J. Math. Anal. Appl. 501 (2021) 125197

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Recent developments in problems with nonstandard growth and nonuniform ellipticity ☆

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ARTICLE INFO

Article history:
Available online 26 March 2021
Submitted by G. Chen

Keywords:
Regularity
Nonstandard growth conditions
Nonuniform ellipticity

ABSTRACT

We provide an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators. Regularity theory is at the center of this paper.

Contents

1. Introduction

In this paper we will outline the most recent trends and advances in variational problems exhibiting nonstandard growth conditions and/or nonuniform ellipticity. This is currently an active field of research that has attracted the attention of researchers, especially during the past few years. Our work here is also in

☆ *Acknowledgments.* This work is supported by the University of Parma via the project "Regularity, Nonlinear Potential Theory and Related Topics". We wish to thank Professor Goong Chen for a very careful reading of a preliminary version of the manuscript and related useful suggestions, that eventually led to a considerable improvement.
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https://doi.org/10.1016/j.jmaa.2021.125197
0022-247X/

connection to this special issue of the Journal of Mathematical Analysis and Applications, which is aimed to collect some recent contributions to the field, with particular emphasis on regularity theory. The topic still contains many open questions and, as we will see in this paper, exhibits interactions with other fields as, for instance, Harmonic Analysis and Function Spaces theory. An overview of some of the latest results in the theory can already be found in Marcellini's contribution included here [170]. That contains also a few streamlined examples of the methods typically employed in regularity theory and it is written by one of the first pioneers of the field [165–168]. Here, we will also outline the context that pertains to the papers featured in this volume, and put them in a proper perspective. Most importantly, this paper can also be considered as a first update of the original survey [173], containing an account of the theory until 2005. In the following, we will basically confine our attention to elliptic problems. For a brief overview of recent results in the evolutionary case we refer to Section 7.

A classical chapter of modern Nonlinear Analysis deals with the regularity of minimizers of integral functional of the Calculus of Variations as

$$W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)\ni w\mapsto \mathcal{F}(w;\Omega):=\int_\Omega [F(x,Dw)-f\cdot w]\,dx\,. \tag{1.1}$$

In this paper we always take $F\colon \Omega\times\mathbb{R}^{N\times n}\to[0,\infty)$ to be a Carathéodory integrand and at least C^1-regular in the second, gradient variables, $\Omega\subset\mathbb{R}^n$ is an open subset, $n\geq 2, N\geq 1$, and $f\in L^n(\Omega;\mathbb{R}^N)$; additional assumptions will be introduced along the way.

The qualitative analysis of minima of $\mathcal{F}$ is related to that of regularity of solutions to nonlinear elliptic equations and systems of the type

$$-\operatorname{div} a(x,Du)=f \qquad \text{in } \Omega\subset\mathbb{R}^n\,, \tag{1.2}$$

where $a\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ is a continuous vector field. The connection comes naturally via the Euler-Lagrange system

$$-\operatorname{div}\partial_z F(x,Du)=f \qquad \text{in } \Omega\subset\mathbb{R}^n\,, \tag{1.3}$$

which is of the type considered in (1.2). Both (1.2) and (1.3) are equations in the scalar case $N=1$.

The study of regularity for both solutions to nonlinear elliptic systems of the type in (1.2), and of minima of integral functionals as in (1.1), is very much a classical topic, for which we refer to the treatises [118,120,122,148,177] and to the more recent survey [173]. At the origin of the theory we find the work of pioneers such as Bernstein, Hopf, Schauder, Caccioppoli, Morrey, Stampacchia, Nirenberg, De Giorgi, Nash, Moser, Ladyzhenskaya, Uraltseva, Maz'ya, Serrin, Krylov, Caffarelli, DiBenedetto. In particular, De Giorgi's original techniques [94] have made a lasting impact on regularity theory in nonlinear elliptic and parabolic problems.

Looking through the literature, the reader can find several notions of ellipticity. In our study, the commonly chosen notion is the positive definiteness of the matrix $\partial_z a(\cdot)$, in the sense that

$$\partial_{z_j^\beta} a_i^\alpha(x,z)\xi_i^\alpha\cdot\xi_j^\beta\equiv\partial_z a(x,z)\xi\cdot\xi\geq 0$$

holds for every choice of matrices $z,\xi\in\mathbb{R}^{N\times n}$ and $x\in\Omega$, provided $\partial_z a(x,z)$ exists. The above condition allows to prove regularity results for solutions provided it is properly reinforced in

$$g_1(x,|z|)|\xi|^2\lesssim\partial_z a(x,z)\xi\cdot\xi\lesssim g_2(x,|z|)|\xi|^2\,, \tag{1.4}$$

where $g_1,g_2\colon\Omega\times(0,\infty)\to(0,\infty)$ are functions bounded by the lowest and highest eigenvalues of $\partial_z a(\cdot)$, from above and below, respectively. In this case, suitable *growth assumptions* on the two functions g_1,g_2 provide quantitative conditions for regularity for solutions. When such conditions are considered in the setting of the variational case in (1.1), they connect to the *growth conditions* of the integrand $F(\cdot)$, establishing a natural linkage between regularity of solutions and coercivity and growth bounds for $F(\cdot)$. This linkage can be rather subtle; see Section 8 for an overview of the possible phenomena.

We recall that degenerate ellipticity occurs when a lower bound of the type

$$g_1(x,|z|) \geq \nu > 0$$

fails to be satisfied; it is important to remark here that uniformly elliptic problems in the sense we are meaning here can be degenerate as well. In this respect, a crucial example of this is given by the p-Laplacian functional

$$w \mapsto \int_\Omega |Dw|^p \, dx \,, \qquad p > 1 \,, \tag{1.5}$$

whose Euler-Lagrange equation (actually, a system) is

$$-\operatorname{div}(|Du|^{p-2}Du) = 0 \,. \tag{1.6}$$

The regularity theory of (1.6) above was initiated by Ural'tseva [209] and Uhlenbeck [208], in the scalar and in the vectorial case, respectively. In both cases, the gradient turns out to be locally Hölder continuous with some exponent (but not with every exponent). The best possible exponent still remains unknown, conjectured to be 1/3 by some researchers.

The functional in (1.5) is the prototype of functionals with standard p-polynomial growth conditions, i.e., those satisfying a double side bound of the type

$$|z|^p \lesssim F(x,z) \lesssim |z|^p + 1 \,, \qquad p > 1 \,. \tag{1.7}$$

There is a large amount literature on such functionals, especially as far as regularity issues are concerned (see [146,162,163], and also [173] for a review).

