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Combined effects of concave–convex nonlinearities and indefinite potential in some elliptic problems

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Abstract. We consider a nonlinear Dirichlet problem driven by the p-Laplacian and a reaction which exhibits the combined effects of concave (that is, sublinear) terms and of convex (that is, superlinear) terms. The concave term is indefinite and the convex term need not satisfy the usual in such cases Ambrosetti–Rabinowitz condition. We prove a bifurcation-type result describing the set of positive solutions as the positive parameter λ varies.

Keywords: indefinite concave term, superlinear perturbation, p-Laplacian, nonlinear regularity, positive solutions

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric elliptic problem

$$\begin{cases} -\Delta_p u(z) = \vartheta(z)u(z)^{q-1} + f\big(z, u(z), \lambda\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{P_λ}$$

where $\lambda > 0$ and $1 < q < p$. Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\big(|Du|^{p-2}Du\big) \quad \text{for all } u \in W_0^{1,p}(\Omega),\ 1 < p < \infty.$$

The perturbation $(z, x) \mapsto f(z, x, \lambda)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x, \lambda)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x, \lambda)$ is continuous), which exhibits $(p-1)$-superlinear growth

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near $+\infty$, without satisfying the usual in such cases (unilateral) Ambrosetti–Rabinowitz condition (*AR*-condition for short). So, in problem (P_λ) we have the combined effects of a concave (that is, of a $(p-1)$-sublinear) nonlinearity which is expressed by the term $\vartheta(z)u^{q-1}$ (recall $1 < q < p$) and of a convex (that is, of a $(p-1)$-superlinear) nonlinearity, expressed by the term $f(z, u, \lambda)$. Hence, we are dealing with a "concave–convex problem". The interesting feature of our work here, is that the concave term $\vartheta(z)u^{q-1}$ is *indefinite*, namely the weight function $\vartheta(\cdot)$ may change sign.

Problems with combined nonlinearities, were first investigated by Ambrosetti, Brezis and Cerami [2], where $p = 2$ (semilinear problem) and the parametric reaction has the form

$$\lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geqslant 0, \text{ with } 1 < q < 2 < r < 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } 2 < N, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

They proved bifurcation type results describing the dependence of the set of positive solutions on the parameter $\lambda > 0$. Their work was extended to nonlinear problems driven by the p-Laplacian, by Garcia Azorero, Manfredi and Peral Alonso [8] and Guo and Zhang [10]. Problems with more general reactions, were studied by Hu and Papageorgiou [11] and Marano and Papageorgiou [14]. Problems with indefinite concave nonlinearities were investigated by de Paiva [6], Li, Wu and Zhou [12], and Papageorgiou and Rădulescu [18] only in the context of semilinear equations (that is, $p = 2$) and with a particular reaction of the form $x \mapsto \vartheta(z)x^{q-1} + \lambda x^{r-1}$ for all $x \geqslant 0$, with $\vartheta \in L^\infty(\Omega)$ and $1 < q < 2 < r < 2^*$. We also refer to the related papers by de Figueiredo, Gossez and Ubilla [5] and Narukawa and Takajo [16].

Using variational methods based on the critical point theory, combined with suitable truncation and comparison techniques, we establish the existence, nonexistence and multiplicity of positive solutions for problem (P_λ) as the parameter $\lambda > 0$ varies.

2. Mathematical background

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\big(1 + \|u_n\|\big)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness type condition on the functional φ, which is needed since the ambient space X need not be locally compact (since, in general X is infinite dimensional). The C-condition is the main tool in proving a deformation theorem, from which one can derive the minimax theory for the critical values of φ. One of the main results in this theory, is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [3], stated here is a slightly more general form (see Gasinski and Papageorgiou [9]).

Theorem 1. *Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,*

$$\max\big\{\varphi(u_0), \varphi(u_1)\big\} < \inf\big[\varphi(u) : \|u - u_0\| = \rho\big] = \eta_\rho$$

and

$$c=\inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1}\varphi\big(\gamma(t)\big)\quad \textit{with } \Gamma=\big\{\gamma\in C\big([0,1],X\big):\ \gamma(0)=u_0,\gamma(1)=u_1\big\}.$$

Then $c\geqslant\eta_\rho$ *and* c *is a critical value of* φ.

In the analysis of problem (P_λ), in addition to the Sobolev space $W_0^{1,p}(\Omega)$ we will also use the Banach space $C_0^1(\overline{\Omega})=\{u\in C^1(\overline{\Omega}):\ u|_{\partial\Omega}=0\}$. This is an ordered Banach space with positive cone $C_+=\{u\in C_0^1(\overline{\Omega}):\ u(z)\geqslant 0 \text{ for all } z\in\overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+=\left\{u\in C_+:\ u(z)>0 \text{ for all } z\in\Omega,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}<0\right\}.$$

Here by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$.

Let $f_0:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function with subcritical growth in $x\in\mathbb{R}$, that is,

$$\big|f_0(z,x)\big|\leqslant a_0(z)\big(1+|x|^{r-1}\big)\quad \text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R},$$

$$\text{with } a_0\in L^\infty(\Omega)_+ \text{ and } 1<r<p^*=\begin{cases}\dfrac{Np}{N-p} & \text{if } p<N,\\ +\infty & \text{if } N\leqslant p.\end{cases}$$

We set $F_0(z,x)=\int_0^x f_0(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\varphi_0: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\varphi_0(u)=\frac{1}{p}\|Du\|_p^p-\int_\Omega F_0\big(z,u(z)\big)\,\mathrm{d}z\quad \text{for all } u\in W_0^{1,p}(\Omega).$$

The next result can be found in Garcia Azero, Manfredi and Peral Alonso [8] and essentially is a consequence of the nonlinear regularity theory of Lieberman [13].

Proposition 2. *Assume that* $u_0\in W^{1,p}(\Omega)$ *is a local* $C_0^1(\overline{\Omega})$*-minimizer of* φ_0*, that is, there exists* $\rho_0>0$ *such that*

$$\varphi_0(u_0)\leqslant\varphi(u_0+h)\quad \textit{for all } h\in C_0^1(\overline{\Omega}) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})}\leqslant\rho_0.$$

Then $u_0\in C_0^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha\in(0,1)$ *and it is also a local* $W_0^{1,p}(\Omega)$*-minimizer of* φ_0*, that is, there exists* $\rho_1>0$ *such that*

$$\varphi_0(u_0)\leqslant\varphi(u_0+h)\quad \textit{for all } h\in W_0^{1,p}(\Omega) \textit{ with } \|h\|\leqslant\rho_1.$$

Hereafter by $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\|=\|Du\|_p\quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega |Du|^{p-2}(Du, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega).$$

The next proposition summarizes the main properties of this map (see, for example, Papageorgiou and Kyritsi [17, p. 314]).

Proposition 3. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), demicontinuous, strictly monotone, hence maximal monotone too and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$).

We recall that the Dirichlet p-Laplacian $(-\Delta_p, W_0^{1,p}(\Omega))$ admits a smallest eigenvalue $\hat{\lambda}_1 > 0$. Sometimes we write $\hat{\lambda}_1(\Omega) > 0$ to emphasize the domain Ω. This eigenvalue is isolated, simple with eigenfunctions of constant sign. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [9, pp. 737–738]), imply that every positive eigenfunction corresponding to $\hat{\lambda}_1 > 0$ belongs in $\operatorname{int} C_+$.

Finally let us fix our notation. So, for $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \qquad u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

Given any measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

(the Nemytski map corresponding to h). Evidently $z \mapsto N_h(u)(z)$ is measurable on Ω. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3. Positive solutions

The hypotheses on the data of problem (P_λ) are the following:

H_1: $\vartheta \in L^\infty(\Omega)$ and if $D_+ = \{z \in \Omega : \vartheta(z) > 0\}$, then $D_+ \neq \emptyset$ and there exists an open set Ω_+ such that $\overline{\Omega}_+ \subseteq D_+$, $\partial\Omega_+$ is C^2 and $\operatorname{ess\,inf}_{\Omega_+} \vartheta = m_+ > 0$.

