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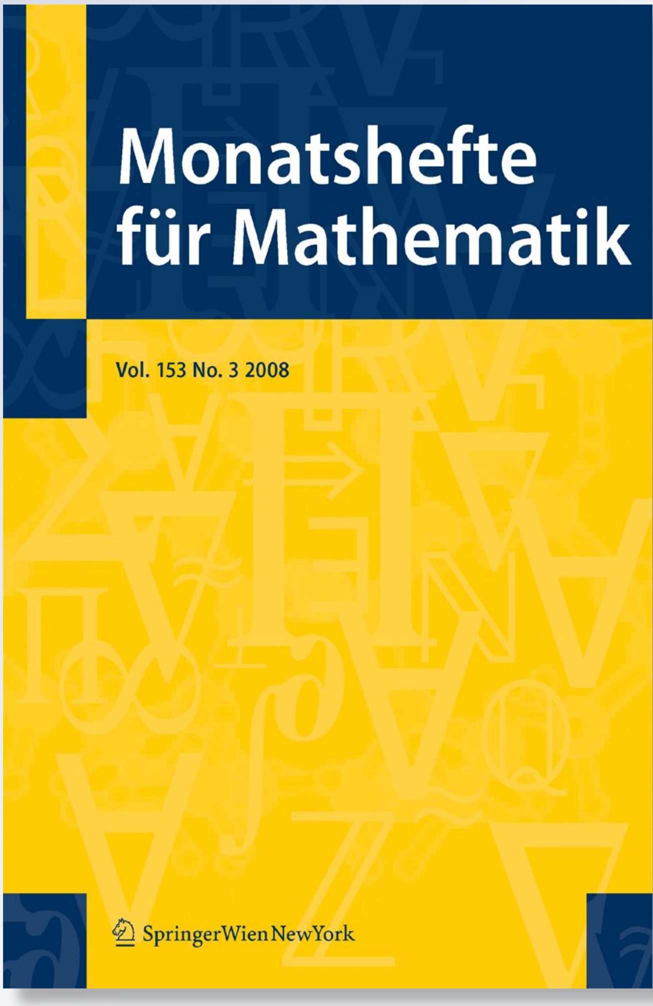

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Abstract Under an appropriate oscillating behavior of the nonlinear term, the existence of a determined open interval of positive parameters for which an eigenvalue non-homogeneous Neumann problem admits infinitely many weak solutions that strongly converges to zero, in an appropriate Orlicz–Sobolev space, is proved. Our approach is based on variational methods. The abstract result of this paper is illustrated by a concrete case.

Keywords Critical point · Weak solutions · Non-homogeneous Neumann problem

Mathematics Subject Classification (2000) 35D05 · 35J60 · 35J70 · 46N20 · 58E05

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1 Introduction

Multiplicity results for quasilinear elliptic partial differential equations involving the p-Laplacian have been broadly investigated in recent years. In this paper we consider more general problems, which involve non-homogeneous differential operators. Problems of this type have been intensively studied in the last few years, due to numerous and relevant applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids), calculus of variations, nonlinear potential theory, the theory of quasi-conformal mappings, differential geometry, geometric function theory, and probability theory. Another recent application which uses non-homogeneous differential operators can be found in the framework of image processing. In that context we refer to the paper by Chen, Levine and Rao [6], where it is studied an energy functional with variable exponent that provides a model for image restoration. The diffusion resulting from the proposed model is a combination of Gaussian smoothing and regularization based on total variation. More exactly, if $\lambda \geq 0$, the version of this problem is to recover an image u, from an observed noisy image I, for which the following adaptive model was proposed

$$\min_{u \in \mathrm{BV} \cap L^2(\Omega)} \int_\Omega \left[\varphi(x, \nabla u) + \frac{\lambda}{2}(u - I)^2 \right] dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain,

$$\varphi(x, r) = \begin{cases} \dfrac{1}{p(x)}|r|^{p(x)}, & \text{for } |r| \leq \beta \\ |r| - \dfrac{\beta \cdot p(x) - \beta^{p(x)}}{p(x)}, & \text{for } |r| > \beta, \end{cases}$$

where $\beta > 0$ is fixed and $\alpha \leq p(x) \leq 2$ for every $x \in \Omega$, for some $\alpha > 1$. The function $p(x)$ involved here depends on the location x in the model. For instance it can be used

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

where $G_\sigma(x) = \dfrac{1}{\sigma}\exp(-|x|^2/(4\sigma^2))$ is the Gaussian filter and $k > 0$ and $\sigma > 0$ are fixed parameters (according to the notation in [6]). For problem (1) Chen, Levine and Rao establish the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in their work.