One of the main aims of this volume is to present recent developments on existence and regularity problems involving functionals where the bulk energy density does not satisfy (1.7). In particular, several contributions are dedicated to functionals satisfying nonstandard growth conditions of (p,q)-type

$$|z|^p \lesssim F(x,z) \lesssim |z|^q + 1 \qquad 1 < p \leq q \,, \tag{1.8}$$

following the terminology introduced in the basic papers of Marcellini [165–167].

These functionals are linked to nonuniformly elliptic problems, as described in Section 3. They emerge in several cases of applications, as, for instance, in the homogenization of strongly anisotropic materials, as pointed out by Zhikov [214–217]; they also occur in relation to non-Newtonian Fluid Mechanics [187] (see also Subsection 3.5). Some of the problems and techniques arising in their study also intersect with different fields from Nonlinear Partial Differential equations as, for example, Nonlinear Potential Theory [55,56,132,141,145,146,172], Function Spaces Theory [96,126,185], Nonlocal Operators [7,8,143] (the last two are also included in this volume). These themes are indeed featured in this volume. Here we will also describe some recent directions concerning, for instance,

- Recent results for nonuniformly elliptic problems, with special emphasis on new gap bounds allowing to prove regularity of minima.
- New conditions on coefficients guaranteeing regularity in the case of nonautonomous integrals.

- Relations between regularity conditions and fundamental properties of the function spaces that are naturally associated with the operators considered. This includes suitable definitions of capacity, Hausdorff dimension, and conditions for controlling basic operators from Harmonic Analysis.
- Relations between regularity of minima, relaxed functionals and the Lavrentiev phenomenon.
- Borderline conditions on data guaranteeing regularity in nonuniformly elliptic problems.

As a note, we recall that the notion of local minimizers used here is quite standard, i.e., the following.

Definition 1. A map $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$ in (1.1) with $f \in L^n(\Omega;\mathbb{R}^N)$ if, for every open subset $\tilde\Omega \Subset \Omega$, we have $\mathcal{F}(u;\tilde\Omega) < \infty$ and $\mathcal{F}(u;\tilde\Omega) \leq \mathcal{F}(w;\tilde\Omega)$ holds for every competitor $w \in u + W^{1,1}_0(\tilde\Omega;\mathbb{R}^N)$.

We will systematically abbreviate "local minimizer" by simply writing "minimizer".

Remark 1. When considering systems of the type in (1.2), we will always consider distributional solutions. In the uniformly elliptic case, these will always be assumed to be also energy solutions. This means that the solution belongs to a function space which is fixed by the coercivity of the operator. For instance, in the case of (1.5), an energy solution is a distributional solutions that also belongs, at least locally, to $W^{1,p}$. See, for example, also the space $W^{1,A(\cdot)}$ in Section 2. The main feature of energy solutions is that the weak formulation of (1.2), i.e.,

$$\int_\Omega a(x,Du)\cdot D\varphi\,dx = \int_\Omega f\cdot\varphi\,dx\,, \tag{1.9}$$

which valid a priori for every test function $\varphi \in C^\infty_0(\Omega)$, can be actually tested for maps φ which are essentially proportional to the solution u. In the nonuniformly elliptic case, the concept of energy solution is not very clear, as it is not always possible to associate to each operator a natural function space leading to a suitably large set of test functions for (1.9). We refer to the work of Zhikov [214–217] for a discussion of such aspects and related density arguments. For this reason, in this paper, when dealing with the nonuniformly elliptic case, we will mostly deal with minimizers. For these, equation (1.9) usually admits more test functions without additional density arguments. There are anyway cases in which a suitable notion of energy solution can be defined. We shall address this point in Remark 5 and Subsection 6.1.

2. The uniformly elliptic case

In the regularity theory of elliptic equations and systems of the type in (1.2), perhaps the most important quantity ruling a priori estimates is the ratio between the highest and the lowest eigenvalue of the matrix $\partial_z a(\cdot)$. In the simplest autonomous case

$$-\operatorname{div} a(Du) = f \qquad \text{in } \Omega\subset\mathbb{R}^n\,, \tag{2.1}$$

which is related to (almost) autonomous functionals of the type

$$w \mapsto \int_\Omega \left[F(Dw) - f\cdot w\right]\,dx\,,$$

the bounds in (1.4) can be written as

$$g_1(|z|)\mathbb{I}_{\rm d} \lesssim \partial_z a(z) \lesssim g_2(|z|)\mathbb{I}_{\rm d}\,. \tag{2.2}$$

As in [20], one is led to introduce the so-called *ellipticity ratio*

$$\mathcal{R}_a(z) := \frac{\text{highest eigenvalue of } \partial_z a(z)}{\text{lowest eigenvalue of } \partial_z a(z)} \,. \tag{2.3}$$

When $\mathcal{R}_a(z)$ remains bounded with respect to $|z|$, for $|z|$ large, this is what is called the *uniformly elliptic case*. Therefore, in view of (2.2), a condition of the type $g_2(t) \lesssim g_1(t)$ for t large, implies the uniform ellipticity of the operators and functionals considered. Most of the functionals with standard p-growth available in the literature are often uniformly elliptic, as the growth conditions in (1.7) turn out to be coupled with analogous, scaled growth conditions on second derivatives of the type

$$\partial_z a(z) \equiv \partial_{zz} F(z) \approx |z|^{p-2}\mathbb{I}_{\mathrm{d}} \tag{2.4}$$

for $|z|$ large, from where *follow* the growth conditions of $F(\cdot)$. An example is indeed given by (1.5), where $F(z) \equiv |z|^p$ and therefore

$$g_1(t) \approx g_2(t) \approx t^{p-2} \,, \tag{2.5}$$

holds, thereby implying uniform ellipticity (yet, the operator in question is degenerate). It is important to note that uniformly elliptic problems do not always obey polynomial growth conditions as in (2.4). A classical example of this situation occurs in the case

$$-\operatorname{div}(\tilde{a}(|Du|)Du) = f \tag{2.6}$$

under the uniformly elliptic assumption

$$\begin{cases} -1 < i_{\mathrm{a}} \leq \dfrac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_{\mathrm{a}} < \infty \quad \text{for every } t > 0 \\ \tilde{a} \colon (0,\infty) \to [0,\infty) \text{ is of class } C^1_{\mathrm{loc}}(0,\infty) \,. \end{cases} \tag{2.7}$$