H_2: $f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a function such that for all $\lambda > 0$, $(z, x) \mapsto f(z, x, \lambda)$ is Carathéodory, $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and

(i) $f(z, x, \lambda) \leqslant a(z, \lambda)(1 + x^{r(\lambda)-1})$ for a.a. $z \in \Omega$ with

$$r(\lambda) \in (p, p^*), \qquad a(\cdot, \lambda) \in L^\infty(\Omega)_+, \qquad \|a(\cdot, \lambda)\|_\infty \to 0 \quad \text{as } \lambda \to 0^+;$$

(ii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda)\,\mathrm{d}s$, then

$$\lim_{x \to +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega$$

and there exist $\tau \in ((r(\lambda) - p)\max\{\frac{N}{p}, 1\}, p^*)$, $\tau > q$ and $\eta_0(\lambda) > 0$ with $\lambda \mapsto \eta_0(\lambda)$ nondecreasing such that

$$\eta_0(\lambda) \leqslant \liminf_{x \to +\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\tau} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) for every $\rho > 0$, there exists $m_\rho(\lambda) > 0$ such that $m_\rho(\lambda) \to +\infty$ as $\lambda \to +\infty$,

$$\inf\bigl[f(z, x, \lambda):\ x \geqslant \rho\bigr] = m_\rho(\lambda) > 0,$$

for a.a. $z \in \Omega$, all $x \geqslant 0$, the map $\lambda \mapsto f(z, x, \lambda)$ is nondecreasing and for every $\xi > 0$, for a.a. $z \in \Omega$, all $x \geqslant \xi$, all $\lambda' > \lambda > 0$, we have $f(z, x, \lambda) \geqslant m_\xi(\lambda)$ with $m_\xi(\cdot)$ nondecreasing, $m_\xi(\lambda) \to +\infty$ as $\lambda \to \infty$ and $f(z, x, \lambda') - f(z, x, \lambda) \geqslant \eta_\xi > 0$;

(iv) for every $\rho > 0$, there exists $\xi_\rho(\lambda) > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z, x, \lambda) + \xi_\rho(\lambda)x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 1. Since we are interested on positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that $f(z, x, \lambda) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$, all $\lambda > 0$. Hypothesis H_2(ii) implies that for a.a. $z \in \Omega$ and all $\lambda > 0$, the perturbation $x \mapsto f(z, x, \lambda)$ is $(p-1)$-superlinear near $+\infty$. However, we do not employ the usual in such cases *AR*-condition. We recall that the *AR*-condition (unilateral version, since we assume that $f(z, x, \lambda) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$, all $\lambda > 0$), says that there exist $\mu = \mu(\lambda) > p$ and $M = M(\lambda) > 0$ such that

(a) $0 < \mu F(z, x, \lambda) \leqslant f(z, x, \lambda)x$ for a.a. $z \in \Omega$, all $x \geqslant M$;
(b) $\operatorname{ess\,inf}_\Omega F(\cdot, M, \lambda) > 0$

(see Ambrosetti and Rabinowitz [3] and Mugnai [15]). Integrating (a) and using (b), we obtain the weaker condition

$$c_1 x^\mu \leqslant F(z, x, \lambda) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant M \text{ and some } c_1 = c_1(\lambda) > 0.$$

Evidently this unilateral growth estimate implies the much weaker condition

$$\lim_{x \to +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Note that our hypothesis H_2(ii) is weaker than the AR-condition. Indeed, suppose that the AR-condition holds (see (a) and (b) above). We may assume that $\mu > (r(\lambda) - p)\max\{\frac{N}{p}, 1\}$. We have

$$\begin{aligned}\frac{f(z,x,\lambda)x - pF(z,x,\lambda)}{x^\mu} &= \frac{f(z,x,\lambda)x - \mu F(z,x,\lambda)}{x^\mu} + (\mu - p)\frac{F(z,x,\lambda)}{x^\mu}\\ &\geqslant (\mu - p)\frac{F(z,x,\lambda)}{x^\mu} \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant M \ \big(\text{see (a)}\big)\\ &\geqslant (\mu - p)c_1 \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant M\\ \Longrightarrow \quad \liminf_{x\to+\infty} \frac{f(z,x,\lambda)x - pF(z,x,\lambda)}{x^\mu} &\geqslant (\mu - p)c_1(\lambda) = \eta_0(\lambda) \quad \text{uniformly for a.a. } z \in \Omega.\end{aligned}$$

So, hypothesis H_2(ii) holds. See the examples that follow for functions which satisfy our hypothesis H_2(ii) but not the AR-condition.

Example 1. The following functions satisfy hypotheses H_2. For the sake of simplicity we drop the z-dependence:

$$\begin{aligned}&f_1(x,\lambda) = \lambda x^{r-1} \quad \text{for all } x \geqslant 0 \text{ with } p < r < p^*,\\ &f_2(x,\lambda) = \xi(\lambda)x^{p-1}\left(\ln(1+x) + \frac{x}{p(x+1)}\right) \quad \text{for all } x \geqslant 0\end{aligned}$$

with $\lambda \mapsto \xi(\lambda)$ strictly increasing on $(0, +\infty)$, $\lim_{\lambda\to\infty} \xi(\lambda) = +\infty$, $\lim_{\lambda\to 0^+} \xi(\lambda) = 0$.

Note that $f_2(\cdot, \lambda)$ does not satisfy the AR-condition.

Let

$$\begin{aligned}&\mathcal{L} = \big\{\lambda > 0\colon \text{ problem } (P_\lambda) \text{ admits a positive solution}\big\},\\ &S(\lambda) = \text{set of positive solutions of problem } (P_\lambda).\end{aligned}$$

First we establish the nonemptiness and a structural property of the set $\mathcal{L}$ of admissible parameters.

Proposition 4. *If hypotheses H_1 and H_2 hold, then $\mathcal{L} \neq \emptyset$, for every $u \in S(\lambda)$ we have $u(z) > 0$ for all $z \in \Omega$ and $\lambda \in \mathcal{L}$ implies $(0, \lambda] \subseteq \mathcal{L}$.*

Proof. We consider the following auxiliary Dirichlet problem

$$-\Delta_p e(z) = 1 \quad \text{in } \Omega,\ e|_{\partial\Omega} = 0. \tag{1}$$

Recalling that A is maximal monotone, strictly monotone and coercive (by virtue of the Poincaré inequality), we see that problem (1) has a unique solution $e \in W_0^{1,p}(\Omega) \setminus \{0\}$ (see, for example, Gasinski and Papageorgiou [9, p. 319]). Acting on (1) with $-e^- \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}&\|De^-\|_p^p = -\int_\Omega e^-\,\mathrm{d}z \leqslant 0\\ &\Longrightarrow \quad e \geqslant 0, \quad e \neq 0.\end{aligned}$$

From the nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [9, pp. 737–738]), we have $e \in \operatorname{int} C_+$.

Claim 1. *There exists $\overline{\lambda} > 0$ such that for all $\lambda \in (0, \overline{\lambda})$, we can find $\xi = \xi(\lambda) > 0$ for which we have*

$$\xi^{q-1}\|\vartheta\|_\infty \|e\|_\infty^{q-1} + \big\|a(\cdot,\lambda)\big\|_\infty\big(1 + \xi^{r-1}\|e\|_\infty^{r-1}\big) < \xi^{p-1}.$$

Arguing by contradiction, suppose that we can find $\lambda_n \downarrow 0$ such that

$$\xi^{p-1} \leqslant \xi^{q-1}\|\vartheta\|_\infty \|e\|_\infty^{q-1} + \big\|a(\cdot,\lambda_n)\big\|_\infty\big(1 + \xi^{r-1}\|e\|_\infty^{r-1}\big)$$

for all $\xi > 0$, all $n \geqslant 1$.

