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Multi-bump solutions for the nonlinear magnetic Choquard equation with deepening potential well

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Abstract

In this paper, using variational methods, we study multiplicity of multi-bump solutions for the following nonlinear magnetic Choquard equation

$$\begin{cases} -(\nabla + \mathrm{i}A(x))^2 u + (\lambda V(x) + 1)u = (\frac{1}{|x|^\mu} * |u|^p)|u|^{p-2}u \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $N \geq 2$, $\lambda > 0$ is a real parameter, $0 < \mu < 2$, i is the imaginary unit, $p \in (2, 2^*(\frac{2(N-\mu)}{2N}))$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = +\infty$, if $N = 2$. The magnetic potential $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function. We show that if the zero set of V has several isolated connected components $\Omega_1, \cdots, \Omega_k$ such that the interior of Ω_j is non-empty and $\partial\Omega_j$ is smooth, then for $\lambda > 0$ large enough, the above equation has at least $2^k - 1$ multi-bump solutions.

MSC: 35J60; 35J20

Keywords: Nonlinear Choquard equation; Magnetic field; Multi-bump solution; Variational methods

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1. Introduction and main results

In this paper, we consider the existence and multiplicity of multi-bump solutions for the following nonlinear magnetic Choquard equation with deepening potential well

$$\begin{cases} -(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u = (\frac{1}{|x|^\mu} * |u|^p)|u|^{p-2}u \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{1.1}$$

where $N \geq 2$, $0 < \mu < 2$, i is the imaginary unit, $\lambda > 0$ is a real parameter, the magnetic potential $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function, $p \in (2, 2^*(\frac{2(N-\mu)}{2N}))$, where $2^* = \frac{2N}{N-2}$ if $N \geq 2$, $2^* = +\infty$, if $N = 2$.

The existence and multiplicity of solutions for the nonlinear Schrödinger with deepening potential well and without magnetic field (that is, if $A = 0$)

$$-\Delta u + (\lambda V(x) + Z(x))u = f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

has been intensively studied. Here, $\lambda > 0$, $V(x)$ and $Z(x)$ satisfies some assumptions. In [18], if $f(t) = t^p$, $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$ and $p \in (1, \infty)$ if $N = 1, 2$, Ding and Tanaka showed problem (1.2) has at least $2^k - 1$ positive multi-bump solutions for λ large enough. After that, for the critical growth case, Alves et al. [2] studied the existence of positive multi-bump solutions for problem (1.2) and $N \geq 3$. For the case $N = 2$, when f has exponential critical growth, Alves and Souto [10] obtained the same results. Moreover, these solutions found in [2] and [10] have the same asymptotic characteristics of those found in [18]. For the nonlocal problems with deepening potential well, Alves and Figueiredo [3] considered the following Kirchhoff problem

$$\begin{cases} M\Big(\int\limits_{\mathbb{R}^3} |\nabla u|^2 dx + \int\limits_{\mathbb{R}^3} (\lambda a(x) + 1)u^2 dx\Big)(-\Delta u + (\lambda a(x) + 1)u) = f(u) \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

Assuming that the nonnegative function $a(x)$ has a potential well with $\mathrm{int}(a^{-1}(0))$ consisting of k disjoint components $\Omega_1, \Omega_2, \cdots, \Omega_k$ and the nonlinearity $f(t)$ has a subcritical growth, they established the existence and multiplicity of positive multi-bump solutions by using variational methods. In [11], Alves and Yang studied the existence of positive multi-bump solutions for the Schrödinger-Poisson system in $\mathbb{R}^3$ with deepening potential well. Recently, Alves et al. [9] studied the existence of multi-bump solutions for the Choquard equation as follows

$$-\Delta u + (\lambda V(x) + 1)u = (\frac{1}{|x|^\mu} * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$

where $\mu \in (0, 3)$, $p \in (2, 6 - \mu)$, the nonnegative continuous function $V(x)$ has a potential well. In previous researches, by using penalization method developed by del Pino and Felmer [16], the researchers were able to overcome the loss of compactness. In [9], the authors avoided the penalization method in [16], because by using this method they are led to assume more restrictions on the constants μ and p. For this reason, by imposing one more condition (see condition

(V) below) on the potential $V(x)$, the authors followed the approach explored by Alves and Nobrega in [8] which showed the existence of multi-bump solution for problem (1.2) driven by the biharmonic operator. For further research about the nonlinear Schrödinger equations with the deepening potential well, we refer to [1], [6], [7], [12], [17], [23], [34] and the references therein.

This paper is motivated by several recent works that appeared in recent years concerning Schrödinger-type equations with magnetic field. For instance, in the local framework, this equation is

$$-(\nabla + iA)^2 u + V(x)u = f(|u|^2)u, \tag{1.3}$$

where the magnetic Schrödinger operator is defined as

$$-(\nabla + iA)^2 u = -\Delta u - 2iA \cdot \nabla u - iu\,\mathrm{div}A + |A|^2 u.$$

As stated in [39], up to correcting the operator by the factor $(1-s)$, it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - iA)^2 u$ in the limit $s \uparrow 1$, where $(-\Delta)^s_A$ is the fractional magnetic operator whose definition can refer to [39]. Thus, up to normalization, we may think the nonlocal case as an approximation of the local case. If $A \equiv 0$, then (1.3) becomes the fractional Schrödinger equation, which was proposed by Laskin [30,31] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. If the interaction between the particles is considered, that is, if $f(u) = (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u$, this kind of problems is usually named Choquard equations.

The nonlinear magnetic Schrödinger equations have been extensively investigated by many authors applying suitable variational and topological methods (see [4,13–15,19,20,24,28,29,37,40] and references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [20]. They used the concentration-compactness principle and minimization arguments to obtain solutions for a local equation for $N = 2, 3$ and $\epsilon > 0$ small. In particular, due to our scope, we would like to mention [4] where the authors use the penalization method and Ljusternik-Schnirelmann category theory to prove the multiplicity and concentration results of solutions for the nonlinear Schrödinger equation with magnetic field

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $f \in C^1$ has the subcritical growth. We point out that if f is only continuous, then the arguments developed in [4] fail. In [25], Ji and Rădulescu used the method of the Nehari manifold, the penalization technique and Ljusternik-Schnirelmann category theory to study the multiplicity and concentration results for the above nonlinear magnetic Schrödinger equation in which the subcritical nonlinearity f is only continuous. After that, Ji and Radulescu [26] continued to study multiplicity and concentration of the solutions for the magnetic Schrödinger equation with critical growth. Tang [37] considered multi-bump solutions of the following problem with critical frequency in which $Z(x) \equiv 0$ and f satisfies subcritical growth

$$-(\nabla + \mathrm{i}A(x))^2 u + (\lambda V(x) + Z(x))u = f(|u|^2)u \quad \text{in } \mathbb{R}^N.$$

Then, Liang and Shi [33] considered multi-bump solutions to the above problem with critical nonlinearity for the case $N \geq 3$. Ji and Rădulescu [27] studied multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$. Ma and Ji [35] considered multi-bump solutions for the Magnetic Schrödinger-Poisson System in $\mathbb{R}^3$. Recently, Alves et al. [5] considered multiple solutions for a nonlinear magnetic Choquard equation by using the penalization method and Ljusternik-Schnirelmann category theory. It is quite natural to consider the multi-bump solutions for the nonlinear Choquard equation with magnetic field and deepening potential well. To the best of our knowledge, this problem has not been studied up to now.

Motivated by [9,18], in the present paper our goal is to prove the existence and multiplicity of multi-bump solutions for problem (1.1). Compared with [9], on one hand, we do not assume the potential $V(x)$ satisfied the following condition:

(V) there exists $M_0 > 0$ such that

$$|\{x \in \mathbb{R}^3 : a(x) \leq M_0\}| < +\infty,$$

which is very important for overcoming the lack of compactness. We shall improve the idea explored by del Pino and Felmer [16] (see also Ding and Tanaka [18]). Since our problem is nonlocal, some estimates are more difficult and complicated. Although our problem has more restrictions on μ and p, we can obtain results completely similar to [18] without the assumption (V), which is better than the results obtained in [9]. On the other hand, as we will see later, due to the appearance of magnetic potential $A(x)$, problem (1.1) cannot be changed into a pure real-valued problem, hence we shall deal with a complex-valued directly, which causes more new difficulties in employing the methods for our problem.

Now we present the general assumptions on the potential V.

- $(V1)$ $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$;
- $(V2)$ The potential well $\Omega = \operatorname{int} V^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and Ω can be decomposed in k connected components $\Omega_1, \cdots, \Omega_k$ with $\operatorname{dist}(\Omega_i, \Omega_j) > 0$, $i \neq j$, $\overline{\Omega} = V^{-1}(0)$.

The main result in this paper is stated below.

Theorem 1.1. *Assume that $(V1)$ and $(V2)$ hold. Then, for any non-empty subset Γ of $\{1, 2, \cdots, k\}$, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, problem (1.1) has a nontrivial solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda \geq \lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges strongly in $H^1_A(\mathbb{R}^N, \mathbb{C})$ to a function u which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$ and the restriction $u|_{\Omega_\Gamma} \in H^{0,1}_A(\Omega_\Gamma)$ is a least energy solution of*

$$-(\nabla + \mathrm{i}A(x))^2 u + u = \Big(\int\limits_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu} dy\Big)|u|^{p-2}u, \quad \text{in } \Omega_\Gamma, \qquad (P)_{\infty,\Gamma}$$

where $\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j$.

