



Nonlinear elliptic equations with variable exponent: Old and new[☆]



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To Professor Enzo L. Mitidieri, with cordial wishes for his 60th birthday

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ABSTRACT

In this survey paper, by using variational methods, we are concerned with the qualitative analysis of solutions to nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \lambda |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded or an exterior domain of \mathbb{R}^N and q is a continuous positive function. The results presented in this paper extend several contributions concerning the Lane–Emden equation and we focus on new phenomena which are due to the presence of variable exponents.

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1. Introduction

One of the reasons of the huge development of the theory of classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ (where $1 \leq p < \infty$) is the description of many phenomena arising in applied sciences. For instance, many materials can be modeled with sufficient accuracy using the function spaces L^p and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as “smart fluids”), this approach is not adequate, but rather the exponent p should be able to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function. Variable exponent Lebesgue spaces appeared in the literature for the first time already in a paper by Orlicz [38]. In the 1950s this study was carried on by Nakano [37], who made the first systematic study of spaces with variable exponent (called modular spaces). Nakano mentioned explicitly variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano [37, p. 284]. Later, the Polish mathematicians investigated the modular function spaces, see Musielak [35]. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the work of Tsenov [45], Sharapudinov [43] and Zhikov [47]. We refer to the monograph by Diening, Harjulehto, Hästö, and Ruzicka [10] for a comprehensive introduction to

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the theory of function spaces with variable exponents and various applications. We also point out the multiple contributions of the Finnish “Research group on variable exponent spaces and image processing” [22], whose main purpose is to study nonlinear potential theory in variable exponent Sobolev spaces.

This paper is motivated by phenomena which are described by nonlinear boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f(x, u), & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded or an exterior domain with smooth boundary.

The interest in studying such problems consists in the presence of the Laplace-type operator with variable exponent $\operatorname{div}(a(x, \nabla u))$. A basic example is the $p(x)$ -Laplace operator, which is defined by

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

The study of differential equations and variational problems involving $p(x)$ -growth conditions is a consequence of their applications. In 1920, E. Bingham was surprised that some paints do not run, like honey. He studied such a behavior and described a strange phenomenon. There are fluids that flow then stop spontaneously (Bingham fluids). Within them, the forces that create flow reach a first threshold. As this threshold is not reached, the fluid flows without deforms as a solid. Invented in the 17th century, the “Flemish medium” makes painting oil thixotropic: it fluidies under pressure of the brush, but freezes as soon as you leave the rest. While the exact composition of the medium Flemish remains unknown, it is known that the bonds form gradually between its components, which is why the picture freezes in a few minutes. Thanks to this wonderful medium, Rubens has painted *La Kermesse* in 24 h.

Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery on electrorheological fluids is due to Willis Winslow, who obtained a US patent on the effect in 1947 and wrote an article published in 1949, see [46]. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium Polymethacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics, we refer to consult Halsey [19]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modeling and the application of variable exponent spaces to these fluids we refer to Acerbi and Mingione [1], Ruzicka [42], Chen, Levine, and Rao [9], Harjulehto, Hästö, Latvala, and Toivanen [20], Molica Bisci and Repovš [34], et al. We also point out the pioneering contributions of Pucci et al. [4,5,3] in the study of Kirchhoff-type problems, including nonlocal problems with variable exponent like

$$M \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)}u = f(x, u).$$

We give in what follows two relevant examples that justify the mathematical study of models involving variable exponents.

Example 1 (*Image Restoration Chen, Levine, Rao [9]*). In image restoration, we consider an input I that corresponds to shades of gray in a domain $\Omega \subset \mathbb{R}^2$. We assume that I is made up of the true image corrupted by noise. Suppose that the noise is additive, that is, $I = T + \eta$ where T is the true image and η is a random variable with zero mean. Thus, the effect of the noise can be eliminated by smoothing the input, since this will cause the effect of the zero-mean random variables at nearby locations to cancel. Smoothing corresponds to minimizing the energy

$$\mathcal{E}_1(u) = \int_{\Omega} (|\nabla u(x)|^2 + |u(x) - I(x)|^2) dx.$$

Unfortunately, smoothing will also destroy the small details from the image, so this procedure is not useful. A better approach is *total variation smoothing*. Since an edge in the image gives rise to a very large gradient, the level sets around the edge are very distinct, so this method does a good job of preserving edges. Total variation smoothing corresponds to minimizing the energy

$$\mathcal{E}_2(u) = \int_{\Omega} (|\nabla u(x)| + |u(x) - I(x)|^2) dx.$$

Unfortunately, total variation smoothing not only preserves edges, but also creates edges where there were none in the original image. This is the *staircase effect*.

Looking at \mathcal{E}_1 and \mathcal{E}_2 , Chen, Levine and Rao suggest that an appropriate energy is

$$\mathcal{E}(u) = \int_{\Omega} (|\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2) dx,$$

where $1 \leq p \leq 2$.

This function should be close to 1 where there are likely to be edges, and close to 2 where there are likely not to be edges. The approximate location of the edges can be determined by just smoothing the input data and looking for where the gradient is large.

Example 2 (*Electrorheological Fluids*). The constitutive equation for the motion of an electrorheological fluid is

$$u_t + \operatorname{div} S(u) + (u \cdot \nabla)u + \nabla \pi = f, \tag{1.2}$$

where $u : \mathbb{R}^{3,1} \rightarrow \mathbb{R}^3$ is the velocity of the fluid at a point in space–time, $\pi : \mathbb{R}^{3,1} \rightarrow \mathbb{R}$ is the pressure, $f : \mathbb{R}^{3,1} \rightarrow \mathbb{R}^3$ represents external forces, and the stress tensor $S : W_{\text{loc}}^{1,1} \rightarrow \mathbb{R}^{3,3}$ is of the form

$$S(u)(x) = \mu(x)[1 + |Du(x)|^2]^{(p(x)-2)/p(x)} Du(x),$$

where $Du = (\nabla u + \nabla u^T)/2$ is the symmetric part of the gradient of u .

We observe that the highest order differential term in (1.2) is

$$\operatorname{div} \left((1 + |Du(x)|^2)^{(p(x)-2)/p(x)} Du(x) \right).$$

The degenerate case corresponds to the Laplace operator with variable exponent.

This survey investigates mathematical models that involve differential operators with variable exponents and we are mainly interested in the *qualitative properties* of solutions for problems with $p(x)$ growth as in (1.1). We point out that even if our results will be formulated in a variational context, our methods and techniques can be applied to systems as well. A particular interest in this work is given to *new phenomena*, which are generated by the presence of one or several variable exponents and which are no longer valid in the classical framework corresponding to homogeneous differential operators like the Laplace or the p -Laplace operator. In particular, we are interested in new spectral properties, for instance the concentration of the spectrum near the origin or at infinity. In all the cases studied in the present paper, a central role is played by various *competition effects* between the terms arising in the equation, as well as by various perturbations that can alter the behavior of the solutions by generating nonexistence properties. We refer to [28,29,16,30,31,18,32,41,33] for related results and complete proofs.

This paper is constructed as follows. In the next section we recall the basic properties of Lebesgue and Sobolev spaces with variable exponent. Section 3 contains a multiplicity result for a class of Dirichlet problems involving a general nonhomogeneous differential operator. Section 4 deals with the existence of a continuous spectrum for a differential operator with two variable exponents. A concentration property at infinity of the spectrum is established in Section 5 in the case of the $p(x)$ -Laplace operator on exterior domains. The case of multiple variable exponents and sign-changing potential is considered in the next section of the paper. In Section 7, by using the Morse theory in combination with local linking theory, we establish several existence results. Finally, we consider the discrete framework and we are concerned with the existence of homoclinic solutions for a class of partial difference equations with variable exponent.

2. Function spaces with variable exponent

With the emergence of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demonstrate their limitations in applications. The class of nonlinear problems with variable exponent growth is a new research field and reflects a new kind of physical phenomena.

In this section we introduce the Lebesgue and Sobolev spaces with variable exponent and we recall their main properties. For more details we refer to the book by Musielak [36], Diening, Harjulehto, Hästö and Ruzicka [10], and the papers by Edmunds et al. [12,11], Kovacik and Rákosník [23].

For any continuous function $h : \bar{\Omega} \rightarrow (1, \infty)$ we denote

$$h^- = \operatorname{ess\,inf}_{x \in \Omega} h(x) \quad \text{and} \quad h^+ = \operatorname{ess\,sup}_{x \in \Omega} h(x).$$

Usually it is assumed that $h^+ < +\infty$, since this condition is known to imply many desirable features for the associated variable exponent Lebesgue space $L^{h(x)}(\Omega)$. This function space is defined by

$$L^{h(x)}(\Omega) = \left\{ u; u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{h(x)} dx < \infty \right\}.$$

On this space we define a norm, the so-called *Luxemburg norm*, by the formula

$$\|u\|_{h(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{h(x)} dx \leq 1 \right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz–Musielak space. For a constant function h the variable exponent Lebesgue space coincides with the standard Lebesgue space.

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and h_1, h_2 are variable exponents so that $h_1(x) \leq h_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{h_2(x)}(\Omega) \hookrightarrow L^{h_1(x)}(\Omega)$.

We denote by $L^{h'(x)}(\Omega)$ the conjugate space of $L^{h(x)}(\Omega)$, where $1/h(x) + 1/h'(x) = 1$. For any $u \in L^{h(x)}(\Omega)$ and $v \in L^{h'(x)}(\Omega)$ the Hölder type inequality

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{h^-} + \frac{1}{h'^-} \right) \|u\|_{h(x)} \|v\|_{h'(x)} \tag{2.1}$$

holds true.

Variable exponent Lebesgue spaces do **not** have the *mean continuity property*: if p is continuous and nonconstant in an open ball B , then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm (see Kovacic and Rákosník [23]).

Most of the problems in the development of the theory of $L^{p(x)}$ spaces arise from the fact that these spaces are virtually never translation invariant. The use of convolution is also limited: the Young inequality

$$\|f * g\|_{L^{p(x)}} \leq C \|f\|_{L^{p(x)}} \|g\|_{L^1}$$

holds if and only if p is constant.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{h(x)}(\Omega)$ space, which is the mapping $\rho_{h(x)} : L^{h(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\rho_{h(x)}(u) = \int_{\Omega} |u(x)|^{h(x)} \, dx.$$

Lebesgue–Sobolev spaces with $h^+ = +\infty$ have been investigated in [11,23]. In such a case we denote $\Omega_{\infty} = \{x \in \Omega; h(x) = +\infty\}$ and define the modular by setting

$$\rho_{h(x)}(u) = \int_{\Omega \setminus \Omega_{\infty}} |u(x)|^{h(x)} \, dx + \text{ess sup}_{x \in \Omega_{\infty}} |h(x)|.$$

If $(u_n), u \in L^{h(x)}(\Omega)$ then the following relations hold true

$$|u|_{h(x)} > 1 \Rightarrow |u|_{h(x)}^{h^-} \leq \rho_{h(x)}(u) \leq |u|_{h(x)}^{h^+} \tag{2.2}$$

$$|u|_{h(x)} < 1 \Rightarrow |u|_{h(x)}^{h^+} \leq \rho_{h(x)}(u) \leq |u|_{h(x)}^{h^-} \tag{2.3}$$

$$|u_n - u|_{h(x)} \rightarrow 0 \Leftrightarrow \rho_{h(x)}(u_n - u) \rightarrow 0. \tag{2.4}$$

Next, we define the variable exponent Sobolev space

$$W^{1,h(x)}(\Omega) = \{u \in L^{h(x)}(\Omega) : |\nabla u| \in L^{h(x)}(\Omega)\}.$$

On $W^{1,h(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{h(x)} = |u|_{h(x)} + |\nabla u|_{h(x)}$$

or

$$\|u\| = \inf \left\{ \mu > 0; \int_{\Omega} \left(\left| \frac{\nabla u(x)}{\mu} \right|^{h(x)} + \left| \frac{u(x)}{\mu} \right|^{h(x)} \right) dx \leq 1 \right\}.$$

We also define $W_0^{1,h(x)}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,h(x)}(\Omega)$. Assuming $h^- > 1$, then the function spaces $W^{1,h(x)}(\Omega)$ and $W_0^{1,h(x)}(\Omega)$ are separable and reflexive Banach spaces. Set

$$\varrho_{h(x)}(u) = \int_{\Omega} (|\nabla u(x)|^{h(x)} + |u(x)|^{h(x)}) \, dx.$$

For all $(u_n), u \in W_0^{1,h(x)}(\Omega)$ the following relations hold

$$\|u\| > 1 \Rightarrow \|u\|^{h^-} \leq \varrho_{h(x)}(u) \leq \|u\|^{h^+} \tag{2.5}$$

$$\|u\| < 1 \Rightarrow \|u\|^{h^+} \leq \varrho_{h(x)}(u) \leq \|u\|^{h^-} \tag{2.6}$$

$$\|u_n - u\| \rightarrow 0 \Leftrightarrow \varrho_{h(x)}(u_n - u) \rightarrow 0. \tag{2.7}$$

Next, we recall some embedding results regarding variable exponent Lebesgue–Sobolev spaces. If $h, \theta : \Omega \rightarrow (1, \infty)$ are Lipschitz continuous and $h^+ < N$ and $h(x) \leq \theta(x) \leq h^*(x)$ for any $x \in \Omega$ where $h^*(x) = Nh(x)/(N - h(x))$, then there exists a continuous embedding $W_0^{1,h(x)}(\Omega) \hookrightarrow L^{\theta(x)}(\Omega)$. Furthermore, assuming that Ω_0 is a bounded subset of Ω , then the embedding $W_0^{1,h(x)}(\Omega_0) \hookrightarrow L^{\theta(x)}(\Omega_0)$ is continuous and compact.

As shown by Zhikov [47], the smooth functions are in general *not* dense in $W^{1,p(x)}(\Omega)$. This property is in relationship with the *Lavrentiev phenomenon*, which asserts that there are variational problems for which the infimum over the smooth functions is strictly greater than the infimum over all functions that satisfy the same boundary conditions. Another formulation asserts that a Lagrangian L exhibits the Lavrentiev phenomenon if the infimum taken over the set of Lipschitzian trajectories $\mathbf{AC}[0, 1]$ is strictly lower than the infimum taken over the set of Lipschitzian trajectories $\mathbf{Lip}[0, 1]$, with fixed boundary conditions. The first example of such a phenomenon is due to Lavrentiev [24], who proved that

$$\inf_{u \in W^{1,1}(0,1), u(0)=0, u(1)=1} \int_0^1 (x - u^3)^2 |u'(x)|^6 dx = 0,$$

while

$$\inf_{u \in W^{1,\infty}(0,1), u(0)=0, u(1)=1} \int_0^1 (x - u^3)^2 |u'(x)|^6 dx > 0.$$

A related example corresponding to the Lagrangian $L(t, x, x') = (x^3 - t^2)x'^6$ on the interval $[0, 1]$ is due to Manià [27]. Moreover, Foss, Hrusa and Mizel [17] gave a physical action of a nonlinear elastic material where the occurrence of the Lavrentiev phenomenon is the occurrence of a meaningful physical event.

If p is logarithmic Hölder continuous (notation: $p \in C^{0, \frac{1}{|\log|\cdot|}}(\overline{\Omega})$), that is,

$$|p(x) - p(y)| \leq \frac{C}{|\log|x - y||} \quad \forall x, y \in \overline{\Omega}, |x - y| \leq 1/2,$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so the Sobolev space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$. Edmunds and Rakosnik [12] derived the same conclusion under a local monotony condition on p .

Since Ω is bounded and $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, then

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega) \text{ [Poincaré inequality]},$$

where $C = C(p, |\Omega|, \text{diam}(\Omega), N)$. Poincaré's inequality holds under a much weaker assumption on p than the Sobolev inequality and embedding, namely if the exponent p is *not too discontinuous*.