These problems are naturally well-posed in the Orlicz space $W^{1,A(\cdot)}(\Omega)$, defined by the function

$$A(t) := \int_0^t \tilde{a}(s)s\, ds \,. \tag{2.8}$$

Under condition $(2.7)_1$, this is essentially the space of maps w whose distributional derivatives are such that $A(|Dw|) \in L^1(\Omega)$ and it turns out that $W^{1,A(\cdot)} \subset W^{1,1}$. Distributional solutions u to (2.6) such that $u \in W^{1,A(\cdot)}$ turn out to be the natural energy solution of the problem. In this case we have that

$$\begin{cases} |\partial_z a(z)| \leq \sqrt{Nn}\max\{1, s_{\mathrm{a}}+1\}\tilde{a}(z) \,, \\ \min\{1, i_{\mathrm{a}}+1\}\tilde{a}(|z|)|\xi|^2 \leq \partial_z a(z)\xi\cdot\xi \,, \end{cases}$$

hold for all $z, \xi \in \mathbb{R}^{N\times n}$, $|z| \neq 0$. It is easily seen that this implies the uniform bound

$$\mathcal{R}_a(z) \lesssim \frac{\max\{1, s_{\mathrm{a}}+1\}}{\min\{1, i_{\mathrm{a}}+1\}} \tag{2.9}$$

and the operator in (2.6) is uniformly elliptic. The case of the p-Laplacian operator in (1.5) is of the type considered in (2.6) where

$$\tilde{a}(t) \equiv t^{p-2}\,, \qquad i_{\rm a} = s_{\rm a} = p-2\,. \tag{2.10}$$

The function $A(\cdot)$ in (2.8) turns out to be $A(t) \equiv t^p/p$. The space $W^{1,A(\cdot)}(\Omega)$ becomes $W^{1,p}(\Omega)$ and we return to the usual energy solutions in the sense of Remark 1.

The literature concerning regularity for such operators is large, starting from the seminal paper of Lieberman [157]. Amongst the recent, more relevant contributions, we quote [15,20,63,64]. For further results we refer to the literature mentioned in the last mentioned papers and to Theorem 14 at the very end of this paper.

Remark 2. In the case of the p-Laplacian functional (1.5), note that by (2.9)-(2.10), when $p \to 1$ the constants implied in (2.5) blow-up at rate $1/(p-1)$. We are approaching the case of linear growth functionals. These are often non-uniformly elliptic in the sense that bounds on second derivatives do not follow the growth of the integrand as, for instance, in (2.4). A typical example is given by the minimal surface functional

$$w \mapsto \int_\Omega \sqrt{1+|Dw|^2}\, dx\,. \tag{2.11}$$

See also [149,190]. More general versions of the previous classic functionals have been treated, and these are of the type

$$w \mapsto \int_\Omega \left(1+|Dw|^m\right)^{1/m}\, dx\,, \qquad m \geq 1\,. \tag{2.12}$$

See, for instance, [182]. They in fact share similar features with almost linear growth functionals of the type

$$w \mapsto \int_\Omega |Dw| \log\left(1+|Dw|\right)\, dx\,. \tag{2.13}$$

Note that when considering functionals as

$$w \mapsto \int_\Omega |Dw|^p \log\left(1+|Dw|\right)\, dx\,, \qquad p > 1\,,$$

we are instead returning to the realm of uniform ellipticity. For recent progress on linear and nearly linear growth functionals as those in (2.12) and (2.13), we refer to [21,22,25]. We note that, while functionals of the type (2.13) are coercive in the Sobolev-Orlicz spaces, still implying that the gradient of minima and competitors belongs to $W^{1,1}$, the natural ambient space of functionals as in (2.11)-(2.12) is BV, the space of functions with bounded variation. In this case the distributional derivatives of u are measures.

3. Nonuniform ellipticity

When the quantity $\mathcal{R}_a(z)$ in (2.3) becomes unbounded as $|z| \to \infty$, or when there is no bound a priori known for $|z|$ large, we are having in hand a *nonuniformly elliptic operator*.

Remark 3. Lipschitz continuity is a focal point of regularity in the nonuniformly elliptic problems considered here. Indeed, if $|Du| \leq M$ for some $M > 0$, then the growth of $\partial_z F(z)$ becomes irrelevant for $|z| \geq M$ and we are back to the realm of uniformly elliptic operators. As a matter of fact, using this type of observations, in [89, Section 6] a method to reduce the proof of Lipschitz estimates for certain classes of nonuniformly elliptic problems to the one for uniformly elliptic problems has been derived.

Nonuniform ellipticity is a classical topic, naturally occurring in a variety of situations. The study of nonuniformly elliptic equations started with some specific model examples, as the minimal surface equation. In the version considered here, the theory finds its roots in the seminal work of Finn [106], Ladyzhenskaya & Uraltseva [148–150] (classical setting), Stampacchia [197], Hartman & Stampacchia [129] (variational integrals), Trudinger [202–206], Ivočkina & A.P. Oskolkov [138,180], Serrin [193] (also minimal surfaces), A.V. Ivanov [135–137], Giusti [121], L. Simon [190], Uraltseva & Urdaletova [210], Lieberman [155,156] (in a setting related to Simon's). In particular Serrin's classical paper [193] contains important theorems concerning equations also linked to minimal surfaces. Ivanov's monograph [137] gives a comprehensive account of the results connected to St. Petersburg's school in the 60/70s.

3.1. The case of autonomous functionals

For the rest of this section we specialize on the scalar case $N = 1$. As mentioned above, functionals with (p, q)-growth as in (1.8) involve the occurrence of nonuniformly elliptic operators. Such conditions often match with similar, scaled ones on the second derivatives of F, i.e.,

$$|z|^{p-2}\mathbb{I}_{\rm d} \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_{\rm d} \qquad \text{for } |z| \text{ large}\,, \tag{3.1}$$

so that the only information we typically have to bound $\mathcal{R}_{\partial_z F}(z)$ is

$$\mathcal{R}_{\partial_z F}(z) \lesssim |z|^{q-p} + 1\ . \tag{3.2}$$

In view of (3.2), gap quantities such as q/p and $q - p$ immediately appear to play a major role in the regularity theory of (p, q)-growth functionals, as they can provide a measure of the rate of non-uniform ellipticity of the functional $F(\cdot)$. For this we refer to Subsection 3.2 for more information on this aspect.

A typical example of autonomous functional with (p, q)-growth is

$$w \mapsto \int_\Omega \left[|Dw|^p + \sum_{i=1}^{n} |D_i w|^{q_i} - fw\right] dx\,, \qquad 1 < p < q_1 \leq \cdots \leq q_n\,. \tag{3.3}$$

This was first considered in the pioneering work of Uraltseva & Urdaletova [210], where the special case of bounded minimizers was considered. For the general case the first results are due to Marcellini [165]. In this case it is $\mathcal{R}_{\partial_z F}(z) \lesssim |z|^{q-p} + 1$, where $q := \max\{q_k\}$, and nonuniform ellipticity occurs when all the exponents are not equal.