Passing to the limit as $n \to \infty$ and using hypothesis H_2(i), we obtain

$$\xi^{p-q} \leqslant \|\vartheta\|_\infty \|e\|_\infty^{q-1} \quad \text{for all } \xi > 0,$$

a contradiction. This proves the claim.

Let $\overline{u} = \xi e \in \operatorname{int} C_+$. Then we have

$$-\Delta_p \overline{u}(z) = \xi^{p-1} > \vartheta(z)\overline{u}(z)^{q-1} + f\big(z, \overline{u}(z), \lambda\big) \quad \text{for a.a. } z \in \Omega \tag{2}$$

(see Claim 1 and hypothesis H_2(i)).

Fix $\lambda \in (0, \overline{\lambda})$ and consider the following Carathéodory function

$$g_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \vartheta(z)x^{q-1} + f(z,x,\lambda) & \text{if } 0 \leqslant x \leqslant \overline{u}(z), \\ \vartheta(z)\overline{u}(z)^{q-1} + f\big(z,\overline{u}(z),\lambda\big) & \text{if } \overline{u}(z) < x. \end{cases} \tag{3}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G_\lambda\big(z, u(z)\big)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3) it is clear that ψ_λ is coercive. Also, using the Sobolev embedding theorem, we can see that ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda) = \inf\big[\psi_\lambda(u) : \ u \in W_0^{1,p}(\Omega)\big]. \tag{4}$$

Let $u \in C_+ \setminus \{0\}$ with $\operatorname{supp} u \subseteq \Omega_+$. Recall that $\overline{u} \in \operatorname{int} C_+$. Hence, we have that $\overline{u}|_{\overline{\Omega}_+} > 0$ (see hypothesis H_1). Therefore, we can find $t \in (0,1)$ small such that $tu \leqslant \overline{u}$. So, using hypothesis H_2(iii) and (3), we have

$$\psi_\lambda(tu) \leqslant \frac{t^p}{p}\|Du\|_p^p - \frac{t^q}{q}\int_\Omega \vartheta(z)u^q\,\mathrm{d}z.$$

Since $\operatorname{supp} u \subseteq \Omega_+$ and $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we infer that

$$\begin{aligned}
&\psi_\lambda(tu) < 0 \\
&\quad \Longrightarrow \quad \psi_\lambda(u_\lambda) < 0 = \psi_\lambda(0) \quad \big(\text{see } (4)\big), \text{ hence } u_\lambda \neq 0.
\end{aligned}$$

From (4), we have

$$\begin{aligned}
&\psi_\lambda'(u_\lambda) = 0 \\
&\quad \Longrightarrow \quad A(u_\lambda) = N_{g_\lambda}(u_\lambda).
\end{aligned} \tag{5}$$

On (5) first we act with $-u_\lambda^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\|Du_\lambda^-\|_p^p = 0 \quad \big(\text{see } (3)\big) \\
&\quad \Longrightarrow \quad u_\lambda \geqslant 0, u_\lambda \neq 0.
\end{aligned}$$

Also, on (5) we act with $(u_\lambda - \overline{u})^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
\big\langle A(u_\lambda), (u_\lambda - \overline{u})^+\big\rangle &= \int_\Omega g_\lambda(z, u_\lambda)(u_\lambda - \overline{u})^+ \,\mathrm{d}z \\
&= \int_\Omega \big[\vartheta(z)\overline{u}^{q-1} + f(z, \overline{u}, \lambda)\big](u_\lambda - \overline{u})^+ \,\mathrm{d}z \quad \big(\text{see } (3)\big) \\
&\leqslant \int_\Omega \xi^{p-1}(u_\lambda - \overline{u})^+ \,\mathrm{d}z \quad (\text{see Claim } 1) \\
&= \big\langle A(\overline{u}), (u_\lambda - \overline{u})^+\big\rangle \quad \big(\text{see } (1)\big) \\
\Longrightarrow \quad & \int_{\{u_\lambda > \overline{u}\}} \big(|Du_\lambda|^{p-2}Du_\lambda - |D\overline{u}|^{p-2}D\overline{u}, Du_\lambda - D\overline{u}\big)_{\mathbb{R}^N} \,\mathrm{d}z \leqslant 0 \\
\Longrightarrow \quad & \big|\{u_\lambda > \overline{u}\}\big|_N = 0, \quad \text{hence } u_\lambda \leqslant \overline{u}.
\end{aligned}$$

So, we have proved that

$$u_\lambda \in [0, \overline{u}] = \big\{u \in W_0^{1,p}(\Omega):\ 0 \leqslant u(z) \leqslant \overline{u}(z) \text{ for a.a. } z \in \Omega\big\}.$$

This fact and (3) imply that $u_\lambda \in S(\lambda)$. The nonlinear regularity theory (see Lieberman [13]), implies that $u_\lambda \in C_+ \setminus \{0\}$. Invoking Harnack's inequality (see Pucci and Serrin [19, p. 163]), we infer that $u_\lambda(z) > 0$ for all $z \in \Omega$.

Now let $\lambda \in \mathcal{L}$ and let $\mu \in (0, \lambda)$, $u_\lambda \in S(\lambda)$. We have

$$\begin{aligned}
-\Delta_p u_\lambda(z) &= \vartheta(z)u_\lambda(z)^{q-1} + f\big(z, u_\lambda(z), \lambda\big) \\
&\geqslant \vartheta(z)u_\lambda(z)^{q-1} + f\big(z, u_\lambda(z), \mu\big) \quad \text{for } \mu \text{ a.a. } z \in \Omega \\
&\quad \big(\text{see hypothesis } H_2(\text{iii})\big).
\end{aligned} \tag{6}$$

We introduce the Carathéodory function $\gamma_\mu(z,x)$ defined by

$$\gamma_\mu(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \vartheta(z)x^{q-1} + f(z,x,\mu) & \text{if } 0 \leqslant x \leqslant u_\lambda(z), \\ \vartheta(z)u_\lambda(z) + f\big(z,u_\lambda(z),\mu\big) & \text{if } u_\lambda(z) < x. \end{cases} \tag{7}$$

We set $\Gamma_\mu(z,x) = \int_0^x \gamma_\mu(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\tau_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau_\mu(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \Gamma_\mu\big(z,u(z)\big)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (7) it is clear that τ_μ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\tau_\mu(u_\mu) = \inf\big[\tau_\mu(u):\ u \in W_0^{1,p}(\Omega)\big].$$

As before, we can show that

$$\tau_\mu(u_\mu) < 0 = \tau_\mu(0), \quad \text{hence } u_\mu \neq 0.$$

Also, we have

$$\begin{aligned} &\tau'_\mu(u_\mu) = 0 \\ &\quad\Longrightarrow \quad A(u_\mu) = N_{\gamma_\mu}(u_\mu). \end{aligned}$$

Acting with $-u_\mu^- \in W_0^{1,p}(\Omega)$ and with $(u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$, as before, we show that

$$\begin{aligned} &u_\mu \in [0,u_\lambda] = \big\{u \in W_0^{1,p}(\Omega):\ 0 \leqslant u(z) \leqslant u_\lambda(z) \text{ for a.a. } z \in \Omega\big\} \\ &\quad\Longrightarrow \quad u_\mu \in S(\mu) \quad \big(\text{see (7)}\big) \\ &\quad\Longrightarrow \quad \mu \in \mathcal{L} \quad \text{and so } (0,\lambda] \subseteq \mathcal{L}. \end{aligned}$$

This completes the proof. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 5. *If hypotheses H_1 and H_2 hold, then $\lambda^* < \infty$.*

Proof. Let $\hat{\lambda}_1^+ = \hat{\lambda}_1(\Omega_+)$ (see Section 2) and take $\beta > \hat{\lambda}_1^+$.