2.1. Some remarks and a striking example

We start with some useful remarks in the framework of function spaces with variable exponent.

Remark 1. If Ω is bounded then the following embeddings hold:

$$C^{0,1}(\overline{\Omega}) \subset W^{1,q}(\Omega) \quad (\text{if } q > N) \subset C^{0, \frac{1}{|\log|\cdot|}}(\overline{\Omega}).$$

Remark 2. If Ω is unbounded, then p is said to be logarithmic Hölder continuous if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x - y||} \quad \forall x, y \in \overline{\Omega}, |x - y| \leq 1/2$$

and

$$|p(x) - p(y)| \leq \frac{C}{|\log(e + |x|)|} \quad \forall x, y \in \Omega, |y| \geq |x|.$$

In such a case we cannot require $p \in W^{1,q}(\Omega)$, since $\int_\Omega |p(x)|^q dx = \infty$.

Let

$$W^{1,(\infty,q(\cdot))}(\Omega) := \{u \in L^\infty(\Omega); |\nabla u| \in L^{q(\cdot)}(\Omega)\},$$

where $N < q_- \leq q_+ < \infty$.

We conclude that if Ω is unbounded then the hypotheses

- (i) $p \in C^{0,1}(\overline{\Omega})$;
- (ii) $p \in W^{1,(\infty,q(\cdot))}(\Omega)$ with $N < q_- \leq q_+ < \infty$;
- (iii) $p \in C^{0, \frac{1}{|\log|\cdot|}}(\overline{\Omega})$

are independent of each other.

Next, we provide the following example related to minimizers of the one-dimensional Dirichlet energy with variable exponent. We say that a function $u \in W^{1,p(x)}(0, 1)$ is a minimizer with boundary values 0 and $a > 0$ if $u(0) = 0, u(1) = a$, and

$$\int_0^1 |u'(y)|^{p(y)} dy \leq \int_0^1 |v'(y)|^{p(y)} dy,$$

for all $v \in W^{1,p(x)}(0, 1)$ with $v(0) = 0, v(1) = a$.

If p is constant, then the minimizer is linear, namely $u(x) = ax$. Let us assume that $p(x) = 3\chi_{(0,1/2)} + 2\chi_{(1/2,1)}$. Assume that u is a minimizer and denote $u(1/2) = b$. Then $u|_{(0,1/2)}$ is the solution of the classical energy integral problem with values 0 and b , and $u|_{(1/2,1)}$ is the solution with boundary values b and a . Thus, these functions are linear. This u has Dirichlet energy $4b^3 + 2(a - b)^2$. The function $b \mapsto 2b^3 + (a - b)^2$ has a minimum at $b = (\sqrt{12a + 1} - 1)/6$, which determines the minimizer of the variable exponent problem. A computation shows that the minimizer is convex if $a > 2/3$, concave if $a < 2/3$ and linear if $a = 2/3$.

3. Combined sublinear perturbations in a problem with variable exponent

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. In this section we study the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1}), & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \\ u \geq 0, & \text{for } x \in \Omega, \end{cases} \tag{3.1}$$

where $1 < \beta < \gamma < \inf_{x \in \overline{\Omega}} p(x)$.

Problem (3.1) is studied in [28], which seems to be the first paper dealing with elliptic equations involving general nonhomogeneous differential operators.

We assume that $a(x, \xi) : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}$, $A = A(x, \xi)$, that is, $a(x, \xi) = \nabla_{\xi} A(x, \xi)$. Suppose that a and A satisfy the following hypotheses:

(A1) The following equality holds

$$A(x, 0) = 0,$$

for all $x \in \overline{\Omega}$.

(A2) There exists a positive constant c_1 such that

$$|a(x, \xi)| \leq c_1(1 + |\xi|^{p(x)-1}),$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

(A3) The following inequality holds

$$0 \leq (a(x, \xi) - a(x, \psi)) \cdot (\xi - \psi),$$

for all $x \in \overline{\Omega}$ and $\xi, \psi \in \mathbb{R}^N$, with equality if and only if $\xi = \psi$.

(A4) There exists $k > 0$ such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k|\xi - \psi|^{p(x)}$$

for all $x \in \overline{\Omega}$ and $\xi, \psi \in \mathbb{R}^N$.

(A5) The following inequalities hold true

$$|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x) A(x, \xi),$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

Examples. 1. Set $A(x, \xi) = \frac{1}{p(x)}|\xi|^{p(x)}$, $a(x, \xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geq 2$. Then we get the $p(x)$ -Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

2. Set $A(x, \xi) = \frac{1}{p(x)}[(1 + |\xi|^2)^{p(x)/2} - 1]$, $a(x, \xi) = (1 + |\xi|^2)^{(p(x)-2)/2}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator

$$\operatorname{div}((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u).$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (3.1) if $u \geq 0$ a.e. in Ω and

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx - \lambda \int_{\Omega} u^{\gamma-1} \phi \, dx + \lambda \int_{\Omega} u^{\beta-1} \phi \, dx = 0$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$.

The main result in this section asserts that problem (3.1) has at least two nontrivial weak solutions, provided that λ is large enough.

Theorem 1. Assume hypotheses (A1)–(A5) are fulfilled. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ problem (3.1) has at least two distinct non-negative, nontrivial weak solutions, provided that $p^+ < \min\{N, Np^-(N - p^-)\}$.

We observe that using Theorem 4.3 in Fan and Zhang [14], problem (3.1) has at least a weak solution in the particular case $a(x, \xi) = |\xi|^{p(x)-1}\xi$. However, the proof in [14] does not state the fact that the solution is non-negative and not even nontrivial in the case when $f(x, 0) = 0$.

3.1. Qualitative properties of the energy functional

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. Define the energy functional $I : E \rightarrow \mathbb{R}$ by

$$I(u) = \int_{\Omega} A(x, \nabla u) \, dx - \frac{\lambda}{\gamma} \int_{\Omega} u_+^{\gamma} \, dx + \frac{\lambda}{\beta} \int_{\Omega} u_+^{\beta} \, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.

We first establish some basic properties of I .

Proposition 2. *The functional I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by*

$$\langle I'(u), \phi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx - \lambda \int_{\Omega} u_+^{\gamma-1} \phi \, dx + \lambda \int_{\Omega} u_+^{\beta-1} \phi \, dx,$$

for all $u, \phi \in E$.

To prove Proposition 2 we define the functional $\Lambda : E \rightarrow \mathbb{R}$ by

$$\Lambda(u) = \int_{\Omega} A(x, \nabla u) \, dx, \quad \forall u \in E.$$

Lemma 3. (i) *The functional Λ is well-defined on E .*

(ii) *The functional Λ is of class $C^1(E, \mathbb{R})$ and*

$$\langle \Lambda'(u), \phi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx,$$

for all $u, \phi \in E$.

Proof. (i) For any $x \in \Omega$ and $\xi \in \mathbb{R}^N$ we have

$$A(x, \xi) = \int_0^1 \frac{d}{dt} A(x, t\xi) \, dt = \int_0^1 a(x, t\xi) \cdot \xi \, dt.$$

Using hypotheses (A2) we get

$$\begin{aligned} A(x, \xi) &\leq c_1 \int_0^1 (1 + |\xi|^{p(x)-1} t^{p(x)-1}) |\xi| \, dt \\ &\leq c_1 |\xi| + \frac{c_1}{p(x)} |\xi|^{p(x)} \\ &\leq c_1 |\xi| + \frac{c_1}{p^-} |\xi|^{p(x)}, \quad \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^N. \end{aligned} \tag{3.2}$$

The above inequality and (A5) imply

$$0 \leq \int_{\Omega} A(x, \nabla u) \, dx \leq c_1 \int_{\Omega} |\nabla u| \, dx + \frac{c_1}{p^-} \int_{\Omega} |\nabla u|^{p(x)} \, dx, \quad \forall u \in E.$$

Using inequality (2.1) and relations (2.2) and (2.3) we deduce that Λ is well defined on E .

(ii) We now argue the existence of the Gâteaux derivative. Let $u, \phi \in E$. Fix $x \in \Omega$ and $0 < |r| < 1$. Then, by the mean value theorem, there exists $\nu \in [0, 1]$ such that

$$|A(x, \nabla u(x) + r \nabla \phi(x)) - A(x, \nabla u)|/|r| = |a(x, \nabla u(x) + \nu r \nabla \phi(x))| |\nabla \phi(x)|.$$

Using condition (A2) we obtain

$$\begin{aligned} |A(x, \nabla u(x) + r \nabla \phi(x)) - A(x, \nabla u)|/|r| &\leq [c_1 + c_1(|\nabla u(x)| + |\nabla \phi(x)|)^{p(x)-1}] |\nabla \phi(x)| \\ &\leq [c_1 + c_1 2^{p^+} (|\nabla u(x)|^{p(x)-1} + |\nabla \phi(x)|^{p(x)-1})] |\nabla \phi(x)|. \end{aligned}$$

Next, by Hölder's inequality, we have

$$\int_{\Omega} c_1 |\nabla \phi| \, dx \leq |c_1|_{\frac{p(x)}{p(x)-1}} \cdot \|\nabla \phi\|_{p(x)}$$

and

$$\int_{\Omega} |\nabla u|^{p(x)-1} |\nabla \phi| \, dx \leq \| |\nabla u|^{p(x)-1} \|_{\frac{p(x)}{p(x)-1}} \cdot \|\nabla \phi\|_{p(x)}.$$

Therefore

$$c_1 [1 + 2^{p^+} (|\nabla u(x)|^{p(x)-1} + |\nabla \phi(x)|^{p(x)-1})] |\nabla \phi(x)| \in L^1(\Omega).$$

It follows from the Lebesgue theorem that

$$\langle \Lambda'(u), \phi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx.$$

In order to establish the continuity of the Gâteaux derivative, we assume that $u_n \rightarrow u$ in E . Let us define $\theta(x, u) = a(x, \nabla u)$. Using hypotheses (A2) and Proposition 2.2 in Fan and Zhang [14], we deduce that $\theta(x, u_n) \rightarrow \theta(x, u)$ in $(L^{q(x)}(\Omega))^N$, where $q(x) = \frac{p(x)}{p(x)-1}$. By inequality (2.1) we obtain

$$|\langle \Lambda'(u_n) - \Lambda'(u), \phi \rangle| \leq |\theta(x, u_n) - \theta(x, u)|_{q(x)} |\nabla \phi|_{p(x)}$$

and so

$$\|\Lambda'(u_n) - \Lambda'(u)\| \leq |\theta(x, u_n) - \theta(x, u)|_{q(x)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The proof of Lemma 3 is complete. \square

The next result generalizes a classical property due to Stampacchia [44]. The proof combines Lebesgue’s dominated convergence theorem and Sobolev-type embeddings.

Lemma 4. *If $u \in E$ then $u_+, u_- \in E$ and*

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \leq 0] \\ \nabla u, & \text{if } [u > 0], \end{cases} \quad \nabla u_- = \begin{cases} 0, & \text{if } [u \geq 0] \\ \nabla u, & \text{if } [u < 0] \end{cases}$$

where $u_{\pm} = \max\{\pm u(x), 0\}$ for all $x \in \Omega$.

By Lemmas 3 and 4 it is clear that Proposition 2 holds true.

We remark that if u is a critical point of I then using Lemma 4 and condition (A5) we have

$$\begin{aligned} 0 &= \langle I'(u), u_- \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla u_- \, dx - \lambda \int_{\Omega} (u_+)^{\gamma-1} u_- \, dx + \lambda \int_{\Omega} (u_+)^{\beta-1} u_- \, dx \\ &= \int_{\Omega} a(x, \nabla u) \cdot \nabla u_- \, dx = \int_{\Omega} a(x, \nabla u_-) \cdot \nabla u_- \, dx \geq \int_{\Omega} |\nabla u_-|^{p(x)} \, dx. \end{aligned}$$

Thus we deduce that $u \geq 0$. It follows that the nontrivial critical points of I are non-negative solutions of problem (3.1).

The above remark shows that we can prove Theorem 1 using the critical points theory. More exactly, we first show that for $\lambda > 0$ large enough, the functional I has a global minimizer $u_1 \geq 0$ such that $I(u_1) < 0$. Next, by means of the Mountain Pass Theorem, a second critical point u_2 with $I(u_2) > 0$ is obtained.

Lemma 5. *The functional Λ is weakly lower semi-continuous.*

Proof. By Corollary III.8 in Brezis [7], it is enough to show that Λ is lower semi-continuous. For this purpose, we fix $u \in E$ and $\epsilon > 0$. Since Λ is convex (by condition (A4)), we deduce that for any $v \in E$ the following inequality holds

$$\int_{\Omega} A(x, \nabla v) \, dx \geq \int_{\Omega} A(x, \nabla u) \, dx + \int_{\Omega} a(x, \nabla u) \cdot (\nabla v - \nabla u) \, dx.$$

Using condition (A2) and inequality (2.1) we have

$$\begin{aligned} \int_{\Omega} A(x, \nabla v) \, dx &\geq \int_{\Omega} A(x, \nabla u) \, dx - \int_{\Omega} |a(x, \nabla u)| |\nabla v - \nabla u| \, dx \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_1 \int_{\Omega} |\nabla(v - u)| \, dx - c_1 \int_{\Omega} |\nabla u|^{p(x)-1} |\nabla(v - u)| \, dx \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_2 \|1\|_{q(x)} \|\nabla(v - u)\|_{p(x)} - c_3 \|\nabla u\|_{q(x)}^{p(x)-1} \|\nabla(v - u)\|_{p(x)} \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_4 \|v - u\| \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - \epsilon \end{aligned}$$

for all $v \in E$ with $\|v - u\| < \delta = \epsilon/c_4$, where c_2, c_3, c_4 are positive constants, and $q(x) = \frac{p(x)}{p(x)-1}$. We conclude that Λ is weakly lower semi-continuous. The proof of Lemma 5 is complete. \square

Standard arguments show that one of the associated Rayleigh quotients has positive infimum. More precisely, we have the following property.

Lemma 6. *There exists $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf_{u \in E, \|u\| > 1} \frac{\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_{\Omega} |u|^{p^-} dx}.$$

Proposition 7. (i) *The functional I is bounded from below and coercive.*

(ii) *The functional I is weakly lower semi-continuous.*

Proof. (i) Since $1 < \beta < \gamma < p^-$ we have

$$\lim_{t \rightarrow \infty} \frac{\frac{1}{\gamma} t^\gamma - \frac{1}{\beta} t^\beta}{t^{p^-}} = 0.$$

Then for any $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\lambda \left(\frac{1}{\gamma} t^\gamma - \frac{1}{\beta} t^\beta \right) \leq \frac{\lambda_1}{2} t^{p^-} + C_\lambda, \quad \forall t \geq 0,$$

where λ_1 is defined in Lemma 6.

Condition (A5) and the above inequality show that for any $u \in E$ with $\|u\| > 1$ we have

$$\begin{aligned} I(u) &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{\lambda_1}{2} \int_{\Omega} |u|^{p^-} dx - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2p^+} \|u\|^{p^-} - C_\lambda \mu(\Omega). \end{aligned}$$

This shows that I is bounded from below and coercive.

(ii) Using Lemma 5 we deduce that Λ is weakly lower semi-continuous. We show that I is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to u in E . Since Λ is weakly lower semi-continuous we have

$$\Lambda(u) \leq \liminf_{n \rightarrow \infty} \Lambda(u_n). \tag{3.3}$$

On the other hand, since E is compactly embedded in $L^\gamma(\Omega)$ and $L^\beta(\Omega)$ it follows that (u_{n+}) converges strongly to u_+ both in $L^\gamma(\Omega)$ and in $L^\beta(\Omega)$. This fact together with relation (3.3) imply

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n).$$

Therefore, I is weakly lower semi-continuous. The proof of Proposition 7 is complete. \square

By Proposition 7 we deduce that there exists $u_1 \in E$ a global minimizer of I . The following result implies that $u_1 \neq 0$, provided that λ is sufficiently large.