Corollary 1.2. *Retain the setting of Theorem 1.1, there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$, problem (1.1) has at least $2^k - 1$ nontrivial solutions.*

The paper is organized as follows. In Section 2 we introduce the functional setting and we give some preliminary results. In Section 3, we study the existence of least energy solution for a nonlocal problem on the bounded domain. In Section 4, we consider an auxiliary problem. The behavior of $(PS)_\infty$ sequence is studied in Section 5. In Sections 6 and 7, we shall build a special minimax value for the energy functional associated to the auxiliary problem and prove Theorem 1.1.

Notation.

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- For any $A \subset \mathbb{R}^N$, A^c denotes the complement of A in $\mathbb{R}^N$;
- $\|\cdot\|$, $|\cdot|_q$, and $\|\cdot\|_{L^\infty(\Lambda)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^N, \mathbb{R})$, $L^q(\mathbb{R}^N, \mathbb{R})$, and $L^\infty(\Lambda, \mathbb{R})$, respectively, where $\Lambda \subset \mathbb{R}^N$;
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.

2. Abstract setting and preliminary results

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas.

For $u : \mathbb{R}^N \to \mathbb{C}$, let us denote by

$$\nabla_A u := (\nabla + iA)u,$$

and

$$H_A^1(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N, \mathbb{R})\}.$$

The space $H_A^1(\mathbb{R}^N, \mathbb{C})$ is a Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \mathrm{Re} \int_{\mathbb{R}^N} \left(\nabla_A u \overline{\nabla_A v} + u\overline{v}\right) dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^N, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover, we denote by $\|u\|_A$ the norm induced by this inner product.

Since $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, on $H_A^1(\mathbb{R}^N, \mathbb{C})$ we have the following diamagnetic inequality (see e.g. [32, Theorem 7.21]):

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Let

$$E_\lambda := \left\{ u \in H^1_A(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda V(x)|u|^2 dx < \infty \right\},$$

with the norm

$$\|u\|^2_\lambda = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2) dx.$$

For any $\lambda > 0$, it is easy to see that $(E_\lambda, \|\cdot\|_\lambda)$ is a Hilbert space and $E_\lambda \subset H^1_A(\mathbb{R}^N, \mathbb{C})$.

Let $K \subset \mathbb{R}^N$ be an open set, we define

$$\begin{aligned}
&H^1_A(K) := \{u \in L^2(K, \mathbb{C}) : |\nabla_A u| \in L^2(K, \mathbb{R})\},\\
&\|u\|_{H^1_A(K)} = \Big(\int_K (|\nabla_A u|^2 + |u|^2) dx \Big)^{\frac{1}{2}},\\
&E_\lambda(K, \mathbb{C}) := \Big\{ u \in H^1_A(K, \mathbb{C}) : \int_K \lambda V(x)|u|^2 dx < \infty \Big\},\\
&\|u\|^2_{\lambda, K} = \int_K (|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2) dx.
\end{aligned}$$

Let $H^{0,1}_A(K, \mathbb{C})$ be the Hilbert space defined by the closure of $C^\infty_0(K, \mathbb{C})$ under the norm $\|u\|_{H^1_A(K)}$.

The diamagnetic inequality (2.1) implies that if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Therefore, the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$ is continuous for $2 \le r \le 2^*$ and the embedding $E_\lambda \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ is compact for $1 \le r < 2^*$.

In order to use the variational methods to study our problem, the following Hardy-Littlewood-Sobolev inequality is very important.

Lemma 2.1. *(Hardy-Littlewood-Sobolev inequality [32]) Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$. If $f \in L^s(\mathbb{R}^N, \mathbb{R})$ and $h \in L^r(\mathbb{R}^N, \mathbb{R})$, then there exists a sharp constant $C(s, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dx dy \le C(s, N, \mu, r)|f|_s |h|_r.$$

The above inequality guarantees that the following holds

$$\Big| \int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * |u|^p \Big) |u|^p dx \Big| < +\infty, \quad \forall u \in E_\lambda. \tag{2.2}$$

By the Hardy-Littlewood-Sobolev inequality, the integral

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{\mu}} * |u|^p\Big)|u|^p dx$$

is finite if $|u|^p \in L^t(\mathbb{R}^N, \mathbb{R})$ for $t > 1$ and

$$\frac{2}{t} + \frac{\mu}{N} = 2,$$

that is, $t = 2N/2N - \mu$. Moreover, once $p \in (2, 2^*(\frac{2N-\mu}{2N}))$ and $\mu \in (0, 2)$, the Sobolev embedding implies that

$$\int_{\mathbb{R}^N} |u|^{pt} dx < \infty, \quad \forall u \in E_\lambda$$

which showing (2.2) holds.

From the above commentaries, the energy functional $J_\lambda : E_\lambda \to \mathbb{R}$ associated with problem (1.1) given by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2)dx - \frac{1}{2p}\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{\mu}} * |u|^p\Big)|u|^p dx$$

is well defined. Furthermore, it is standard to prove that $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ with

$$J'_\lambda(u)\phi = \mathrm{Re}\Big(\int_{\mathbb{R}^N} (\nabla_A u\overline{\nabla_A \phi} + (\lambda V(x) + 1)u\overline{\phi})dx - \int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{\mu}} * |u|^p\Big)|u|^{p-2}u\overline{\phi}dx\Big), \quad \forall u, \phi \in E_\lambda.$$

Hence, the weak solutions of problem (1.1) are the critical points of J_λ.

In view of $(V1)$, for any open set $K \subset \mathbb{R}^N$, it is easy to see that

$$\|u\|_{2,K}^2 \le \int_K \Big(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2\Big)dx \quad \text{for all } u \in E_\lambda(K, \mathbb{C}) \text{ and } \lambda > 0$$

where $\|u\|_{2,K}^2 = \int_K |u|^2 dx$. The following property is an immediate consequence of the above consideration.

Lemma 2.2. *There exist $\delta_0, \nu_0 > 0$ with $\delta_0 \sim 1$ and $\nu_0 \sim 0$ such that for any open set $K \subset \mathbb{R}^N$*

$$\delta_0\|u\|_{\lambda,K}^2 \le \|u\|_{\lambda,K}^2 - \nu_0\|u\|_{2,K}^2, \quad \textit{for all } u \in E_\lambda(K, \mathbb{C}) \textit{ and } \lambda > 0.$$

3. The problem $(P)_{\infty,\Gamma}$

To prove Theorem 1.1, we need to study the existence of least energy solution for problem $(P)_{\infty,\Gamma}$. The main idea is to prove that the energy functional associated with nonlocal problem $(P)_{\infty,\Gamma}$ given by

$$I_\Gamma(u) = \frac{1}{2}\int\limits_{\Omega_\Gamma} (|\nabla_A u|^2 + |u|^2)dx - \frac{1}{2p}\int\limits_{\Omega_\Gamma}\Big(\int\limits_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu}dy\Big)|u|^p dx,$$

assumes a minimum value on the set

$$\mathcal{M}_\Gamma = \{u \in \mathcal{N}_\Gamma : I'_\Gamma(u)[u_j] = 0 \text{ and } u_j \neq 0, \forall j \in \Gamma\}$$

where $u_j = u|_{\Omega_j}$ and $\mathcal{N}_\Gamma$ is the corresponding Nehari manifold defined by

$$\mathcal{N}_\Gamma = \{u \in H_A^{0,1}(\Omega_\Gamma)\backslash\{0\} : I'_\Gamma(u)[u] = 0\}.$$

More precisely, we will prove that there is $\omega \in \mathcal{M}_\Gamma$ such that

$$I_\Gamma(\omega) = \inf_{u \in \mathcal{M}_\Gamma} I_\Gamma(u).$$

Then, we use a deformation lemma to prove that ω is a critical point of I_Γ, and so, ω is a least energy solution for $(P)_{\infty,\Gamma}$.

In what follows, in order to show the details of the existence of least energy solution for problem $(P)_{\infty,\Gamma}$, we will only consider $\Gamma = \{1,2\}$ for simplicity. Thus,

$$\Omega_\Gamma = \Omega_1 \bigcup \Omega_2,$$

$$\mathcal{N}_\Gamma = \{u \in H_A^{0,1}(\Omega_\Gamma)\backslash\{0\} : I'_\Gamma(u)[u] = 0\},$$

and

$$\mathcal{M}_\Gamma = \{u \in \mathcal{N}_\Gamma : I'_\Gamma(u)[u_j] = 0 \text{ and } u_1, u_2 \neq 0\}$$

where $u_j = u|_{\Omega_j}$, $j = 1, 2$.

In order to show that the set $\mathcal{M}_\Gamma$ is not empty, we need the following lemma, whose proof is similar to that of Lemma 2.1 in [9].

Lemma 3.1. *Let $u \in H_A^{0,1}(\Omega_\Gamma)$ with $u_j \neq 0$ for $j = 1, 2$, then there exists $(s,t) \in (0,+\infty)^2$ such that $su_1 + tu_2 \in \mathcal{M}_\Gamma$ which means $\mathcal{M}_\Gamma \neq \emptyset$ and $c_\Gamma = \inf_{u\in\mathcal{M}_\Gamma} I_\Gamma(u) > 0$.*

Now we give a useful lemma below.