Our main purpose in this paper is to study the non-homogeneous Neumann problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$N^f_{\alpha,\lambda}$}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, while $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, λ is a positive parameter and $\alpha : (0, \infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0\,, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

We point out that a related Neumann problem has been recently studied in [17], where it is established the existence of at least one nontrivial solution. The main goal of this paper is to establish the existence of a precise interval of positive parameters λ such that, under natural assumptions, problem $(N^f_{\alpha,\lambda})$ admits a sequence of pairwise distinct solutions that strongly converges to zero in the Orlicz–Sobolev space $W^1L_\Phi(\Omega)$.

The interest in analyzing this kind of problems is motivated by some recent advances in the study of eigenvalue problems involving non-homogeneous operators in the divergence form; see, for instance, the papers [7–12,19] and [13–16].

Moreover, an overview on Orlicz–Sobolev spaces is given in, for instance, the monograph of Rao and Ren [19] and the references given therein.

The main tool in order to prove our multiplicity result is the following critical points theorem obtained in [4] that we recall here in a convenient form. This result is a refinement of the Variational Principle of Ricceri, contained in [20].

Theorem 1.1 ([4, Theorem 2.1]) *Let X be a reflexive real Banach space, let $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that J is strongly continuous, sequentially weakly lower semicontinuous and coercive and I is sequentially weakly upper semicontinuous. For every $r > \inf_X J$, put*

$$\varphi(r) := \inf_{u\in J^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in J^{-1}(]-\infty,r[)} I(v)\right) - J(u)}{r - J(u)},$$

and $\delta := \liminf\limits_{r\to(\inf_X J)^+} \varphi(r)$.

Then, if $\delta < +\infty$, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds: either

(c_1) there is a global minimum of J which is a local minimum of $g_\lambda := J - \lambda I$, or

(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of g_λ which weakly converges to a global minimum of J, with $\lim_{n\to+\infty} J(u_n) = \inf_X J$.

The above theorem assures the existence of a sequence of pairwise distinct critical points for Gâteaux differentiable functionals under assumptions that, when we consider the energy functional associated to $(N^f_{\alpha,\lambda})$, are satisfied just assuming an appropriate oscillating behavior on the potential of the nonlinearity at zero. The main question in applying Theorem 1.1 is to find sufficient conditions in order that the positive constant δ is finite. For this reason, in our approach, we will require that the space $W^1L_\Phi(\Omega)$ is embedded in $C^0(\overline{\Omega})$.

Finally we point out that, by using a similar approach, for the p-Laplacian operator, the existence of a well determined open interval of positive parameters for which the problem ($N^f_{\alpha,\lambda}$) admits infinitely many weak solutions in $W^{1,p}(\Omega)$, was proved in the recent paper [2].

The plan of the paper is as follows. In Sect. 2 we introduce our notation and the abstract Orlicz–Sobolev spaces setting. Sect. 3 is devoted to main theorem and finally, in Sect. 4, as an application, of the obtained results, we prove that, for every $\lambda > 0$, there exists a sequence of pairwise distinct solutions that strongly converges to zero in $W^1L_\Phi(\Omega)$, for a class of elliptic problems whose prototype is

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right) + \dfrac{|u|^{p-2}}{\log(1+|u|)}u = \lambda f(x,u) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{N^f_λ}$$

where $p > N + 1$.