Claim 2. *There exists $\lambda_0 > 0$ big such that*

$$\vartheta(z)x^{q-1} + f(z,x,\lambda) \geqslant \beta x^{p-1} \quad \textit{for a.a. } z \in \Omega_+,\ \textit{all } x \geqslant 0,\ \textit{all } \lambda \geqslant \lambda_0.$$

Since $q < p$, we can find $\delta > 0$ small such that

$$\vartheta(z)x^{q-1} \geqslant \beta x^{p-1} \quad \text{for a.a. } z \in \Omega_+, \text{ all } 0 \leqslant x \leqslant \delta \text{ (see hypothesis } H_1). \tag{8}$$

Note that hypothesis H_2(ii) implies that for all $\lambda > 0$, we have

$$\lim_{x \to +\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

So, we can find $M = M(\lambda) > 0$ such that

$$f(z, x, \lambda) \geqslant \beta x^{p-1} \quad \text{for a.a. } z \in \Omega \text{ all } x \geqslant M. \tag{9}$$

Let $\delta > 0$ be as in (8) and $\lambda > 0$. By virtue of hypothesis H_2(iii), we have

$$\inf\bigl[f(z, x, \lambda) \colon\ x \geqslant \delta\bigr] = m_\delta(\lambda) > 0,$$

with $\lambda \mapsto m_\delta(\lambda)$ nondecreasing and $m_\delta(\lambda) \to +\infty$ as $\lambda \to +\infty$. So, we can choose $\lambda_0 \geqslant 1$ big such that

$$\begin{aligned} &m_\delta(\lambda) \geqslant \beta M^{p-1} \quad \text{for all } \lambda \geqslant \lambda_0 \\ &\quad \Longrightarrow \quad f(z, x, \lambda) \geqslant \beta x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } \delta \leqslant x \leqslant M, \text{ all } \lambda \geqslant \lambda_0. \end{aligned} \tag{10}$$

Combining (8), (9), (10), we conclude that Claim 2 holds.

Take $\lambda > \lambda_0$ and assume that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S(\lambda)$ such that

$$-\Delta_p u_\lambda(z) = \vartheta(z)u_\lambda(z)^{q-1} + f\bigl(z, u_\lambda(z), \lambda\bigr) \geqslant \beta u_\lambda(z)^{q-1} \quad \text{for a.a. } z \in \Omega_+$$

(see Claim 2).

Let $\hat{u}_+ \in \operatorname{int} C_+(\Omega_+)$ be the principal, L^p-normalized (that is, $\|\hat{u}_+\|_{L^p(\Omega_+)} = 1$) positive eigenfunction for $(-\Delta_p, W_0^{1,p}(\Omega_+))$. From Proposition 4, we know that

$$u_\lambda|_{\overline{\Omega}_+} > 0.$$

So, we can find $t \in (0, 1)$ small such that

$$t\hat{u}_+(z) \leqslant u_\lambda(z) \quad \text{for all } z \in \overline{\Omega}_+. \tag{11}$$

We have

$$\begin{aligned} &-\Delta_p(t\hat{u}_+)(z) = \hat{\lambda}_1^+(t\hat{u}_+)(z)^{p-1} < \beta(t\hat{u}_+)(z)^{p-1} \quad \text{for a.a. } z \in \Omega_+ \\ &\bigl(\text{recall that } \hat{\lambda}_1^+ < \beta\bigr). \end{aligned} \tag{12}$$

We consider the following Carathéodory function

$$k_\beta(z,x)=\begin{cases}\beta(t\hat{u}_+)(z)^{p-1} & \text{if } x<t\hat{u}_+(z),\\ \beta x^{p-1} & \text{if } t\hat{u}_+(z)\leqslant x\leqslant u_\lambda(z),\ (z,x)\in\Omega_+\times\mathbb{R},\\ \beta u_\lambda(z)^{p-1} & \text{if } u_\lambda(z)<x\end{cases}\tag{13}$$

(see (11)).

We set $K_\beta(z,x)=\int_0^x k_\beta(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\chi_\beta : W_0^{1,p}(\Omega_+)\to\mathbb{R}$ defined by

$$\chi_\beta(u)=\frac{1}{p}\|Du\|_{L^p(\Omega_+,\mathbb{R}^N)}-\int_{\Omega_+}K_\beta\bigl(z,u(z)\bigr)\,\mathrm{d}z\quad\text{for all } u\in W_0^{1,p}(\Omega_+).$$

From (13) it is clear that χ_β is coercive and also it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u}_\beta\in W_0^{1,p}(\Omega_+)$ such that

$$\begin{aligned}&\chi_\beta(\tilde{u}_\beta)=\inf\bigl[\chi_\beta(u):\ u\in W_0^{1,p}(\Omega_+)\bigr]\\ &\quad\Longrightarrow\quad \chi_\beta'(\tilde{u}_\beta)=0\\ &\quad\Longrightarrow\quad \hat{A}(\tilde{u}_\beta)=N_{k_\beta}(\tilde{u}_\beta)\quad\bigl(\text{with } \hat{A}=A|_{W_0^{1,p}(\Omega_+)}\bigr).\end{aligned}\tag{14}$$

On (14), first we act with $(t\hat{u}_+-\tilde{u}_\beta)^+\in W_0^{1,p}(\Omega_+)$. Then

$$\begin{aligned}\bigl\langle A(\tilde{u}_\beta),(t\hat{u}_+-\tilde{u}_\beta)^+\bigr\rangle&=\int_{\Omega_+}k_\beta(z,\tilde{u}_\beta)(t\hat{u}_+-\tilde{u}_\beta)^+\,\mathrm{d}z\\ &=\int_{\Omega_+}\beta(t\hat{u}_+)^{p-1}(t\hat{u}_+-\tilde{u}_\beta)^+\,\mathrm{d}z\\ &\geqslant\int_{\Omega_+}\hat{\lambda}_1^+(t\hat{u}_+)^{p-1}(t\hat{u}_+-\tilde{u}_\beta)^+\,\mathrm{d}z\quad\bigl(\text{see (12)}\bigr)\\ &=\bigl\langle\hat{A}(t\hat{u}_+),(t\hat{u}_+-\tilde{u}_\beta)^+\bigr\rangle\\ \Longrightarrow\quad&\bigl\langle\hat{A}(t\hat{u}_+)-\hat{A}(\tilde{u}_\beta),(t\hat{u}_+-\tilde{u}_\beta)^+\bigr\rangle\leqslant 0\\ \Longrightarrow\quad&\bigl|\{t\hat{u}_+>\tilde{u}_\beta\}_{\Omega_+}\bigr|_N=0,\quad\text{hence } t\hat{u}_+\leqslant\tilde{u}_\beta \text{ in } \Omega_+.\end{aligned}$$

Next, on (14) we act with $(\tilde{u}_\beta-\tilde{u}_\lambda)^+\in W_0^{1,p}(\Omega_+)$ where $\tilde{u}_\lambda=u_\lambda|_{\Omega_+}$ (recall that $u_\lambda|_{\overline{\Omega}_+}>0$). We have

$$\begin{aligned}&\bigl\langle\hat{A}(\tilde{u}_\beta),(\tilde{u}_\beta-\tilde{u}_\lambda)^+\bigr\rangle\\ &\quad=\int_{\Omega_+}k_\beta(z,\tilde{u}_\beta)(\tilde{u}_\beta-\tilde{u}_\lambda)^+\,\mathrm{d}z\\ &\quad=\int_{\Omega_+}\beta\tilde{u}_\lambda^{p-1}(\tilde{u}_\beta-\tilde{u}_\lambda)^+\,\mathrm{d}z\end{aligned}$$

$$\leqslant \int_{\Omega_+} \bigl[\vartheta(z)\tilde{u}_\lambda^{q-1} + f(z, \tilde{u}_\lambda, \lambda)\bigr](\tilde{u}_\beta - \tilde{u}_\lambda)^+ \, \mathrm{d}z \quad \text{(see Claim 2)}$$
$$= \bigl\langle \hat{A}(\tilde{u}_\beta) - \hat{A}(\tilde{u}_\lambda), (\tilde{u}_\beta - \tilde{u}_\lambda)^+ \bigr\rangle \quad \bigl(\text{since } u_\lambda \in S(\lambda)\bigr)$$
$$\Longrightarrow \quad \bigl|\{\tilde{u}_\beta > \tilde{u}_\lambda\}_{\Omega_+}\bigr|_N = 0, \quad \text{hence } \tilde{u}_\beta \leqslant \tilde{u}_\lambda \text{ in } \Omega_+.$$