We also mention the very recent, remarkable paper [33], where the functional

$$w \mapsto \int_\Omega \sum_{i=1}^{n} |D_i w|^{q_i}\, dx \tag{3.4}$$

is considered. Note that in this case even a lower bound (3.1) is not satisfied as we are faced with a so-called orthotropic structure. See Theorem 5.

3.2. Gap bounds for (p, q)-growth, autonomous functionals

When $f \equiv 0$, the regularity theory in the autonomous case

$$w \mapsto \int_\Omega F(Dw)\, dx \tag{3.5}$$

is well developed, especially for the one of (p,q)-growth functionals as in (1.8). In many cases, the first step in regularity to prove that minimizers u, that obviously belong to $W^{1,p}_{\rm loc}$ by the lower bound in (1.8), actually belong to $W^{1,q}_{\rm loc}$. This allows to test the weak form of the Euler-Lagrange equation by functions that are proportional to u. Eventually, this opens the way for proving the Lipschitz continuity of minima. As mentioned above, Lipschitz continuity is a focal point of regularity for the kind of nonuniformly elliptic problems considered here.

In the (p,q) setting, the crucial condition to prove the Lipschitz continuity of minima is the assumption of a bound of the type

$$\frac{q}{p} < 1 + {\rm o}(n)\,, \tag{3.6}$$

where

$${\rm o}(n) \approx \frac{1}{n} \tag{3.7}$$

for n large. Conditions of such a type are necessary and sufficient for regularity [23,119,133,166], under suitable convexity conditions. In fact, different assumptions lead to different bounds, but all sharing the same asymptotic (3.7), when $n \to \infty$. An example of this situation is given by the following.

Theorem 1 *(Bella & Schäffner [23]). Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a minimizer of functional (3.5), where $F\colon \mathbb{R}^n \to \mathbb{R}$ is C^2-regular, and satisfies*

$$\begin{cases} \nu(|z|^2+1)^{p/2} \lesssim F(z) \lesssim (|z|^2+1)^{q/2}\,, \\ |\partial^2 F(z)| \lesssim L(|z|^2+1^2)^{(q-2)/2}\,, \qquad\qquad 2 \le p \le q \\ \nu(|z|^2+1)^{(p-2)/2}|\xi|^2 \lesssim \partial^2 F(z)\xi\cdot\xi\,, \end{cases} \tag{3.8}$$

for every choice of $z, \xi \in \mathbb{R}^n$. Assume that

$$\frac{q}{p} < 1 + \min\left\{1, \frac{2}{n-1}\right\}\,. \tag{3.9}$$

Then Du is locally bounded in Ω.

The one in (3.9) is, up to now, the best bound found in the autonomous case under general (p,q)-growth conditions in order to get the Lipschitz regularity, and improves on the one originally found by Marcellini in [166], which was

$$\frac{q}{p} < 1 + \frac{2}{n}\,.$$

It would be interesting to check if a Sobolev embedding exponent related bound of the type

$$\frac{q}{p} < \frac{n}{n-p}\,, \qquad p < n$$

would be sufficient to imply the local Lipschitz continuity. In this respect some advance is marked in [49] under additional assumptions. In relation to Theorem 1, we may mention also some recent, interesting vectorial version of Schäffner [188], where local $W^{1,q}$-regularity of minima is proved under the same bound in (3.9). Applications to partial regularity are also provided in [188].

At the end of this section we note that, as low order regularity of minimizers of functionals with (p, q)-growth, the picture becomes more complete, and actually boundedness can be obtained directly for nonautonomous integrals of the type

$$w \mapsto \mathcal{F}_0(w, \Omega) := \int_\Omega F(x, Dw)\, dx\,, \tag{3.10}$$

under very weak assumptions on continuity of coefficients. We indeed have

Theorem 2 *(Hirsch & Schäffner [133]). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be minimizer of the functional in (3.10), where $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is Carathéodory regular and satisfies*

$$\begin{cases} |z|^p \lesssim F(x,z) \lesssim |z|^q + 1 \\ F(x, 2z) \lesssim F(x,z) + 1\,, \end{cases}$$

for $(x, z) \in \Omega \times \mathbb{R}^n$ (in the Carathéodory sense), and where $1 < p \leq q$. Assume that

$$\frac{q}{p} \leq 1 + \frac{q}{n-1}\,. \tag{3.11}$$

Then u is locally bounded in Ω.

A rather relevant fact here is that (3.11) is optimal in view of Marcellini's examples [166]. Note that in this case no regularity is assumed on the coefficients, that is, no smoothness is assumed on the partial function $x \mapsto F(x, z)$. In this respect, there is no difference with the case $p = q$. The situation changes completely when considering higher regularity of minima of nonautonomous functionals, as we are going to see in Section 4. Theorem 2 brings to a conclusion a line of research devoted to find optimal bounds guaranteeing that minimizers are locally bounded; in this respect see also the recent papers [70–72].

3.3. Interpolative gap bounds

Better, still non-dimensional bounds can be obtained assuming that solutions are a priori more regular (before reaching Lipschitz continuity). The first result in this direction is again in the pioneering paper [210]. A general result is the following.

Theorem 3 *(Choe [61]). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of functional (3.5), where $F\colon \mathbb{R}^n \to \mathbb{R}$ is C^2-regular and satisfies (8.7) for all $z, \xi \in \mathbb{R}^n$. Assume that*

$$1 < p \leq q < p + 1\,. \tag{3.12}$$

Then Du is locally bounded in Ω.

The bound in (3.12) has also been considered in [48], in the vectorial case, where $W^{1,q}$-local regularity of minima has been obtained. The one in (3.12) is a bound of interpolative type, and its effectiveness relies on hidden forms of interpolation inequalities embedded in the PDE estimates, exploiting the a priori assumption $u \in L^\infty$. They find a precise counterpart in the nonautonomous case; see Section 4.

The phenomenon is even more general, as assuming more and more a priori regularity on minima leads to better and better gap bounds. The following, vectorial result, offers an example.

Theorem 4 *(De Filippis & Mingione [92]). Let $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)\cap C^{0,\gamma}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of functional (3.5), where $0<\gamma<1$ and $F\colon \mathbb{R}^n \to \mathbb{R}$ is C^2-regular, and satisfies (3.8) for all $z,\xi \in \mathbb{R}^{N\times n}$. Assume that*

$$q < p + \frac{2\gamma}{\vartheta(1-\gamma)}\,, \qquad \textit{where} \;\; \vartheta := \begin{cases} 1 & \textit{if} \;\; p \geq 2 \\ 2/p & \textit{if} \;\; 1<p<2\,. \end{cases} \tag{3.13}$$

Then $u \in W^{1,q}_{\rm loc}(\Omega;\mathbb{R}^N)$.