The next property asserts that the energy functional achieves negative values for big values of the parameter.

Proposition 8. *There exists $\lambda^* > 0$ such that $\inf_E I < 0$ for all $\lambda > \lambda^*$.*

Proof. Let $\Omega_1 \subset \Omega$ be a compact subset, large enough and $u_0 \in E$ be such that $u_0(x) = t_0$ in Ω_1 and $0 \leq u_0(x) \leq t_0$ in $\Omega \setminus \Omega_1$, where $t_0 > 1$ is chosen such that

$$\frac{1}{\gamma} t_0^\gamma - \frac{1}{\beta} t_0^\beta > 0.$$

We have

$$\begin{aligned} \frac{1}{\gamma} \int_{\Omega} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega} u_0^\beta dx &\geq \frac{1}{\gamma} \int_{\Omega_1} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega_1} u_0^\beta dx - \frac{1}{\beta} \int_{\Omega \setminus \Omega_1} u_0^\beta dx \\ &\geq \frac{1}{\gamma} \int_{\Omega_1} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega_1} u_0^\beta dx - \frac{1}{\beta} t_0^\beta \mu(\Omega \setminus \Omega_1) > 0 \end{aligned}$$

and thus $I(u_0) < 0$ for $\lambda > 0$ large enough. \square

Since Proposition 8 holds true it follows that $u_1 \in E$ is a nontrivial weak solution of problem (3.1).

Fix $\lambda \geq \lambda^*$. Set

$$g(x, t) = \begin{cases} 0, & \text{for } t < 0 \\ t^{\gamma-1} - t^{\beta-1}, & \text{for } 0 \leq t \leq u_1(x) \\ u_1(x)^{\gamma-1} - u_1(x)^{\beta-1}, & \text{for } t > u_1(x) \end{cases}$$

and

$$G(x, t) = \int_0^t g(x, s) \, ds.$$

Define the functional $J : E \rightarrow \mathbb{R}$ by

$$J(u) = \int_{\Omega} A(x, \nabla u) \, dx - \lambda \int_{\Omega} G(x, u) \, dx.$$

The same arguments as those used for functional I imply that $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), \phi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx - \lambda \int_{\Omega} g(x, u) \phi \, dx,$$

for all $u, \phi \in E$.

On the other hand, we point out that if $u \in E$ is a critical point of J then $u \geq 0$. The proof can be carried out as in the case of functional I .

Lemma 9. *If u is a critical point of J then $u \leq u_1$.*

Proof. We have

$$\begin{aligned} 0 &= \langle J'(u) - J'(u_1), (u - u_1)_+ \rangle \\ &= \int_{\Omega} (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla (u - u_1)_+ \, dx - \lambda \int_{\Omega} [g(x, u) - (u_1^{\gamma-1} - u_1^{\beta-1})](u - u_1)_+ \, dx \\ &= \int_{[u > u_1]} (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla (u - u_1) \, dx. \end{aligned}$$

By condition (A3) we deduce that the above equality holds if and only if $\nabla u = \nabla u_1$. It follows that $\nabla u(x) = \nabla u_1(x)$ for all $x \in \omega := \{y \in \Omega; u(y) > u_1(y)\}$. Hence

$$\int_{\omega} |\nabla(u - u_1)|^{p(x)} \, dx = 0$$

and thus

$$\int_{\Omega} |\nabla(u - u_1)_+|^{p(x)} \, dx = 0.$$

By relation (2.3) we obtain

$$\|(u - u_1)_+\| = 0.$$

Since $u - u_1 \in E$ by Lemma 4 we have that $(u - u_1)_+ \in E$. Thus we obtain that $(u - u_1)_+ = 0$ in Ω , that is, $u \leq u_1$ in Ω . The proof of Lemma 9 is complete. \square

In the following we determine a critical point $u_2 \in E$ of J such that $J(u_2) > 0$ via the mountain pass theorem. By the above lemma we will deduce that $0 \leq u_2 \leq u_1$ in Ω . Therefore

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma} u_2^{\gamma} - \frac{1}{\beta} u_2^{\beta}$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

More exactly we find

$$I(u_2) > 0 = I(0) > I(u_1) \quad \text{and} \quad I'(u_2) = 0.$$

This shows that u_2 is a weak solution of problem (3.1) such that $0 \leq u_2 \leq u_1$, $u_2 \neq 0$ and $u_2 \neq u_1$.

In order to find u_2 described above we prove that the energy J satisfies the following geometric property (see [28] for details).

Lemma 10. *There exists $\rho \in (0, \|u_1\|)$ and $a > 0$ such that $J(u) \geq a$, for all $u \in E$ with $\|u\| = \rho$.*

Lemma 11. *The functional J is coercive.*

The proof combines our assumption (A5) with relation (2.2) and inequality (2.1).

The following result yields a sufficient condition which ensures that a weakly convergent sequence in E converges strongly, too.

Lemma 12. *Assume that the sequence (u_n) converges weakly to u in E and*

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \leq 0.$$

Then (u_n) converges strongly to u in E .

Proof. Using relation (3.2) we have that there exists a positive constant c_5 such that

$$A(x, \xi) \leq c_5(|\xi| + |\xi|^{p(x)}), \quad \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^N.$$

The above inequality implies

$$A(x, \nabla u_n) \leq c_5(|\nabla u_n| + |\nabla u_n|^{p(x)}), \quad \forall x \in \overline{\Omega}, n. \quad (3.4)$$

The fact that u_n converges weakly to u in E implies that there exists $R > 0$ such that $\|u_n\| \leq R$ for all n . By relation (3.4), inequalities (2.1)–(2.3) we deduce that $\{\int_{\Omega} A(x, \nabla u_n) \, dx\}$ is bounded. Then, up to a subsequence, we deduce that $\int_{\Omega} A(x, \nabla u_n) \, dx \rightarrow c$. By Lemma 5 we obtain

$$\int_{\Omega} A(x, \nabla u) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} A(x, \nabla u_n) \, dx = c.$$

On the other hand, since A is convex, we have

$$\int_{\Omega} A(x, \nabla u) \, dx \geq \int_{\Omega} A(x, \nabla u_n) \, dx + \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u - \nabla u_n) \, dx.$$

Next, by the hypothesis $\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \leq 0$, we conclude that $\int_{\Omega} A(x, \nabla u) \, dx = c$.

Taking into account that $(u_n + u)/2$ converges weakly to u in E and using Lemma 5 we have

$$c = \int_{\Omega} A(x, \nabla u) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} A\left(x, \nabla \frac{u_n + u}{2}\right) \, dx. \quad (3.5)$$

We assume by contradiction that u_n does not converge to u in E . Then by (2.4) it follows that there exist $\epsilon > 0$ and a subsequence (u_{n_m}) of (u_n) such that

$$\int_{\Omega} |\nabla(u_{n_m} - u)|^{p(x)} \, dx \geq \epsilon, \quad \forall m. \quad (3.6)$$

By condition (A4) we have

$$\frac{1}{2}A(x, \nabla u) + \frac{1}{2}A(x, \nabla u_{n_m}) - A\left(x, \nabla \frac{u + u_{n_m}}{2}\right) \geq k|\nabla(u_{n_m} - u)|^{p(x)}. \quad (3.7)$$

Relations (3.6) and (3.7) yield

$$\frac{1}{2} \int_{\Omega} A(x, \nabla u) \, dx + \frac{1}{2} \int_{\Omega} A(x, \nabla u_{n_m}) \, dx - \int_{\Omega} A\left(x, \nabla \frac{u + u_{n_m}}{2}\right) \, dx \geq k \int_{\Omega} |\nabla(u_{n_m} - u)|^{p(x)} \, dx \geq k\epsilon.$$

Letting $m \rightarrow \infty$ in the above inequality we obtain

$$c - k\epsilon \geq \limsup_{m \rightarrow \infty} \int_{\Omega} A\left(x, \nabla \frac{u + u_{n_m}}{2}\right) \, dx$$

and that is a contradiction with (3.5). It follows that u_n converges strongly to u in E and Lemma 12 is proved. \square

3.2. Existence of multiple solutions

Using Lemma 10 and the mountain pass theorem (see Ambrosetti and Rabinowitz [2]) we deduce that there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \rightarrow c > 0 \quad \text{and} \quad J'(u_n) \rightarrow 0 \tag{3.8}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0, 1], E); \gamma(0) = 0, \gamma(1) = u_1\}.$$

By relation (3.8) and Lemma 11 we obtain that (u_n) is bounded and thus passing eventually to a subsequence, still denoted by (u_n) , we may assume that there exists $u_2 \in E$ such that u_n converges weakly to u_2 . Since E is compactly embedded in $L^i(\Omega)$ for any $i \in [1, p^-]$, it follows that u_n converges strongly to u_2 in $L^i(\Omega)$ for all $i \in [1, p^-]$. Thus, as $n \rightarrow \infty$,

$$\langle \Lambda'(u_n) - \Lambda'(u_2), u_n - u_2 \rangle = \langle J'(u_n) - J'(u_2), u_n - u_2 \rangle + \lambda \int_{\Omega} [g(x, u_n) - g(x, u_2)](u_n - u_2) \, dx = o(1).$$

By Lemma 12 we deduce that u_n converges strongly to u_2 in E and using relation (3.8) we find

$$J(u_2) = c > 0 \quad \text{and} \quad J'(u_2) = 0.$$

Therefore, $J(u_2) = c > 0$ and $J'(u_2) = 0$. By Lemma 9 we deduce that $0 \leq u_2 \leq u_1$ in Ω . Therefore

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma} u_2^{\gamma} - \frac{1}{\beta} u_2^{\beta}$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

We conclude that u_2 is a critical point of I and thus a solution of (3.1). Furthermore, $I(u_2) = c > 0$ and $I(u_2) > 0 > I(u_1)$. Thus u_2 is not trivial and $u_2 \neq u_1$. The proof of Theorem 1 is now complete. \square

4. Subcritical Lane–Emden equations with multiple variable exponents

4.1. Previous results and statement of the problem

The standard Lane–Emden equation is

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This paper has been intensively studied, especially after the pioneering paper by Ambrosetti and Rabinowitz [2]. The structure of solutions is well understood at this stage and the behavior of solutions strongly depends on the values of q with respect to the critical Sobolev exponent. In the present paper we point out some striking properties of solutions in the case of the presence of several variable exponents. In particular, we will point out some concentration properties of the spectrum, which are generated by the nonstandard nonlinearities in relationship with the nonhomogeneous differential operator.

Consider the following nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda |u|^{q(x)-2} u, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega. \end{cases} \tag{4.1}$$

In the case when $p(x) = q(x)$ on $\overline{\Omega}$, Fan, Zhang and Zhao [15] established the existence of infinitely many eigenvalues for problem (4.1) by using an argument based on the Lyusternik–Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that Λ is discrete, $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).

In the case when $\min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth Mihăilescu and Rădulescu [29] used Ekeland’s variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. If $\max_{x \in \overline{\Omega}} p(x) < \min_{x \in \overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth a mountain-pass argument, then any $\lambda > 0$ is an eigenvalue of problem (4.1).

Assuming that $\max_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ then the energy functional associated to problem (4.1) has a nontrivial minimum for any positive λ , see [14, Theorem 4.3]. Using now the main result in [29], we obtain two positive constants λ^* and λ^{**} such that any $\lambda \in (0, \lambda^*) \cup (\lambda^{**}, \infty)$ is an eigenvalue of problem (4.1).

In this section we are concerned with the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u = \lambda|u|^{q(x)-2}u, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega. \end{cases} \tag{4.2}$$

We impose the following hypotheses:

$$1 < p_2(x) < \min_{y \in \overline{\Omega}} q(y) \leq \max_{y \in \overline{\Omega}} q(y) < p_1(x), \quad \forall x \in \overline{\Omega} \tag{4.3}$$

and

$$\max_{y \in \overline{\Omega}} q(y) < p_2^*(x), \quad \forall x \in \overline{\Omega}, \tag{4.4}$$

where $p_2^*(x) := \frac{Np_2(x)}{N-p_2(x)}$ if $p_2(x) < N$ and $p_2^*(x) = +\infty$ if $p_2(x) \geq N$.

Under these conditions, we show a concentration property of the spectrum at infinity. More precisely, we argue that there are two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (4.2) while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (4.2). An important consequence of our study is that, under hypotheses (3.1) and (4.4), we have

$$\inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} > 0.$$

That fact is proved by using the Lagrange multiplier theorem. The absence of homogeneity will be balanced by the fact that assumptions (3.1) and (4.4) yield

$$\lim_{\|u\|_{p_1(x)} \rightarrow 0} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} = \infty$$

and

$$\lim_{\|u\|_{p_1(x)} \rightarrow \infty} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} = \infty,$$

where $\|\cdot\|_{p_1(x)}$ stands for the norm in the variable exponent Sobolev space $W_0^{1,p_1(x)}(\Omega)$.

4.2. Concentration at infinity

Since $p_2(x) < p_1(x)$ for any $x \in \overline{\Omega}$ it follows that $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$. Thus, a solution for a problem of type (4.2) will be sought in the variable exponent space $W_0^{1,p_1(x)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (4.2) if there exists $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v dx - \lambda \int_{\Omega} |u|^{q(x)-2}uv dx = 0,$$

for all $v \in W_0^{1,p_1(x)}(\Omega)$. We point out that if λ is an eigenvalue of problem (4.2) then the corresponding eigenfunction $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ is a weak solution of problem (4.2).

Define a first Rayleigh quotient by

$$\lambda_1 := \inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx}. \tag{4.5}$$

The main result in this section is given by the following theorem. This result points out the importance of the quotient defined in (4.5) and of a second Rayleigh quotient denoted by λ_0 .

Theorem 13. Assume that conditions (4.3) and (4.4) are fulfilled. Then $\lambda_1 > 0$. Moreover, any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (4.2). Furthermore, there exists a positive constant λ_0 such that $\lambda_0 \leq \lambda_1$ and any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (4.2).

Proof. Let E denote the generalized Sobolev space $W_0^{1,p_1(x)}(\Omega)$. We denote by $\|\cdot\|$ the norm on $W_0^{1,p_1(x)}(\Omega)$ and by $\|\cdot\|_1$ the norm on $W_0^{1,p_2(x)}(\Omega)$.

Define the functionals $J, I, J_1, I_1 : E \rightarrow \mathbb{R}$ by

$$\begin{aligned} J(u) &= \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx, \\ I(u) &= \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx, \\ J_1(u) &= \int_{\Omega} |\nabla u|^{p_1(x)} dx + \int_{\Omega} |\nabla u|^{p_2(x)} dx, \\ I_1(u) &= \int_{\Omega} |u|^{q(x)} dx. \end{aligned}$$

Standard arguments imply that $J, I \in C^1(E, \mathbb{R})$ and for all $u, v \in E$,

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v dx, \\ \langle I'(u), v \rangle &= \int_{\Omega} |u|^{q(x)-2} uv dx. \end{aligned}$$

We split the proof of Theorem 13 into four steps.

Step 1. We show that $\lambda_1 > 0$.