So, finally we have

$$t\hat{u}_+(z) \leqslant \tilde{u}_\beta(z) \leqslant u_\lambda(z) \quad \text{for all } z \in \overline{\Omega}_+.$$

From (13) and (14) it follows that

$$-\Delta_p \tilde{u}_\beta(z) = \beta \tilde{u}_\beta(z)^{p-1} \quad \text{for a.a. } z \in \Omega_+, \tilde{u}_\beta|_{\Omega_+} = 0, \tilde{u}_\beta \geqslant 0,$$

a contradiction since $\beta > \hat{\lambda}_1^+$ (recall that every nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega_+)))$, has nodal (that is, sign changing) eigenfunctions, see [9].

This means that $\lambda^* \leqslant \lambda_0 < \infty$. □

In what follows, for every $\lambda > 0$, by $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ we denote the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \frac{1}{q}\int_\Omega \vartheta(z)u^+(z)^q \, \mathrm{d}z - \int_\Omega F\bigl(z, u(z), \lambda\bigr) \, \mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$.

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Proposition 6. *If hypotheses H_1 and H_2 hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ such that $\lambda_n \to (\lambda^*)^-$ as $n \to \infty$ and for every $n \geqslant 1$, let $u_n \in S(\lambda_n)$. We may assume that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \geqslant 1. \tag{15}$$

Indeed, if $\lambda < \tilde{\lambda} < \lambda^*$ and $\tilde{u} \in S(\tilde{\lambda})$, then by virtue of hypothesis H_2(iii), we have

$$\begin{aligned} -\Delta_p \tilde{u}(z) &= \vartheta(z)\tilde{u}(z)^{q-1} + f\bigl(z, \tilde{u}(z), \tilde{\lambda}\bigr) \\ &\geqslant \vartheta(z)\tilde{u}(z)^{q-1} + f\bigl(z, \tilde{u}(z), \lambda\bigr) \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{16}$$

(see hypothesis H_2(iii)).

Then reasoning as in the proof of Proposition 4, we introduce the following truncation of the reaction of problem (P_λ):

$$w_\lambda(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \vartheta(z)x^{q-1} + f(z, x, \lambda) & \text{if } 0 \leqslant x \leqslant \tilde{u}(z), \\ \vartheta(z)\tilde{u}(z)^{q-1} + f\bigl(z, \tilde{u}(z), \lambda\bigr) & \text{if } \tilde{u}(z) < x. \end{cases} \tag{17}$$

This is a Carathéodory function. We set $W_\lambda(z,x) = \int_0^x w_\lambda(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\hat{\psi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega W_\lambda\bigl(z, u(z)\bigr)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Again, $\hat{\psi}_\lambda$ is coercive (see (17)) and sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda(u_\lambda) = \inf\bigl[\hat{\psi}_\lambda(u):\ u \in W_0^{1,p}(\Omega)\bigr]. \tag{18}$$

As in the proof of Proposition 4, we show that

$$\hat{\psi}_\lambda(u_\lambda) < 0 = \hat{\psi}_\lambda(0), \quad \text{hence } u_\lambda \neq 0.$$

From (18), we have

$$\begin{aligned} &\hat{\psi}_\lambda'(u_\lambda) = 0 \\ &\quad\Longrightarrow \quad A(u_\lambda) = N_{w_\lambda}(u_\lambda). \end{aligned} \tag{19}$$

On (19) first we act with $-u_\lambda^- \in W_0^{1,p}(\Omega)$ and obtain $u_\lambda \geqslant 0$, $u_\lambda \neq 0$ (see (17)). Then we act with $(u_\lambda - \tilde{u})^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} &\bigl\langle A(u_\lambda), (u_\lambda - \tilde{u})^+\bigr\rangle \\ &\quad = \int_{\Omega_+} w_\lambda(z, u_\lambda)(u_\lambda - \tilde{u})^+\,\mathrm{d}z \\ &\quad = \int_{\Omega_+} \bigl[\vartheta(z)\tilde{u}^{q-1} + f(z, \tilde{u}, \lambda)\bigr](u_\lambda - \tilde{u})^+\,\mathrm{d}z \quad \bigl(\text{see (17)}\bigr) \\ &\quad \leqslant \bigl\langle A(\tilde{u}), (u_\lambda - \tilde{u})^+\bigr\rangle \quad \bigl(\text{see (16)}\bigr) \\ &\quad \Longrightarrow \quad \bigl\langle A(u_\lambda) - A(\tilde{u}), (u_\lambda - \tilde{u})^+\bigr\rangle \leqslant 0 \\ &\quad \Longrightarrow \quad \bigl|\{u_\lambda > \tilde{u}\}\bigr|_N = 0, \quad \text{hence } u_\lambda \leqslant \tilde{u}. \end{aligned}$$

So, we have

$$\begin{aligned} &u_\lambda \in [0, \tilde{u}] = \bigl\{u \in W_0^{1,p}(\Omega):\ 0 \leqslant u(z) \leqslant \tilde{u}(z) \text{ for a.a. } z \in \Omega\bigr\} \\ &\quad \Longrightarrow \quad u_\lambda \in S(\lambda) \quad \bigl(\text{see (17)}\bigr) \quad \text{and} \quad \varphi_\lambda(u_\lambda) < 0 \quad (\text{since } \varphi_\lambda|_{[0,\tilde{u}]} = \hat{\psi}_\lambda|_{[0,\tilde{u}]}). \end{aligned}$$

This proves that we can always assume that (15) holds.

From (15) we have

$$\|Du_n\|_p^p - \frac{p}{q}\int_\Omega \vartheta(z)u_n^q\,\mathrm{d}z - \int_\Omega pF(z, u_n, \lambda_n)\,\mathrm{d}z < 0 \quad \text{for all } n \geqslant 1. \tag{20}$$

Also, since $u_n \in S(\lambda_n)$ for all $n \geqslant 1$, we have

$$A(u_n) = \vartheta(z)u_n^{q-1} + N_{f_{\lambda_n}}(u_n), \quad n \geqslant 1, \tag{21}$$

where $f_{\lambda_n}(z,x) = f(z,x,\lambda_n)$. On (21) we act with $u_n \in W_0^{1,p}(\Omega)$ and obtain

$$-\|Du_n\|_p^p + \int_\Omega \vartheta(z)u_n^q \,\mathrm{d}z + \int_\Omega f(z,u_n,\lambda_n)u_n \,\mathrm{d}z = 0 \quad \text{for all } n \geqslant 1. \tag{22}$$

Adding (20) and (22), we obtain

$$\int_\Omega \big[f(z,u_n,\lambda_n)u_n - pF(z,u_n,\lambda_n)\big]\,\mathrm{d}z \leqslant \left(\frac{p}{q} - 1\right)\int_\Omega \vartheta(z)u_n^q \,\mathrm{d}z \quad \text{for all } n \geqslant 1. \tag{23}$$

It is clear that in hypothesis H_2(i) without any loss of generality, we may assume that $\lambda \to r(\lambda)$ and $\lambda \to \|a(\cdot,\lambda)\|_\infty$ are both nondecreasing in $(0,+\infty)$. Then hypotheses H_2(i)(ii), imply that we find $c_2, c_3 > 0$ such that

$$c_3x^\tau - c_4 \leqslant f(z,x,\lambda_n)x - pF(z,x,\lambda_n) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant 0,\ \text{all } n \geqslant 1. \tag{24}$$