In Theorem 4 we note that the asymptotic of the gap bound is of the type

$$q < p + \mathrm{O}(\gamma)\,, \qquad \mathrm{O}(\gamma) \approx \frac{1}{1-\gamma}\,. \tag{3.14}$$

This reconnects with what is stated in Remark 3: once we know that the gradient is bounded – this formally corresponds to $\gamma = 1$ – the problem becomes one of uniform ellipticity. Indeed in the limit $\gamma \to 1$, in (3.7) no bound is required on q/p.

Another interesting feature of interpolative gap bounds here is that, assuming more structural properties on the functional, it progressively leads to the improvement of (3.12) (still keeping fixed the assumption of bounded minimizers). For instance, some higher integrability results can be proved by means of a maximum principle valid for functionals depending on minors of the Jacobian (here in the vectorial case). These involve the bound $q < p+2$ (see [173] for more precise references).

When coming to the limiting case of very specific structures, bounds might simply disappear, as the following shows.

Theorem 5 *(Bousquet & Brasco [33]). Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a locally bounded minimizer of the functional in (3.4), where $q_1 \geq 2$. Then Du is locally bounded in Ω.*

Additional, recent results under special splitting structures are due to Bildhauer & Fuchs [28] in this volume. In this case these authors consider the interesting situation of two dimensional ($\Omega \subset \mathbb{R}^2$) variational integrals of the type

$$w \mapsto \int_\Omega [F_1(D_1 w) + F_2(D_2 w)]\,dx\,.$$

3.4. (p,q)-growth conditions and models from Nonlinear Elasticity

In the vectorial case $w\colon \Omega \to \mathbb{R}^N$, $N>1$, functionals with (p,q) growth conditions have been initially considered also related to problems arising from Nonlinear Elasticity. In [12], Ball has pointed out the relevance of the concept of Quasiconvexity (originally introduced by Morrey [178]) in Materials Science, and in particular, the role of the so-called polyconvex functionals. These are integral functionals whose integrand is a convex function of the minors of the Jacobian matrix $\{D_i u^\alpha\}$. A typical polyconvex functional is

$$w \mapsto \int_\Omega [|Dw|^p + |\det(Du)|^s\,]\,dx\,, \qquad s>1\,. \tag{3.15}$$

Integrands $F(\cdot)$ in (3.15) satisfy global growth conditions of the type

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1\,, \qquad q = sn\,. \tag{3.16}$$

Cavitation phenomena can be then interpreted in terms of the Lavrentiev phenomena, and in turn this leads to the need of introducing suitably relaxed functionals and the related the Lavrentiev gap functionals. This approach has been pursued originally by Marcellini [164] and Zhikov [214–217]. See also the interesting issues raised (and solved) in [13,111,112]. In Section 5, we expand on relaxed functionals, nonuniform ellipticity and regularity. We only mention that interpolative gap bounds as those considered in Subsection 3.3 are useful when considering functionals such as in (3.15), where the ratio $q/p \equiv sn/p$ can be much larger than one and might not obey (3.6)-(3.7). Moreover, when considering problems as in (3.15), the specific structure of the integrand comes into the play. In this respect, examples of polyconvex functionals, and therefore with nonstandard growth conditions, have been treated in this issue too, see [152]; see also the recent [69]. More general quasiconvex functionals in the setting of (p, q)-growth conditions have been treated in [191,192].

Let us also mention that, in the models considered by Ball, functionals such as (3.15) are not considered to be realistic. Indeed, in order to take into account the non interpretability of the matter one has to confine to competitors w such that when $\det(Dw) > 0$ and moreover, the energy density must be such that $F(z) \to \infty$ when $\det(z) \to 0$.

3.5. Non-Newtonian fluids and nonstandard growth conditions

These are fluids modeled via operators that different from the Laplacian. The equations considered generalize the Navier-Stokes system and are of the type

$$\begin{cases} u_t + \operatorname{div} u \otimes u - \operatorname{div} a(\mathcal{E}(u)) = g - D\pi \\ \operatorname{div} u = 0\,. \end{cases}$$

Here $u\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $\mathcal{E}(u)$ is the symmetric part of the gradient and π is the pressure. In the classical Navier-Stokes case $a(\mathcal{E}(u)) \equiv \mathcal{E}(u)$. In the Non-Newtonian case, instead, $a(\cdot)$ can be a general, nonlinear monotone operator. For instance, one can have $a(z) \equiv |z|^{p-2}z$. A general, basic account of the related mathematical theory can be found in the treatise [161]. In recent years there have been proposed several models for such fluids involving vector fields with nonstandard growth conditions. See [115,116], where $a(z)$ comes from a potential with nearly linear growth, i.e. $a(z) \equiv \partial_z F(z)$ and $F(z) \approx |z| \log(e+|z|)$. In other words, the underlying energy is the one in (2.13). Fluids with potentials having (p, q)-growth conditions have been considered in the stationary case, see, for instance [26], where a partial regularity is proposed. Moreover, a popular theory of so-called Electrorheological fluids has been considered for two decades. See [2,187] and related references. In this case we a have a nonautonomous model, i.e., $a(\cdot) \equiv a(x, z) \equiv |z|^{p(x)-2}z$. For this we refer to Section 4. This special issue also contains recent studies on the Non-Newtonian fluids, see [24].

3.6. Fractional operators, subelliptic problems

During the past decade, we have witnessed a tumultuous development concerning problems where ellipticity and coercivity properties are formulated using fractional spaces and energies. This has intensified since the basic paper of Caffarelli & Silvestre [46]. Equations involved might be in some way considered as problems with weaker ellipticity conditions and, as such, they naturally share a few properties with those of nonuniform ellipticity. Recently there has been an increasing interest in bridging the theory of nonlocal operators with the one of problems with nonstandard growth conditions. After the analog of the nonlocal p-Laplacian has been considered (see, for example, [34,35,76,77,113]), several papers have been devoted to the study of nonlocal operators having non polynomial growth with respect to the underlying Gagliardo norm. See [7,8,107,143,212], also in the present volume.

Another interesting direction is the one of nonuniformly elliptic operators on manifolds and, in particular, on Lie groups. Very little is known about this case. In the setting of (p, q)-growth conditions a first results, for $p = 2$, has been treated in [108] and is restricted to the special, yet meaningful case of the Heisenberg group.

