = r^{-d} \frac{\partial}{\partial r} \left(r^d \frac{\partial u}{\partial r} \right) + r^{-2} \Delta_h u,$$

the equation $(\mathfrak{F}_\lambda)$ reduces to

$$-\Delta_h w + s(1-s-d) w = \lambda K(\sigma) f(w), \quad \sigma \in \mathbb{S}^d, \; w \in H_1^2(\mathbb{S}^d);$$

see also Kristály and Rădulescu [14].

From Remark 4.2, we have the following result.

Corollary 4.2. *Assume that d and s are two constants such that $1 - d < s < 0$. Further, let $K \in C^\infty(\mathbb{S}^d)$ be a positive function and $f : \mathbb{R} \to \mathbb{R}$ as in Corollary 4.1. Then, for each parameter λ belonging to*

$$\Lambda^{s,d}_{(\gamma,\delta)} := \left] \frac{s(1-s-d)\omega_d \delta^2}{2F(\delta)\|K\|_{L^1(\mathbb{S}^d)}}, \frac{s(1-s-d)\omega_d}{2\|K\|_\infty \left(a_1 \frac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-2} \right)} \right[,$$

the following problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\}, \qquad (\mathfrak{F}_\lambda)$$

admits at least three distinct solutions.

Proof of Corollary 4.2. Let us choose $(M, g) = (\mathbb{S}^d, h)$, and $\alpha(\sigma) := s(1-s-d)$ for every $\sigma \in \mathbb{S}^d$ in Corollary 4.1. Clearly $\alpha \in C^\infty(\mathbb{S}^d)$ and, thanks to $1 - d < s < 0$, α to be positive on $\mathbb{S}^d$. Thus, for every $\lambda \in \Lambda^{s,d}_{(\gamma,\delta)}$, the problem

$$-\Delta_h w + s(1-s-d) w = \lambda K(\sigma) f(w), \quad \sigma \in \mathbb{S}^d, \; w \in H_1^2(\mathbb{S}^d),$$

has at least three distinct solutions $w_\lambda^i \in H_1^2(\mathbb{S}^d), i \in \{1, 2, 3\}$. On account of (13), the elements $u_\lambda^i(x) = |x|^s w_\lambda^i(x/|x|)$, $i \in \{1, 2, 3\}$, are solutions of $(\mathfrak{F}_\lambda)$. □

5. Some examples

In the next example, problem (P_λ) admits three nontrivial solutions owing to Theorem 3.1, while [14, Theorem 1.1] as well as [15, Theorem 9.4, p. 222] cannot be applied.

Example 5.1. Let $(\mathcal{M}, g)$ be a compact d-dimensional ($d \geq 3$) Riemannian manifold without boundary, fix $q \in]2, 2d/(d-2)[$ and let $K \in C^\infty(\mathcal{M})$ be a positive function. Moreover, let $h : \mathbb{R} \to \mathbb{R}$ be the locally Lipschitz continuous function defined by

$$h(t) := \begin{cases} 1 + |t|^{q-1} & \text{if } |t| \leq r \\ \dfrac{(1+r^2)(1+r^{q-1})}{1+t^2} & \text{if } |t| > r, \end{cases}$$

where r is a fixed constant such that

$$r > \max\left\{\left(\frac{2}{\mathrm{Vol}_g(\mathcal{M})}\right)^{1/2}, q^{\frac{1}{q-2}}\left(\frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}}(K_1+K_2)\right)^{\frac{1}{q-2}}\right\}. \tag{14}$$

Clearly $h(0) \neq 0$ and $h(t) \leq (1 + |t|^{q-1})$ for every $t \in \mathbb{R}$. Hence, the condition (h_1) is satisfied for $a_1 = a_2 = 1$. Moreover, one has that $\lim_{|t|\to\infty} h(t)/t = 0$. Finally, owing to (14), it follows that

$$\frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}}(K_1+K_2) < \frac{r^{q-2}}{q}.$$

Therefore,

$$\frac{\int_0^r h(t)dt}{r^2} = \frac{r^{q-2}}{q} + \frac{1}{r} > \frac{\|K\|_\infty}{\|K\|_{L^1(\mathcal{M})}}(K_1+K_2),$$

and condition (h_2) holds choosing $\delta = r$.

Consequently, from Theorem 3.1, for each parameter

$$\lambda \in \left]\frac{qr^2\mathrm{Vol}_g(\mathcal{M})}{2(qr+r^q)\|K\|_{L^1(\mathcal{M})}}, \frac{\mathrm{Vol}_g(\mathcal{M})}{2\|K\|_\infty(K_1+K_2)}\right[,$$

the following problem

$$-\Delta_g w + w = \lambda K(\sigma)h(w), \quad \sigma \in \mathcal{M}, \; w \in H_1^2(\mathcal{M}),$$

possesses at least three nontrivial solutions.

The next example follows directly by Corollary 4.1. Moreover, the existence of at least three nontrivial solutions for a class of Emden–Fowler equations is achieved.

