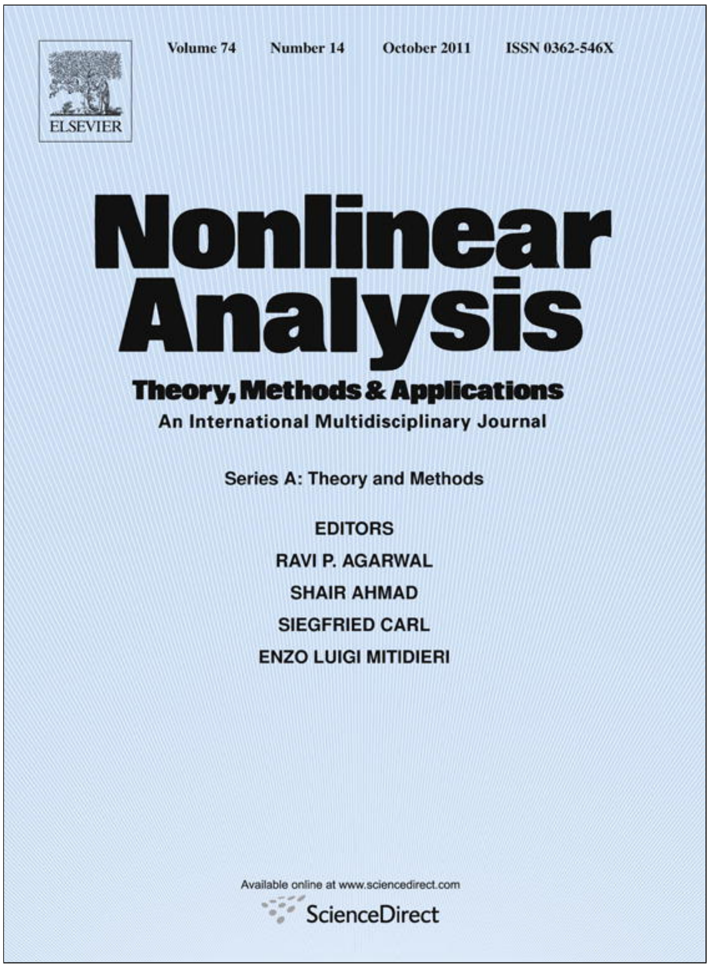
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Existence of three solutions for a non-homogeneous Neumann problem through Orlicz–Sobolev spaces

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ABSTRACT

The aim of this paper is to establish a multiplicity result for an eigenvalue non-homogeneous Neumann problem which involves a nonlinearity fulfilling a nonstandard growth condition. Precisely, a recent critical points result for differentiable functionals is exploited in order to prove the existence of a determined open interval of positive eigenvalues for which the problem admits at least three weak solutions in an appropriate Orlicz–Sobolev space.

1. Introduction

In this paper, we study the following non-homogeneous Neumann problem

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$N^f_{\alpha,\lambda}$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, λ is a positive parameter and $\alpha : (0,\infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

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The interest in analyzing these kinds of problems is motivated by some recent advances in the study of eigenvalue problems involving non-homogeneous operators in the divergence form. The study of such problems has been stimulated by recent advances in elasticity (see [1]), fluid dynamics (see [2–4]), calculus of variations and differential equations with nonstandard growth (see [5–8]). Another relevant application which uses operators of this type can be found in the framework of image processing. In that context we refer to the paper by Chen et al. [9]. In [9], the authors study a functional with variable exponent, which provides a model for image restoration. The diffusion resulting from the proposed model is a combination of Gaussian smoothing and regularization based on total variation. More exactly, the following adaptive model was proposed

$$\min_{I=u+v,u\in \mathrm{BV}\cap L^2(\Omega)} \int_\Omega \varphi(x, \nabla u)\, \mathrm{d}x + \lambda \cdot \|u\|^2_{L^2(\Omega)}, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain,

$$\varphi(x,r) = \begin{cases} \dfrac{1}{p(x)}|r|^{p(x)}, & \text{for } |r| \le \beta \\ |r| - \dfrac{\beta\cdot p(x) - \beta^{p(x)}}{p(x)}, & \text{for } |r| > \beta, \end{cases}$$

where $\beta > 0$ is fixed and $1 < \alpha \le p(x) \le 2$. The function $p(x)$ involved here depends on the location x in the model. For instance it can be used

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

where $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/(4\sigma^2))$ is the Gaussian filter and $k > 0$ and $\sigma > 0$ are fixed parameters (according to the notation in [9]). For problem (1) Chen, Levine and Rao establish the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in the paper.