Using (24) in (23) and recalling that $\tau > q$ (see hypothesis H_2(ii)), we infer that

$$\{u_n\}_{n\geqslant 1} \subseteq L^\tau(\Omega) \quad \text{is bounded.} \tag{25}$$

It is clear from hypothesis H_2(iii) that without any loss of generality, we may assume that $\tau < r_1 = r(\lambda_1) \leqslant r(\lambda_n)$ for all $n \geqslant 1$ (see hypothesis H_1(i)). Suppose $N \neq p$. Then we can find $t \in (0,1)$ such that

$$\frac{1}{r_1} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{26}$$

Invoking the interpolation inequality (see, for example, Gasinski and Papageorgiou [9, p. 905]), we have

$$\begin{aligned} &\|u_n\|_r \leqslant \|u_n\|_\tau^{1-t}\|u_n\|_{p^*}^t \quad \text{for all } n \geqslant 1 \\ &\quad\Longrightarrow \quad \|u_n\|_r^r \leqslant M_1\|u_n\|^{tr} \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1 \end{aligned} \tag{27}$$

(see (25) and use the Sobolev embedding theorem).

From hypothesis H_2(i), we have

$$f(z,x,\lambda_n)x \leqslant c_4\big(1 + x^r\big) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant 0,\ \text{some } c_4 > 0.$$

From (22), (25) and since $\tau > q$ (see H_2(ii)) and $\vartheta \in L^\infty(\Omega)$ (see H_1)

$$\begin{aligned} \|Du_n\|_p^p &\leqslant c_5\big(1 + \|u_n\|_r^r\big) \quad \text{for some } c_5 > 0,\ \text{all } n \geqslant 1 \\ &\leqslant c_6\big(1 + \|u_n\|^{tr}\big) \quad \text{for some } c_6 > 0,\ \text{all } n \geqslant 1 \ \big(\text{see (27)}\big). \end{aligned} \tag{28}$$

From (26) and our hypothesis on τ (see H_2(iii)), it follows that $tr < p$. Hence from (28) we infer that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.}$$

If $N = p$, then since $p^* = +\infty$ and $W_0^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for all $q \in [1, \infty)$, then the above argument works, if we replace p^* by $\eta > r > p$ big. Then again we reach the same conclusion.

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r_*}(\Omega)\ \big(r_* = r(\lambda^*)\big) \text{ as } n \to \infty. \tag{29}$$

On (21) we act with $u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (29). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0$$
$$\implies \quad u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \tag{30}$$

(see Proposition 3).

Passing to the limit as $n \to \infty$ in (21) and using (30), we obtain

$$A(u) = \vartheta(z)u^{q-1} + N_{f_{\lambda^*}}(u). \tag{31}$$

We need to show that $u \neq 0$, because then $u \in S(\lambda^*)$, that is $\lambda^* \in \mathcal{L}$. To this end, we consider the following auxiliary Dirichlet problem

$$-\Delta_p u(z) = \vartheta(z)u(z)^{q-1} \quad \text{in } \Omega_+, \qquad u|_{\partial\Omega_+} = 0, \quad u > 0. \tag{32}$$

Since $q < p$, a straightforward application of the direct method (as before), establishes that problem (32) admits a nontrivial solution $\overline{u} \in W_0^{1,p}(\Omega)$, $\overline{u} \geqslant 0$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [9, pp. 737–738]), imply that $\overline{u} \in \operatorname{int} C_+$. Moreover, Theorem 2 of Diaz and Saa [7], implies that $\overline{u} \in \operatorname{int} C_+$ is the unique positive solution of (32).

Now, let $\lambda \in \mathcal{L}$ and $u_\lambda \in S(\lambda)$. We introduce the following Carathéodory function

$$j(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \vartheta(z)x^{q-1} & \text{if } 0 \leqslant x \leqslant u_\lambda(z),\ (z,x) \in \Omega_+ \times \mathbb{R}, \\ \vartheta(z)u_\lambda(z)^{q-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{33}$$

Let $J(z,x) = \int_0^x j(z,s)\,\mathrm{d}s$ and consider the C^1-functional $J : W_0^{1,p}(\Omega_+) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\|Du\|_{L^p(\Omega_+,\mathbb{R}^N)}^p - \int_{\Omega_+} J\big(z, u(z)\big)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p}(\Omega_+).$$

As before, (33) implies that $J(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega_+)$ such that

$$J(\tilde{u}) = \inf\bigl[J(u):\ u \in W_0^{1,p}(\Omega_+)\bigr]. \tag{34}$$

Since $q < p$ and $u_\lambda|_{\overline{\Omega}_+} > 0$ (recall $u_\lambda \in \operatorname{int} C_+$), as in previous similar cases, we have

$$J(\tilde{u}) < 0 = J(0), \quad \text{hence } \tilde{u} \neq 0.$$

From (34), we have

$$\begin{aligned} &J'(\tilde{u}) = 0, \\ &\quad\Longrightarrow\quad A(\tilde{u}) = N_j(\tilde{u}). \end{aligned} \tag{35}$$

On (35) we act with $-\tilde{u}^- \in W_0^{1,p}(\Omega_+)$ and with $(\tilde{u} - u_\lambda)^+ \in W_0^{1,p}(\Omega_+)$ (here we use the fact that $u_\lambda|_{\overline{\Omega}_+} > 0$) and we obtain

$$\begin{aligned} &0 < \tilde{u}(z) \leqslant u_\lambda(z) \quad \text{for all } z \in \overline{\Omega}_+, \tilde{u} \neq 0 \\ &\quad\Longrightarrow\quad \tilde{u} \quad \text{is a positive solution of (32)} \\ &\quad\Longrightarrow\quad \tilde{u} = \overline{u} \quad \bigl(\text{recall that } \overline{u} \text{ is the unique positive solution of (32)}\bigr) \\ &\quad\Longrightarrow\quad \overline{u} \leqslant u_\lambda \quad \text{in } \Omega_+. \end{aligned}$$

So, we have

$$\begin{aligned} &\overline{u}(z) \leqslant u_n(z) \quad \text{for all } z \in \overline{\Omega}_+, \ \text{all } n \geqslant 1 \\ &\quad\Longrightarrow\quad \overline{u}(z) \leqslant u(z) \quad \text{for all } z \in \Omega_+ \ \bigl(\text{see (30)}\bigr) \\ &\quad\Longrightarrow\quad u \neq 0. \end{aligned}$$

Therefore, $u \in S(\lambda^*)$ and so $\lambda^* \in \mathcal{L}$. $\square$

Next, we look for additional positive solutions for problem (P_λ). To this end, we consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) = \vartheta^+(z)u(z)^{q-1} + f\bigl(z, u(z), \lambda\bigr) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{Au_λ}$$

Reasoning as in the proofs of Propositions 4, 5 and 6 (with Ω_+ replaced by Ω) we obtain the following proposition.

Proposition 7. *If hypotheses H_1 and H_2 hold, then there exists $\lambda_0^* \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda_0^*]$ problem (Au_λ) has at least one positive solution $\tilde{u}_\lambda \in \operatorname{int} C_+$.*

Remark 2. Note that in this case, the solution $\tilde{u}_\lambda \in C_+ \setminus \{0\}$ satisfies

$$-\Delta_p \tilde{u}_\lambda(z) = \vartheta^+(z)\tilde{u}_\lambda(z)^{q-1} + f\big(z, \tilde{u}_\lambda(z), \lambda\big) \geqslant 0 \quad \text{for a.a. } z \in \Omega \ \big(\text{see hypotheses } H_2(\text{iii})\big)$$
$$\Longrightarrow \quad \tilde{u}_\lambda \in \operatorname{int} C_+ \quad (\text{see Gasinski and Papageorgiou [9, p. 738]}).$$

We can use Proposition 7, to produce a multiplicity result for the positive solutions of problem (P_λ) with $\lambda \in (0, \lambda_0^*]$.