Example 5.2. Consider three positive constants a_1, a_2, ϵ, fix $q \in]2, 2d/(d-2)[$, where $d \geq 3$, and let $\alpha, K \in C^\infty(\mathbb{S}^d)$ be two positive functions. Define $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(t) := \begin{cases} a_1 + a_2|t|^{q-1} & \text{if } |t| \leq \max\{\kappa_\alpha, \rho\} + \epsilon \\ a_1 + a_2(\max\{\kappa_\alpha, \rho\} + \epsilon)^{q-1} & \text{if } |t| > \max\{\kappa_\alpha, \rho\} + \epsilon, \end{cases}$$

where

$$\kappa_\alpha := \left(\frac{2}{\|\alpha\|_{L^1(\mathbb{S}^d)}}\right)^{1/2},$$

and

$$\rho := \left(\frac{\|K\|_\infty \|\alpha\|_{L^1(\mathbb{S}^d)}}{a_2\|K\|_{L^1(\mathbb{S}^d)}}\right)^{\frac{1}{q-2}} \left(\frac{qa_1\kappa_1^1}{\sqrt{2}\min\left\{1, \min\limits_{\sigma\in\mathbb{S}^d}\alpha(\sigma)^{1/2}\right\}} + \frac{2^{\frac{q-2}{2}}a_2\kappa_q^q}{\min\left\{1, \min\limits_{\sigma\in\mathbb{S}^d}\alpha(\sigma)^{q/2}\right\}}\right)^{\frac{1}{q-2}}.$$

From Corollary 4.1, for each parameter

$$\lambda \in \left]\frac{q(\max\{\kappa_\alpha,\rho\}+\epsilon)^2\|\alpha\|_{L^1(\mathbb{S}^d)}}{2(qa_1(\max\{\kappa_\alpha,\rho\}+\epsilon) + a_2(\max\{\kappa_\alpha,\rho\}+\epsilon)^q)\|K\|_{L^1(\mathbb{S}^d)}}, \frac{q\|\alpha\|_{L^1(\mathbb{S}^d)}}{2\|K\|_{L^1(\mathbb{S}^d)}\rho^{q-2}}\right[,$$

the following problem

$$-\Delta_h w + \alpha(\sigma)w = \lambda K(\sigma)g(w), \quad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d),$$

possesses at least three nontrivial solutions.

Finally, let s be a constant such that $d - 1 < s < 0$. As consequence of Corollary 4.2, we obtain that, for every

$$\lambda \in \left] \frac{sq(1-s-d)(\max\{\kappa,\rho\}+\epsilon)^2\omega_d}{2(qa_1(\max\{\kappa,\rho\}+\epsilon)+a_2(\max\{\kappa,\rho\}+\epsilon)^q)\|K\|_{L^1(\mathbb{S}^d)}}, \frac{sq(1-s-d)\omega_d}{2\|K\|_{L^1(\mathbb{S}^d)}\rho^{q-2}} \right[,$$

where

$$\kappa := \left(\frac{2}{s(1-s-d)\omega_d}\right)^{1/2},$$

the problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|)g(|x|^{-s}u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\},$$

admits at least three distinct nontrivial solutions.

Acknowledgements

The authors express their gratitude to the anonymous referees for useful comments and remarks.

V. Rădulescu acknowledges the support through Grant CNCSIS PCCE–8/2010 "Sisteme diferenţiale în analiza neliniară şi aplicaţii".

References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, in: Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[2] T. Aubin, Nonlinear Analysis on Manifolds. Monge–Ampeère Equations, Grundlehren der Mathematischen Wissenschaften, in: Fundamental Principles of Mathematical Sciences, vol. 252, Springer-Verlag, New York, 1982.
[3] M. Berger, La géometrie métrique des variétés Riemanniennes, in Elie Cartan et les Mathématiques d'Aujourd'Hui", Astérisque, Société Mathématique de France 1985, pp. 9–66.
[4] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka J. Math. 12 (1960) 21–37.
[5] T. Aubin, Équations différentielles non linéaires et probleème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976) 269–296.
[6] R.M. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. 20 (1984) 479–495.
[7] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa 13 (1959) 115–162.
[8] A. Cotsiolis, D. Iliopoulos, Équations elliptiques non linéaires sur $\mathcal{S}^n$, Le problème de Nirenberg, C. R. Acad. Sci. Paris, Sér. I Math. 313 (1991) 607–609.
[9] A. Cotsiolis, D. Iliopoulos, Équations elliptiques non linéaires à croissance de Sobolev sur-critique, Bull. Sci. Math. 119 (1995) 419–431.
[10] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, New York, 1999.
[11] J.L. Kazdan, F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geom. 10 (1975) 113–134.
[12] J.L. Vázquez, L. Véron, Solutions positives d'équations elliptiques semi-linéaires sur des variétés riemanniennes compactes, C. R. Acad. Sci. Paris, Sér. I Math. 312 (1991) 811–815.
[13] J.M. Lee, T.H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. 17 (1987) 37–91.
[14] A. Kristály, V. Rădulescu, Sublinear eigenvalue problems on compact Riemannian manifolds with applications in Emden–Fowler equations, Studia Math. 191 (2009) 237–246.
[15] A. Kristály, V. Rădulescu, Cs. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, in: Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 2010.
[16] G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. TMA 54 (2003) 651–665.
[17] G. Bonanno, G. Molica Bisci, Three weak solutions for Dirichlet problems, preprint (2010).
[18] G. Bonanno, S.A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal. 89 (2010) 1–10.
[19] G. Bonanno, P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, J. Differential Equ. 244 (2008) 3031–3059.
[20] W. Beckner, Sharp sobolev inequalities on the sphere and the Moser–Trudinger inequality, Ann. of Math. 138 (1993) 213–242.
[21] A. Kristály, Asymptotically critical problems on higher-dimensional spheres, Discrete Contin. Dyn. Syst. 23 (2009) 919–935.
[22] A. Kristály, W. Marzantowicz, Multiplicity of symmetrically distinct sequences of solutions for a quasilinear problem in $\mathbb{R}^N$, Nonlinear Diff. Eqns. Appl. (NoDEA) 15 (2008) 209–226.
[23] M.F. Bidaut-Véron, L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, Invent. Math. 106 (1991) 489–539.