Lemma 3.2. *Let $\{\omega_n\}$ be a bounded sequence in $\mathcal{M}_\Gamma$ with $\omega_n \rightharpoonup \omega$ in $H_A^{0,1}(\Omega_\Gamma)$. If $\int_{\Omega_j}(|\nabla_A \omega_{n,j}|^2 + |\omega_{n,j}|^2)dx \nrightarrow 0$, then $\omega_j \neq 0$, where $\omega_{n,j} = \omega_n|_{\Omega_j}$ and $\omega_j = \omega|_{\Omega_j}$ for $j = 1, 2$.*

The proof of this lemma is similar to that of Lemma 2.2 in [9], here we also omit it.

Now, our main goal is to present the following result, for its proof, we may refer to Theorem 1.1 in [9].

Theorem 3.1. *$(P)_{\infty,\Gamma}$ possesses a least energy solution u which is nonzero on each component Ω_j of Ω_Γ, $j \in \Gamma$.*

4. An auxiliary problem

In this section, we shall work with an auxiliary problem adapting the idea explored by del Pino and Felmer in [16] (see also [18]).

For $t \geq 0$, if we set $f(t) = t^{(p-2)/2}$, then problem (1.1) may be rewritten as

$$-(\nabla + \mathrm{i}A(x))^2 u + (\lambda V(x) + 1)u = \frac{p}{2}\Big(\frac{1}{|x|^\mu} * F(|u|^2)\Big) f(|u|^2)u \quad x \in \mathbb{R}^N,$$

where $F(t) = \int_0^t f(s)ds = \frac{2}{p} t^{p/2}$. Let $\nu_0 > 0$ be a constant given in Lemma 2.2, and $a > 0$ verifying $a^{(p-2)/2} = \nu_0$. Moreover, we set $\tilde{f}, \widetilde{F} : \mathbb{R}^+ \to \mathbb{R}$ given by

$$\tilde{f}(t) = \begin{cases} t^{(p-2)/2}, & 0 \leq t \leq a, \\ \nu_0, & t \geq a, \end{cases}$$

and

$$\widetilde{F}(t) = \int_0^t \tilde{f}(s)ds.$$

Note that

$$\widetilde{f}(t) \leq t^{(p-2)/2}, \quad t \geq 0.$$

From now on, fix a non-empty subset $\Gamma \subset \{1, \cdots, k\}$ and for each $j \in \Gamma$, we fix a bounded open subset Ω'_j with smooth boundary such that

$$\overline{\Omega_j} \subset \Omega'_j, \qquad \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset \quad \text{for all} \quad i \neq j,$$

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \qquad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j,$$

$$\chi_\Gamma(x) := \begin{cases} 1 & \text{for} \quad x \in \Omega'_\Gamma, \\ 0 & \text{for} \quad x \notin \Omega'_\Gamma, \end{cases}$$

the function

$$g(x,t)=\chi_\Gamma(x)t^{(p-2)/2}+(1-\chi_\Gamma(x))\tilde{f}(t) \tag{4.1}$$

$$G(x,t)=\int_0^t g(x,s)ds=\chi_\Gamma(x)\frac{2}{p}t^{p/2}+(1-\chi_\Gamma(x))\tilde{F}(t), \tag{4.2}$$

where χ_Γ is the characteristic function on Γ and $\tilde{F}(t)=\int_0^t \tilde{f}(s)dx$.

From (4.1) and (4.2), it is easy to prove that g is a Carathéodory function satisfying the following properties:

- (g_1) $g(x,t)=0$ for each $t\le 0$;
- (g_2) $\lim_{t\to 0^+} g(x,t)=0$ uniformly in $x\in\mathbb{R}^N$;
- (g_3) $g(x,t)\le t^{(p-2)/2}$ for all $t\ge 0$;
- (g_4) $0\le G(x,t)\le g(x,t)t$, for each $x\in\Omega'_\Gamma$, $t>0$;
- (g_5) $0\le G(x,t)\le g(x,t)t\le \nu_0 t$, for each $x\in(\Omega'_\Gamma)^c$, $t>0$;
- (g_6) for each $x\in\Omega'_\Gamma$, the function $t\mapsto g(x,t)$ is strictly increasing in $t\in(0,+\infty)$ and for each $x\in(\Omega'_\Gamma)^c$, the function $t\mapsto g(x,t)$ is strictly increasing in $(0,a)$.

Now we consider the *auxiliary* problem

$$-(\nabla+\mathrm{i}A(x))^2u+(\lambda V(x)+1)u=\frac{p}{2}(\frac{1}{|x|^\mu}*G(x,|u|^2))g(x,|u|^2)u \quad x\in\mathbb{R}^N. \tag{4.3}$$

Note that, if u is a solution of problem (4.3) with

$$|u(x)|^2\le a \quad \text{for all } x\in(\Omega'_\Gamma)^c,$$

then u is a solution of (1.1).

The functional associated to problem (4.3) is

$$\Phi_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_A u|^2+(\lambda V(x)+1)|u|^2)dx-\frac{p}{8}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*G(x,|u|^2)\Big)G(x,|u|^2)dx$$

defined in E_λ. It is standard to prove that $\Phi_\lambda\in C^1(E_\lambda,\mathbb{R})$ and its critical points are the weak solutions of the auxiliary problem (4.3).

Lemma 4.1. *For all $u\in E_\lambda\backslash\{0\}$, there holds*

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*G(x,|u|^2)\Big)g(x,|u|^2)|u|^2dx-\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*G(x,|u|^2)\Big)G(x,|u|^2)dx\ge 0.$$

Proof. From (g_4) and (g_5), we may obtain the conclusion of the lemma directly. □

Now we show that the functional Φ_λ satisfies the Mountain Pass Geometry [38].

Lemma 4.2. *For any fixed $\lambda > 0$, the functional Φ_λ satisfies the following properties:*
(i) there exist $\beta, r > 0$ such that $\Phi_\lambda(u) \geq \beta$ if $\|u\|_\lambda = r$;
(ii) there exists $e \in E_\lambda$ with $\|e\|_\lambda > r$ such that $\Phi_\lambda(e) < 0$.

Proof. Let us prove (i).

By $(g_1) - (g_3)$ and the Hardy-Littlewood-Sobolev inequality, we have that

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|_\lambda^2 - C\|u\|_\lambda^{2p}.$$

Thus, there exist $\beta, r > 0$ such that $\Phi_\lambda(u) \geq \beta$ if $\|u\|_\lambda = r$.

To prove (ii), let us fix $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ with $\mathrm{supp}(\varphi) \subset \Omega_\Gamma$. Observe that

$$G(x, |\varphi|^2) = F(|\varphi|^2) = \frac{2}{p}|\varphi|^p.$$

Let

$$\mathfrak{g}(t) = \mathfrak{F}\left(\frac{t\varphi}{\|\varphi\|_\lambda}\right) > 0, \quad \text{for } t > 0,$$

where

$$\mathfrak{F}(u) = \frac{p}{8}\int\limits_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * F(|u|^2)\Big)F(|u|^2)dx.$$

By a direct computation, one has then

$$\frac{\mathfrak{g}'(t)}{\mathfrak{g}(t)} \geq \frac{2p}{t} \text{ for all } t > 0.$$

Integrating this over $[1, s\|\varphi\|_\lambda]$ with $s > \frac{1}{\|\varphi\|_\lambda}$ we find

$$\mathfrak{F}(s\varphi) \geq \mathfrak{F}\left(\frac{\varphi}{\|\varphi\|_\lambda}\right)\|\varphi\|_\lambda^{2p}s^{2p}.$$

Therefore,

$$\Phi_\lambda(s\varphi) \leq C_1 s^2 - C_2 s^{2p} \quad \text{for } s > \frac{1}{\|\varphi\|_\lambda},$$

and (ii) holds for $e = s\varphi$ for $s > 0$ large enough. $\square$

Since $\mathrm{supp}(\varphi) \subset \Omega_\Gamma$, it is easy to obtain the existence of constant $d > 0$, independent of λ and a such that $\max_{s>0}\Phi_\lambda(s\varphi) < d$.

Lemma 4.3. *If (u_n) is a $(PS)_c$ sequence to Φ_λ with $c \in [0, d]$, then (u_n) is bounded and there exists $n_0 \in N$ such that $\|u_n\|_\lambda^2 \leq 4(d+1)$ for all $n \geq n_0$.*

Proof. Since $\Phi_\lambda(u_n) \to c$ and $\Phi'_\lambda(u_n) \to 0$, from Lemma 4.1, we have

$$c + o_n(1)\|u_n\|_\lambda \geq \Phi_\lambda(u_n) - \frac{1}{4}\Phi'_\lambda(u_n)[u_n]$$
$$\geq (\frac{1}{2} - \frac{1}{4})\|u_n\|^2_\lambda.$$

From this, we know that (u_n) is bounded in E_λ. Moreover, from $c \in [0, d]$, there exists $n_0 \in N$ such that $\|u_n\|^2_\lambda \leq 4(d+1)$ for all $n \geq n_0$. □

Before proving the next lemma, we give some notations. In what follows,

$$\mathcal{B} = \Big\{u \in E_\lambda : \|u\|^2_\lambda \leq 4(d+1)\Big\}$$

and

$$K(u)(x) = \frac{1}{|x|^\mu} * G(x, |u|^2).$$

Using the above notations, we can show the following estimates.