Since for any $x \in \overline{\Omega}$ we have $p_1(x) > q^+ \geq q(x) \geq q^- > p_2(x)$ we deduce that for any $u \in E$,

$$2(|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)}) \geq |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.$$

Integrating the above inequalities we find

$$2 \int_{\Omega} (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}) dx \geq \int_{\Omega} (|\nabla u|^{q^+} + |\nabla u|^{q^-}) dx, \quad \forall u \in E \tag{4.6}$$

and

$$\int_{\Omega} (|u|^{q^+} + |u|^{q^-}) dx \geq \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \tag{4.7}$$

By Sobolev embeddings, there exist positive constants λ_{q^+} and λ_{q^-} such that

$$\int_{\Omega} |\nabla u|^{q^+} dx \geq \lambda_{q^+} \int_{\Omega} |u|^{q^+} dx, \quad \forall u \in W_0^{1,q^+}(\Omega) \tag{4.8}$$

and

$$\int_{\Omega} |\nabla u|^{q^-} dx \geq \lambda_{q^-} \int_{\Omega} |u|^{q^-} dx, \quad \forall u \in W_0^{1,q^-}(\Omega). \tag{4.9}$$

Using again the fact that $q^- \leq q^+ < p_1(x)$ for any $x \in \overline{\Omega}$ we deduce that E is continuously embedded in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (4.8) and (4.9) hold true for any $u \in E$.

Using inequalities (4.8), (4.9) and (4.7) it is clear that there exists a positive constant μ such that

$$\int_{\Omega} (|\nabla u|^{q^+} + |\nabla u|^{q^-}) dx \geq \mu \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \tag{4.10}$$

Next, inequalities (4.10) and (4.6) yield

$$\int_{\Omega} (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}) dx \geq \frac{\mu}{2} \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \tag{4.11}$$

By relation (4.11) we deduce that

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0 \tag{4.12}$$

and thus,

$$J_1(u) \geq \lambda_0 I_1(u), \quad \forall u \in E. \tag{4.13}$$

The above inequality yields

$$p_1^+ \cdot J(u) \geq J_1(u) \geq \lambda_0 I_1(u) \geq \lambda_0 I(u) \quad \forall u \in E. \tag{4.14}$$

The last inequality assures that $\lambda_1 > 0$ and thus, step 1 is verified.

Step 2. We show that λ_1 is an eigenvalue of problem (4.2).

Lemma 14. *The following relations hold true:*

$$\lim_{\|u\| \rightarrow \infty} \frac{J(u)}{I(u)} = \infty \tag{4.15}$$

and

$$\lim_{\|u\| \rightarrow 0} \frac{J(u)}{I(u)} = \infty. \tag{4.16}$$

Proof. Since E is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\| \geq c_1 \cdot |u|_{q^+}, \quad \forall u \in E \tag{4.17}$$

and

$$\|u\| \geq c_2 \cdot |u|_{q^-}, \quad \forall u \in E. \tag{4.18}$$

For any $u \in E$ with $\|u\| > 1$ by relations (2.2), (4.7), (4.17), (4.18) we infer

$$\frac{J(u)}{I(u)} \geq \frac{\frac{\|u\|^{p_1^-}}{p_1^+}}{\frac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\frac{\|u\|^{p_1^-}}{p_1^+}}{\frac{c_1^{-q^+} \|u\|^{q^+} + c_2^{-q^-} \|u\|^{q^-}}{q^-}}.$$

Since $p_1^- > q^+ \geq q^-$, passing to the limit as $\|u\| \rightarrow \infty$ in the above inequality we deduce that relation (4.15) holds true.

Next, let us remark that since $p_1(x) > p_2(x)$ for any $x \in \overline{\Omega}$, the space $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$. Thus, if $\|u\| \rightarrow 0$ then $\|u\|_1 \rightarrow 0$.

The above remarks enable us to affirm that for any $u \in E$ with $\|u\| < 1$ small enough we have $\|u\|_1 < 1$.

On the other hand, since (4.4) holds true we deduce that $W_0^{1,p_2(x)}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_1 \geq d_1 \cdot |u|_{q^+}, \quad \forall u \in W_0^{1,p_2(x)}(\Omega) \tag{4.19}$$

and

$$\|u\|_1 \geq d_2 \cdot |u|_{q^-}, \quad \forall u \in W_0^{1,p_2(x)}(\Omega). \tag{4.20}$$

Thus, for any $u \in E$ with $\|u\| < 1$ small enough, relations (2.3), (4.7), (4.19), (4.20) imply

$$\frac{J(u)}{I(u)} \geq \frac{\frac{\int_{\Omega} |\nabla u|^{p_2(x)} dx}{p_2^+}}{\frac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\frac{\|u\|_1^{p_2^+}}{p_2^+}}{\frac{d_1^{-q^+} \|u\|_1^{q^+} + d_2^{-q^-} \|u\|_1^{q^-}}{q^-}}.$$

Since $p_2^+ < q^- \leq q^+$, passing to the limit as $\|u\| \rightarrow 0$ (and thus, $\|u\|_1 \rightarrow 0$) in the above inequality we deduce that relation (4.16) holds true. The proof of Lemma 14 is complete.

Lemma 15. *There exists $u \in E \setminus \{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.*

Proof. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for λ_1 , that is,

$$\lim_{n \rightarrow \infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{4.21}$$

By relation (4.15) it is clear that $\{u_n\}$ is bounded in E . Since E is reflexive it follows that there exists $u \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in E . On the other hand, similar arguments as those used in the proof of Lemma 3.4 in [28] show that the functional J is weakly lower semi-continuous. Thus, we find

$$\liminf_{n \rightarrow \infty} J(u_n) \geq J(u). \tag{4.22}$$

By the compact embedding theorem for spaces with variable exponent and assumption $1 \leq \max_{y \in \overline{\Omega}} q(y) < p_1(x)$ for all $x \in \overline{\Omega}$ (see (4.3)) it follows that E is compactly embedded in $L^{q(x)}(\Omega)$. Thus, $\{u_n\}$ converges strongly in $L^{q(x)}(\Omega)$. Then, by relation (2.4) it follows that

$$\lim_{n \rightarrow \infty} I(u_n) = I(u). \tag{4.23}$$

Relations (4.22) and (4.23) imply that if $u \neq 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u is not trivial. Assume by contradiction the contrary. Then u_n converges weakly to 0 in E and strongly in $L^{q(x)}(\Omega)$. In other words, we will have

$$\lim_{n \rightarrow \infty} I(u_n) = 0. \tag{4.24}$$

Letting $\epsilon \in (0, \lambda_1)$ be fixed by relation (4.21) we deduce that for n large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon)I(u_n) < J(u_n) < (\lambda_1 + \epsilon)I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (4.24) holds true we find

$$\lim_{n \rightarrow \infty} J(u_n) = 0.$$

That fact combined with relation (2.4) implies that actually u_n converges strongly to 0 in E , that is, $\lim_{n \rightarrow \infty} \|u_n\| = 0$. By this information and relation (4.16) we get

$$\lim_{n \rightarrow \infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \neq 0$. The proof of Lemma 15 is complete.

By Lemma 15 we conclude that there exists $u \in E \setminus \{0\}$ such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w \in E \setminus \{0\}} \frac{J(w)}{I(w)}. \tag{4.25}$$

Then, for any $v \in E$ we have

$$\left. \frac{d}{d\epsilon} \frac{J(u + \epsilon v)}{I(u + \epsilon v)} \right|_{\epsilon=0} = 0.$$

A simple computation yields

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx \cdot I(u) - J(u) \cdot \int_{\Omega} |u|^{q(x)-2} uv \, dx = 0, \quad \forall v \in E. \tag{4.26}$$

Relation (4.26) combined with the fact that $J(u) = \lambda_1 I(u)$ and $I(u) \neq 0$ implies the fact that λ_1 is an eigenvalue of problem (4.2). Thus, step 2 is verified.

Step 3. We show that any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (4.2).

Let $\lambda \in (\lambda_1, \infty)$ be arbitrary but fixed. Define $T_\lambda : E \rightarrow \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly, $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T'_\lambda(u), v \rangle = \langle J'(u), v \rangle - \lambda \langle I'(u), v \rangle, \quad \forall u \in E.$$

Thus, λ is an eigenvalue of problem (4.2) if and only if there exists $u_\lambda \in E \setminus \{0\}$ a critical point of T_λ .

With similar arguments as in the proof of relation (4.15) we can show that T_λ is coercive, that is, $\lim_{\|u\| \rightarrow \infty} T_\lambda(u) = \infty$. On the other hand, as we have already remarked, similar arguments as those used in the proof of Lemma 3.4 in [28] show that the functional T_λ is weakly lower semi-continuous. Thus there exists $u_\lambda \in E$ a global minimum point of T_λ and thus, a critical point of T_λ . In order to conclude that step 4 holds true it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ , or λ is an eigenvalue of problem (4.2). Thus, step 3 is verified.

Step 4. Any $\lambda \in (0, \lambda_0)$, where λ_0 is given by (4.12), is not an eigenvalue of problem (4.2).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (4.2) it follows that there exists $u_\lambda \in E \setminus \{0\}$ such that

$$\langle J'(u_\lambda), v \rangle = \lambda \langle I'(u_\lambda), v \rangle, \quad \forall v \in E.$$

Thus, for $v = u_\lambda$ we find

$$\langle J'(u_\lambda), u_\lambda \rangle = \lambda \langle I'(u_\lambda), u_\lambda \rangle,$$

that is,

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E \setminus \{0\}$ assures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information yields

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 13 is now complete.

5. Eigenvalue problems with variable exponents on exterior domains

The result established in Section 4 can be extended to other classes of nonlinear equations with variable exponent. We give an example in this section in the framework of nonhomogeneous differential operators with a single variable exponent.

We are concerned with the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u + |u|^{q(x)-2} u = \lambda g(x) |u|^{r(x)-2} u & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \quad (5.1)$$

where Ω is a smooth exterior domain in \mathbb{R}^N , that is, Ω is the complement of a bounded domain with Lipschitz boundary. The mappings $p, q, r : \overline{\Omega} \rightarrow [2, \infty)$ are Lipschitz continuous functions while $g : \overline{\Omega} \rightarrow [0, \infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \geq 0$ is a real number. We assume that the functions p, q and r satisfy the hypotheses

$$2 \leq p^- \leq p^+ < N, \quad (5.2)$$

$$p^+ < r^- \leq r^+ < q^- \leq q^+ < \frac{Np^-}{N - p^-}. \quad (5.3)$$

Furthermore, we assume that the function $g(x)$ satisfies the hypotheses

$$g \in L^\infty(\Omega) \cap L^{p_0(x)}(\Omega), \quad (5.4)$$

where $p_0(x) = p^*(x)/(p^*(x) - r^-)$ for any $x \in \overline{\Omega}$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (5.1) if there exists $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv + |u|^{q(x)-2} uv) dx - \lambda \int_\Omega g(x) |u|^{r(x)-2} uv dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Define the Rayleigh quotients

$$\lambda_1 := \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx + \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx}{\int_{\Omega} \frac{g(x)}{r(x)} |u|^{r(x)} \, dx}$$

and

$$\lambda_0 := \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx + \int_{\Omega} |u|^{q(x)} \, dx}{\int_{\Omega} g(x) |u|^{r(x)} \, dx}.$$

Theorem 16. Let Ω be an exterior domain with Lipschitz boundary in \mathbb{R}^N , where $N \geq 3$. Suppose that $p, q, r : \overline{\Omega} \rightarrow [2, \infty)$ are Lipschitz continuous functions and $g : \overline{\Omega} \rightarrow [0, \infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g > 0$ in Ω_0 . Assume conditions (5.2), (4.3), and (5.4) are fulfilled.

Then

$$0 < \lambda_0 \leq \lambda_1.$$

Furthermore, each $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (5.1) while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (5.1).

Sketch of the proof. Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$.

Define the functionals $J_1, I_1, J_0, I_0 : E \rightarrow \mathbb{R}$ by

$$J_1(u) = \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx + \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx,$$

$$I_1(u) = \int_{\Omega} \frac{g(x)}{r(x)} |u|^{r(x)} \, dx,$$

$$J_0(u) = \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx + \int_{\Omega} |u|^{q(x)} \, dx,$$

$$I_0(u) = \int_{\Omega} g(x) |u|^{r(x)} \, dx.$$

Standard arguments imply that $J_1, I_1 \in C^1(E, \mathbb{R})$ and for all $u, v \in E$,

$$\langle J_1'(u), v \rangle = \int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv + |u|^{q(x)-2} uv) \, dx,$$

$$\langle I_1'(u), v \rangle = \int_{\Omega} g(x) |u|^{r(x)-2} uv \, dx.$$

For any $\lambda > 0$ we also define the functional $T_\lambda : E \rightarrow \mathbb{R}$ by

$$T_\lambda(u) = J_1(u) - \lambda \cdot I_1(u), \quad \forall u \in E.$$

It is clear that λ is an eigenvalue for problem (5.1) if and only if there exists $u_\lambda \in E \setminus \{0\}$ a critical point of the functional T_λ .

The proof of Theorem 16 is divided into the following steps.

Step 1. We have λ_0 and $\lambda_1 > 0$. This follows with energy estimates and similar arguments as in Section 4.

Step 2. Any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (5.1).

Step 3. Any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue for problem (5.1). This property relies on the following auxiliary results first prove two auxiliary results.

Lemma 17. Assume that the hypotheses of Theorem 16 are satisfied and s is a real number such that

$$r^+ < s < (p^-)^*,$$

where $(p^-)^* = Np^- / (N - p^-)$. Then $g \in L^{\frac{s}{s-r^-}}(\Omega) \cap L^{\frac{s}{s-r^+}}(\Omega)$ and

$$\int_{\Omega} g(x) |u|^{r(x)} \leq |g|_{\frac{s}{s-r^-}} |u|_s^{r^-} + |g|_{\frac{s}{s-r^+}} |u|_s^{r^+}, \quad \forall u \in E. \tag{5.5}$$

Lemma 18. For any $\lambda > 0$ we have

$$\lim_{\|u\| \rightarrow \infty} T_\lambda(u) = \infty.$$

Returning to the proof of Step 3, we fix $\lambda \in (\lambda_1, \infty)$. By Lemma 18 we deduce that $\lim_{\|u\| \rightarrow \infty} T_\lambda(u) = \infty$, that is, T_λ is coercive. On the other hand, standard arguments show that the functional T_λ is weakly lower semi-continuous. Thus there exists $u_\lambda \in E$ a global minimum point of T_λ and hence, a critical point of T_λ . In order to conclude that step 3 holds true it

is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J_1(u)}{I_1(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that

$$J_1(v_\lambda) < \lambda I_1(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ , or λ is an eigenvalue of problem (5.1). Thus, step 3 is verified.

Step 4. λ_1 is an eigenvalue of problem (5.1).

This property follows by using the following two auxiliary results.

We begin by proving two auxiliary results.