3.7. Uniformly ellipticity vs (p, q)-growth conditions

It is often the case that several uniformly elliptic integrals are treated as functionals with (p, q)-growth conditions, simply because they cannot be posed within the framework of functionals with standard polynomial p-growth as in (1.7). This approach could be somehow misleading, and sometimes not very effective, as regularity for functionals with (p, q)-growth needs gap bounds of the type in (3.6) in order to be developed. In fact, there are integrals whose Euler-Lagrange equation is of the type in (2.6) but fail to satisfy any gap bound, as in the following.

Theorem 6 *(De Filippis & Leonetti [87]). Given $1 < p < q$, there exists a function $A(\cdot)$ of the type in (2.8), such that*

- *Conditions (2.7) holds for some $-1 < i_{\rm a} \leq s_{\rm a}$.*
- *Condition $|z|^p - 1 \lesssim A(z) \lesssim |z|^q + 1$ holds.*
- *Whenever $1 < p_1 < q_1$ are such that $q_1 - p_1 < q - p$, condition $|z|^{p_1} \lesssim A(z) \lesssim |z|^{q_1} + 1$ fails to hold.*

Nevertheless, minimizers of the functional

$$w \mapsto \int_\Omega A(|Dw|)\, dx$$

are locally Lipschitz continuous; see the results of [20,91] for local estimates and [63,64] for global ones. An explicit a priori estimate is given in Theorem 14. In conclusion, regularity theory of (p, q)-growth functionals does not apply efficiently in this case, whereas exploiting more carefully the specific structure of the functional in question leads to regularity of minima. We have already encountered a similar situation in Theorem 5.

3.8. Faster growth rates

Other types of nonuniformly elliptic functionals might exhibit faster growth conditions than the polynomial ones. A basic example is

$$w \mapsto \int_\Omega \exp(|Dw|^p)\, dx\,, \qquad p > 1\,, \tag{3.17}$$

considered in [78,98,158,167–169]. Actually, much larger families of functionals with arbitrary fast growth conditions can be treated, like

$$w \mapsto \int_\Omega \Big[\exp(\exp(\ldots\exp(|Dw|^p)\ldots))\Big]\, dx\,, \qquad p > 1\,. \tag{3.18}$$

For this we have the following.

Theorem 7 *(Marcellini [167]). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional in (3.18), then Du is locally bounded in Ω.*

Note that this result holds in the vectorial case $N > 1$, too, as in fact stated in theorem; compare also Theorem 13. Priori to this, Lieberman [158] proved the local Lipschitz regularity of minimizers of the functional (3.17) in the case $p = 2$. Lieberman's proof for this latter case is very special and does not seem to allow for further extensions. Another example is given by the functional

$$w \mapsto \int_\Omega \left[\exp(A_0|Dw|^p) + \sum_{i=1}^n \exp(A_i|D_i w|^p) - fw\right] dx \,,$$

where again it is $p \geq 1$ and $0 < A_0 \leq A_1 \leq \ldots \leq A_n$. This one above, in a sense, combines the features of the functionals in (3.3) and (3.17) and has been first explicitly considered in [20], where the Lipschitz continuity has been proved in the case

$$\frac{A_n}{A_0} < 1 + \frac{2}{n}$$

and assuming that f belongs to the Lorentz space $L(n,1)$ (provided $n > 2$); see also Section 8. Let us note that in the case of the functional in (3.18), we have [20, (6.13)]:

$$\mathcal{R}_{\partial_z F}(z) \lesssim [\exp(\exp(\ldots\exp(|z|^p)\ldots))]^{1+\delta} + 1 \qquad \text{for every } \delta > 0 \tag{3.19}$$

where, if $k \geq 2$ is the number of the exponentials involved in (3.18), the number in (3.19) is $k-1$ (it is zero in the case of (3.17)). In the case of one exponential we simply have $\mathcal{R}_{\partial_z F}(z) \lesssim t^p + 1$. Therefore, we face a faster growth for the ellipticity ratio too, with non-polynomial rate when at least two exponentials are composed.

3.9. Vectorial case, Uhlenbeck structures and partial regularity

The Lipschitz regularity results for minima presented up to now are mostly concerned with the scalar case $N = 1$. When dealing with the vectorial case, everywhere local regularity fails in general, as already known in most classical settings. We refer to [173] for a comprehensive discussion, counterexamples, and references.

There are now two ways to recover the Lipschitz and higher regularity. The first is to restrict to integrands with additional, special structure features. A structure guaranteeing for everywhere regularity is the quasidiagonal one also called the Uhlenbeck structure after the original work [208]. This means that, in (1.1), we take integrands

$$F(x, Dw) \equiv \tilde{F}(x, |Dw|)\,. \tag{3.20}$$

For instance, we have already seen that Theorem 7 holds in the vectorial case, too. Results in the vectorial case, and in the nonuniformly elliptic setting, can be found in [20,78,91,100]. We refer to Section 8 for a presentation of the most recent results from [91]. We also remark that special structures as in (3.20) have a certain number of additional properties. Most notably, they allow for certain peculiar test functions that are usually taken in the scalar case, but are otherwise not matching the ellipticity properties of $\partial_z F(\cdot)$ in the case no additional structure is considered and the problem is vectorial. Truncations, widely used since the pioneering work of De Giorgi [94], are a chief example in this respect. Much general assumptions can be considered, also in order to get existence results for large families of problems via truncations. See [151] for a pioneering contribution and [97] for applications to problems admitting very weak solutions. An example is provided in this volume; see [65].

The second way to deal with regularity for vectorial problems involves the concept of partial regularity. This means that solutions are proved to be regular only outside a negligible closed subset. We again refer to [173] for a discussion for the uniformly elliptic setting. Partial regularity has been treated at length in the nonuniformly elliptic case. An overview of the methods is available in [25]. Some results in the nonautonomous case can be found in [27]. Some of these results depend on the possibility of the construction of proper liftings in functions spaces, like, for example, in [191,192]. More recent results in this direction can be found in [50] in this volume. We also mention that some new gap bounds for partial regularity have been obtained in [188].