Lemma 4.4. *There exists $k_0 > 0$ large such that*

$$\frac{\sup_{u\in\mathcal{B}} |K(u)(x)|_{L^\infty(\mathbb{R}^N)}}{k_0} < \frac{1}{2}.$$

Proof. Notice that

$$|G(x,t)| \leq |F(t)| = \frac{2}{p}t^{p/2}, \quad \forall s \in \mathbb{R}^+,$$

thus,

$$|K(u)(x)| \leq \frac{2}{p}\Big|\int\limits_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\mu}dy\Big| = \frac{2}{p}\Big|\int\limits_{|x-y|\leq 1} \frac{|u|^p}{|x-y|^\mu}dy\Big| + \frac{2}{p}\Big|\int\limits_{|x-y|>1} \frac{|u|^p}{|x-y|^\mu}dy\Big|.$$

From the Sobolev's embedding, there exists $C_1 > 0$ such that

$$|K(u)(x)| \leq \frac{2}{p}\Big|\int\limits_{|x-y|\leq 1} \frac{|u|^p}{|x-y|^\mu}dy\Big| + C_1.$$

Since $\mu \in (0, 2)$ and $p \in (2, \frac{2(N-\mu)}{N-2})$, we fix $s \in (\frac{N}{N-\mu}, \frac{2N}{(N-2)p})$. Then, by the Hölder inequality,

$$\int\limits_{|x-y|\le 1} \frac{|u|^p}{|x-y|^\mu} dy \le \Big(\int\limits_{|x-y|\le 1} |u|^{sp} dy \Big)^{1/s} \Big(\int\limits_{|x-y|\le 1} \frac{1}{|x-y|^{\frac{s\mu}{s-1}}} dy \Big)^{(s-1)/s}$$
$$\le C_2 \Big(\int\limits_{|r|\le 1} |r|^{N-1-\frac{s\mu}{s-1}} dy \Big)^{(s-1)/s}.$$

As $N-1-\frac{s\mu}{s-1} > -1$, there exists $C_3 > 0$ such that

$$\int\limits_{|x-y|\le 1} \frac{|u|^p}{|x-y|^\mu} dy \le C_3, \quad x \in \mathbb{R}^N.$$

Thus, there exists a constant $k_0 > 0$ such that

$$\frac{\sup_{u\in\mathcal{B}} |K(u)(x)|_{L^\infty(\mathbb{R}^N)}}{k_0} < \frac{1}{2}. \quad \square$$

Lemma 4.5. *Let (u_n) be a $(PS)_c$ sequence to Φ_λ with $c \in [0, d]$. Then, for any $\xi > 0$, there exists $R(\xi) > 0$, if $R > R(\xi)$ there holds*

$$\limsup_n \int\limits_{(B_R(0))^c} (|\nabla_A u_n|^2 + (\lambda V(x)+1)|u_n|^2) dx \le \xi.$$

Proof. From Lemma 4.3, we have

$$\|u_n\|_\lambda^2 \le 4(d+1), \quad \text{for } n \text{ large}.$$

Thus, we can assume that there exists $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ for all $1 \le r < 2^*$ as $n \to +\infty$.

Now, we take $R > 0$ such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$. Let $\phi_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{\frac{R}{2}}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0 \le \phi_R \le 1, \quad \text{and} \quad |\nabla \phi_R| \le C/R,$$

where $C > 0$ is a constant independent of R. Once the sequence $(\phi_R u_n)$ is bounded in E_λ, one has

$$\Phi'_\lambda(u_n)[u_n \phi_R] = o_n(1),$$

and

$$\overline{\nabla_A(u_n\phi_R)} = \overline{u_n} \nabla \phi_R + \phi_R \overline{\nabla_A u_n}.$$

Therefore,

$$o_n(1) = \Phi'_\lambda(u_n)[u_n\phi_R] = \int\limits_{\mathbb{R}^N} (|\nabla_A u_n|^2\phi_R + (\lambda V(x)+1)|u_n|^2\phi_R)dx$$
$$+ \operatorname{Re}\left(\int\limits_{\mathbb{R}^N} \overline{u_n}\nabla_A u_n \nabla\phi_R dx\right) - \frac{p}{2}\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * G(x,|u_n|^2)\Big)g(x,|u_n|^2)|u_n|^2\phi_R dx.$$

If $\nu_0 > 0$ small enough, the above inequalities, (g_5) and Lemma 4.4 imply that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}\Big(|\nabla_A u_n|^2\phi_R + (\lambda V(x)+1)|u_n|^2\phi_R\Big)dx =&\frac{p}{2}\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * G(x,|u_n|^2)\Big)g(x,|u_n|^2)|u_n|^2\phi_R dx\\
&- \operatorname{Re}\left(\int\limits_{\mathbb{R}^N} \overline{u_n}\nabla_A u_n\nabla\phi_R dx\right) + o_n(1)\\
\le&\frac{p}{2}\int\limits_{\mathbb{R}^N} \sup_{u\in\mathcal{B}}|K(u)(x)|_{L^\infty(\mathbb{R}^N)}\nu_0|u_n|^2\phi_R dx\\
&+ \frac{C}{R}\|u_n\|_\lambda^2 + o_n(1)\\
\le&\frac{k_0\nu_0 p}{4}\int\limits_{\mathbb{R}^N}|u_n|^2\phi_R dx + \frac{C}{R}\|u_n\|_\lambda^2 + o_n(1)\\
\le&\frac{1}{2}\int\limits_{\mathbb{R}^N}(\lambda V(x)+1)|u_n|^2\phi_R dx + \frac{C_1}{R} + o_n(1).
\end{aligned}$$

So, for any $\xi > 0$, we can choose $R > 0$ large enough such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$ and

$$\limsup_n \int\limits_{(B_R(0))^c} (|\nabla_A u_n|^2 + (\lambda V(x)+1)|u_n|^2)dx \le \xi.$$

This completes the proof. □

Lemma 4.6. *The functional Φ_λ satisfies the $(PS)_c$ condition for any $c \in [0,d]$.*

Proof. Let $(u_n) \subset E_\lambda$ be a $(PS)_c$ sequence for Φ_λ at the level $c \in [0,d]$. From Lemma 4.3, we have that there exists $n_0 \in N$ such that $\|u_n\|_\lambda^2 \le 4(d+1)$ for all $n \ge n_0$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in E_λ, $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$, for all $1 \le r < 2^*$, $|u_n| \to |u|$ for a.e. $x \in \mathbb{R}^N$. Moreover, it is easy to show that $\Phi'_\lambda(u) = 0$.

From Lemma 4.4, we know that there exists $C > 0$ such that

$$|K(u_n)|_{L^\infty(\mathbb{R}^N)} \le C, \ \forall n \in N.$$

Thus, for any fixed $R > 0$, the Sobolev's compact embedding, the subcritical growth of g and a variant of the Lebesgue Dominated Convergence Theorem imply that

$$\lim_{n\to\infty} \int_{B_R} K(u_n)g(x,|u_n|^2)|u_n|^2 dx \to \int_{B_R} K(u)g(x,|u|^2)|u|^2 dx. \tag{4.4}$$

Moreover, from Lemma 4.5, we know that for any $\xi > 0$, there exists $R_\xi > 0$ such that for all $R > R_\xi$

$$\limsup_n \int_{(B_R(0))^c} (|\nabla_A u_n|^2 + (\lambda V(x)+1)|u_n|^2)dx \le \xi.$$

Consequently, by the Sobolev's embedding and the subcritical growth of g, we obtain

$$\limsup_n \left| \int_{(B_R(0))^c} K(u_n)g(x,|u_n|^2)|u_n|^2 dx \right| \le C_2\xi. \tag{4.5}$$

Since

$$\left| \int_{\mathbb{R}^N} K(u)g(x,|u|^2)|u|^2 dx \right| \le C,$$

we have

$$\left| \int_{(B_R(0))^c} K(u)g(x,|u|^2)|u|^2 dx \right| \le \xi, \tag{4.6}$$

for $R > 0$ large enough.