Lemma 19. *The following relation holds true*

$$\lim_{\|u\| \rightarrow 0} \frac{J_0(u)}{I_0(u)} = +\infty.$$

Lemma 20. *Assume $\{u_n\}$ converges weakly to u in E . Then the following relations hold true*

$$\lim_{n \rightarrow \infty} I_0(u_n) = I_0(u), \tag{5.6}$$

$$\lim_{n \rightarrow \infty} \langle I'_1(u_n), u_n - u \rangle = 0. \tag{5.7}$$

Returning to the proof of Step 4, let $\lambda_n \searrow \lambda_1$. By Step 3 we deduce that for each n there exists $u_n \in E \setminus \{0\}$ such that

$$\langle J'_1(u_n), v \rangle = \lambda_n \cdot \langle I'_1(u_n), v \rangle, \quad \forall v \in E. \tag{5.8}$$

Taking $v = u_n$ we find

$$J_0(u_n) = \lambda_n \cdot I_0(u_n). \tag{5.9}$$

Passing to the limit as $n \rightarrow \infty$ in relation (5.9) and taking into account that relation (5.6) holds true we deduce

$$\lim_{n \rightarrow \infty} J_0(u_n) = \lambda_1 \cdot I_0(u),$$

and thus, the sequence $\{J_0(u_n)\}$ is bounded in \mathbb{R} . That remark, the definition of J_0 and relations (2.5) and (2.6) imply that the sequence $\{u_n\}$ is bounded in E . Since E is a reflexive Banach space it follows that there exists $u \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in E . Then by relations (5.6) and (5.7) it follows that

$$\lim_{n \rightarrow \infty} I_0(u_n) = I_0(u)$$

and

$$\lim_{n \rightarrow \infty} \langle I'_1(u_n), u_n - u \rangle = 0.$$

On the other hand, by Lemma 4.2 in [25] we find that for any $\theta \geq 2$ and any $\xi, \eta \in \mathbb{R}^N$ we have

$$\frac{2}{2^{\theta-1} - 1} |\xi - \eta|^\theta \leq \theta (|\xi|^{\theta-2} \xi - |\eta|^{\theta-2} \eta) \cdot (\xi - \eta). \tag{5.10}$$

Using inequality (5.10) and the above relations we deduce that there exist two positive constants L_1 and L_2 such that

$$\begin{aligned} L_1 \int_\Omega (|\nabla(u_n - u)|^{p(x)} + |u_n - u|^{p(x)}) \, dx &\leq \int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) \cdot (\nabla u_n - \nabla u) \, dx \\ &\quad + \int_\Omega (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) \cdot (u_n - u) \, dx \end{aligned}$$

and

$$L_2 \int_\Omega |u_n - u|^{q(x)} \, dx \leq \int_\Omega (|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u) \cdot (u_n - u) \, dx.$$

Adding the two relations above, using relations (5.8) and (5.7) and the fact that $\{u_n\}$ converges weakly to u in E we deduce that

$$\begin{aligned} L_1 \int_{\Omega} (|\nabla(u_n - u)|^{p(x)} + |u_n - u|^{p(x)}) \, dx &\leq \langle J'_1(u_n) - J'_1(u), u_n - u \rangle \\ &= |\langle J'_1(u_n), u_n - u \rangle| + |\langle J'_1(u), u_n - u \rangle| \\ &= |\lambda_n \cdot \langle I'_1(u_n), u_n - u \rangle| + |\langle J'_1(u), u_n - u \rangle| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

The above inequalities and relations (2.4) and (2.7) show that u_n converges strongly to u in E . Then passing to the limit as $n \rightarrow \infty$ in (5.8) it follows that

$$\langle J'_1(u), v \rangle = \lambda_1 \cdot \langle I'_1(u), v \rangle, \quad \forall v \in E.$$

Thus, u is a critical point for T_{λ_1} . In order to prove that λ_1 is an eigenvalue for problem (5.1) it remains to show that $u \neq 0$. Indeed, passing to the limit as $n \rightarrow \infty$ in (5.9) we find

$$\lim_{n \rightarrow \infty} \frac{J_0(u_n)}{I_0(u_n)} = \lambda_1.$$

On the other hand, if we assume by contradiction that $u = 0$ then we have $u_n \rightarrow 0$ in E , or $\|u_n\| \rightarrow 0$. But by Lemma 19 we deduce that

$$\lim_{n \rightarrow \infty} \frac{J_0(u_n)}{I_0(u_n)} = \infty,$$

which represents a contradiction. Consequently, $u \neq 0$ and thus, λ_1 is an eigenvalue for problem (5.1).

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 16 is now complete. \square

We notice that a similar result as those of Theorem 16 can be proved with similar arguments for the problem

$$\begin{cases} -\Delta u + u + |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u & \text{for } x \in \Omega \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where Ω is a smooth exterior domain in \mathbb{R}^2 . The mappings q and $r : \overline{\Omega} \rightarrow [2, \infty)$ are still Lipschitz continuous functions while $g : \overline{\Omega} \rightarrow [0, \infty)$ is a function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \geq 0$ is a real number. This time conditions (5.2)–(5.4) should be replaced by the following conditions

$$2 < r^- \leq r^+ < q^- \leq q^+ < \infty$$

and

$$g \in L^\infty(\Omega) \cap L^1(\Omega).$$

6. Nonlinear eigenvalue problems with variable exponent and sign-changing potential

In this section we discuss some combined effects in a class of nonlinear eigenvalue problems with several variable exponents and sign-changing potential. We are concerned with the study of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u + V(x)|u|^{m(x)-2}u = \lambda(|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary.

We assume that $p_1, p_2, q_1, q_2, m : \overline{\Omega} \rightarrow (1, \infty)$ are continuous functions satisfying the following hypotheses:

$$\max_{\overline{\Omega}} p_2 < \min_{\overline{\Omega}} q_2 \leq \max_{\overline{\Omega}} q_2 \leq \min_{\overline{\Omega}} m \leq \max_{\overline{\Omega}} m \leq \min_{\overline{\Omega}} q_1 \leq \max_{\overline{\Omega}} q_1 < \min_{\overline{\Omega}} p_1, \tag{6.2}$$

$$\max_{\overline{\Omega}} q_1 < p_2^*(x) := \begin{cases} \frac{Np_2(x)}{N - p_2(x)} & \text{if } p_2(x) < N \\ +\infty & \text{if } p_2(x) \geq N. \end{cases} \tag{6.3}$$

We assume that the potential $V : \Omega \rightarrow \mathbb{R}$ satisfies

$$V \in L^{r(x)}(\Omega), \quad \text{with } r \in C(\overline{\Omega}) \quad \text{and} \quad r(x) > \frac{N}{\min_{\overline{\Omega}} m} \quad \forall x \in \overline{\Omega}. \tag{6.4}$$

Condition (6.2) which describes the competition between the growth rates involved in Eq. (6.1) represents the key of the present study since it establishes a balance between all the variable exponents involved in the problem. Such a balance is essential since our setting assumes a non-homogeneous eigenvalue problem for which a minimization technique based on the Lagrange multiplier theorem cannot be applied in order to find (principal) eigenvalues (unlike the case offered by the homogeneous operators). Thus, in the case of nonlinear non-homogeneous eigenvalue problems the classical theory used in the homogeneous case does not work entirely, but some of its ideas can still be useful and some particular results can still be obtained in some aspects while in other aspects entirely new phenomena can occur. To focus on our case, condition (6.2) together with conditions (6.3) and (6.4) implies

$$\lim_{\|u\|_{p_1(x)} \rightarrow 0} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_{\Omega} \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\int_{\Omega} \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_{\Omega} \frac{1}{q_2(x)} |u|^{q_2(x)} dx} = \infty$$

and

$$\lim_{\|u\|_{p_1(x)} \rightarrow \infty} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_{\Omega} \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\int_{\Omega} \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_{\Omega} \frac{1}{q_2(x)} |u|^{q_2(x)} dx} = \infty,$$

where $\|\cdot\|_{p_1(x)}$ stands for the norm in the variable exponent Sobolev space $W_0^{1,p_1(x)}(\Omega)$. In other words, the absence of homogeneity is balanced by the behavior (actually, the blow-up) of the Rayleigh quotient associated to problem (6.1) in the origin and at infinity. The consequences of the above remarks is that the infimum of the Rayleigh quotient associated to problem (6.1) is a real number, that is,

$$\inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_{\Omega} \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\int_{\Omega} \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_{\Omega} \frac{1}{q_2(x)} |u|^{q_2(x)} dx} \in \mathbb{R}, \tag{6.5}$$

and it will be attained for a function $u_0 \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$. Moreover, the value in (6.5) represents an eigenvalue of problem (6.1) with the corresponding eigenfunction u_0 . However, at this stage we cannot say if the eigenvalue described above is the lowest eigenvalue of problem (6.1) or not, even if we are able to show that any λ small enough is not an eigenvalue of (6.1). At the moment this remains an open question. On the other hand, we can prove that any λ larger than the value given by relation (6.5) is also an eigenvalue of problem (6.1). Thus, we conclude that problem (6.1) possesses a continuous family of eigenvalues.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (6.1) if there exists $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v dx + \int_{\Omega} V(x) |u|^{m(x)-2} uv dx - \lambda \int_{\Omega} (|u|^{q_1(x)-2} + |u|^{q_2(x)-2}) uv dx = 0,$$

for all $v \in W_0^{1,p_1(x)}(\Omega)$. We point out that if λ is an eigenvalue of problem (6.1) then the corresponding eigenfunction $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ is a weak solution of problem (6.1).

For each potential $V \in L^{r(x)}(\Omega)$ we define

$$E(V) := \inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_{\Omega} \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\int_{\Omega} \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_{\Omega} \frac{1}{q_2(x)} |u|^{q_2(x)} dx}$$

and

$$F(V) := \inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p_1(x)} dx + \int_{\Omega} |\nabla u|^{p_2(x)} dx + \int_{\Omega} V(x) |u|^{m(x)} dx}{\int_{\Omega} |u|^{q_1(x)} dx + \int_{\Omega} |u|^{q_2(x)} dx}.$$

Thus, we can define a function $E : L^{r(x)}(\Omega) \rightarrow \mathbb{R}$.

The first result of this section is given by the following theorem.

Theorem 21. Assume that conditions (6.2)–(6.4) are fulfilled. Then $E(V)$ is an eigenvalue of problem (6.1). Moreover, there exists $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ an eigenfunction corresponding to the eigenvalue $E(V)$ such that

$$E(V) = \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_{\Omega} \frac{V(x)}{m(x)} |u|^{m(x)} dx}{\int_{\Omega} \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_{\Omega} \frac{1}{q_2(x)} |u|^{q_2(x)} dx}.$$

Furthermore, $F(V) \leq E(V)$, each $\lambda \in (E(V), \infty)$ is an eigenvalue of problem (6.1), while each $\lambda \in (-\infty, F(V))$ is not an eigenvalue of problem (6.1).

Next, we show that on each convex, bounded and closed subset of $L^{r(x)}(\Omega)$ the function E defined above is bounded from below and attains its minimum. The result is the following:

Theorem 22. Assume that conditions (6.2)–(6.4) are fulfilled. Assume that S is a convex, bounded and closed subset of $L^{r(x)}(\Omega)$. Then there exists $V_\star \in S$ which minimizes $E(V)$ on S , that is,

$$E(V_\star) = \inf_{V \in S} E(V).$$

6.1. Concentration of the spectrum

We describe the main steps in the proof of Theorem 21. Let X denote the generalized Sobolev space $W_0^{1,p_1(x)}(\Omega)$. We denote by $\|\cdot\|$ the norm on $W_0^{1,p_1(x)}(\Omega)$ and by $\|\cdot\|_1$ the norm on $W_0^{1,p_2(x)}(\Omega)$.

Define the functionals $J_V, I : X \rightarrow \mathbb{R}$ by

$$J_V(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx + \int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx,$$

$$I(u) = \int_\Omega \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_\Omega \frac{1}{q_2(x)} |u|^{q_2(x)} dx.$$

We notice that for any V satisfying condition (6.4) we have

$$J_V(u) = J_0(u) + \int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx, \quad \forall u \in X,$$

where J_0 is obtained in the case when $V = 0$ in Ω .

Standard arguments imply that $J_V, I \in C^1(X, \mathbb{R})$ and for all $u, v \in X$,

$$\langle J'_V(u), v \rangle = \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v dx + \int_\Omega V(x) |u|^{m(x)-2} uv dx,$$

$$\langle I'(u), v \rangle = \int_\Omega |u|^{q_1(x)-2} uv dx + \int_\Omega |u|^{q_2(x)-2} uv dx.$$

The proof of Theorem 21 is based on some auxiliary results.

Lemma 23. Assume that conditions (6.2)–(6.4) are fulfilled. Then for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\left| \int_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx \right| \leq \epsilon \int_\Omega \left(\frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx \right) + C_\epsilon |V|_{r(x)} \int_\Omega (|u|^{m^-} + |u|^{m^+}) dx,$$

for all $u \in X$.

Lemma 24. The following relations hold true:

$$\lim_{\|u\| \rightarrow \infty} \frac{J_V(u)}{I(u)} = \infty \tag{6.6}$$

and

$$\lim_{\|u\| \rightarrow 0} \frac{J_V(u)}{I(u)} = \infty. \tag{6.7}$$

Lemma 25. There exists $u \in X \setminus \{0\}$ such that $\frac{J_V(u)}{I(u)} = E(V)$.

We refer to [32] for detailed proofs of Lemmas 23–25.

Returning to the proof of Theorem 21, we deduce by Lemma 25 that there exists $u \in X \setminus \{0\}$ such that

$$\frac{J_V(u)}{I(u)} = E(V) = \inf_{w \in X \setminus \{0\}} \frac{J_V(w)}{I(w)}. \tag{6.8}$$

Then, for any $w \in X$ we have

$$\left. \frac{d}{d\epsilon} \frac{J_V(u + \epsilon w)}{I(u + \epsilon w)} \right|_{\epsilon=0} = 0.$$

A simple computation yields

$$\langle J'_V(u), w \rangle I(u) - J_V(u) \langle I'(u), w \rangle = 0, \tag{6.9}$$

for all $w \in X$. Relation (6.9) combined with the fact that $J_V(u) = E(V) \cdot I(u)$ and $I(u) \neq 0$ implies the fact that $E(V)$ is an eigenvalue of problem (6.1).

Next, we show that any $\lambda \in (E(V), \infty)$ is an eigenvalue of problem (6.1).

Let $\lambda \in (E(V), \infty)$ be arbitrary but fixed. Define $T_{V,\lambda} : X \rightarrow \mathbb{R}$ by

$$T_{V,\lambda}(u) = J_V(u) - \lambda I(u).$$

Clearly, $T_{V,\lambda} \in C^1(X, \mathbb{R})$ with

$$\langle T'_{V,\lambda}(u), v \rangle = \langle J'_V(u), v \rangle - \lambda \langle I'(u), v \rangle, \quad \forall u \in X.$$

Thus, λ is an eigenvalue of problem (6.1) if and only if there exists $u_\lambda \in X \setminus \{0\}$ a critical point of $T_{V,\lambda}$.

With similar arguments as in the proof of relation (6.6) we can show that $T_{V,\lambda}$ is coercive, i.e. $\lim_{\|u\| \rightarrow \infty} T_{V,\lambda}(u) = \infty$. On the other hand, as we have already remarked, similar arguments as those used in the proof of Lemma 3.4 in [28] show that the functional $T_{V,\lambda}$ is weakly lower semi-continuous. Thus there exists $u_\lambda \in X$ a global minimum point of $T_{V,\lambda}$ and thus, a critical point of $T_{V,\lambda}$. It is enough to show that u_λ is not trivial. Indeed, since $E(V) = \inf_{u \in X \setminus \{0\}} \frac{J_V(u)}{I(u)}$ and $\lambda > E(V)$ it follows that there exists $v_\lambda \in X$ such that

$$J_V(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_{V,\lambda}(v_\lambda) < 0.$$

Thus,

$$\inf_X T_{V,\lambda} < 0$$

and we conclude that u_λ is a nontrivial critical point of $T_{V,\lambda}$, or λ is an eigenvalue of problem (6.1).

Finally, we prove that each $\lambda < F(V)$ is not an eigenvalue of problem (6.1). With that end in view we assume by contradiction that there exists $\lambda < F(V)$ an eigenvalue of problem (6.1). It follows that there exists $u_\lambda \in X \setminus \{0\}$ such that

$$\langle J'_V(u_\lambda), u_\lambda \rangle = \lambda \langle I'(u_\lambda), u_\lambda \rangle.$$

Since $u_\lambda \neq 0$ we have $\langle I'(u_\lambda), u_\lambda \rangle > 0$. Using that fact and the definition of $F(V)$ it follows that the following relation holds true

$$\langle J'_V(u_\lambda), u_\lambda \rangle = \lambda \langle I'(u_\lambda), u_\lambda \rangle < F(V) \langle I'(u_\lambda), u_\lambda \rangle \leq \langle J'_V(u_\lambda), u_\lambda \rangle.$$

Obviously, this is a contradiction. We deduce that each $\lambda \in (-\infty, F(V))$ is not an eigenvalue of problem (6.1). Furthermore, it is clear that $E(V) \geq F(V)$. The proof of Theorem 21 is complete. \square

6.2. More about a Rayleigh quotient with variable exponents

In this section we give the proof of Theorem 22, which yields a sufficient condition in order to achieve the infimum of the Rayleigh quotient $E(V)$.