4. Nonautonomous integrals and notions of nonuniform ellipticity

Let us start out pointing out two basic, dual facts:

- In the uniformly elliptic, nonautonomous case, the presence of coefficients is often reduced to be a perturbation of a fixed autonomous integrand. In more general cases it is reduced to that of a family of autonomous, "frozen" integrands with uniform ellipticity properties. This is, for example, the case of product structures as

$$w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega \mathfrak{c}(x)\tilde{F}(Dw)\,dx\,, \qquad 0<\nu \leq \mathfrak{c}(\cdot) \leq L\,,$$

  where uniform ellipticity is described via

$$\partial_{zz}\tilde{F}(z) \approx g(|z|)\mathbb{I}_{\mathrm{d}}$$

  for a suitable function $g\colon (0,\infty) \to (0,\infty)$. In the chief model case given by the p-Laplacian operator (1.6), $g(t) \equiv t^{p-2}$. This fact leads to certain proper extensions of classical linear facts related to perturbation such as Schauder estimates [162,163], Calderón-Zygmund theory [99], and nonlinear potential estimates [145,146].
- In the autonomous, nonuniformly elliptic case, nonuniform ellipticity of functionals stems directly from the way the corresponding integrands depend on the gradient variable.

In the nonautonomous, nonuniformly elliptic case, both the previous bullet points might fail. The interesting thing is the appearance of a class of integrals where the nonuniform ellipticity stems directly from a subtle interaction between the coefficients and gradient variable. In particular, it may happen that there are integrands $F(\cdot)$ with (p,q)-growth conditions, such that whenever we fix $x_0 \in \Omega$, the integrand $z \mapsto F(x_0,z)$ is uniformly elliptic in the sense of Section 2, i.e.,

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) \lesssim 1\,. \tag{4.1}$$

Nevertheless, all the typical regularity features of uniformly elliptic integrals fail. For instance, the Calderón-Zygmund theory and Schauder theory are not valid in general [102,110], as well as perturbation methods. The crucial point here is that the very concept of nonuniform ellipticity must be *properly revisited and redefined* in the nonautonomous case in order to capture some of the peculiar phenomena appearing in that situation.

In this respect, crucial examples have been provided by Zhikov [214–217]. Amongst them, we single out two especially relevant functionals, that is

$$w \mapsto \int_\Omega |Dw|^{p(x)}\,dx\,, \tag{4.2}$$

for $p(x) > 1$, and

$$w \mapsto \int_\Omega [|Dw|^p + a(x)|Dw|^q]\,dx \tag{4.3}$$

with $1 < p \le q$, $0 \le a(\cdot) \in L^\infty(\Omega)$. Zhikov originally considered them in the setting of homogenization of strongly anisotropic materials and to show new examples of the Lavrentiev phenomenon. In the setting of nonautonomous functionals, the first represents a model where the transition between different growth exponents of the gradient is slower (when, for example, $p(\cdot)$ is continuous). The second one is in a sense the most extreme case of transition between two different growth exponents due to the presence of the space variable x.

The functional in (4.2) has been investigated with extraordinary intensity in the last years, and a vast amount of literature has been devoted to it; we refer, for instance, to [96,126,173,183–185] for a list of references and results from the various perspectives: PDE, Harmonic Analysis, Function Spaces theory, and applications.

As for the double phase functional (4.3), a complete regularity theory has been provided only recently (see [17] for results and related references), after the first higher integrability results were originally were obtained in [102].

In both cases (4.2) and (4.3), when fixing x_0, we obtain an integrand $z \mapsto F(x_0, z) := F_0(z)$ which is uniformly elliptic in the sense of (4.1), i.e., for the double phase we have

$$\mathcal{R}_{\partial_z F_0}(z) \approx \frac{\max\{1, q-1\}}{\min\{1, p-1\}} \frac{|z|^{p-2} + a(x_0)|z|^{q-2}}{|z|^{p-2} + a(x_0)|z|^{q-2}} \lesssim 1\,. \tag{4.4}$$

On the other hand, the behavior of functionals (4.2) and (4.3) is not comparable to those of uniformly elliptic ones. For instance, the Schauder theory does not hold in general, and in general it is possible to find counterexamples to continuity of minima when $a(\cdot)$ is Hölder continuous [102,110].

In the both cases of (4.2) and (4.3), the regularity of minimizers depends in a subtle way on the interaction between the growth conditions of $F(\cdot)$ with respect to the gradient variable z, and its regularity with respect to the x-variable. To better highlight this fact, let us consider the p-Laplacian functional with coefficients

$$w \mapsto \int_\Omega \mathfrak{c}(x)|Dw|^p\,dx \qquad 0 < \nu \le \mathfrak{c}(x) \le L\,. \tag{4.5}$$

In this case, the Hölder continuity of $c(\cdot)$ implies the Hölder continuity of the gradient. This is essentially a nonlinear analog of Schauder theory [162,163] when $p \neq 2$. When $p = 2$, this is precisely the content of Schauder theory. Now, in view of (4.4), we are lead to think that the same result holds in the case of the double phase functional in (4.3). That is, consideration of a Hölder continuous coefficient $a(\cdot)$ leads to gradient Hölder continuity. This is not the case by the counterexamples given in [102,110]. Specifically, as in [110], it is possible to construct a Hölder continuous coefficient $a(\cdot) \in C^{0,\alpha}$, such that there exists bounded but yet discontinuous minimizers of (4.3) provided

$$p < n < n + \alpha < q\,. \tag{4.6}$$

It obviously follows that such minimizers cannot belong to $W^{1,q}$. Moreover, the results of [110] show that such “bad” minimizers develop singularities on a fractal set of the Cantor type, whose Hausdorff dimension can be made arbitrarily close to $n-p$. Therefore, minimizers can be nearly as “bad” as any other competitor.

Towards a common understanding of functionals (4.2) and (4.3) and beyond, we need to develop a different, more general notion of non-uniform ellipticity. Specifically, following [91], for any given ball $B \subset \Omega$, we consider the nonlocal quantity

$$\mathfrak{R}_{\partial_z F}(z, B) := \frac{\sup_{x\in B} \text{ highest eigenvalue of } \partial_{zz} F(x, z)}{\inf_{x\in B} \text{ lowest eigenvalue of } \partial_{zz} F(x, z)} \,. \tag{4.7}$$

This is more efficient for the detection and quantification of nonuniform ellipticity than the natural pointwise (frozen) version of (2.3)

$$\mathfrak{R}_{\partial_z F}(x, z) := \mathcal{R}_{\partial_z F(x,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x, z)}{\text{lowest eigenvalue of } \partial_{zz} F(x, z)} \tag{4.8}$$

which is in some sense more rigid. As a matter of fact, the quantity in (4.7) reveals to play a key role when performing integral estimates for certain nonautonomous problems. Needless to say, it is

$$\mathfrak{R}_{\partial_z F}(x, z) \lesssim \mathfrak{R}_{\partial_z F}(z, B)$$

holds for every $x \in B$. Let us see how this fits the situations of (4.2) and (4.3) (for large $|z| > 1$).