2 Orlicz–Sobolev spaces setting

This section summarizes those aspects of the theory of Orlicz–Sobolev spaces, which will be needed here. This types of space provides an appropriate venue for the analysis of quasilinear elliptic partial differential equations with rapidly or slowly growing principal parts. Set

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds, \qquad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an N–function. The function $\Phi^\star$ is called the complementary function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \geq 0\}, \qquad \text{for all } t \geq 0.$$

We observe that $\Phi^\star$ is also an N–function and the following Young's inequality holds true:

$$st \leq \Phi(s) + \Phi^\star(t), \qquad \text{for all } s, t \geq 0.$$

Assume that Φ satisfying the following structural hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty; \tag{Φ_0}$$

$$N < p_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{Φ_1}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N–function Φ (see for instance [1] and [7]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_\Omega u(x)v(x)\,dx;\ \int_\Omega \Phi^\star(|v(x)|)\,dx \leq 1\right\} < \infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\,dx \leq 1\right\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem $(N^f_{\alpha,\lambda})$, defined by

$$W^1L_\Phi(\Omega) = \left\{u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \ldots, N\right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi,$$

see [1] and [7].

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$.

Further, one has

Lemma 2.1 *On $W^1L_\Phi(\Omega)$ the norms*

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi,$$
$$\|u\|_{2,\Phi} = \max\{\||\nabla u|\|_\Phi, \|u\|_\Phi\},$$
$$\|u\| = \inf\left\{\mu > 0;\ \int\limits_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right] dx \le 1\right\},$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|.$$

Moreover the following relations hold true

Lemma 2.2 *Let $u \in W^1L_\Phi(\Omega)$. Then*

$$\int\limits_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,dx \ge \|u\|^{p_0}, \qquad \text{if } \|u\| > 1;$$
$$\int\limits_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,dx \ge \|u\|^{p^0}, \qquad \text{if } \|u\| < 1.$$

For a proof of the previous two results see, respectively, Lemma 2.1 and 2.2 of the paper [13].

Moreover, we say that $u \in W^1L_\Phi(\Omega)$ is a weak solution for problem ($N^f_{\alpha,\lambda}$) if

$$\int\limits_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\,dx + \int\limits_\Omega \alpha(|u(x)|)u(x)v(x)\,dx$$
$$-\lambda\int_\Omega f(x,u(x))v(x)\,dx = 0,$$

for every $v \in W^1L_\Phi(\Omega)$.

Finally, the following Lemma will be useful in the sequel.

Lemma 2.3 *Let $u \in W^1L_\Phi(\Omega)$ and assume that*

$$\int\limits_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,dx \le r, \tag{2}$$

for some $0 < r < 1$. Then, one has $\|u\| < 1$.

For the proof see, for instance, [5].

3 Main result

Here and in the sequel "meas(Ω)" denotes the Lebesgue measure of the set Ω. From hypothesis (Φ_1), by Lemma D.2 in [7] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume $p_0 > N$ we deduce that $W^{1,p_0}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, one has that $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\,\|u\|_{1,\Phi}, \quad \forall\, u \in W^1L_\Phi(\Omega), \tag{3}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$. A concrete estimation of a concrete upper bound for the constant c remains an open question.

Let

$$A := \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^0}}, \qquad B := \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p_0}}.$$

Our main result is the following.

Theorem 3.1 *Let $f : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, Φ be a Young function satisfying the structural hypotheses (Φ_0)–(Φ_1) and let ϱ be a positive constant such that*

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^{p_0}} < \varrho. \tag{Φ_ϱ}$$

Further, assume that

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^0}} < \frac{1}{(2c)^{p^0}\varrho\,\mathrm{meas}(\Omega)} \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p_0}}. \tag{h_0}$$

Then, for every λ belonging to

$$\left]\frac{\varrho\,\mathrm{meas}(\Omega)}{B}, \frac{1}{(2c)^{p^0}A}\right[,$$

the problem $(N^f_{\alpha,\lambda})$ admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

Proof Let us put $X := W^1L_\Phi(\Omega)$. Hypothesis (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2-condition (at infinity), see [1, p. 232] and [7]. In particular, both (Φ,Ω) and $(\Phi^\star,\Omega)$ are Δ-regular, see [1, p. 232]. Consequently, the spaces