From (4.4), (4.5) and (4.6), one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(u_n)g(x,|u_n|^2)|u_n|^2 dx \to \int_{\mathbb{R}^N} K(u)g(x,|u|^2)|u|^2 dx.$$

Finally, since $\Phi'_\lambda(u) = 0$, we have

$$o_n(1) = \Phi'_\lambda(u_n)[u_n] = \|u_n\|^2_\lambda - \frac{p}{2}\int_{\mathbb{R}^N} K(u_n)g(x,|u_n|^2)|u_n|^2 dx = \|u_n\|^2_\lambda - \|u\|^2_\lambda + o_n(1).$$

Thus, the sequence (u_n) strongly converges to u in E_λ. $\square$

5. The $(PS)_\infty$ condition

Now we study the behavior of a $(PS)_{\infty,c}$ sequence, that is, a sequence $(u_n) \subset H^1_A(\mathbb{R}^N, \mathbb{C})$ satisfying

$$
\begin{aligned}
&u_n \in E_{\lambda_n} \text{ and } \lambda_n \to \infty,\\
&\Phi_{\lambda_n}(u_n) \to c,\\
&\|\Phi'_{\lambda_n}(u_n)\|_{E^*_{\lambda_n}} \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Proposition 5.1. *Let (u_n) be a $(PS)_{\infty,c}$ sequence with $c \in [0, d]$. Then, for some subsequence, still denoted by (u_n), there exists $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ such that*

$$
u_n \rightharpoonup u \ \ in \ H^1_A(\mathbb{R}^N, \mathbb{C}).
$$

Moreover,

(i) $\|u_n - u\|_{\lambda_n} \to 0$*, and so* $\|u_n - u\|_A \to 0$*;*
(ii) $u \equiv 0$ *in* $\mathbb{R}^N \backslash \Omega_\Gamma$ *and* $u|_{\Omega_\Gamma}$ *is a solution of*

$$
\begin{cases}
-(\nabla + \mathrm{i}A(x))^2 u + u = \Big(\displaystyle\int_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu} dy\Big)|u|^{p-2}u, & in\ \Omega_\Gamma,\\
u \in H^{0,1}_A(\Omega_\Gamma);
\end{cases}
$$

(iii) u_n *also satisfies*

$$
\begin{aligned}
&\lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^2 dx \to 0,\\
&\|u_n\|^2_{\lambda_n, \mathbb{R}^N \backslash \Omega'_\Gamma} \to 0,\\
&\Phi_{\lambda_n}(u_n) \to \frac{1}{2}\int_{\Omega_\Gamma} (|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2) dx - \frac{1}{2p}\int_{\Omega_\Gamma}\Big(\int_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu} dy\Big)|u|^p dx.
\end{aligned}
$$

The proof of this proposition is similar to that of Proposition 4.1 in [9], here we omit it.

Proposition 5.2. *Let (u_λ) be a family of nontrivial solutions of problem (4.3) with $0 \le \Phi_\lambda(u_\lambda) \le d$ Then, there exists $\lambda^* > 0$ such that*

$$
\||u_\lambda|\|^2_{L^\infty(\mathbb{R}^N \backslash \Omega'_\Gamma)} \le a, \qquad \forall \lambda \ge \lambda^*.
$$

In particular, u_λ solves the original problem (1.1) for any $\lambda \ge \lambda^*$.

Proof. We use the notation $B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$. Since $u_\lambda \in E_\lambda$ is a critical point of $\Phi_\lambda(u)$, that is, u_λ satisfies the following equation

$$-(\nabla + \mathrm{i}A(x))^2 u_\lambda + (\lambda V(x) + 1)u_\lambda = \frac{p}{2}(\frac{1}{|x|^\mu} * G(x, |u_\lambda|^2))g(x, |u_\lambda|^2)u_\lambda, \quad x \in \mathbb{R}^N.$$

By Kato's inequality

$$\Delta|u_\lambda| \geq \mathrm{Re}\Big(\frac{\overline{u_\lambda}}{|u_\lambda|}(\nabla + iA(x))^2 u_\lambda(x)\Big),$$

there holds

$$\Delta|u_\lambda(x)| - (\lambda V(x) + 1)|u_\lambda(x)| + \frac{p}{2}(\frac{1}{|x|^\mu} * G(x, |u_\lambda|^2))g(x, |u_\lambda|^2)|u_\lambda(x)| \geq 0, \quad x \in \mathbb{R}^N.$$

From $0 \leq \Phi_\lambda(u_\lambda) \leq d$ and Lemma 4.4, we know that there exists $C > 0$ such that

$$|K(u_\lambda)|_{L^\infty(\mathbb{R}^N)} \leq C, \ \forall \lambda \in \lambda^*.$$

We also have that there exists $C_1 > 0$ such that

$$g(x, t) \leq t^{\frac{p-2}{2}}, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Since $|u_\lambda| \geq 0$ and $(\lambda V(x) + 1) \geq 1$, from the above inequalities, we have

$$\Delta|u_\lambda(x)| - |u_\lambda(x)| + C|u_\lambda(x)|^{p-2}|u_\lambda(x)| \geq 0, \quad x \in \mathbb{R}^N.$$

Using the subsolution estimate (see [22] Theorem 8.17), there exists a constant $C(r) > 0$ such that for $1 < q < 2$

$$\sup_{y \in B_r(x)} |u_\lambda(y)| \leq C(r)\Big(\int\limits_{B_{2r}(x)} |u_\lambda|^q dy\Big)^{1/2}.$$

By Proposition 5.1, for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that

$$u_{\lambda_{n_i}} \to u \in H_A^{0,1}(\Omega_\Gamma, \mathbb{C}) \quad \text{in } H_A^1(\mathbb{R}^N, \mathbb{C}).$$

In particular,

$$u_{\lambda_{n_i}} \to 0 \quad \text{in } L^2(\mathbb{R}^N \backslash \overline{\Omega_\Gamma}, \mathbb{C}).$$

Since $\lambda_n \to \infty$ is arbitrary, we have

$$u_\lambda \to 0 \quad \text{in } L^2(\mathbb{R}^N \backslash \overline{\Omega_\Gamma}, \mathbb{C}) \text{ as } \lambda \to \infty.$$

Thus, choosing $r \in (0, \mathrm{dist}(\Omega_\Gamma, \mathbb{R}^N \backslash \Omega'_\Gamma))$, we have uniformly in $x \in \mathbb{R}^N \backslash \Omega'_\Gamma$ that

$$\begin{aligned}|u_\lambda(y)| &\le C(r)\|u_\lambda\|_{L^q(B_{2r}(x))}\\ &\le C(r)|B_{2r}(x)|^{\frac{2-q}{2q}}\|u_\lambda\|_{L^2(\mathbb{R}^N\setminus\overline{\Omega_\Gamma})}\\ &\to 0.\end{aligned}$$

This finishes the proof. □

6. A special minimax value for Φ_λ

In this section, without loss generality, we consider $\Gamma = \{1, \cdots, l\}$ with $l \le k$. Now, we introduce the functional in $H^1_A(\Omega'_\Gamma, \mathbb{C})$

$$\Phi_{\lambda,\Gamma}(u) = \frac{1}{2}\int\limits_{\Omega'_\Gamma}(|\nabla_A u|^2 + (\lambda V(x)+1)|u|^2)dx - \frac{1}{2p}\int\limits_{\Omega'_\Gamma}\Big(\int\limits_{\Omega'_\Gamma}\frac{|u|^p}{|x-y|^\mu}dy\Big)|u|^p dx.$$

We denote by $c_{\lambda,\Gamma}$ given by

$$c_{\lambda,\Gamma} = \inf_{u\in\mathcal{M}'_\Gamma}\Phi_{\lambda,\Gamma}(u)$$

where

$$\mathcal{M}'_\Gamma = \{u \in \mathcal{N}'_\Gamma : \Phi'_{\lambda,\Gamma}(u)[u_j] = 0 \text{ and } u_j \ne 0,\ j \in \Gamma\}$$

where $u_j = u|_{\Omega'_j}$ and

$$\mathcal{N}'_\Gamma = \{u \in H^1_A(\Omega'_\Gamma)\setminus\{0\} : \Phi'_{\lambda,\Gamma}(u)[u] = 0\}.$$

Using the same arguments in Section 3, we know that there exists $w_{\lambda,\Gamma} \in H^1_A(\Omega'_\Gamma, \mathbb{C})$ verifying

$$\Phi_{\lambda,\Gamma}(w_{\lambda,\Gamma}) = c_{\lambda,\Gamma} \quad \text{and} \quad \Phi'_{\lambda,\Gamma}(w_{\lambda,\Gamma}) = 0.$$

Moreover, we have the following assertions.

Lemma 6.1. *The following assertions hold:*

(i) $0 < c_{\lambda,\Gamma} \le c_\Gamma$, *for any* $\lambda > 0$;
(ii) $c_{\lambda,\Gamma} \to c_\Gamma$ *as* $\lambda \to \infty$.

For the proof of this lemma, we refer to that of Lemma 5.1 in [9].

In what follows, we denote by $\omega \in H^{0,1}_A(\Omega_\Gamma)$ the least energy solution obtained in Section 3, that is

$$\omega \in \mathcal{M}_\Gamma, \quad I_\Gamma(w) = c_\Gamma \quad \text{and} \quad I'_\Gamma(w) = 0.$$

Considering the function

$$G(s_1, s_2, \cdots, s_l) = I_\Gamma(s_1^{\frac{1}{p}}\omega_1 + s_2^{\frac{1}{p}}\omega_2 + \cdots + s_l^{\frac{1}{p}}\omega_l). \tag{6.1}$$

It is obvious that

$$\begin{aligned} I_\Gamma(s_1^{\frac{1}{p}}\omega_1 + s_2^{\frac{1}{p}}\omega_2 + \cdots + s_l^{\frac{1}{p}}\omega_l) = &\sum_{j=1}^{l} \frac{s_j^{\frac{2}{p}}}{2} \int_{\Omega_j} (|\nabla_A \omega_j|^2 + |\omega_j|^2)dx \\ &- \frac{1}{2p} \int_{\Omega_\Gamma} \Big(\int_{\Omega_\Gamma} \frac{\sum_{j=1}^{l} s_j|\omega_j|^p}{|x-y|^\mu} dy \Big) \Big(\sum_{j=1}^{l} s_j|\omega_j|^p \Big) dx. \end{aligned}$$