Proposition 8. *If hypotheses H_1 and H_2 hold and $\lambda \in (0, \lambda_0^*]$, then problem (P_λ) has at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in C_+ \setminus \{0\}, \qquad u_\lambda \neq \hat{u}_\lambda,$$
$$0 < u_\lambda(z) \leqslant \hat{u}_\lambda(z) \quad \textit{for all } z \in \Omega.$$

Proof. Let $\mu \in (\lambda, \lambda_0^*)$. From Proposition 4, we know that $\lambda, \mu \in \mathcal{L}$ and we can find $u_\lambda \in S(\lambda) \subseteq C_+$ and $\hat{u}_\lambda \in \operatorname{int} C_+$ solution of (Au_μ). We claim that we can have

$$u_\lambda \leqslant \tilde{u}_\mu. \tag{36}$$

Indeed note that

$$\begin{aligned} -\Delta_p \tilde{u}_\mu(z) &= \vartheta^+(z)\tilde{u}_\mu(z)^{q-1} + f\big(z, \tilde{u}_\mu(z), \mu\big) \\ &\geqslant \vartheta(z)\tilde{u}_\mu(z)^{q-1} + f\big(z, \tilde{u}_\mu(z), \lambda\big) \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{37}$$

(see hypothesis H_2(iii)).

We consider the following truncation of the reaction of problem (P_λ).

$$\hat{g}_\lambda(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \vartheta(z)x^{q-1} + f(z, x, \lambda) & \text{if } 0 \leqslant x \leqslant \tilde{u}_\mu(z), \\ \vartheta(z)\tilde{u}_\mu(z)^{q-1} + f\big(z, \tilde{u}_\mu(z), \lambda\big) & \text{if } \tilde{u}_\mu(z) < x. \end{cases} \tag{38}$$

This is a Carathéodory function.

We set $\hat{G}_\lambda(z, x) = \int_0^x \hat{g}_\lambda(z, s)\,\mathrm{d}s$ and introduce the C^1-functional $\hat{\psi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \hat{G}_\lambda\big(z, u(z)\big)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\hat{\psi}_\lambda$ is coercive (see (38)) and sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda(u_\lambda) = \inf\big[\hat{\psi}_\lambda(u) : \ u \in W_0^{1,p}(\Omega)\big]. \tag{39}$$

Since $q < p$, as before (see, for example, the proof of Proposition 4), we have

$$\hat{\psi}_\lambda(u_\lambda) < 0 = \hat{\psi}_\lambda(0), \quad \text{hence } u_\lambda \neq 0.$$

From (39) we have

$$\begin{aligned} &\hat{\psi}'_\lambda(u_\lambda) = 0 \\ &\implies \quad A(u_\lambda) = N_{\hat{g}_\lambda}(u_\lambda). \end{aligned} \tag{40}$$

On (40) we act with $-u_\lambda^- \in W_0^{1,p}(\Omega)$ and with $(u_\lambda - \tilde{u}_\mu)^+ \in W_0^{1,p}(\Omega)$. As in the proof of Proposition 4, using this time (37), we show that

$$u_\lambda \in [0, \tilde{u}_\mu] = \{u \in W_0^{1,p}(\Omega):\ 0 \leqslant u(z) \leqslant \tilde{u}_\mu(z) \text{ for a.a. } z \in \Omega\}.$$

Hence $u_\lambda \in S(\lambda) \subseteq C_+$ (see (38)) and $u_\lambda(z) > 0$ for all $z \in \Omega$ (by Harnack's inequality, see Pucci and Serrin [19, p. 163]). Therefore (36) holds.

We introduce the following truncation of the reaction of problem (P_λ):

$$k_\lambda(z, x) = \begin{cases} \vartheta(z) u_\lambda(z)^{q-1} + f(z, u_\lambda(z), \lambda) & \text{if } x \leqslant u_\lambda(z), \\ \vartheta(z) x^{q-1} + f(z, x, \lambda) & \text{if } u_\lambda(z) < x. \end{cases} \tag{41}$$

This is a Carathéodory function.

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s)\, ds$ and consider the C^1-functional $\tau_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau_\lambda(u) = \frac{1}{p} \|Du\|_p^p - \int_\Omega K_\lambda(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From the proof of Proposition 6, we know that

$$\tau_\lambda \text{ satisfies the } C\text{-condition.} \tag{42}$$

We truncate $k_\lambda(z, \cdot)$ as follows:

$$\hat{k}_\lambda(z, x) = \begin{cases} k_\lambda(z, x) & \text{if } x \leqslant \tilde{u}_\mu(z), \\ k_\lambda(z, \tilde{u}_\mu(z)) & \text{if } \tilde{u}_\mu(z) < x. \end{cases} \tag{43}$$

This too is a Carathéodory function. We set $\hat{K}_\lambda(z, x) = \int_0^x \hat{k}_\lambda(z, s)\, ds$ and consider the C^1-functional $\hat{\tau}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\tau}_\lambda(u) = \frac{1}{p} \|Du\|_p^p - \int_\Omega \hat{K}_\lambda(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (43) it is clear that $\hat{\tau}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\tau}_\lambda(\overline{u}_\lambda) = \inf\bigl[\hat{\tau}_\lambda(u):\ u \in W_0^{1,p}(\Omega)\bigr] \\ &\quad\Longrightarrow \quad \hat{\tau}_\lambda'(\overline{u}_\lambda) = 0 \\ &\quad\Longrightarrow \quad A(\overline{u}_\lambda) = N_{\hat{k}_\lambda}(\overline{u}_\lambda). \end{aligned} \tag{44}$$

On (44) we act, first with $(u_\lambda - \overline{u}_\lambda) \in W_0^{1,p}(\Omega)$ and then with $(\overline{u}_\lambda - \tilde{u}_\mu)^+ \in W_0^{1,p}(\Omega)$. As in the proof of Proposition 5, we show that

$$\overline{u}_\lambda \in [u_\lambda, \tilde{u}_\mu] = \bigl\{u \in W_0^{1,p}(\Omega):\ u_\lambda(z) \leqslant u(z) \leqslant \tilde{u}_\mu(z) \text{ for a.a. } z \in \Omega\bigr\}.$$

If $\overline{u}_\lambda \neq u_\lambda$, then this is the desired second positive solution of problem (P_λ) and $u_\lambda \leqslant \overline{u}_\lambda = \hat{u}_\lambda$.

So, we may assume that $\overline{u}_\lambda = u_\lambda$. Let $\rho = \|\tilde{u}_\mu\|_\infty$ (recall that $\tilde{u} \in \operatorname{int} C_+$, see Proposition 7) and let $\xi_\rho = \xi_\rho(\mu) > 0$ be as postulated by hypothesis H_2(iv) for the perturbation $f(z,\cdot,\mu)$. We have

$$\begin{aligned} &-\Delta_p u_\lambda(z) + \xi_\rho u_\lambda(z)^{p-1} \\ &\quad= \vartheta(z)u_\lambda(z)^{q-1} + f\bigl(z, u_\lambda(z), \lambda\bigr) + \xi_\rho u_\lambda(z)^{p-1} \\ &\quad= \vartheta(z)u_\lambda(z)^{q-1} + f\bigl(z, u_\lambda(z), \mu\bigr) + \xi_\rho u_\lambda(z)^{p-1} + \bigl[f\bigl(z, u_\lambda(z), \lambda\bigr) - f\bigl(z, u_\lambda(z), \mu\bigr)\bigr] \\ &\quad\leqslant \vartheta^+(z)u_\lambda(z)^{q-1} + f\bigl(z, u_\lambda(z), \mu\bigr) + \xi_\rho u_\lambda(z)^{p-1} - \delta_{(\lambda',\lambda)}(z) \\ &\qquad \text{with } \delta_{(\lambda',\lambda)}(z) = f\bigl(z, u_\lambda(z), \mu\bigr) - f\bigl(z, u_\lambda(z), \lambda\bigr) \\ &\quad\leqslant \vartheta^+(z)\tilde{u}_\mu(z) + f\bigl(z, \tilde{u}_\mu(z), \mu\bigr) + \xi_\rho \tilde{u}_\mu(z)^{p-1} \quad \bigl(\text{see hypothesis } H_2(\text{iv})\bigr) \\ &\quad= -\Delta_p \tilde{u}_\mu(z) + \xi_\rho \tilde{u}_\mu(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{45}$$