Our approach in this paper relies on adequate variational methods in Orlicz–Sobolev spaces. Such spaces originated with Nakano [10] and were developed by Musielak and Orlicz [11]. Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Dankert [12], Donaldson and Trudinger [13], and O'Neill [14] (see also [15] for an excellent account of those works). Orlicz–Sobolev spaces have been used in the last decades to model various phenomena. This kind of spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, the theory of quasiconformal mappings, differential geometry, geometric function theory, and probability theory. These spaces consists of functions that have weak derivatives and satisfy certain integrability conditions. The first general existence result using the theory of monotone operators in Orlicz–Sobolev spaces were in [16–18]. Recent works that put the problem into this framework are contained in [19,20,6,21–25,7,8]. Concerning the boundary value problems with Neumann boundary condition we point out the existence and multiplicity results obtained in [26] and, very recently, in [27].

The aim of this paper is to establish a precise interval, of values of the parameter λ, for which the eigenvalue non-homogeneous Neumann problem $(N^f_{\alpha,\lambda})$ admits at least three weak solutions. The precise notion of weak solutions for the problem $(N^f_{\alpha,\lambda})$ will be given in Section 2.

As an example we present a special case of our results; see Example 2.1 and Corollary 3.1.

Theorem 1.1. *Let $p > N + 1$ and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative (not identically zero) continuous function. Assume that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{g(t)}{|t|^s} = 0, \tag{h_0^∞}$$

for some positive constant $s < p - 2$. Further, let $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function.

Then, for each

$$\lambda > \frac{\operatorname{meas}(\Omega)}{\|h\|_{L^1(\Omega)}} \inf_{\delta\in S} \frac{\Phi(\delta)}{\int_0^\delta g(t)\, \mathrm{d}t},$$

where

$$\Phi(\delta) := \int_0^\delta \frac{t|t|^{p-2}}{\log(1+|t|)}\, \mathrm{d}t \quad \text{and} \quad S := \left\{\delta > 0 : \int_0^\delta g(t)\, \mathrm{d}t > 0\right\},$$

the following non-homogeneous Neumann problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right)+\dfrac{|u|^{p-2}}{\log(1+|u|)}u=\lambda h(x)g(u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ^{hg}}$$

admits at least two non-trivial weak solutions in $W^1L_\Phi(\Omega)$.

For the p-Laplacian operator (homogeneous case) there is a wide literature based on the abstract framework of the seminal papers [28–30] that deal with multiplicity results for such a problem in the case $p > N$. We refer, for instance, to [31–36] and references therein for details. We point out that our result is also new in this setting; see Remark 3.4 and Theorem 3.3.

The main tool is a critical point theorem that we recall here in a convenient form. This result has been obtained in [37] and it is a more precise version of Theorem 3.2 of [38].

Theorem 1.2. *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$J(0) = I(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < J(\bar{x})$, such that:

(a$_1$) $\frac{\sup_{J(x)\le r} I(x)}{r} < \frac{I(\bar{x})}{J(\bar{x})}$;

(a$_2$) *for each* $\lambda \in \Lambda_r := \left]\frac{J(\bar{x})}{I(\bar{x})}, \frac{r}{\sup_{J(x)\le r} I(x)}\right[$ *the functional* $g_\lambda := J - \lambda I$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional g_λ has at least three distinct critical points in X.

When we say that the derivative of J admits a continuous inverse on X^* we mean that there exists a continuous operator $T : X^* \to X$ such that $T(J'(x)) = x$ for all $x \in X$.

The plan of the paper is as follows. In the next section we introduce our abstract framework. The last section is devoted to our multiplicity result.