Let S be a convex, bounded and closed subset of $L^{r(x)}(\Omega)$ and

$$E_* := \inf_{V \in S} E(V).$$

Let $(V_n) \subset S$ be a minimizing sequence for E_* , that is,

$$E(V_n) \rightarrow E_*, \quad \text{as } n \rightarrow \infty.$$

Then (V_n) is a bounded sequence, hence there exists $V_* \in L^{r(x)}(\Omega)$ such that V_n converges weakly to V_* in $L^{r(x)}(\Omega)$. Moreover, since S is convex and closed it is also weakly closed (see e.g. Theorem III.7 in Brezis [7]) and consequently $V_* \in S$.

Next, we show that $E(V_*) = E_*$. Indeed, by Theorem 21 we deduce that for each positive integer n there exists $u_n \in X \setminus \{0\}$ such that

$$\frac{J_{V_n}(u_n)}{I(u_n)} = E(V_n). \tag{6.10}$$

Since $(E(V_n))$ is a bounded sequence we have

$$\frac{J_{V_n}(u_n)}{I(u_n)} \geq \beta \frac{J_0(u_n)}{I(u_n)} - C, \quad \text{for any } n,$$

where C is a positive constant, we infer that (u_n) is bounded in X and it cannot contain a subsequence converging to 0 (otherwise we obtain a contradiction by applying Lemma 24). Thus, there exists $u_0 \in X \setminus \{0\}$ such that (u_n) converges weakly to u_0 in X . Using the Rellich–Kondrachov theorem we deduce that (u_n) converges strongly to u_0 in $L^{s(x)}(\Omega)$ for any $s(x) \in C(\overline{\Omega})$ satisfying $1 < s(x) < \frac{Np_1(x)}{N-p_1(x)}$ for any $x \in \overline{\Omega}$. In particular, using conditions (6.2)–(6.4), we deduce that (u_n) converges

to u_0 in $L^{m(x)}(\Omega)$ and in $L^{m(x) \cdot r'(x)}(\Omega)$ where $r'(x) = \frac{r(x)}{r(x)-1}$. Using that information, Hölder's inequality and the fact that $V_\star \in L^{r(x)}(\Omega)$ and (V_n) is bounded in $L^{r(x)}(\Omega)$ we find

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_n|^{m(x)} dx = \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_0|^{m(x)} dx \tag{6.11}$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{V_n(x)}{m(x)} |u_n|^{m(x)} - \frac{V_n(x)}{m(x)} |u_0|^{m(x)} \right) dx = 0. \tag{6.12}$$

On the other hand, since (V_n) converges weakly to V_\star in $L^{r(x)}(\Omega)$ and $u_0 \in L^{m(x) \cdot r'(x)}(\Omega)$, where $r'(x) = \frac{r(x)}{r(x)-1}$, we deduce

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{V_n(x)}{m(x)} |u_0|^{m(x)} dx = \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_0|^{m(x)} dx. \tag{6.13}$$

Combining the equality

$$\begin{aligned} \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_n|^{m(x)} dx - \int_{\Omega} \frac{V_n(x)}{m(x)} |u_n|^{m(x)} dx &= \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_n|^{m(x)} dx - \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_0|^{m(x)} dx \\ &\quad + \int_{\Omega} \frac{V_\star(x)}{m(x)} |u_0|^{m(x)} dx - \int_{\Omega} \frac{V_n(x)}{m(x)} |u_0|^{m(x)} dx \\ &\quad + \int_{\Omega} \frac{V_n(x)}{m(x)} |u_0|^{m(x)} dx - \int_{\Omega} \frac{V_n(x)}{m(x)} |u_n|^{m(x)} dx, \end{aligned}$$

with relations (6.11)–(6.13) we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{V_\star(x)}{m(x)} |u_n|^{m(x)} - \frac{V_n(x)}{m(x)} |u_n|^{m(x)} \right) dx = 0. \tag{6.14}$$

Since

$$E(V_\star) = \inf_{u \in X \setminus \{0\}} \frac{J_{V_\star}(u)}{I(u)},$$

it follows that

$$E(V_\star) \leq \frac{J_{V_\star}(u_n)}{I(u_n)}.$$

Combining the above inequality and equality (6.10) we obtain

$$E(V_\star) \leq \frac{J_{V_\star}(u_n) - J_{V_n}(u_n)}{I(u_n)} + E(V_n).$$

Taking into account the result of relation (6.14), the fact that $I(u_n)$ is bounded and does not converge to 0 and $(E(V_n))$ converges to E_\star then passing to the limit as $n \rightarrow \infty$ in the last inequality we infer that

$$E(V_\star) \leq E_\star.$$

But using the definition of E_\star and the fact that $V_\star \in S$ we conclude that actually

$$E(V_\star) = E_\star.$$

The proof of Theorem 22 is complete. \square

7. Morse theory and local linking for a degenerate problem

In this section we are concerned with the study of the following nonlinear problem

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2} u := f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary. Let $p \in C(\overline{\Omega})$ and $1 < p_- := \min_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p_+ := \max_{x \in \overline{\Omega}} p(x) < \infty$ and $F(x, t) = \int_0^t f(x, s) ds$, $\mathcal{F}(x, t) = f(x, t)t - p_+ F(x, t)$.

We assume that the reaction term $f(x, u)$ satisfies the following hypotheses:

(H1) $f \in C(\bar{\Omega} \times \mathbb{R})$ with $f(x, 0) = 0$ and there exists $C_1 > 0$ such that

$$|f(x, t)| \leq C_1(1 + |t|^{q(x)-1}), \quad \forall (x, t) \in \Omega \times \mathbb{R},$$

where $q(x) \in C(\bar{\Omega})$, $1 < q(x) < p^*(x)$ for all $x \in \bar{\Omega}$ and $p^* = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \geq N$;

(H2) $\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^{p_+}} = +\infty$ uniformly for $x \in \bar{\Omega}$;

(H3) there exists $\theta \geq 1$ such that $\theta \mathcal{F}(x, t) \geq \mathcal{F}(x, st)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$;

(H4) there exists $\nu > 0$ such that

$$\frac{f(x, t)}{|t|^{p_+ - 2}t} \text{ is increasing in } t \geq \nu \text{ and decreasing in } t \leq -\nu;$$

(H5) there are small constants r and R with $0 < r < R$ such that

$$C_2|t|^{\alpha(x)} \leq p(x)F(x, t) \leq C_3|t|^{p(x)}, \quad \text{for } t \in \mathbb{R} \text{ with } r \leq |t| \leq R, \text{ a.e. } x \in \Omega, \quad (7.2)$$

where C_2 and C_3 are constants with $0 < C_2 < C_3 < 1$, $\alpha(x) \in C(\bar{\Omega})$ and $1 < \alpha(x) < p(x)$. Moreover, there exists $C_4 > 0$ such that

$$F(x, t) \geq -C_4|t|^{p_+} \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}. \quad (7.3)$$

The assumption (H2) implies that the problem (7.1) is superlinear at infinity. A lot of works concerning superlinear elliptic boundary value problem have been done by using the usual Ambrosetti–Rabinowitz condition, that is,

(AR) there exist $\mu > p_+$ and $M > 0$ such that

$$0 < p_+F(x, t) \leq f(x, t)t \quad \text{for all } x \in \Omega \text{ and } |t| \geq M. \quad (7.4)$$

From (7.4) it follows that for some $a, b > 0$

$$F(x, t) \geq a|t|^\mu - b \quad \text{for } (x, t) \in \Omega \times \mathbb{R}, \quad (7.5)$$

which is a stronger assumption than our condition (H2).

Let us consider the following function (for simplicity we drop the x -dependence):

$$f(x, t) = |t|^{p_+ - 2}t \left(p_+ \log(1 + |t|) + \frac{|t|}{1 + |t|} \right).$$

Then $F(x, t) = |t|^{p_+} \log(1 + |t|)$ and f does not satisfy the Ambrosetti–Rabinowitz condition, but it satisfies our conditions (H2) and (H3). Furthermore, we can show that this function fulfills all hypotheses (H1)–(H5).

We prove that the above hypotheses provide sufficient conditions for the existence of one or more nontrivial solutions of problem (7.1).

Theorem 26. *Assume that (H1), (H2), (H3) and (H5) hold. Then the problem (7.1) has at least one nontrivial weak solution in $W_0^{1, p(x)}(\Omega)$.*

Theorem 27. *Assume that (H1), (H2), (H4) and (H5) hold. Then the problem (7.1) has at least one nontrivial weak solution in $W_0^{1, p(x)}(\Omega)$.*

Before the statement of the next result, we recall some known properties about the eigenvalues of the nonhomogeneous differential operator $-\Delta_{p(x)}$ in $W_0^{1, p(x)}(\Omega)$. We say that λ is an eigenvalue of $-\Delta_{p(x)}$ with Dirichlet if the problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

has nonzero solution. Fan, Zhang and Zhao [15] obtained the principal eigenvalue $\lambda_* > 0$ by introducing the following condition:

(P) there exists a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that for any $x \in \Omega$, $c(t) = p(x + tl)$ is monotone in $t \in I_x = \{t : x + tl \in \Omega\}$.

Theorem 28. *Assume that conditions (P), (H1), (H5) are fulfilled and*

(H6) $\limsup_{|t| \rightarrow \infty} \frac{p(x)F(x, t)}{|t|^{p(x)}} < \frac{p_-}{p_+} \lambda_* + 1$ uniformly on $x \in \bar{\Omega}$;

(H7) $F(x, t) \geq 0$ for all $x \in \Omega$ and $|t| \leq r$.

Then the problem (7.1) has at least two nontrivial weak solutions in $W_0^{1, p(x)}(\Omega)$.

From the assumption (H1) we deduce that the energy functional $\Phi : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ given by

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \int_{\Omega} F(x, u) dx$$

is well defined and of class C^1 . The derivative of Φ at u is given by

$$\langle \Phi'(u), v \rangle = \int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v + |u|^{p(x)-2} u \cdot v) dx - \int_{\Omega} f(x, u) v dx$$

for $v \in W_0^{1,p(x)}(\Omega)$.

Set

$$I(u) = \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx, \quad J(u) = \int_{\Omega} F(x, u) dx.$$

Then $\Phi(u) = I(u) - J(u)$.

The function $u \in X$ is called a weak solution of problem (7.1) if for any $\phi \in X$

$$\int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi + |u|^{p(x)-2} u \cdot \phi) dx - \int_{\Omega} f(x, u) \phi dx = 0.$$

The functional Φ satisfies the (C) condition if for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset X$ such that $\Phi(u) \rightarrow c, (1 + \|u_n\|) \|\Phi'(u_n)\|_{X^*} \rightarrow 0$ has a convergent subsequence. The functional Φ satisfies the (PS) condition if any sequence $\{u_n\} \subset X$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \rightarrow 0$ has a convergent subsequence.

Standard arguments show that the energy functional Φ satisfies the (C) condition under the hypotheses of Theorems 26 and 27 or 28.

7.1. Computation of critical groups

Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R}), K = \{u \in X : \Phi'(u) = 0\}$, then the q th critical group of Φ at an isolated critical point $u \in K$ with $\Phi(u) = c$ is defined by

$$C_q(\Phi, u) := H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad q \in \mathbb{N} := \{0, 1, 2, \dots\},$$

where $\Phi^c = \{u \in X : \Phi(u) \leq c\}, U$ is any neighborhood of u , containing the unique critical point, H_* is the singular relative homology with coefficients in an Abelian group G .

We say that $u \in K$ is a homological nontrivial critical point of Φ if at least one of its critical groups is nontrivial.

The following critical point result will be used in the sequel.

Proposition 29 (See [35, Theorem 2.1]). *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the (PS) condition and is bounded from below. If Φ has a critical point that is homological nontrivial and is not a minimizer of Φ , then Φ has at least three critical points.*

If Φ satisfies the condition (C) and the critical values of Φ are bounded from below by some $a < \inf \Phi(K)$, then the critical groups of Φ at infinity were introduced by Bartsch and Li [6] as

$$C_q(\Phi, \infty) := H_q(X, \Phi^a), \quad q \in \mathbb{N}. \tag{7.6}$$

If Φ satisfies the condition (C), then Φ satisfies the deformation condition. By the deformation lemma, the right-hand side of (4.1) does not depend on the choice of a .

Remark 30. Morse theory [8] tells us that if $K = \{0\}$ then $C_q(\Phi, \infty) = C_q(\Phi, 0)$ for all $q \in \mathbb{N}$. It follows that if $C_q(\Phi, \infty) \neq C_q(\Phi, 0)$ for some $q \in \mathbb{N}$, then Φ must have a nontrivial critical point. So, we must compute the critical groups at zero and at infinity.

For the proofs of our theorems, in what follows we may assume that Φ has only finitely many critical points. Since Φ satisfies the condition (C), then the critical groups $C_q(\Phi, \infty)$ at infinity make sense.

Theorem 31. *Suppose that Φ satisfies (H1), (H2) and (H3). Then $C_q(\Phi, \infty) = 0$ for all $q \in \mathbb{N}$.*

Proof. Let $S = \{u \in X : \|u\| = 1\}$. For $u \in S$, by Fatou lemma and (H2) we have

$$\lim_{t \rightarrow +\infty} \int_{\Omega} \frac{F(x, tu)}{|t|^{p_+}} dx \geq \int_{\Omega} \lim_{t \rightarrow +\infty} \frac{F(x, tu)}{|tu|^{p_+}} |u|^{p_+} dx = +\infty.$$

Therefore

$$\Phi(tu) \leq \frac{t^{p_+}}{p_-} - \int_{\Omega} F(x, tu) dx \leq t^{p_+} \left(\frac{1}{p_-} - \int_{\Omega} \frac{F(x, tu)}{|t|^{p_+}} dx \right) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

Choose $a < \min\{\inf_{\|u\| \leq 1} \Phi(u), 0\}$, then for any $u \in S$, there exists $t_0 > 1$ such that $\Phi(t_0u) \leq a$. By (H3), we have

$$\mathcal{F}(x, m) \geq 0 \quad \text{for } (x, m) \in \Omega \times \mathbb{R}. \tag{7.7}$$

Therefore, if

$$\Phi(tu) = \int_{\Omega} \frac{t^{p(x)}}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \int_{\Omega} F(x, tu) dx \leq a,$$

then

$$\int_{\Omega} (t^{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})) dx \leq p_+ a + \int_{\Omega} p_+ F(x, tu) dx.$$

Using relation (7.7), we obtain

$$\begin{aligned} \frac{d}{dt} \Phi(tu) &= \frac{1}{t} \left[\int_{\Omega} t^{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \int_{\Omega} f(x, tu) t u dx \right] \\ &\leq \frac{1}{t} \left[p_+ a - \int_{\Omega} \mathcal{F}(x, tu) dx \right] < 0. \end{aligned}$$

Then by the implicit function theorem, there exists a unique $T \in C(S, \mathbb{R})$ such that $\Phi(T(u)u) = a$. Next, we use the function T to construct a strong deformation retract from $X \setminus \{0\}$ to Φ^a . Therefore, we deduce

$$C_q(\Phi, \infty) = H_q(X, \Phi^a) = H_q(X, X \setminus \{0\}) = 0, \quad \forall q \in \mathbb{N}.$$

The proof is completed. \square

Theorem 32. *Suppose that Φ satisfies (H1), (H2) and (H4). Then $C_q(\Phi, \infty) = 0$ for all $q \in \mathbb{N}$.*

The proof is standard and we omit the details.