$L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see Adams [1, p. 241 and p. 247]. Now, define the functionals $J,\ I : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega (\Phi(|\nabla u(x)|) + \Phi(|u(x)|))\,dx \quad \text{and} \quad I(u) = \int_\Omega F(x, u(x))\,dx,$$

where $F(x,\xi) := \int_0^\xi f(x,t)dt$ for every $(x,\xi) \in \overline{\Omega} \times \mathbb{R}$ and put

$$g_\lambda(u) := J(u) - \lambda I(u), \quad u \in X.$$

The functionals J and I satisfy the regularity assumptions of Theorem 1.1. Indeed, similar arguments as those used in [10, Lemma 3.4] and [7, Lemma 2.1] imply that $J,\ I \in C^1(X,\mathbb{R})$ with the derivatives given by

$$\langle J'(u), v\rangle = \int_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \alpha(|u(x)|)u(x)v(x)\,dx,$$
$$\langle I'(u), v\rangle = \int_\Omega f(x,u(x))v(x)\,dx,$$

for any $u, v \in X$.
Moreover, owing that Φ is convex, it follows that J is a convex functional, hence one has that J is sequentially weakly lower semicontinuous. Finally we observe that J is a coercive functional. Indeed, by Lemma 2.2, we deduce that for any $u \in X$ with $\|u\| > 1$ we have $J(u) \geq \|u\|^{p_0}$. On the other hand the fact that X is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $I' : X \to X^\star$ is compact. Consequently, the functional $I : X \to \mathbb{R}$ is sequentially weakly (upper) continuous, see Zeidler [21, Corollary 41.9]. Let us observe that $u \in X$ is a weak solution of problem $(N^f_{\alpha,\lambda})$ if u is a critical point of the functional g_λ. Hence, we can seek for weak solutions of problem $(N^f_{\alpha,\lambda})$ by applying Theorem 1.1. Now, let $\{c_n\}$ be a real sequence such that $\lim_{n\to\infty} c_n = 0$ and

$$\lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\leq c_n} F(x,t)\,dx}{c_n^{p^0}} = A.$$

Put $r_n = \left(\frac{c_n}{2c}\right)^{p^0}$ for all $n \in \mathbb{N}$. Then, by Lemmas 2.3 and 2.2, we have

$$\{v \in W^1L_\Phi(\Omega) : J(v) < r_n\} \subseteq \left\{v \in W^1L_\Phi(\Omega) : \|v\| < \frac{c_n}{2c}\right\}.$$

Due to (3) and Lemma 2.1, we have

$$|v(x)| \leq \|v\|_\infty \leq c\|v\|_{1,\Phi} \leq 2c\|v\| \leq c_n, \quad \forall x \in \overline{\Omega}.$$

Hence

$$\left\{v \in W^1L_\Phi(\Omega) : \|v\| < \frac{c_n}{2c}\right\} \subseteq \left\{v \in W^1L_\Phi(\Omega) : |v| \leq c_n\right\}.$$

Taking into account that $J(u_0) = 0$ and $\int_\Omega F(x, u_0(x))\, dx = 0$, where $u_0(x) = 0$ for all $x \in \Omega$, for all $n \in \mathbb{N}$ one has

$$\varphi(r_n) = \inf_{J(u)<r_n} \frac{\sup_{J(v)<r_n} \int_\Omega F(x, v(x))\, dx - \int_\Omega F(x, u(x))\, dx}{r_n - J(u)} \leq \frac{\sup_{J(v)<r_n} \int_\Omega F(x, v(x))\, dx}{r_n}$$
$$\leq \frac{\int_\Omega \max_{|t|\leq c_n} F(x,t)\, dx}{r_n} = (2c)^{p^0} \frac{\int_\Omega \max_{|t|\leq c_n} F(x,t)\, dx}{c_n^{p^0}}.$$

Therefore, since from the assumption (h_0) one has $A < +\infty$, we obtain

$$\delta \leq \liminf_{n\to\infty} \varphi(r_n) \leq (2c)^{p^0} A < +\infty.$$

Now, take

$$\lambda \in \left] \frac{\varrho\, \mathrm{meas}(\Omega)}{B}, \frac{1}{(2c)^{p^0} A} \right[.$$

At this point we will show that 0, that is the unique global minimum of J, is not a local minimum of g_λ. For this goal, let $\{\zeta_n\}$ be a real sequence of positive numbers such that $\lim_{n\to\infty} \zeta_n = 0$ and

$$\lim_{n\to\infty} \frac{\int_\Omega F(x, \zeta_n)\, dx}{\zeta_n^{p_0}} = B. \tag{4}$$

For each $n \in \mathbb{N}$, put $w_n(x) := \zeta_n$, for all $x \in \Omega$. Clearly $w_n \in W^1L_\Phi(\Omega)$, for each $n \in \mathbb{N}$ and w_n strongly converges to zero. Hence

$$J(w_n) = \int_\Omega (\Phi(|\nabla w_n(x)|) + \Phi(|w_n(x)|))\, dx = \int_\Omega \Phi(\zeta_n)\, dx = \Phi(\zeta_n)\, \mathrm{meas}(\Omega),$$

for every $n \in \mathbb{N}$.