2. Abstract framework

Let $\varphi : \mathbb{R} \to \mathbb{R}$ as in Introduction and consider the functions

$$\Phi(t) = \int_0^t \varphi(s)\,ds, \qquad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an N-function. The function $\Phi^\star$ is called the complementary function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s); s \ge 0\}, \quad \text{for all } t \ge 0.$$

We observe that $\Phi^\star$ is also an N-function and the following Young's inequality holds true:

$$st \le \Phi(s) + \Phi^\star(t), \quad \text{for all } s, t \ge 0.$$

Assume that Φ satisfying the following structural hypotheses

$$1 < \liminf_{t\to\infty}\frac{t\varphi(t)}{\Phi(t)} \le p^0 := \sup_{t>0}\frac{t\varphi(t)}{\Phi(t)} < \infty; \tag{Φ_0}$$

$$N < p_0 := \inf_{t>0}\frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty}\frac{\log(\Phi(t))}{\log(t)}. \tag{Φ_1}$$

Further, we also assume that the function

$$[0,\infty) \ni t \to \Phi(\sqrt{t}) \tag{Φ_2}$$

is convex.

Example 2.1. Let $p > N + 1$. Define

$$\varphi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t \quad \text{for } t \neq 0, \text{ and } \varphi(0) = 0,$$

and

$$\Phi(t) = \int_0^t \varphi(s)\, \mathrm{d}s.$$

By [20, Example 3, p. 243] one has

$$p_0 = p - 1 < p^0 = p = \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (Φ_0) and (Φ_1) are verified. Further, by direct computations, the function $[0, \infty) \ni t \mapsto \Phi(\sqrt{t})$ is convex. Hence, also condition (Φ_2) is fulfilled.

The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ (see for instance [15,19]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup \left\{ \int_\Omega u(x)v(x)\, \mathrm{d}x; \int_\Omega \Phi^\star(|v(x)|)\, \mathrm{d}x \leq 1 \right\} < \infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf \left\{ k > 0; \int_\Omega \Phi\left(\frac{u(x)}{k}\right) \mathrm{d}x \leq 1 \right\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem $(N^f_{\alpha,\lambda})$, defined by

$$W^1L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega); \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), i = 1, \ldots, N \right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|\,|\nabla u|\,\|_\Phi + \|u\|_\Phi,$$

see [15,19,17]. Further, one has

Lemma 2.1. *On $W^1L_\Phi(\Omega)$ the norms*

$$\|u\|_{1,\Phi} = \|\,|\nabla u|\,\|_\Phi + \|u\|_\Phi,$$
$$\|u\|_{2,\Phi} = \max\{\|\,|\nabla u|\,\|_\Phi, \|u\|_\Phi\},$$
$$\|u\| = \inf \left\{ \mu > 0; \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right) \right] \mathrm{d}x \leq 1 \right\},$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\|.$$

Moreover the following relations hold true

Lemma 2.2. *Let $u \in W^1L_\Phi(\Omega)$. Then*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\, \mathrm{d}x \geq \|u\|^{p_0}, \quad \textit{if } \|u\| > 1;$$
$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\, \mathrm{d}x \geq \|u\|^{p^0}, \quad \textit{if } \|u\| < 1.$$

For a proof of the previous two results see, respectively, Lemmas 2.2 and 2.3 of the paper [27].
These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$.
Moreover, we say that $u \in W^1L_\Phi(\Omega)$ is a *weak solution* for problem $(N^f_{\alpha,\lambda})$ if

$$\int_\Omega \alpha(|\nabla u(x)|)\nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x + \int_\Omega \alpha(|u(x)|)u(x)v(x)\, \mathrm{d}x - \lambda \int_\Omega f(x, u(x))v(x)\, \mathrm{d}x = 0,$$

for every $v \in W^1L_\Phi(\Omega)$.
Finally, the following lemma will be useful in what follows.

Lemma 2.3. *Let $u \in W^1L_\Phi(\Omega)$ and assume that*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \le r, \tag{2}$$

for some $0 < r < 1$. Then, one has, $\|u\| < 1$.

Proof. By definition,

$$\|u\| = \inf\left\{\mu > 0;\ \int_\Omega \left[\Phi\left(\frac{|u|}{\mu}\right) + \Phi\left(\frac{|\nabla u|}{\mu}\right)\right]\mathrm{d}x \le 1\right\},$$

for every $u \in W^1L_\Phi(\Omega)$. Then, if (2) holds, it follows that $\|u\| \le 1$.