Arguing as in [21], one has

$$\int_{\Omega_\Gamma} \Big(\int_{\Omega_\Gamma} \frac{\sum_{j=1}^{l} s_j|\omega_j|^p}{|x-y|^\mu} dy \Big) \Big(\sum_{j=1}^{l} s_j|\omega_j|^p \Big) dx = \int_{\Omega_\Gamma} \Big[\frac{1}{|x|^{\mu/2}} * \Big(\sum_{j=1}^{l} s_j|\omega_j|^p \Big) \Big]^2 dx.$$

As $s \mapsto s^{2/p}$ is concave and $s \mapsto s^2$ is strictly convex, we conclude the function (6.1) is strictly concave with $\nabla G(1, 1, \cdots, 1) = (0, 0, \cdots, 0)$. Hence, $(1, 1, \cdots, 1)$ is the unique global maximum point of G on $[0, +\infty)^l$ with $G(1, 1, \cdots, 1) = c_\Gamma$. In the sequel, we denote by $\omega \in H_A^{0,1}(\Omega_\Gamma)$ the least energy solution for $(P)_{\infty,\Gamma}$, that is

$$\omega \in \mathcal{M}_\Gamma, \quad I_\Gamma(w_\Gamma) = c_\Gamma \quad \text{and} \quad I'_\Gamma(w_\Gamma) = 0.$$

Since $p > 2$, there are $r > 0$ small enough and $R > 0$ large enough such that

$$I'_\Gamma(\sum_{j=1, j\neq i}^{l} t_j\omega_j + R\omega_i)[R\omega_i] < 0, \text{ for } i \in \Gamma, \ \forall t_j \in [r, R] \text{ and } j \neq i, \tag{6.2}$$

$$I'_\Gamma(\sum_{j=1, j\neq i}^{l} t_j\omega_j + r\omega_i)[r\omega_i] > 0, \text{ for } i \in \Gamma, \ \forall t_j \in [r, R] \text{ and } j \neq i, \tag{6.3}$$

and

$$I_\Gamma(\sum_{j=1}^{l} t_j\omega_j) < c_\Gamma, \ \forall (t_1, t_2, \cdots, t_l) \in \partial([r, R]^l),$$

where $\omega_j = \omega|_{\Omega_j}$, $j \in \Gamma$. Using the above inequalities, we can define the maps

$$\gamma_0(t_1, t_2, \cdots, t_l)(x) = \sum_{j=1}^{l} t_j\omega_j(x) \in H_A^{0,1}(\Omega_\Gamma) \quad \forall (t_1, t_2, \cdots, t_l) \in [r, R]^l, \tag{6.4}$$

$$\Lambda_* = \Big\{ \gamma \in C\Big([r, R]^l, E_\lambda \backslash \{0\} \Big) : \gamma = \gamma_0 \text{ on } \partial([r, R]^l) \Big\},$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma \in \Lambda_*} \max_{(t_1,\cdots,t_l)\in[r,R]^l} \Phi_\lambda(\gamma(t_1,\cdots,t_l)),$$

where $R > 1 > r > 0$ are the positive constants obtained in (6.2) and (6.3). We remark that $\gamma_0 \in \Lambda_*$, so $\Lambda_* \neq \emptyset$ and $b_{\lambda,\Gamma}$ is well defined.

Lemma 6.2. *For any $\gamma \in \Lambda_*$, there exists $(t_1, t_2, \cdots, t_l) \in [r, R]^l$ such that*

$$\Phi'_{\lambda,\Gamma}(\gamma(t_1,t_2,\cdots,t_l))[\gamma_j(t_1,t_2,\cdots,t_l)] = 0,$$

where $\gamma_j(t_1,t_2,\cdots,t_l) = \gamma(t_1,t_2,\cdots,t_l)|_{\Omega'_j}$, $j \in \Gamma$.

Proof. Since $p > 2$ and $\gamma = \gamma_0$ on $\partial([r, R]^l)$, using (6.2), (6.3) and Miranda's Theorem [36], we obtain the conclusion of the lemma. $\square$

Proposition 6.1. *The following facts hold*

(*i*) $0 < c_{\lambda,\Gamma} \leq b_{\lambda,\Gamma} \leq c_\Gamma$ *for all* $\lambda > 0$;
(*ii*) $\Phi_\lambda(\gamma(t_1,t_2,\cdots,t_l)) < c_\Gamma$ *for all* $\lambda > 0$, $\gamma \in \Lambda_*$ *and* $(t_1,t_2,\cdots,t_l) \in \partial([r, R]^l)$.

Proof. Since γ_0 defined in (6.4) belongs to Λ_*, we have

$$\begin{aligned} b_{\lambda,\Gamma} &\leq \max_{(s_1,s_2,\cdots,s_l)\in[r,R]^l} \Phi_\lambda(\gamma_0(s_1,s_2,\cdots,s_l)) \\ &= \max_{(s_1,s_2,\cdots,s_l)\in[r,R]^l} I_\Gamma(\sum_{j=1}^{l} t_j\omega_j) = c_\Gamma. \end{aligned}$$

Fixing $(t_1,t_2,\cdots,t_l) \in [r, R]^l$ given in Lemma 6.2 and recalling that

$$c_{\lambda,\Gamma} = \inf_{u\in\mathcal{M}'_\Gamma} \Phi_{\lambda,\Gamma}(u)$$

where

$$\mathcal{M}'_\Gamma = \{u \in \mathcal{N}'_\Gamma : \Phi'_{\lambda,\Gamma}(u)[u_j] = 0 \text{ and } u_j \neq 0,\ j \in \Gamma\}.$$

It follows that

$$\Phi_{\lambda,\Gamma}(\gamma(t_1,t_2,\cdots,t_l)) \geq c_{\lambda,\Gamma}.$$

On the other hand, let $d \geq c_\Gamma$ and ν_0 small, using Lemma 4.3 and Lemma 4.4, we have that

$$\Phi_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma}(u) \geq 0 \quad \text{for all } u \in H^1_A(\mathbb{R}^N\setminus\Omega'_\Gamma)$$

which leads to

$$\Phi_\lambda(\gamma(t_1, t_2, \cdots, t_l)) \geq \Phi_{\lambda,\Gamma}(\gamma(t_1, t_2, \cdots, t_l)), \quad \forall (t_1, t_2, \cdots, t_l) \in [r, R]^l.$$

Thus

$$\max_{(s_1,s_2,\cdots,s_l)\in[r,R]^l} \Phi_\lambda(\gamma(s_1, s_2, \cdots, s_l)) \geq \Phi_{\lambda,\Gamma}(\gamma(t_1, t_2, \cdots, t_l)) \geq c_{\lambda,\Gamma}.$$

From the definition of $b_{\lambda,\Gamma}$, we can obtain

$$b_{\lambda,\Gamma} \geq c_{\lambda,\Gamma}.$$

This completes the proof of (i).

Since $\gamma(t_1, t_2, \cdots, t_l) = \gamma_0(t_1, t_2, \cdots, t_l)$ on $\partial([r, R]^l)$, we have

$$\Phi_\lambda(\gamma_0(t_1, t_2, \cdots, t_l)) = I_\Gamma(\sum_{j=1}^{l} t_j\omega_j).$$

From (6.2) and (6.3), we derive

$$\Phi_\lambda(\gamma_0(t_1, t_2, \cdots, t_l)) \leq c_\Gamma - \epsilon,$$

for some $\epsilon > 0$. This completes the proof of (ii). □

Corollary 6.1. *The following claims hold:*

(i) $b_{\lambda,\Gamma} \to c_\Gamma$ *as* $\lambda \to \infty$;
(ii) $b_{\lambda,\Gamma}$ *is a critical value of* Φ_λ *for large* λ.

Proof. (i) For all $\lambda > 0$, $c_{\lambda,\Gamma} \leq b_{\lambda,\Gamma} \leq c_\Gamma$. From Lemma 6.1, $c_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$, thus, $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.

(ii) By choosing $d > c_\Gamma$, for $\lambda > 0$ large enough $b_{\lambda,\Gamma} \in [0, d]$, using the fact that Φ_λ verifies that $(PS)_c$ condition with $c \in [0, d]$, we can use well known arguments involving deformation lemma [38] to conclude that $b_{\lambda,\Gamma}$ is a critical level to Φ_λ for large λ. □

7. Proof of the main theorem

To prove Theorem 1.1, we need to find nontrivial solutions u_λ for $\lambda > 0$ large enough, which converges to a least energy solution of $(P)_{\infty,\Gamma}$ as $\lambda \to \infty$. To this end, we will show two propositions which together with Propositions 5.1 and 5.2 will imply that Theorem 1.1 holds.