Moreover, from hypothesis (Φ_ϱ), taking into account that $\lim_{n\to\infty} w_n = 0$, one has that there exists $\delta > 0$ and $\nu_0 \in \mathbb{N}$ such that $w_n \in]0, \delta[$ and

$$\Phi(w_n) < \varrho w_n^{p_0},$$

for every $n \geq \nu_0$.

If $B < +\infty$, let $\epsilon \in \left]\dfrac{\varrho \operatorname{meas}(\Omega)}{\lambda B}, 1\right[$. By (4) there exists ν_ϵ such that

$$\int_\Omega F(x, \zeta_n)\, dx > \epsilon B \zeta_n^{p_0}, \quad \forall n > \nu_\epsilon.$$

Hence,

$$\begin{aligned} g_\lambda(w_n) &= J(w_n) - \lambda I(w_n) \leq \varrho w_n^{p_0} \operatorname{meas}(\Omega) - \lambda \epsilon B w_n^{p_0} \\ &= w_n^{p_0}(\varrho \operatorname{meas}(\Omega) - \lambda \epsilon B) < 0, \end{aligned}$$

for every $n \geq \max\{\nu_0, \nu_\epsilon\}$.

On the other hand, if $B = +\infty$ let us consider $M > \dfrac{\varrho \operatorname{meas}(\Omega)}{\lambda}$. By (4) there exists ν_M such that

$$\int_\Omega F(x, \zeta_n)\, dx > M \zeta_n^{p_0}, \quad \forall n > \nu_M.$$

Moreover,

$$\begin{aligned} g_\lambda(w_n) &= J(w_n) - \lambda I(w_n) \leq \varrho w_n^{p_0} \operatorname{meas}(\Omega) - \lambda M w_n^{p_0} \\ &= w_n^{p_0}(\varrho \operatorname{meas}(\Omega) - \lambda M) < 0, \end{aligned}$$

for every $n \geq \max\{\nu_0, \nu_M\}$.

Hence $g_\lambda(w_n) < 0$ for every n sufficiently large. Since $g_\lambda(0) = J(0) - \lambda I(0) = 0$, this means that 0 is not a local minimum of g_λ. Then, owing to J has 0 as unique global minimum, Theorem 1.1 ensures the existence of a sequence $\{v_n\}$ of pairwise distinct critical points of the functional g_λ, such that $\lim_{n\to\infty} J(v_n) = 0$. By Lemma 2.2 we have $\|v_n\|^{p^0} \leq J(v_n)$ for every n sufficiently large. Then $\lim_{n\to\infty} \|v_n\| = 0$ and this completes the proof. □

Remark 3.1 We point out that, by using inequality (3), Theorem 3.1 guarantees the existence of a sequence of weak solutions of problem $(N^f_{\alpha,\lambda})$ that strongly converges to zero in $C^0(\overline{\Omega})$.

Remark 3.2 Let $f(x, t) := h(x)g(t)$, where $h : \overline{\Omega} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions, with $\min_{\overline{\Omega}} h(x) > 0$ and $\min_{\mathbb{R}} g(t) \geq 0$.
Condition (h_0) reads as follows

$$\liminf_{\xi\to 0^+} \frac{G(\xi)}{\xi^{p^0}} < \frac{1}{(2c)^{p^0} \varrho \operatorname{meas}(\Omega)} \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^{p_0}}, \qquad (\mathrm{h}_0')$$

where $G(\xi) := \int_0^\xi g(t)\, dt$.