Hence, the result is attained proving that if $u \in W^1L_\Phi(\Omega)$ such that (2) holds, then $\|u\| \ne 1$. We first observe that

$$\Phi(t) \ge \tau^{p^0}\Phi(t/\tau), \quad \forall t > 0 \text{ and } \tau \in]0, 1[. \tag{3}$$

Arguing by contradiction, assume that there exists $u \in W^1L_\Phi(\Omega)$ with $\|u\| = 1$ and such that (2) holds. Let us take $\xi \in (0, 1)$. Using relation (3) we have

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \ge \xi^{p^0}\left[\int_\Omega [\Phi(|v(x)|) + \Phi(|\nabla v(x)|)]\,\mathrm{d}x\right], \tag{4}$$

where $v(x) := u(x)/\xi$, for all $x \in \Omega$. We have $\|v\| = 1/\xi > 1$. By the first inequality in Lemma 2.2 we deduce that

$$\int_\Omega [\Phi(|v(x)|) + \Phi(|\nabla v(x)|)]\,\mathrm{d}x \ge \|v\|^{p^0} > 1. \tag{5}$$

Relations (4) and (5) show that

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \ge \xi^{p^0}.$$

Letting $\xi \nearrow 1$ in the above inequality we obtain

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \ge 1, \tag{6}$$

that contradicts condition (2). The proof is complete. □

3. Main result

Here and in what follows "meas(Ω)" denotes the Lebesgue measure of the set Ω. From hypothesis (Φ_1), by Lemma D.2 in [19] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume $p_0 > N$ we deduce that $W^{1,p_0}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, one has that $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|_{1,\Phi}, \quad \forall u \in W^1L_\Phi(\Omega), \tag{7}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}}|u(x)|$. A direct estimation of the constant c remains an open question.

Now, assuming that the growth of $f(x,\cdot)$ is $(p_0 - 1)$-sublinear at infinity the main result reads as follows

Theorem 3.1. *Let Φ be a Young function satisfying the structural hypotheses (Φ_0)–(Φ_2) and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

(h_1) *There exist two positive constants γ and δ, with $\gamma < 2c$ such that*

$$\Phi(\delta) > \kappa_\Omega^\Phi \gamma^{p^0},$$

and

$$\frac{\int_\Omega \max_{|\xi|\le\gamma} F(x,\xi)\,\mathrm{d}x}{\gamma^{p^0}} < \kappa_\Omega^\Phi \frac{\int_\Omega F(x,\delta)\,\mathrm{d}x}{\Phi(\delta)},$$

where $\kappa_\Omega^\Phi := \frac{1}{(2c)^{p^0}\mathrm{meas}(\Omega)}$ and c is defined in (7);

(h_2) *There exist $c_0 > 0$ and $0 < s < p_0 - 1$ such that $|f(x,t)| \le c_0(1 + |t|^s)$ for every $(x,t) \in \Omega \times \mathbb{R}$.*

Then, for each parameter λ belonging to

$$\Lambda_{(\gamma,\delta)} := \left]\frac{\Phi(\delta)\mathrm{meas}(\Omega)}{\int_\Omega F(x,\delta)\,\mathrm{d}x}, \frac{\gamma^{p^0}}{(2c)^{p^0}\int_\Omega \max\limits_{|\xi|\le\gamma} F(x,\xi)\,\mathrm{d}x}\right[,$$

the problem $(N^f_{\alpha,\lambda})$ possesses at least three distinct weak solutions in $W^1L_\Phi(\Omega)$.

Proof. Let us put $X := W^1L_\Phi(\Omega)$. Hypothesis (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2-condition (at infinity), see [15, p. 232] and [19]. In particular, both (Φ, Ω) and $(\Phi^\star, \Omega)$ are Δ-regular, see [15, p. 232]. Consequently, the spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see [15, p. 241 and p. 247]. Now, define the functionals $J, I : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega (\Phi(|\nabla u(x)|) + \Phi(|u(x)|))\,\mathrm{d}x \quad \text{and} \quad I(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x,$$

where $F(x,\xi) := \int_0^\xi f(x,t)\,\mathrm{d}t$ for every $(x,\xi) \in \Omega \times \mathbb{R}$ and put