As in [21], since X is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$\begin{aligned} f_n(e_m) &= \delta_{n,m} = \begin{cases} 1, & n \neq m \\ 0, & n = m \end{cases} \\ X &= \overline{\text{span}\{e_n : n = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{f_n : n = 1, 2, \dots\}}. \end{aligned}$$

For $k = 1, 2, \dots$, we denote $Y_k = \text{span}\{e_1, e_2, \dots, e_k\}$, hence Y_k has a closed complementing subspace Z_k in X . Thus, $X = Y_k \oplus Z_k$ (see [7]).

Lemma 33. *Assume that $\varphi : X \rightarrow \mathbb{R}$ is weakly-strongly continuous, $\varphi(0) = 0$, $\rho > 0$ is a given positive number. Set $\eta_k = \sup_{u \in Z_k, \|u\| \leq \rho} |\varphi(u)|$. Then $\eta_k \rightarrow 0$ as $k \rightarrow \infty$.*

We refer to Fan and Han [13, Lemma 3.3] for the proof of this result.

In order to compute the critical groups at zero, we need the following classical linking theorem.

Proposition 34. *Assume that Φ has a critical point $u = 0$ with $\Phi(0) = 0$. Suppose that Φ has a local linking at 0 with respect to $X = V \oplus W$, $k = \dim V < \infty$, that is, there exists $\rho > 0$ small such that*

$$\begin{cases} \Phi(u) \leq 0, & u \in V, \|u\| \leq \rho; \\ \Phi(u) > 0, & u \in W, 0 < \|u\| \leq \rho. \end{cases}$$

Then $C_k(\Phi, 0) \neq 0$, hence 0 is a homological nontrivial critical point of Φ .

Theorem 35. *Suppose that Φ satisfies (H1) and (H5). Then there exists $k_0 \in \mathbb{N}$ such that $C_{k_0}(\Phi, 0) \neq 0$.*

Proof. Since $f(x, 0) = 0$, the zero function 0 is a critical point of Φ . So we only need to prove that Φ has a local linking at 0 with respect to $X = Y_k \oplus Z_k$. We take two steps:

Step 1. Take $u \in Y_k$. Since Y_k is finite dimensional, we have that for given $R > 0$, there exists $0 < \rho < 1$ small such that

$$u \in Y_k, \quad \|u\| < \rho \Rightarrow |u(x)| < R, \quad \forall x \in \Omega.$$

For $0 < r < R$, let $\Omega_1 = \{x \in \Omega : |u(x)| < r\}$, $\Omega_2 = \{x \in \Omega : r \leq |u(x)| \leq R\}$, $\Omega_3 = \{x \in \Omega : |u(x)| > R\}$. Then $\Omega = \cup_{i=1}^3 \Omega_i$ and Ω_i are pairwise disjoint. For the sake of simplicity, let $G(x, u) = F(x, u) - \frac{C_2}{p(x)} |u|^{p(x)}$. Therefore, from (H5)

it follows that

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) F(x, u) dx \\ &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \int_{\Omega} \frac{C_2}{p(x)} |u|^{\alpha(x)} dx - \int_{\Omega_1} G(x, u) dx - \int_{\Omega_2} G(x, u) dx - \int_{\Omega_3} G(x, u) dx \\ &\leq \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \frac{C_2}{p_+} \int_{\Omega} |u|^{\alpha(x)} dx - \int_{\Omega_1} G(x, u) dx. \end{aligned}$$

In terms of the assumptions on $\alpha(x)$ and Ω is a bounded domain in \mathbb{R}^N , the embedding $L^{p(x)}(\Omega) \rightarrow L^{\alpha(x)}(\Omega)$ is continuous. This implies that there exists a constant $C > 1$ such that

$$\|u\|_{\alpha(x)} \leq C \|u\|_{p(x)} \leq C \|u\|_{1,p(x)} \leq 2C \|u\| \leq 2C\rho.$$

If $\rho \leq \frac{1}{2C}$, then $\|u\|_{\alpha(x)} \leq 1$. Note that the norms on Y_k are equivalent to each other, $\|u\|_{\alpha(x)}$ is equivalent to $\|u\|$. Since $\alpha(x), p(x) \in C(\bar{\Omega})$ and $\alpha(x) < p(x)$, for each $x \in \Omega$, there exists an open subset $B_\delta(x)$ of $\bar{\Omega}$ such that

$$\alpha_x := \sup_{x \in B_\delta(x)} \alpha(x) < \inf_{x \in B_\delta(x)} p(x) := p_x.$$

Then $\{B_\delta(x)\}_{x \in \bar{\Omega}}$ is an open covering of $\bar{\Omega}$. Since $\bar{\Omega}$ is compact, there is a finite subcovering $\{B_\delta(x_i)\}_{i=1}^m$. We can use all the hyperplanes, for each of which there exists at least one hypersurface of some $\{B_\delta(x_i)\}_{i=1}^m$ lying on it, to divide $\{B_\delta(x_i)\}_{i=1}^m$ into finite open hypercube $\{\mathcal{Q}_{j=1}^n\}$ which mutually have no common points. It is obvious that $\bar{\Omega} = \cup_{j=1}^n \bar{\mathcal{Q}}_j$ and

$$\alpha_{j+} := \sup_{x \in \mathcal{Q}_j} \alpha(x) < \inf_{x \in \mathcal{Q}_j} p(x) := p_{j-}.$$

Notice that $\int_{\Omega_1} G(x, u) dx \rightarrow 0$ as $r \rightarrow 0$ and $\|u\|_{\mathcal{Q}_j} \leq \|u\|$. Therefore, there is a constant $C > 0$ such that for $0 < \rho < 0$ small and r sufficiently small

$$\begin{aligned} \Phi(u) &\leq \sum_{j=1}^n \left[\int_{\mathcal{Q}_j} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \frac{C_2}{p_-} \int_{\mathcal{Q}_j} |u|^{\alpha(x)} dx \right] + \int_{\Omega_1} G(x, u) dx \\ &\leq \sum_{j=1}^n (\|u\|_{\mathcal{Q}_j}^{p_{j-}} - C \|u\|_{\mathcal{Q}_j}^{\alpha_{j+}}) - \int_{\Omega_1} G(x, u) dx \leq 0. \end{aligned}$$

Step 2. By hypothesis (H1) and Young’s inequality, there exists $C > 0$ such that

$$|F(x, u)| \leq \frac{C_3}{p_+} |u|^{p(x)} + C |u|^{s(x)}, \quad \text{for all } x \in \Omega \text{ and } |u| \geq R, \tag{4.3}$$

where $s(x) \in C(\bar{\Omega})$ and $p(x) < s(x) < p^*(x)$. For the sake of simplicity, let $H(x, u) = F(x, u) - \frac{C_3}{p(x)} |u|^{p(x)}$. Therefore, if $u \in Z_k$ and $\|u\| \leq 1$, from (H5) and (4.3) we deduce

$$\begin{aligned} \Phi(u) &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) F(x, u) dx \\ &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - \int_{\Omega} \frac{C_3}{p(x)} |u|^{p(x)} dx - \int_{\Omega_1} H(x, u) dx - \int_{\Omega_2} H(x, u) dx - \int_{\Omega_3} H(x, u) dx \\ &\geq \frac{1 - C_3}{p_+} \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - C \int_{\Omega} |u|^{s(x)} dx - \int_{\Omega_1} H(x, u) dx. \end{aligned}$$

We next consider the functional $\varphi : X \rightarrow \mathbb{R}, \varphi(u) = \int_{\Omega} |u|^{s(x)} dx$. We already know that the embedding $X \rightarrow L^{s(x)}(\Omega)$ is compact. Hence by Lemma 33, we have

$$\eta_k = \sup_{u \in Z_k, \|u\| \leq 1} |\varphi(u)| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{4.4}$$

Note that $\int_{\Omega_1} H(x, u) dx \rightarrow 0$ as $r \rightarrow 0$. Therefore, using the same argument as in Step 1, we obtain

$$\begin{aligned} \Phi(u) &\geq \sum_{i=1}^l \left[\frac{1 - C_3}{p_+} \int_{\mathcal{Q}_i} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx - C \int_{\mathcal{Q}_i} |u|^{s(x)} dx \right] + \int_{\Omega_1} H(x, u) dx \\ &\geq \sum_{i=1}^l \left[\frac{1 - C_3}{p_+} \|u\|_{\mathcal{Q}_i}^{p_{i+}} - C \eta_k \|u\|_{\mathcal{Q}_i}^{s_{i-}} \right] - \int_{\Omega_1} H(x, u) dx. \end{aligned}$$

From (4.4) we know that there exists $k_0 \in \mathbb{N}$ such that $\eta_{k_0} \leq \frac{1-C_3}{2Cp_+}$. Then we get

$$\Phi(u) \geq \sum_{i=1}^l \frac{1-C_3}{2Cp_+} (\|u\|_{Q_i}^{p_{i+}} - \|u\|_{Q_i}^{s_{i-}}) - \int_{\Omega_1} H(x, u) dx > 0$$

as $0 < \rho < 1$ small and r sufficiently small. Thus, there exists $k_0 \in \mathbb{N}$ such that $\Phi(u) > 0$ as $u \in Z_{k_0}$ and $0 < \|u\| \leq \rho$. Combining Steps 1 and 2, we complete the proof of Theorem 35 due to Proposition 34. \square

Remark 36. From the proof of Theorem 35, we deduce that the conclusion of Theorem 35 still holds under the assumptions (H1), (H5) and (H7).

Next, by Theorems 31 and 35, we have $C_{k_0}(\Phi, \infty) \neq C_{k_0}(\Phi, 0)$ for some $k_0 \in \mathbb{N}$. Then Theorem 26 follows immediately from the fact that Φ satisfies the (C) condition and Remark 30.

By Theorems 32 and 35, we have $C_{k_0}(\Phi, \infty) \neq C_{k_0}(\Phi, 0)$ for some $k_0 \in \mathbb{N}$. Then Theorem 27 follows immediately from the fact that Φ satisfies the (C) condition and Remark 30.

We know that Φ satisfies the (PS) condition and is bounded from below. By the assumption (H7) and Remark 36, the trivial solution $u = 0$ is homologically nontrivial and is not a minimizer. The conclusion follows from Proposition 29.

8. Difference equations with variable exponent

Partial difference equations usually describe the evolution of certain phenomena over the course of time. In this section, we consider a discrete problem involving variable exponents. In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics and conversely a beautiful description of such phenomena can be found in Lovász [26]. The modeling and simulation of certain nonlinear problems from economics, biological neural networks, optimal control and others enforced in a natural manner the rapid development of the theory of difference equations. Elementary but relevant examples of partial difference equations are concerned with heat diffusion, heat control, temperature distribution, population growth, cellular neural networks, etc.

Let $T > 0$ be a given natural number and let $p(\cdot), q(\cdot) : \mathbb{Z} \rightarrow [2, \infty), V(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$ be three T -periodic functions and $f(k, t) : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function in $t \in \mathbb{R}$ and T -periodic in k .

Let Δ denote the difference operator, namely

$$\Delta u(k) = u(k + 1) - u(k),$$

for each $k \in \mathbb{Z}$. We denote by $\Delta_{p(\cdot)}^2$ the $p(\cdot)$ -Laplace difference operator, that is,

$$\Delta_{p(k-1)}^2 u(k-1) = |\Delta u(k)|^{p(k)-2} \Delta u(k) - |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1), \tag{8.1}$$

for all $k \in \mathbb{Z}$.

This section is devoted to the study of the difference non-homogeneous equations of type

$$\begin{cases} \Delta_{p(k-1)}^2 u(k-1) - V(k)|u(k)|^{q(k)-2}u(k) + f(k, u(k)) = 0 & \text{for } k \in \mathbb{Z}, \\ u(k) \rightarrow 0 & \text{as } |k| \rightarrow \infty. \end{cases} \tag{8.2}$$

The goal of the present paper is to establish the existence of nontrivial *homoclinic solutions* for problem (8.2). Poincaré [39] called a trajectory $x(t)$ a *homoclinic orbit* (or *doubly asymptotic trajectory*) if it is asymptotic to a constant as $|t| \rightarrow \infty$. Since we are seeking solutions $u(k)$ for problem (8.2) satisfying $\lim_{|k| \rightarrow \infty} u(k) = 0$, we are interested in finding nontrivial homoclinic solutions for problem (8.2).

Set

$$\begin{aligned} p^+ &:= \sup_{k \in \mathbb{Z}} p(k) & p^- &:= \inf_{k \in \mathbb{Z}} p(k) \\ q^+ &:= \sup_{k \in \mathbb{Z}} q(k) & q^- &:= \inf_{k \in \mathbb{Z}} q(k), \end{aligned}$$

and we assume that

$$1 < q^- \leq q^+ < p^- \leq p^+. \tag{8.3}$$

We also assume that the T -periodic function V satisfies the supplementary conditions

- (V1) $0 < V_0 := \min\{V(0), \dots, V(T-1)\}$;
- (V2) $V_0 < q^+$.

while the continuous function $f = f(k, t) : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ which is assumed to be T -periodic in k verifies

(F1) there exist $\alpha > p^+$ and $r > 0$ such that

$$\alpha F(k, t) := \alpha \int_0^t f(k, s) ds \leq tf(k, t), \quad \forall k \in \mathbb{Z}, t \neq 0,$$

and

$$F(k, t) > 0, \quad \forall k \in \mathbb{Z}, t \geq r;$$

(F2) $f(k, t) = o(|t|^{q^+-1})$ as $|t| \rightarrow 0$.

For $p : \mathbb{Z} \rightarrow (1, \infty)$, define the function space

$$\ell^{p(\cdot)} := \left\{ u : \mathbb{Z} \rightarrow \mathbb{R}; \rho_{p(\cdot)}(u) := \sum_{k \in \mathbb{Z}} |u(k)|^{p(k)} < \infty \right\}.$$

On $\ell^{p(\cdot)}$ we introduce the Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0; \sum_{k \in \mathbb{Z}} \left| \frac{u(k)}{\mu} \right|^{p(k)} \leq 1 \right\}.$$

Then the following relations hold true

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}, \tag{8.4}$$

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+}, \tag{8.5}$$

$$|u|_{p(\cdot)} \rightarrow 0 \Leftrightarrow \rho_{p(\cdot)}(u) \rightarrow 0. \tag{8.6}$$

We also consider the space

$$\ell^\infty = \left\{ u : \mathbb{Z} \rightarrow \mathbb{R}; |u|_\infty := \sup_{k \in \mathbb{Z}} |u(k)| < \infty \right\}.$$

Proposition 37. Assume condition (8.3) is fulfilled. Then $\ell^{q(\cdot)} \subset \ell^{p(\cdot)}$.

Proof. If $\sum_{k \in \mathbb{Z}} |u(k)|^{p(k)} < \infty$ then there exists $S > 0$ such that

$$|u(k)|^{q(k)} \leq 1, \quad \forall |k| > S.$$

It follows that

$$|u(k)| \leq 1, \quad \forall |k| > S.$$

By relation (8.3) we infer that $q(k) < p(k)$ for all $k \in \mathbb{Z}$. That fact and the above inequality assure that

$$|u(k)|^{p(k)} \leq |u(k)|^{q(k)}, \quad \forall |k| > S,$$

and the proof is complete. \square

By Proposition 37, relation (8.3) and the hypotheses on functions V and f we infer that the natural space where we should seek homoclinic solutions for (8.2) is $\ell^{q(\cdot)}$. Thus, we say that $u \in \ell^{q(\cdot)}$ is a *homoclinic solution* for (8.2) if

$$\sum_{k \in \mathbb{Z}} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) + \sum_{k \in \mathbb{Z}} V(k) |u(k)|^{q(k)-2} u(k) v(k) - \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k) = 0,$$

for all $v \in \ell^{q(\cdot)}$ and $\lim_{|k| \rightarrow \infty} u(k) = 0$.