For every $K \subseteq \Omega$ compact, we have $u_\lambda|_K \geqslant \xi > 0$ (recall $u_\lambda(z) > 0$ for all $z \in \Omega$, see Proposition 4). So, by hypothesis H_2(iii) we have

$$\delta_{\lambda,\mu}|_K \geqslant m_\xi > 0.$$

Then from (45) and Proposition 2.6 of Arcoya and Ruiz [4] (recall that $\tilde{u}_\mu \in \operatorname{int} C_+$), we infer that

$$\tilde{u}_\mu - u_\lambda \in \operatorname{int} C_+. \tag{46}$$

We claim that u_λ is a local $C_0^1(\overline{\Omega})$-minimizer of the functional τ_λ. Indeed, if this is not the case, we can find $\{u_n\}_{n\geqslant 1} \subseteq C_0^1(\overline{\Omega})$ such that

$$u_n \to u_\lambda \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{and} \quad \tau_\lambda(u_n) < \tau_\lambda(u_\lambda) \quad \text{for all } n \geqslant 1. \tag{47}$$

From (46) and (47) it follows that we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &u_n \leqslant \tilde{u}_\mu \quad \text{for all } n \geqslant n_0 \\ &\quad\Longrightarrow \quad u_n^+ \leqslant \tilde{u}_\mu \quad \text{for all } n \geqslant n_0 \text{ (recall } \tilde{u}_\mu \in \operatorname{int} C_+). \end{aligned}$$

Also, we have

$$\hat{\tau}_\lambda\big(u_n^+\big) = \tau_\lambda\big(u_n^+\big) \leqslant \tau_\lambda(u_n) < \tau_\lambda(u_\lambda) \quad \text{for all } n \geqslant n_0$$

which contradicts the fact that $u_\lambda = \overline{u}_\lambda$ is a global minimizer of $\hat{\tau}_\lambda$.

Since u_λ is a local $C_0^1(\overline{\Omega})$-minimizer of τ_λ. We may assume that u_λ is an isolated critical point of τ_λ, or otherwise we already have a sequence of distinct positive solutions, since the critical set of τ_λ is in $[u_\lambda) = \{u \in W_0^{1,p}(\Omega)\colon\ u_\lambda(z) \leqslant u(z)$ for a.a. $z \in \Omega\}$ (see (41)). Therefore, we can find $\rho \in (0, 1)$ small such that

$$\tau_\lambda(u_\lambda) < \inf\big[\tau_\lambda(u)\colon\ \|u - u_\lambda\| = \rho\big] = m_\lambda \tag{48}$$

(see Aizicovici, Papageorgiou and Staicu [1, Proof of Proposition 29]).

Moreover, by virtue of hypothesis H_2(ii), if $u \in \operatorname{int} C_+$, then

$$\tau_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{49}$$

Because of (42), (48) and (49), we can apply Theorem 1 (the mountain pass theorem) and find $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\tau_\lambda'(\hat{u}_\lambda) = 0 \quad \text{and} \quad m_\lambda \leqslant \tau(\hat{u}_\lambda). \tag{50}$$

From (49), (50) and since the critical set of τ_λ is in $[u_\lambda)$, it follows that

$$\hat{u}_\lambda \in S(\lambda) \subseteq C_+ \quad \text{and} \quad \hat{u}_\lambda \neq u_\lambda,\ u_\lambda \leqslant \hat{u}_\lambda.$$

Finally from Proposition 4, we have

$$0 < u_\lambda(z) \leqslant \hat{u}_\lambda(z) \quad \text{for all } z \in \Omega.$$

The proof is now complete. □

So, summarizing the situation for problem (P_λ), we can state the following result describing the set of positive solutions as the parameter $\lambda > 0$ varies.

Theorem 9. *If hypotheses H_1 and H_2 hold, then*

(a) *there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*]$ problem (P_λ) has at least one positive solution $u_\lambda \in C_+$ with $u_\lambda(z) > 0$ for all $z \in \Omega$ and for $\lambda > \lambda^*$ there are no positive solutions*;

(b) *there exists $\lambda_0^* \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda_0^*)$ problem (P_λ) has at least two positive solutions*

$$u_\lambda,\ \hat{u}_\lambda \in C_+,\ u_\lambda \leqslant \hat{u}_\lambda,\ u_\lambda \neq \hat{u}_\lambda,\ u_\lambda(z) > 0 \quad \textit{for all } z \in \Omega.$$

Remark 3. It will be interesting to know if $\lambda_0^* = \lambda^*$. Also, it is not clear to us if this result can be extended to Neumann problems. A careful inspection of the proofs reveals that they fail in the Neumann case.

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References

[1] S. Aizicovici, N.S. Papageorgiou and V. Staicu, *Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints*, Memoirs Amer. Math. Soc., Vol. 196, 2008.
[2] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.
[3] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
[4] D. Arcoya and D. Ruiz, The Ambrosetti–Prodi problem for the p-Laplace operator, *Commun. Partial Diff. Equations* **31** (2006), 849–865.
[5] D.G. de Figueiredo, J.-P. Gossez and P. Ubilla, Local "superlinearity" and "sublinearity" for the p-Laplacian, *J. Funct. Anal.* **257** (2009), 721–752.
[6] F.O. de Paiva, Nonnegative solutions of elliptic problems with sublinear indefinite nonlinearity, *J. Funct. Anal.* **261** (2011), 2569–2586.
[7] J.I. Diaz and J.E. Saa, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, *C. R. Acad. Sci. Paris* **305** (1987), 521–524.
[8] J. Garcia Azero, J. Manfredi and I. Peral Alonso, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp. Math.* **2** (2000), 385–404.
[9] L. Gasinski and N.S. Papageorgiou, *Nonlinear Analysis*, Chapman and Hall/CRC, Boca Raton, FL, 2006.
[10] Z. Guo and Z. Zhang, $W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations, *J. Math. Anal. Appl.* **286** (2003), 32–50.
[11] S. Hu and N.S. Papageorgiou, Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin, *Tohoku Math. J.* **62** (2010), 137–162.
[12] S. Li, S. Wu and H. Zhou, Solutions to semilinear elliptic problems with combined nonlinearities, *J. Differential Equations* **185** (2002), 200–224.
[13] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), 1203–1219.
[14] S. Marano and N.S. Papageorgiou, Positive solutions to a Dirichlet problem with p-Laplacian and concave–convex nonlinearity depending on a parameter, *Commun. Pure Appl. Anal.* **12** (2013), 815–829.
[15] D. Mugnai, Addendum to multiplicity of critical points in presence of linking: Applications to a superlinear boundary value problem, *NoDEA (Nonlin. Differential Equations Appl.)* **11** (2004), 379–391; and A comment on the generalized Ambrosetti–Rabinowitz condition, *NoDEA (Nonlin. Differential Equations Appl.)* **19** (2011), 299–301.
[16] K. Narukawa and Y. Takajo, Existence of nonnegative solutions for quasilinear elliptic equations with indefinite critical nonlinearities, *Nonlinear Anal.* **74** (2011), 5793–5813.
[17] N.S. Papageorgiou and S. Kyritsi, *Handbook of Applied Analysis*, Springer, New York, 2009.
[18] N.S. Papageorgiou and V.D. Rădulescu, Multiple solutions for elliptic equations with sign changing weight, *Kyoto J. Math.*, to appear.
[19] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.