$$g_\lambda(u) := J(u) - \lambda I(u), \quad u \in X.$$

The functionals J and I satisfy the regularity assumptions of Theorem 1.2. Indeed, similar arguments as those used in [21, Lemma 3.4] and [19, Lemma 2.1] imply that $J, I \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\langle J'(u), v\rangle = \int_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega \alpha(|u(x)|)u(x)v(x)\,\mathrm{d}x,$$
$$\langle I'(u), v\rangle = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x,$$

for any $u, v \in X$.
Moreover, owing that Φ is convex, it follows that J is a convex functional, hence one has that J is sequentially weakly lower semicontinuous. On the other hand the fact that X is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $I' : X \to X^\star$ is compact.
Let us observe that $u \in X$ is a weak solution of problem $(N^f_{\alpha,\lambda})$ if u is a critical point of the functional g_λ. Hence, we can seek for weak solutions of problem $(N^f_{\alpha,\lambda})$ by applying Theorem 1.2.
Now, we observe that the technical assumption (Φ_2) ensures a Clarkson type inequality for the function Φ, i.e.

$$\frac{1}{2}\left[\int_\Omega \Phi(|\nabla u|)\mathrm{d}x + \int_\Omega \Phi(|\nabla v|)\mathrm{d}x\right] \ge \int_\Omega \Phi\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right)\mathrm{d}x + \int_\Omega \Phi\left(\left|\frac{\nabla u - \nabla v}{2}\right|\right)\mathrm{d}x,$$

for any $u, v \in W^1L_\Phi(\Omega)$. From this, it follows that the functional $J' : W^1L_\Phi(\Omega) \to (W^1L_\Phi(\Omega))^*$ has a continuous inverse operator on $(W^1L_\Phi(\Omega))^*$, where $(W^1L_\Phi(\Omega))^*$ denotes the dual space of $(W^1L_\Phi(\Omega))$. See, for instance, Lemma 3.2 of [27].
Now, it is clear that $I(0) = J(0) = 0$ and $J(u) \ge 0$ for every $u \in X$. At this point one may choose $\overline{w} := \delta \in X$ and $r := \gamma^{p^0}/(2c)^{p^0}$. From (h_1), we have

$$J(\overline{w}) = \int_\Omega (\Phi(|\nabla\overline{w}(x)|) + \Phi(|\overline{w}(x)|))\,\mathrm{d}x = \int_\Omega \Phi(\delta)\,\mathrm{d}x = \Phi(\delta)\mathrm{meas}(\Omega) > \frac{\gamma^{p^0}}{(2c)^{p^0}}.$$

By Lemmas 2.3 and 2.2, one has

$$\{u \in W^1L_\Phi(\Omega) : J(u) \le r\} \subseteq \left\{u \in W^1L_\Phi(\Omega) : \|u\| \le \frac{\gamma}{2c}\right\}.$$

Moreover, for every $u \in W^1L_\Phi(\Omega)$, due to (7) and Lemma 2.1, we have

$$|u(x)| \le \|u\|_\infty \le c\|u\|_{1,\Phi} \le 2c\|u\| \le \gamma, \quad \forall x \in \Omega.$$

Hence

$$\left\{u \in W^1L_\Phi(\Omega) : \|u\| \le \frac{\gamma}{2c}\right\} \subseteq \left\{u \in W^1L_\Phi(\Omega) : \|u\|_\infty \le \gamma\right\},$$

and one has

$$\frac{\sup\limits_{u\in J^{-1}(]-\infty,r])} I(u)}{r} \le (2c)^{p^0}\frac{\int_\Omega \max\limits_{|\xi|\le\gamma} F(x,\xi)\,\mathrm{d}x}{\gamma^{p^0}}.$$

Moreover, owing that

$$\frac{I(\overline{w})}{J(\overline{w})} = \frac{\int_\Omega F(x,\delta)\,dx}{\text{meas}(\Omega)\Phi(\delta)},$$

from (h_1) it follows that

$$\frac{\sup_{J(u)\le r} I(u)}{r} < \frac{I(\overline{w})}{J(\overline{w})},$$

i.e. condition (a_1) is verified.