Hereafter, we denote by

$$\Theta = \left\{ u \in E_\lambda : \|u\|_{\lambda,\Omega'_j} > \frac{r\tau}{2}, \ \forall j \in \Gamma \right\},$$

where r was fixed in (6.2) and τ is the positive constant such that

$$\int_{\Omega_j} (|\nabla_A u_j|^2 + |u_j|^2)dx > \tau,\ \forall u \in \Upsilon_\Gamma = \{u \in \mathcal{M}_\Gamma : I_\Gamma(u) = c_\Gamma\} \text{ and } \forall j \in \Gamma.$$

Furthermore, we denote the set

$$\Phi_\lambda^{c_\Gamma} = \{u \in E_\lambda : \Phi_\lambda(u) \le c_\Gamma\}.$$

Fixing $\kappa = \frac{r\tau}{8}$, and for small $\zeta > 0$, we define

$$A_\zeta^\lambda = \{u \in \Theta_{2\kappa} : \|u\|^2_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma} \le \zeta,\ |\Phi_\lambda(u) - c_\Gamma| \le \zeta\}.$$

Notice that $w \in A_\zeta^\lambda \cap \Phi_\lambda^{c_\Gamma}$ which shows that $A_\zeta^\lambda \cap \Phi_\lambda^{c_\Gamma} \neq \emptyset$. We have the following uniform estimate of $\|\Phi'_\lambda(u)\|$ in the set $(A_{2\zeta}^\lambda \setminus A_\zeta^\lambda) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 7.1. *For each $\zeta > 0$, there exist $\lambda^* > 0$ large enough and $\sigma_0 > 0$ independent of λ such that*

$$\|\Phi'_\lambda(u)\| \ge \sigma_0 \quad \textit{for } \lambda \ge \lambda^* \textit{ and } u \in (A_{2\zeta}^\lambda \setminus A_\zeta^\lambda) \cap \Phi_\lambda^{c_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and $u_n \in (A_{2\zeta}^{\lambda_n} \setminus A_\zeta^{\lambda_n}) \cap \Phi_{\lambda_n}^{c_\Gamma}$ such that

$$\|\Phi'_{\lambda_n}(u_n)\| \to 0.$$

Since $u_n \in A_{2\zeta}^{\lambda_n}$, we know that $(\|u_n\|_{\lambda_n})$ and $(\Phi_{\lambda_n}(u_n))$ are both bounded. Then, passing to a subsequence if necessary, we can assume that $(\Phi_{\lambda_n}(u_n))$ converges. Thus, choosing appropriately $d > 0$ large, from Proposition 5.1, there exists $u \in H_A^{0,1}(\Omega_\Gamma)$ such that u is a solution for

$$-(\nabla + \mathrm{i}A(x))^2 u + u = \Big(\int_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu} dy\Big)|u|^{p-2}u, \quad \text{in } \Omega_\Gamma,$$

and

$$u_n \to u \text{ in } H_A^1(\mathbb{R}^N, \mathbb{C}), \quad \|u_n\|^2_{\lambda_n,\mathbb{R}^N\setminus\Omega'_\Gamma} \to 0, \quad \Phi_{\lambda_n}(u_n) \to I_\Gamma(u).$$

Since $(u_n) \subset \Theta_{2\kappa}$, we derive that

$$\int_{\Omega'_j} (|\nabla_A u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2)dx > \frac{r\tau}{4}, \quad \forall j \in \Gamma.$$

Let $n \to +\infty$, we have the inequality

$$\int_{\Omega_j} (|\nabla_A u|^2 + |u|^2)dx \ge \frac{r\tau}{4} > 0, \quad \forall j \in \Gamma,$$

which yields $u|_{\Omega_j} \neq 0$, $j = 1, \cdots, l$ and $I'_\Gamma(u) = 0$. Consequently, $I_\Gamma(u) \geq c_\Gamma$. However, from the fact that $\Phi_{\lambda_n}(u_n) \leq c_\Gamma$ and $\Phi_{\lambda_n}(u_n) \to I_\Gamma(u)$ as $n \to +\infty$, we derive that $I_\Gamma(u) = c_\Gamma$, and so, $u \in \Upsilon_\Gamma$ Thus, for n large enough

$$\int\limits_{\Omega_j} (|\nabla_A u_n|^2 + |u_n|^2)dx > \frac{r\tau}{2}, \quad |\Phi_{\lambda_n}(u_n) - c_\Gamma| \leq \zeta, \text{ for any } j \in \Gamma.$$

So, $u_n \in A_\zeta^{\lambda_n}$ for large n, which is a contradiction to $u_n \in (A_{2\zeta}^{\lambda_n} \backslash A_\zeta^{\lambda_n})$. Thus, we complete the proof. $\square$

In the sequel, ζ_1, ζ^* denote the following numbers

$$\min_{(t_1,\cdots,t_l)\in\partial([r,R]^l)} |I_\Gamma(\gamma_0(t_1, \cdots, t_l)) - c_\Gamma| = \zeta_1 > 0$$

and

$$\zeta^* = \min\{\zeta_1/2, \kappa, \rho/2\},$$

where $\kappa = \frac{r\tau}{8}$ was given before and

$$\rho = 4R^2 c_\Gamma,$$

where $R > 0$ was fixed in (6.2). Moreover, for each $s > 0$, B_s^λ denotes the set

$$B_s^\lambda = \{u \in E_\lambda : \|u\|_\lambda \leq s\}, \text{ for } s > 0.$$

Proposition 7.2. *Let $\zeta \in (0, \zeta^*)$ and $\lambda^* > 0$ large enough given in the previous proposition. Then, for $\lambda \geq \lambda^*$, there is a solution u_λ of problem (4.3) satisfying $u_\lambda \in A_\zeta^\lambda \cap \Phi_\lambda^{c_\Gamma} \cap B_{2\rho+1}^\lambda$.*

Proof. For $\lambda \geq \lambda^*$, assume that there are no critical points in $A_\zeta^\lambda \cap \Phi_\lambda^{c_\Gamma} \cap B_{2\rho+1}^\lambda$. Since Φ_λ verifies the $(PS)_c$ condition with $0 \leq c \leq d$, there exists a constant $d_\lambda > 0$ such that

$$\|\Phi'_\lambda(u)\| \geq d_\lambda \quad \text{for all} \quad u \in A_\zeta^\lambda \cap \Phi_\lambda^{c_\Gamma} \cap B_{2\rho+1}^\lambda.$$

By Proposition 7.1, we have

$$\|\Phi'_\lambda(u)\| \geq \sigma_0 \quad \text{for all } u \in (A_{2\zeta}^\lambda \backslash A_\zeta^\lambda) \cap \Phi_\lambda^{c_\Gamma},$$

where $\sigma_0 > 0$ is independent of λ. In what follows, $\Psi : E_\lambda \to \mathbb{R}$ be a continuous functional verifying

$$\begin{aligned}
&\Psi(u) = 1 && \text{for } u \in A_{3\zeta/2}^\lambda \cap \Upsilon_\kappa \cap B_{2\rho}^\lambda, \\
&\Psi(u) = 0 && \text{for } u \notin A_{2\zeta}^\lambda \cap \Upsilon_{2\kappa} \cap B_{2\rho+1}^\lambda, \\
&0 \leq \Psi(u) \leq 1 && \text{for } \forall u \in E_\lambda,
\end{aligned}$$

and $H : \Phi_\lambda^{c_\Gamma} \to E_\lambda$ verifies

$$H(u) := \begin{cases} -\Psi(u)\frac{Y(u)}{\|Y(u)\|}, & u \in A_{2\zeta}^\lambda \cap B_{2\rho+1}^\lambda, \\ 0, & u \notin A_{2\zeta}^\lambda \cap B_{2\rho+1}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{K} = \{u \in E_\lambda : \Phi_\lambda'(u) \neq 0\}$. Observe that H is well defined, since $\Phi_\lambda'(u) \neq 0$, for $u \in A_{2\zeta}^\lambda \cap \Phi_\lambda^{c_\Gamma}$. The following inequality

$$\|H(u)\| \leq 1, \ \ \forall \lambda \geq \lambda^* \text{ and } u \in \Phi_\lambda^{c_\Gamma},$$

guarantees the deformation flow $\eta : [0, \infty) \times \Phi_\lambda^{c_\Gamma} \to \Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt} = H(\eta) \quad \text{and} \quad \eta(0, u) = u \in \Phi_\lambda^{c_\Gamma},$$

verifies

$$\frac{d}{dt}\Phi_\lambda(\eta(t,u)) \leq -\Psi(\eta(t,u))\|\Phi_\lambda'(\eta(t,u))\| \leq 0, \tag{7.1}$$

$$\|\frac{d\eta}{dt}\|_\lambda = \|H(\eta)\|_\lambda \leq 1,$$

$$\eta(t,u) = u \quad \text{for all } t \geq 0 \text{ and } u \in \Phi_\lambda^{c_\Gamma} \backslash (A_{2\mu}^\lambda \cap B_{2\rho+1}^\lambda). \tag{7.2}$$

We now study two paths, which are relevant for what follows:

(1) The path $(t_1, \cdots, t_l) \to \eta(t, \gamma_0(t_1, \cdots, t_l))$, where $(t_1, \cdots, t_l) \in [r, R]^l$.

Since $\zeta \in (0, \zeta^*)$, we have that

$$\gamma_0(t_1, \cdots, t_l) \notin A_{2\zeta}^\lambda, \ \forall (t_1, \cdots, t_l) \in \partial([r, R]^l),$$

and

$$\Phi_\lambda(\gamma_0(t_1, \cdots, t_l)) < c_\Gamma \quad \forall (t_1, \cdots, t_l) \in \partial([r, R]^l).$$

From (7.2), it follows that

$$\eta(t, \gamma_0(\mathbf{t})) = \gamma_0(t_1, \cdots, t_l) \quad \forall (t_1, \cdots, t_l) \in \partial([r, R]^l).$$

So, $\eta(t, \gamma_0(t_1, \cdots, t_l) \in \Gamma_*$ for all $t \geq 0$.