Hence, putting

$$A^* := \liminf_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p^0}} \quad \text{and} \quad B^* := \limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p_0}},$$

for every

$$\lambda \in \left] \frac{\varrho \, \mathrm{meas}(\Omega)}{\|h\|_{L^1(\Omega)} B^*}, \frac{1}{(2c)^{p^0} \|h\|_{L^1(\Omega)} A^*} \right[,$$

Theorem 3.1 assures that the problem $(N^f_{\alpha,\lambda})$ admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

4 Application

Define

$$\varphi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t \quad \text{for } t \neq 0, \quad \text{and} \quad \varphi(0) = 0.$$

Let $\Phi(t) := \int_0^t \varphi(s)\, ds$ and consider the space $W^1L_\Phi(\Omega)$. By [8, Example 3, p. 243] one has

$$p_0 = p - 1 < p^0 = p = \liminf_{t \to \infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (Φ_0) and (Φ_1) are verified.
Moreover also condition (Φ_ϱ) holds owing to

$$\lim_{t \to 0^+} \frac{1}{t^{p-1}} \int_0^t \frac{s|s|^{p-2}}{\log(1+|s|)}\, ds = \frac{1}{p-1}.$$

From the previous observations, by using Theorem 3.1 and taking into account Remark 3.2, it follows that

Theorem 4.1 *Let $p > N+1$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function with potential $G(\xi) := \int_0^\xi g(t)\, dt$. Moreover, let $h : \overline{\Omega} \to \mathbb{R}$ be a continuous and positive function.*

Assume that

$$\liminf_{\xi \to 0^+} \frac{G(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p-1}} = +\infty. \tag{h$_0''$}$$

Then, for each $\lambda > 0$, *the Neumann problem*

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\right) + \dfrac{|u|^{p-2}}{\log(1+|u|)}u = \lambda h(x)g(u) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{N_λ^{hg}}$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

Following Omari and Zanolin in [18] we construct a concrete example of positive continuous function $g : \mathbb{R} \to \mathbb{R}$ such that its potential G satisfies the growth condition (h_0'') near to zero. Precisely, let $\{s_n\}$, $\{t_n\}$ and $\{\delta_n\}$ be sequences defined by

$$s_n := 2^{-\frac{n!}{2}}, \qquad t_n := 2^{-2n!}, \qquad \delta_n := 2^{-(n!)^2}.$$

Observe that, definitively, one has

$$s_{n+1} < t_n < s_n - \delta_n.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $g(t) = 0$, for every $t \leq 0$, $g(t) > 0$ for every $t > 0$ and

$$g(t) := 2^{-4n!}, \qquad \forall t \in [s_{n+1}, s_n - \delta_n],$$

for n large.
Then

$$\frac{G(s_n)}{s_n^5} \leq \frac{g(s_{n+1})s_n + g(s_n)\delta_n}{s_n^5},$$

and

$$\frac{G(t_n)}{t_n^4} \geq \frac{g(s_{n+1})(t_n - s_{n+1})}{t_n^4}.$$

Owing to

$$\lim_{n\to\infty} \frac{g(s_{n+1})s_n + g(s_n)\delta_n}{s_n^5} = \lim_{n\to\infty} \frac{2^{-\frac{9}{2}n!} + 2^{-4(n-1)!-(n!)^2}}{2^{-\frac{5}{2}n!}} = 0,$$

and

$$\lim_{n\to\infty} \frac{g(s_{n+1})(t_n - s_{n+1})}{t_n^4} = \lim_{n\to\infty} \frac{2^{-2n!} - 2^{-\frac{(n+1)!}{2}}}{2^{-4n!}} = +\infty,$$

it follows that

$$\lim_{n\to\infty} \frac{G(s_n)}{s_n^5} = 0, \qquad \lim_{n\to\infty} \frac{G(t_n)}{t_n^4} = +\infty.$$

Hence condition (h_0'') holds.

Then, let Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and consider the Young function

$$\Phi(t) := \int_0^t \frac{s|s|^3}{\log(1+|s|)}\, ds.$$

From the previous computations Theorem 4.1 ensures that, for each $\lambda > 0$, the following non-homogeneous Neumann problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{|\nabla u|^3}{\log(1+|\nabla u|)}\nabla u\right) + \dfrac{|u|^3}{\log(1+|u|)}u = \lambda h(x)g(u) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \qquad (N_\lambda^{hg})$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

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