Finally, we prove that for every $\lambda > 0$, the functional g_λ is coercive. Indeed, by Lemma 2.2 we deduce that for any $u \in X$ with $\|u\| > 1$ we have $J(u) \ge \|u\|^{p_0}$. Hence J is coercive. On the other hand, by (h_2), one has that there exists a positive constant c_1 such that

$$\int_\Omega F(x,u(x))\,dx \le c_1(\|u\|_\infty + \|u\|_\infty^{s+1}), \quad \forall u \in X.$$

Since X is compactly embedded into $C^0(\overline{\Omega})$ and, due to Lemma 2.1, it follows that there exists $c_2 > 0$ such that

$$g_\lambda(u) = J(u) - \lambda I(u) \ge \|u\|^{p_0} - \lambda c_2(\|u\| + \|u\|^{s+1}),$$

for every $u \in X$ and $\|u\| > 1$.

Since $1 < s + 1 < p_0$ it follows that

$$\lim_{\|u\|\to+\infty} g_\lambda(u) = +\infty, \quad \forall \lambda > 0.$$

Hence g_λ is a coercive functional for every positive parameter, in particular, for every $\lambda \in \Lambda_{(\gamma,\delta)} \subseteq \left]\frac{J(\overline{w})}{I(\overline{w})}, \frac{r}{\sup_{J(u)\le r} I(u)}\right[$. Then, also condition (a_2) holds. Since all the assumptions of Theorem 1.2 are satisfied. Then, for each $\lambda \in \Lambda_{(\gamma,\delta)}$, the functional g_λ has at least three distinct critical points that are weak solutions of the problem $(N^f_{\alpha,\lambda})$. The proof is complete. □

Remark 3.1. In our setting, as pointed out in [27], the presence of the eigenvalue $\lambda > 0$ in $(N^f_{\alpha,\lambda})$ is indispensable. Moreover, we point out that in the cited paper, the existence of a localized interval of parameters for which the problem $(N^f_{\alpha,\lambda})$ admits at least two non-trivial solutions is established under the more restrictive assumption that the nonlinearity f is such that $f(x,t)\,t \le 0$, for every $x \in \Omega$ and $t \in [-\delta, \delta]$, for some positive constant δ. From this, clearly, $f(x, 0) = 0$ for every $x \in \Omega$. Thus, under this hypothesis, $u = 0$ can always considered a solution of problem $(N^f_{\alpha,\lambda})$.

A particular case of Theorem 3.1 is the following one.

Theorem 3.2. *Let Φ be a Young function satisfying the structural hypotheses (Φ_0)–(Φ_2) and let $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function. Further, let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function. Set $G(\xi) := \int_0^\xi g(t)\,dt$ and assume that the following conditions hold*

(h_1) *There exist two positive constants γ and δ, with $\gamma < 2c$ such that*

$$\Phi(\delta) > \kappa_\Omega^\Phi \gamma^{p^0},$$

and

$$\frac{G(\gamma)}{\gamma^{p^0}} < \kappa_\Omega^\Phi \frac{G(\delta)}{\Phi(\delta)},$$

where $\kappa_\Omega^\Phi := \frac{1}{(2c)^{p^0}\text{meas}(\Omega)}$ and c is defined in (7);

(h_2) *There exist $c_0 > 0$ and $0 < s < p_0 - 1$ such that $g(t) \le c_0(1 + |t|^s)$ for every $t \in \mathbb{R}$.*

Then, for each parameter λ belonging to

$$\Lambda_{(\gamma,\delta)} := \left]\frac{\Phi(\delta)\text{meas}(\Omega)}{\|h\|_{L^1(\Omega)}G(\delta)}, \frac{\gamma^{p^0}}{(2c)^{p^0}\|h\|_{L^1(\Omega)}G(\gamma)}\right[,$$

the following problem

$$\begin{cases} -\text{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda h(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (N^{hg}_{\alpha,\lambda})$$

possesses at least three distinct weak solutions in $W^1L_\Phi(\Omega)$.

Remark 3.2. The same conclusion of Theorem 3.2 holds under the assumption that $h : \Omega \to \mathbb{R}$ is a bounded measurable function with $\operatorname{ess\,inf}_{x\in\Omega} h(x) \geq 0$ and $\int_\Omega h(x)\, dx > 0$.

A direct consequence of the previous result reads as follows.