The main result of this section is given by the following theorem.

Theorem 38. Assume hypotheses (8.3), (V1)–(V2) and (F1)–(F2) are fulfilled. Then problem (1.1) possesses at least a nontrivial homoclinic solution. Moreover, given a nontrivial homoclinic solution u of problem (8.2), there exist two integers S_1 and S_2 with $S_1 \leq S_2$ such that for all $k > S_2$ and all $k < S_1$ the sequence $u(k)$ is strictly monotone.

8.1. Homoclinic solutions and qualitative properties

The basic idea in proving Theorem 38 is to consider the associate energetic functional of problem (8.2) and to show that it possesses a nontrivial critical point by using the mountain-pass lemma.

Set

$$\phi_{p(t)}(t) := |t|^{p(t)-2} t \quad \Phi_{p(t)}(t) := \frac{|t|^{p(t)}}{p(t)}.$$

Note that

$$\Delta_{p(k-1)}^2 u(k-1) = \Delta(\phi_{p(k-1)}(\Delta u(k-1))).$$

Next, we introduce the functional $A : \ell^{q(\cdot)} \rightarrow \mathbb{R}$ defined by

$$A(u) := \sum_{k \in \mathbb{Z}} \Phi_{p(k-1)}(\Delta u(k-1)) + \sum_{k \in \mathbb{Z}} V(k) \Phi_{q(k)}(u(k)).$$

We define the energy functional associate to problem (8.2) as $J : \ell^{q(\cdot)} \rightarrow \mathbb{R}$ defined by

$$J(u) := A(u) - \sum_{k \in \mathbb{Z}} F(k, u(k)).$$

Standard arguments show that $J \in C^1(\ell^{q(\cdot)}, \mathbb{R})$ with the derivative given by

$$\langle J'(u), v \rangle = \sum_{k \in \mathbb{Z}} [|\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) + V(k) |u(k)|^{q(k)-2} u(k) v(k) - f(k, u(k)) v(k)],$$

for all $u, v \in \ell^{q(\cdot)}$.

We point out that on $\ell^{q(\cdot)}$ we can introduce an equivalent norm with $\|\cdot\|_{q(\cdot)}$, namely

$$\|u\|_{q(\cdot)} := \inf \left\{ \mu > 0; \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} \left| \frac{u(k)}{\mu} \right|^{q(k)} \leq 1 \right\}.$$

By straightforward computation we show that

$$\|u\|_{q(\cdot)} < 1 \Rightarrow \|u\|_{q(\cdot)}^{q^+} \leq \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |u(k)|^{q(k)} \leq \|u\|_{q(\cdot)}^{q^-}, \quad (8.7)$$

$$\|u\|_{q(\cdot)} > 1 \Rightarrow \|u\|_{q(\cdot)}^{q^-} \leq \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |u(k)|^{q(k)} \leq \|u\|_{q(\cdot)}^{q^+}, \quad (8.8)$$

$$\|u\|_{q(\cdot)} \rightarrow 0 \Leftrightarrow \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |u(k)|^{q(k)} \rightarrow 0. \quad (8.9)$$

The next result shows that J has a mountain-pass geometry.

Lemma 39. Assume the hypotheses of Theorem 38 are fulfilled. Then there exist $\rho > 0$ and $\nu > 0$ and $e \in \ell^{q(\cdot)}$ with $\|e\|_{q(\cdot)} > \rho$ such that

- (i) $J(u) \geq \nu$ for all $u \in \ell^{q(\cdot)}$ with $\|u\|_{q(\cdot)} > \rho$;
- (ii) $J(e) < 0$.

We recall that given $c \in \mathbb{R}$, we say that a sequence $(u(k)) \subset \ell^{q(\cdot)}$ satisfies the Palais–Smale $(PS)_c$ condition if

$$J(u(k)) \rightarrow c \quad \text{and} \quad J'(u(k)) \rightarrow 0.$$

Lemma 40. Assume the hypotheses of Theorem 38 are fulfilled. Then, there exist $c > 0$ and a bounded $(PS)_c$ sequence for J in $\ell^{q(\cdot)}$.

Returning to the proof of Theorem 38, assume that $\{u_n\}$ is the sequence given by Lemma 40. Then for each $n \in \mathbb{N}$ the sequence $\{|u_n(k)|; k \in \mathbb{Z}\} \subset \ell^{q(\cdot)}$ is bounded and $|u_n(k)| \rightarrow 0$ as $|k| \rightarrow \infty$.

Assume that $\{|u_n(k)|\}_{k \in \mathbb{Z}}$ achieves its maximum in $k_n \in \mathbb{Z}$. Then there exists $j_n \in \mathbb{Z}$ such that $j_n T \leq k_n < (j_n + 1)T$. Define

$$w_n(k) := u_n(k - j_n T).$$

Then $\{|w_n(k)|\}_{k \in \mathbb{Z}}$ attains its maximum in

$$i_n := k_n - j_n T \in [0, T].$$

The T -periodicity of $p(\cdot)$, $q(\cdot)$ and $V(\cdot)$ implies

$$\sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |u_n(k)|^{q(k)} = \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |w_n(k)|^{q(k)},$$

and $J(u_n) = J(w_n)$. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\ell^{q(\cdot)}$ the above estimates and relations (8.7) and (8.8) yield that $\{w_n\}_{n \in \mathbb{N}}$ is bounded in $\ell^{q(\cdot)}$, too. Then there exists $w \in \ell^{q(\cdot)}$ such that w_n converges weakly to w in $\ell^{q(\cdot)}$ as $n \rightarrow \infty$.

We claim that $w_n(k) \rightarrow w(k)$ as $n \rightarrow \infty$ for each $k \in \mathbb{Z}$. Indeed, defining the test function $v_m \in \ell^{q(\cdot)}$ by

$$v_m(j) := \begin{cases} 1 & \text{if } j = m \\ 0 & \text{if } j \neq m, \end{cases}$$

and taking into account the weak convergence of w_n to w in $\ell^{q(\cdot)}$ we find

$$\lim_{n \rightarrow \infty} w_n(k) = \lim_{n \rightarrow \infty} \langle w_n, v_k \rangle = \langle w, v_k \rangle = w(k),$$

for all $k \in \mathbb{Z}$. The claim is clear now.

Next, we point out that for each $v \in \ell^{q(\cdot)}$ we have

$$|J'(w_n, v)| = |J'(u_n, v(\cdot + j_n T))| \leq \|J'(u_n)\|_* \|v\|_{q(\cdot)},$$

which implies $J'(w_n) \rightarrow 0$ as $n \rightarrow \infty$. It follows that for each $v \in \ell^{q(\cdot)}$ we have as $n \rightarrow \infty$

$$\sum_{k \in \mathbb{Z}} [\phi_{p(k-1)}(\Delta w_n(k-1)) \Delta v(k-1) + V(k) \phi_{q(k)}(w_n(k)) v(k) - f(k, w_n(k)) v(k) \rightarrow 0] \rightarrow 0. \tag{8.10}$$

Consider $v \in \ell^{q(\cdot)}$ has compact support, hence there exist $a, b \in \mathbb{Z}, a < b$ such that $v(k) = 0$ if $k \in \mathbb{Z} \setminus [a, b]$ and $v(k) \neq 0$ if $k \in \{a+1, b-1\}$. The set of compact support functions, denoted by $\ell_0^{q(\cdot)}$, is dense in $\ell^{q(\cdot)}$. Indeed, for all $v \in \ell^{q(\cdot)}$ we can define $v_n \in \ell_0^{q(\cdot)}$ by $v_n(j) = 0$ if $|j| \geq n+1$ and $v_n(j) = v(j)$ if $|j| \leq n$ and we have $\sum_{j \in \mathbb{Z}} \frac{V(j)}{q(j)} |v(j) - v_n(j)| \rightarrow 0$ as $n \rightarrow \infty$, or, by relation (8.9), $\|v - v_n\|_{q(\cdot)} \rightarrow 0$ as $n \rightarrow \infty$.

Now, for each $v \in \ell_0^{q(\cdot)}$ in (8.10) taking into account the finite sums and the continuity of $f(k, \cdot)$ we obtain by passing to the limit as $n \rightarrow \infty$ that

$$\sum_{k \in \mathbb{Z}} [\phi_{p(k-1)}(\Delta w(k-1)) \Delta v(k-1) + V(k) \phi_{q(k)}(w(k)) v(k) - f(k, w(k)) v(k) \rightarrow 0] \rightarrow 0.$$

We found that w is a critical point of J and consequently a solution of problem (8.2).

We show that $w \neq 0$. Assume by contradiction that $w = 0$. Therefore

$$\|u_n\|_\infty = \|w_n\|_\infty = \max\{|w_n(k)|; k \in \mathbb{Z}\} \rightarrow 0,$$

as $n \rightarrow \infty$. On the other hand, condition (F2) implies that for a given $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$\begin{cases} |F(k, t)| \leq \epsilon |t|^{q^+} \\ |f(x, t)t| \leq \epsilon |t|^{q^+}, \end{cases} \tag{8.11}$$

for all $k \in \{0, 1, \dots, T-1\}$ and all $|t| < \delta$. The above inequalities show that for every $k \in \{0, 1, \dots, T-1\}$ there exists M_k such that for $n > M_k$ we have

$$\|w_n(k)\| < \delta.$$

Since $i_n \in \{0, 1, \dots, T-1\}$ it follows that for $n > M := \max\{M_n; k \in \{0, 1, \dots, T-1\}\}$ and every $k \in \mathbb{Z}$ we have

$$\|w_n(k)\| \leq \|w_n(i_n)\| < \delta < 1.$$

That fact and relation (8.11) imply

$$|F(k, w_n(k))| \leq \epsilon |w_n(k)|^{q^+} \leq \epsilon |w_n(k)|^{q(k)},$$

and

$$|f(k, w_n(k)) w_n(k)| \leq \epsilon |w_n(k)|^{q^+} \leq \epsilon |w_n(k)|^{q(k)}.$$

We infer that for each $n > M$ and every $k \in \mathbb{Z}$ the following estimates hold true

$$\begin{aligned} 0 &< q^- J(w_n) = q^- \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |\Delta w_n(k-1)|^{p(k-1)} + q^- \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |w_n(k)|^{q(k)} - q^- \sum_{k \in \mathbb{Z}} F(k, w_n(k)) \\ &\leq \sum_{k \in \mathbb{Z}} |\Delta w_n(k-1)|^{p(k-1)} + \sum_{k \in \mathbb{Z}} V(k) |w_n(k)|^{q(k)} - \sum_{k \in \mathbb{Z}} f(k, w_n(k)) w_n(k) \\ &\quad - \sum_{k \in \mathbb{Z}} (q^- F(k, w_n(k)) - f(k, w_n(k)) w_n(k)) \\ &\leq \langle J'(w_n), w_n \rangle + q^- \sum_{k \in \mathbb{Z}} F(k, w_n(k)) + \sum_{k \in \mathbb{Z}} |f(k, w_n(k)) w_n(k)| \\ &\leq \langle J'(w_n), w_n \rangle + q^- \epsilon \sum_{k \in \mathbb{Z}} |w_n(k)|^{q(k)} + \epsilon \sum_{k \in \mathbb{Z}} |w_n(k)|^{q(k)} \\ &\leq \langle J'(w_n), w_n \rangle + \left(q^- \epsilon \frac{q^+}{V_0} + \epsilon \frac{q^+}{V_0} \right) \sum_{k \in \mathbb{Z}} \frac{V(k)}{q(k)} |w_n(k)|^{q(k)} \\ &\leq \|J'(w_n)\|_* \|w_n\|_{q(\cdot)} + \epsilon \frac{q^+(q^-+1)}{V_0} \left[\|w_n\|_{q(\cdot)}^{q^+} + \|w_n\|_{q(\cdot)}^{q^-} \right]. \end{aligned}$$

Taking into account that $\|u_n\|_{q(\cdot)}$ is bounded, $J'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\epsilon > 0$ is arbitrary we find by the above estimates a contradiction with $J(w_n) \rightarrow c > 0$ as $n \rightarrow \infty$. Thus, w is a nontrivial solution of problem (8.2).

Next, let u be a nonzero homoclinic solution of problem (8.2). Assume that it attains positive local maximums and negative local minimums at infinitely many points k_n . In particular we can assume that $\{|k_n|\} \rightarrow \infty$. Consequently,

$$\Delta_{p(k_n-1)}^2 u(k_n - 1) u(k_n) \leq 0,$$

and

$$u(k_n) \rightarrow 0,$$

as $n \rightarrow \infty$. Using that facts and multiplying in (8.2) by $u(k_n)/|u(k_n)|^{q(k_n)}$, we have

$$\begin{aligned} \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^{q(k_n)}} &\geq \frac{\Delta_{p(k_n-1)}^2 u(k_n - 1) u(k_n)}{|u(k_n)|^{q(k_n)}} + \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^{q(k_n)}} \\ &= V(k_n) \geq V_0 > 0. \end{aligned} \tag{8.12}$$

Using (8.12) and condition (F2) we deduce

$$0 = \lim_{n \rightarrow \infty} \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^{q(k_n)}} \geq V_0 > 0,$$

which represents a contradiction. Consequently u does not attain positive local maximums and negative local minimums at infinitely many points.

Assume now that u vanishes at infinitely many points l_n . By condition (F2) we find that $\Delta_{p(l_n-1)}^2 u(l_n - 1) = 0$ and, consequently, $u(l_n - 1)u(l_n + 1) < 0$. Therefore it has an unbounded sequence of positive local maximums and negative local minimums, in contradiction with the previous assertion.

We proved that, for $|k|$ large enough, function u has constant sign and it is strictly monotone. The proof of Theorem 38 is complete. \square

We observe that the homogeneous problem

$$\begin{cases} \Delta_{p(k-1)}^2 u(k - 1) - V(k)|u(k)|^{q(k)-2}u(k) = 0 & \text{for } k \in \mathbb{Z}, \\ u(k) \rightarrow 0 & \text{as } |k| \rightarrow \infty, \end{cases} \tag{8.13}$$

has only the trivial solution. Indeed, if u is positive or negative, let k_0 be the point of its positive maximum or negative minimum. Then $\Delta_{p(k_0-1)}^2 u(k_0 - 1)u(k_0) \leq 0$ and

$$0 = \Delta_{p(k_0-1)}^2 u(k_0 - 1)u(k_0) - V(k_0)|u(k_0)|^{q(k_0)} < 0,$$

which is a contradiction. The same conclusion can be made if u is sign-changing.

Note that under the assumptions of Theorem 38, then for every $\lambda > 0$ we can establish with the same approach the existence of a nontrivial solution for the eigenvalue problem

$$\begin{cases} \Delta_{p(k-1)}^2 u(k - 1) - V(k)|u(k)|^{q(k)-2}u(k) + \lambda f(k, u(k)) = 0 & \text{for } k \in \mathbb{Z}, \\ u(k) \rightarrow 0 & \text{as } |k| \rightarrow \infty. \end{cases} \tag{8.14}$$

We can prove that if in addition to conditions (F1) and (F2) the following condition holds:

(F3) $f(k, t) \geq 0$ for any $t < 0$ and all $k \in \mathbb{Z}$,

then the homoclinic solution of the problem (8.2) is positive.

A comprehensive treatment of nonlinear partial differential equations with variable exponent can be found in the monograph by Rădulescu and Repovš [40].

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