(2) The path $(t_1, \cdots, t_l) \to \gamma_0(t_1, \cdots, t_l)$, where $(t_1, \cdots, t_l) \in [r, R]^l$.

Since $\text{supp}(\gamma_0(t_1, \cdots, t_l)) \subset \overline{\Omega_\Gamma}$ for all $(t_1, \cdots, t_l) \in [r, R]^l$, then $\Phi_\lambda(\gamma_0(t_1, \cdots, t_l))$ does not depend on $\lambda \geq 0$. On the other hand,

$$\Phi_\lambda(\gamma_0(t_1, \cdots, t_l))) \leq c_\Gamma \quad \forall \ (t_1, \cdots, t_l) \in [r, R]^l,$$

and

$$\Phi_\lambda(\gamma_0(t_1, \cdots, t_l)) = c_\Gamma \quad \text{if and only if} \quad t_j = 1, \ \forall j \in \Gamma.$$

Thus, we have that

$$m_0 := \sup\{\Phi_\lambda(u) : u \in \gamma_0([r,R]^l)\backslash A_\zeta^\lambda\}$$

is independent of $\lambda \geq 0$ and $m_0 < c_\Gamma$. Now, observing that there exists $K_* > 0$ such that

$$|\Phi_\lambda(u) - \Phi_\lambda(v)| \leq K_* \|u - v\|_\lambda, \quad \forall\, u, v \in B_{2\rho}^\lambda,$$

we claim that if $T > 0$ is large enough, the estimate below holds

$$\max_{(t_1,\cdots,t_l)\in[r,R]^l} \Phi_\lambda(\eta(T, \gamma_0(t_1,\cdots,t_l))) < \max\{m_0, c_\Gamma - \frac{1}{2K_*}\sigma_0\mu\}. \tag{7.3}$$

In fact, writing $u = \gamma_0(t_1,\cdots,t_l)$, $(t_1,\cdots,t_l) \in [r,R]^l$, if $u \notin A_\mu^\lambda$, then by (7.1), we have

$$\Phi_\lambda(\eta(t,u)) \leq \Phi_\lambda(\eta(0,u)) = \Phi_\lambda(u) \leq m_0, \quad \forall t \geq 0.$$

On the other hand, if $u \in A_\zeta^\lambda$, by setting $\tilde{\eta}(t) = \eta(t,u)$, $\tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}$ and $T = \frac{\sigma_0\mu}{2K_*\tilde{d}_\lambda} > 0$. Now we distinguish two cases:

(1) $\tilde{\eta}(t) \in A_{3\zeta/2}^\lambda \cap \Theta_\kappa \cap B_{2\rho}^\lambda$ for $\forall\, t \in [0,T]$.

(2) $\tilde{\eta}(t_0) \notin A_{3\zeta/2}^\lambda \cap \Theta_\kappa \cap B_{2\rho}^\lambda$ for some $t_0 \in [0,T]$.

If case (1) holds, we have $\Psi(\tilde{\eta}(t)) \equiv 1$ and $\|\Phi_\lambda'(\tilde{\eta}(t))\| \geq \tilde{d}_\lambda$ for all $t \in [0,T]$. Thus, by (7.1), we have

$$\begin{aligned}
\Phi_\lambda(\tilde{\eta}(T)) =& \Phi_\lambda(u) + \int_0^T \frac{d}{ds}\Phi_\lambda(\tilde{\eta}(s))ds \\
\leq& c_\Gamma - \int_0^T \tilde{d}_\lambda ds \\
=& c_\Gamma - \tilde{d}_\lambda T \\
\leq& c_\Gamma - \frac{\sigma_0\mu}{2K_*}.
\end{aligned}$$

If the case (2) holds. In this case we have the following situations:

(i) There exists $t_2 \in [0,T]$ such that $\tilde{\eta}(t_2) \notin \Theta_\kappa$, and thus, for $t_1 = 0$ it yields that

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\|_\lambda \geq \kappa > \zeta,$$

because $\tilde{\eta}(t_1) = u \in \Theta$.

(ii) There exists $t_2 \in [0,T]$ such that $\tilde{\eta}(t_2) \notin B_{2\rho}^\lambda$, so that for $t_1 = 0$, we obtain

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\|_\lambda \geq \rho > \zeta,$$

because $\tilde{\eta}(t_1) = u \in B_\rho^\lambda$.

(iii) $\tilde{\eta}(t) \in \Theta_\kappa \cap B^\lambda_{2\rho}$, and there are $0 \le t_1 < t_2 \le T$ such that $\tilde{\eta}(t) \in A^\lambda_{3\zeta/2} \backslash A^\lambda_\zeta$ for all $t \in [t_1, t_2]$ with

$$|\Phi_\lambda(\tilde{\eta}(t_1)) - c_\Gamma| = \zeta \ \text{ and } \ |\Phi_\lambda(\tilde{\eta}(t_2)) - c_\Gamma| = \frac{3\zeta}{2}.$$

From the definition of K_*, we have

$$\begin{aligned}\|\tilde{\eta}(t_2) - \tilde{\eta}(t)\|_\lambda &\ge \frac{1}{K_*}\Big|\Phi_\lambda(\tilde{\eta}(t_2)) - \Phi_\lambda(\tilde{\eta}(t_1))\Big| \\ &\ge \frac{1}{2K_*}\zeta.\end{aligned}$$

By the mean value theorem and $t_2 - t_1 \ge \frac{1}{2K_*}\zeta$, we have

$$\begin{aligned}\Phi_\lambda(\tilde{\eta}(T)) =&\Phi_\lambda(u) + \int_0^T \frac{d}{ds}\Phi_\lambda(\tilde{\eta}(s))ds \\ \le&\Phi_\lambda(u) - \int_0^T \Psi(\tilde{\eta}(s))\|\Phi'_\lambda(\tilde{\eta}(s))\|ds \\ \le&c_\Gamma - \int_{t_1}^{t_2} \sigma_0 ds \\ =&c_\Gamma - \sigma_0(t_2 - t_1) \\ \le&c_\Gamma - \frac{\sigma_0\zeta}{2K_*},\end{aligned}$$

and so (7.3) is proved.

Fixing $\widehat{\eta}(t_1, \cdots, t_l) = \eta(T, \gamma_0(t_1, \cdots, t_l))$, we have that $\widehat{\eta}(t_1, \cdots, t_l) \in \Theta_{2\kappa}$, and so $\widehat{\eta}(t_1, \cdots, t_l)|_{\Omega'_j} \ne 0$ for all $j \in \Gamma$. Thus, $\widehat{\eta} \in \Gamma_*$ and

$$b_{\lambda,\Gamma} \le \max_{(t_1,\cdots,t_l)\in[r,R]^l} \Phi_\lambda(\widehat{\eta}(t_1, \cdots, t_l)) \le c_\Gamma - \frac{\sigma_0\zeta}{2K_*}, \ \forall\lambda \ge \lambda^*,$$

which contradicts Corollary 6.1(i) $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.

Thus, we can conclude that Φ_λ has a critical point $u_\lambda \in A^\lambda_\zeta$ for λ large enough. □

Proof of Theorem 1.1. By choosing $d > 0$ appropriately, from Proposition 7.2, there exists a family of nontrivial solutions (u_λ) to problem (4.3) verifying the following properties:

$$\begin{aligned}&\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \ \ \forall n \in N, \\ &\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^N\backslash\Omega'_\Gamma} \to 0, \\ &\Phi_{\lambda_n}(u_{\lambda_n}) \to b \le c_\Gamma \le d.\end{aligned}$$

Thus, from Proposition 5.1, we have

$$u_{\lambda_n} \to u \quad \text{in} \quad H_A^1(\mathbb{R}^N) \text{ with } u \in H_A^{0,1}(\Omega_\Gamma),$$

and $u|_{\Omega_j} \neq 0$, $\forall j \in \Gamma$.

From Proposition 5.2, u_λ is a solution for problem (1.1) and $\lambda > 0$ large. Moreover, by Proposition 5.1, we know that $u = 0$ on $(\Omega_\Gamma)^c$ and $u|_{\Omega_\Gamma} \in H_A^{0,1}(\Omega_\Gamma)$ is a nontrivial solution of

$$-(\nabla + \mathrm{i}A(x))^2 u + u = \Big(\int_{\Omega_\Gamma} \frac{|u|^p}{|x-y|^\mu} dy\Big)|u|^{p-2}u, \quad \text{in } \Omega_\Gamma,$$

where $\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j$. Thus, $I_\Gamma(u) \geq c_\Gamma$.

On the other hand, we also know that

$$\Phi_{\lambda_n}(u_{\lambda_n}) \to I_\Gamma(u)$$

implying that

$$I_\Gamma(u) = b \text{ and } b \geq c_\Gamma.$$

Since $b \leq c_\Gamma$, we deduce that

$$I_\Gamma(u) = c_\Gamma$$

which showing that u is a least energy solution of $(P)_{\infty,\Gamma}$. We complete the proof of Theorem 1.1. □

Remark 7.1. Since $\Omega = \bigcup_{j=1}^k \Omega_j$, where k is a finite positive integer, we may choose $d > 0$ large appropriately in Lemma 4.3, so that the conclusion of Corollary 1.2 holds.

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