Corollary 3.1. *Let $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative (not identically zero) and continuous function such that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p^0-1}} = 0. \tag{ℓ_0}$$

Further, assume that condition (h_2) *holds.*
Then, for each

$$\lambda > \frac{\operatorname{meas}(\Omega)}{\|h\|_{L^1(\Omega)}} \inf_{\delta\in S} \frac{\Phi(\delta)}{G(\delta)},$$

where

$$S := \{\delta > 0 : G(\delta) > 0\},$$

the problem $(N^{hg}_{\alpha,\lambda})$ *possesses at least three distinct weak solutions in* $W^1L_\Phi(\Omega)$.

Proof. Fix $\lambda > \frac{\operatorname{meas}(\Omega)}{\|h\|_{L^1(\Omega)}} \inf_{\delta\in S} \frac{\Phi(\delta)}{G(\delta)}$. Then, there exists $\bar\delta$ such that $G(\bar\delta) > 0$ and $\lambda > \frac{\operatorname{meas}(\Omega)\Phi(\bar\delta)}{\|h\|_{L^1(\Omega)} G(\bar\delta)}$. By using condition (ℓ_0) one has

$$\lim_{\xi\to 0^+} \frac{G(\xi)}{\xi^{p^0}} = 0.$$

Therefore, we can find a positive constant $\bar\gamma$ such that

$$\bar\gamma < \min\left\{2c, \left(\frac{\Phi(\bar\delta)}{\kappa^\Phi_\Omega}\right)^{1/p^0}\right\},$$

and

$$\frac{G(\bar\gamma)}{\bar\gamma^{p^0}} < \min\left\{\kappa^\Phi_\Omega \frac{G(\bar\delta)}{\Phi(\bar\delta)}, \frac{1}{(2c)^{p^0}\|h\|_{L^1(\Omega)}\lambda}\right\}.$$

Hence

$$\lambda \in \Lambda_{(\bar\gamma,\bar\delta)} := \left]\frac{\operatorname{meas}(\Omega)\Phi(\bar\delta)}{\|h\|_{L^1(\Omega)} G(\bar\delta)}, \frac{\bar\gamma^{p^0}}{(2c)^{p^0}\|h\|_{L^1(\Omega)} G(\bar\gamma)}\right[.$$

All the hypotheses of Theorem 3.2 are satisfied and the problem $(N^{hg}_{\alpha,\lambda})$ admits at least three distinct weak solutions. The proof is complete. $\square$

Remark 3.3. We point out that Theorem 1.1 in Introduction is a particular case of Corollary 3.1, taking into account Example 2.1.

Here, we give a concrete example of application of Corollary 3.1.

Example 3.1. Let Ω be a non-empty bounded open subset of the Euclidean Space $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, Φ be a Young function that satisfy hypotheses (Φ_0)–(Φ_2). Define $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(t) := \begin{cases} 0 & \text{if } t < 0 \\ t^{p^0} & \text{if } 0 \leq t \leq 1 \\ t^s & \text{if } t > 1, \end{cases}$$

where $s \in]0, p_0 - 1[$. Further, let $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function. From Corollary 3.1, for each parameter

$$\lambda > \frac{\operatorname{meas}(\Omega)}{\|h\|_{L^1(\Omega)}} \inf_{\delta > 0} \frac{\Phi(\delta)}{G(\delta)},$$

the following non-homogeneous Neumann problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda h(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (N^{hg}_{\alpha,\lambda})$$

possesses at least two non-trivial weak solutions in $W^1L_\Phi(\Omega)$.
In particular, let $\Omega \subset \mathbb{R}^3$ with $\mathrm{meas}(\Omega) = 1$. Consider $g : \mathbb{R} \to \mathbb{R}$ defined as follows

$$g(t) := \begin{cases} 0 & \text{if } t < 0 \\ t^5 & \text{if } 0 \le t \le 1 \\ t^2 & \text{if } t > 1, \end{cases}$$

andlet $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function with $\|h\|_{L^1(\Omega)} = 1$.
Set $\Phi(\delta) := \int_0^\delta \frac{t|t|^3}{\log(1+|t|)}\,\mathrm{d}t$. One has, for every $\delta > 0$, that

$$\frac{\Phi(\delta)}{G(\delta)} := \begin{cases} 6\dfrac{\int_0^\delta \frac{t|t|^3}{\log(1+|t|)}\,\mathrm{d}t}{\delta^6} & \text{if } 0 \le \delta \le 1, \\ 6\dfrac{\int_0^\delta \frac{t|t|^3}{\log(1+|t|)}\,\mathrm{d}t}{2\delta^3 - 1} & \text{if } \delta > 1. \end{cases}$$

Moreover, by direct computations, owing to the function $\frac{\Phi(\delta)}{G(\delta)}$ attains its minimum in $\delta_0 \approx 1.189089126$, it follows that

$$\inf_{\delta>0} \frac{\Phi(\delta)}{G(\delta)} \approx 1.804670144.$$

Then, for instance, the following non-homogeneous Neumann problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^3}{\log(1+|\nabla u|)}\nabla u\right) + \dfrac{|u|^3}{\log(1+|u|)}u = 2h(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P^{hg}_2)$$

admits at least two non-trivial weak solutions in $W^1L_\Phi(\Omega)$.

Remark 3.4. If $\varphi(t) := |t|^{p-2}t$, with $p > 1$, one has $p_0 = p^0 = p$, and the Orlicz–Sobolev space $W^1L_\Phi(\Omega)$ coincides with $W^{1,p}(\Omega)$. It is clear if $p > N$, $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Let $\kappa > 0$ such that

$$\|u\|_\infty \le \kappa\|u\|_{W^{1,p}(\Omega)},$$

for every $u \in W^{1,p}(\Omega)$, where

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x + \int_\Omega |u(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

In this setting the existence of three weak solutions in $W^{1,p}(\Omega)$ of problem (N^f_λ) can be obtained assuming that f is a Carathéodory function, $(p-1)$-sublinear at infinity, and such that

(h_1') *There exist two positive constants γ and δ, such that*

$$\delta > \left(\frac{1}{\kappa\,\mathrm{meas}(\Omega)^{1/p}}\right)\gamma,$$

and

$$\frac{\int_\Omega \max_{|\xi|\le\gamma} F(x,\xi)\mathrm{d}x}{\gamma^p} < \frac{1}{\kappa^p\mathrm{meas}(\Omega)}\frac{\int_\Omega F(x,\delta)\mathrm{d}x}{\delta^p}.$$

Moreover, the interval of parameters assume the following form

$$\Lambda_{(\gamma,\delta)} := \left]\frac{\delta^p\mathrm{meas}(\Omega)}{p\int_\Omega F(x,\delta)\,\mathrm{d}x}, \frac{\gamma^p}{p\kappa^p\int_\Omega \max_{|\xi|\le\gamma} F(x,\xi)\,\mathrm{d}x}\right[.$$

Therefore, arguing in a similar way of the proof of Theorem 3.1, we can obtain the following result that guarantees the existence of a precise interval, of values of the parameter λ, for which a homogeneous Neumann problem involving the p-Laplacian, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, admits at least three non-trivial weak solutions.

Theorem 3.3. *Let $h : \Omega \to \mathbb{R}$ be a bounded measurable and positive function. Further, let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with $g(0) \neq 0$ and such that the following conditions hold*

(h_1') *There exists two positive constants γ and δ such that $\delta > \left(\frac{1}{\kappa\,\mathrm{meas}(\Omega)^{1/p}}\right)\gamma$, and*

$$\frac{G(\gamma)}{\gamma^p} < \frac{1}{\kappa^p \mathrm{meas}(\Omega)}\frac{G(\delta)}{\delta^p};$$

(h_2') *Assume that*

$$\lim_{|t|\to\infty}\frac{g(t)}{|t|^\beta} = 0,$$

for some $0 \leq \beta < (p-1)$.

Then, for each parameter λ belonging to

$$\Lambda_{(\delta,\gamma)} := \left]\frac{\delta^p \mathrm{meas}(\Omega)}{p\|h\|_{L^1(\Omega)}G(\delta)}, \frac{\gamma^p}{p\kappa^p\|h\|_{L^1(\Omega)}G(\gamma)}\right[,$$

the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda h(x) g(u) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{N_λ^{hg}}$$

possesses at least three non-trivial weak solutions in $W^{1,p}(\Omega)$.

Remark 3.5. Respect to result contained in [31], Theorem 3.3 gives a more precise interval of parameters for which an homogeneous Neumann problem admits at least three weak solutions. Moreover observe that, in the case treated in Theorem 3.3, when Ω is convex, an explicit upper bound for the constant κ is ensured as pointed out in [34, Remark 1].

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