As for (4.2), we have

$$\mathfrak{R}_{\partial_z F}(z, B) \approx |z|^{p_M(B)-p_m(B)} \,,$$

where $p_M(B)$ and $p_m(B)$ denote, respectively, the sup and the inf of $p(\cdot)$ on the ball B. In this case the required balancing condition implying that minimizers are not discontinuous is

$$\limsup_{\mathtt{r}(B)\to 0} \, [p_M(B) - p_m(B)] \log \left(\frac{1}{\mathtt{r}(B)}\right) < \infty \,, \tag{4.9}$$

where $\mathtt{r}(B)$ denotes the radius of the ball B. This condition is essentially sharp [173]; it also connects to the lack of the Lavrentiev phenomenon [214–217]. See also Remark 4.

Specifically, once (4.9) is assumed, minimizers of (4.2) are locally Hólder continuous. Reinforcement of (4.9) in

$$\limsup_{\mathtt{r}(B)\to 0} \, [p_M(B) - p_m(B)] \log \left(\frac{1}{\mathtt{r}(B)}\right) = 0 \tag{4.10}$$

implies that minimizers of (4.2) are locally Hölder continuous with every exponent. Finally, when the exponent function $p(\cdot)$ is itself Hölder continuous, that is, when

$$p_M(B) - p_m(B) \lesssim [\mathtt{r}(B)]^{\beta} \qquad \text{for some } \beta \in (0, 1] \tag{4.11}$$

holds, then the gradient of minima is locally Hölder continuous, too. This is the maximal regularity, and indeed, it meets the conditions of the standard case (4.5), where $p(\cdot)$ is a constant function. That is, in order to get local Hölder continuity of the minimizers, coefficients and dependence on x must be Hölder continuous. See [173, Section 7] for an overview on the regularity theory of elliptic problems with a variable growth exponent. See also [96,183,185].

Looking at (4.3), and considering a ball B such that $B \cap \{a(x) = 0\} \neq 0$, we have that

$$\mathfrak{R}_{\partial_z F}(z, B) \approx \|a\|_{L^\infty(B)} |B| |z|^{q-p} + 1 \,. \tag{4.12}$$

It follows that $\mathfrak{R}_{\partial_z F}(z,B) \to \infty$ when $|z| \to \infty$, and the functional in (4.3) belongs to the realm of nonuniformly elliptic integrals. Now, in order to prove regularity theorems, the delicate balance between the growth with respect $|z|$ on one side, and the regularity with respect to x on the other, is this time encoded by the condition

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}\,, \qquad a(\cdot) \in C^{0,\alpha}(\Omega)\,, \qquad \alpha \in (0,1]\,. \tag{4.13}$$

Condition (4.13) allows to correct the growth of $\mathfrak{R}_{\partial_z F}(z,B)$ with respect to $|z|$, using the smallness of $a(x)$ in $|B|$. This explains the occurrence of the bound in (4.13): when considering small balls touching $\{a(x) = 0\}$, the magnitude of α, and therefore the smallness of $\|a\|_{L^\infty(B)}$, helps compensating the growth of $\mathfrak{R}_{\partial_z F}(z,B)$ with respect to $|z|$ by (4.12). Notice that the bound in (4.13) is of the type already encountered in (3.6), but exhibits a further, delicate and precise dependence on α. Condition (4.13) allows to prove local Hölder continuity of the gradient of minimizers of (4.3) [17], so that we again obtain the maximal regularity. Most importantly, the bound in (4.13) is sharp; this can be inferred by looking at the conditions (4.6) that are implying the presence of discontinuous minimizers, as shown in [110].

In conclusion, we may say that assumptions (4.9)-(4.11) and (4.13) bring back the functionals (4.2) and (4.3), respectively, to the realm of functionals with standard p-growth. This analogy goes on in several respects. For instance, in Subsection 8.1, we will see that when considering non-homogeneous problems of the type

$$w \mapsto \int_\Omega [|Dw|^p + a(x)|Dw|^q - f \cdot w]\, dx\,, \tag{4.14}$$

the sharp assumptions to impose on f in order to have that minima are locally Lipschitz continuous are essentially the same as of the p-Laplacian functional. The same happens when considering the functional $w \mapsto \int_\Omega [|Dw|^{p(x)} - f \cdot w]\, dx$.

Remark 4. A refinement of condition (4.9), involving iterated logarithms, has been found by Zhikov for the proof of first density of smooth functions and then higher integrability results [218,219]. The effectiveness for regularity of a similar borderline correction, this time applied on the modulus of continuity of the function $a(\cdot)$, has been observed in [124] for the double phase case. See also [3].

4.1. Borderline cases, mixed variatonal structures

In many regularity problems where coefficients are required to be Hölder continuous, it happens that when passing to borderline cases, Hölder continuity can be replaced by a logarithmic modulus of continuity. This is the case here, too. A catch between functionals (4.2) and (4.3), is given by the functional

$$w \mapsto \int_\Omega [|Dw|^p + a(x)|Dw|^p \log(e + |Dw|)]\, dx\,, \tag{4.15}$$

first considered in [16]. In such a borderline case, Hölder continuity of minimizers is ensured by the occurrence of the same logarithmic modulus of continuity (4.9) working for the functional (4.2), that is,

$$\limsup_{\mathtt{r}(B)\to 0}\, [a_M(B) - a_m(B)] \log\left(\frac{1}{\mathtt{r}(B)}\right) < \infty\,, \tag{4.16}$$

where this time $a_M(B)$ and $a_m(B)$ denote the sup and the inf of $a(\cdot)$ in the ball B, respectively. Eventually, exactly as in the variable exponent case, when the limit in (4.16) is instead equal to zero, minimizers of

(4.15) are locally Hölder continuous with every exponent. Finally, if $a(\cdot)$ is itself Hölder continuous - this time not requiring any quantification as in (4.13) - gradients of minimizers of (4.15) are locally Hölder continuous too. For the sharpness of condition (4.16) and related issues about the Lavrentiev phenomenon for the functional (4.15), see also [11].

Further perturbation can be made by considering additional coefficients, that is considering functionals of the type

$$w \mapsto \int_\Omega \mathfrak{c}(x,w)[|Dw|^p + a(x)|Dw|^p \log(e+|Dw|)]\,dx\,, \qquad 0<\nu\le \mathfrak{c}(\cdot)\le L$$

under suitable conditions on the coefficients $\mathfrak{c}(\cdot)$. The outcome is a complete analog of the standard p-theory in [162,163]. Theorems of this type can be found in [67] of this volume. The analogy with operators with p-growth can be carried out in several respects. For instance, from this volume we mention the paper [43], where existence and regularity theorems have been developed for problems ruled by elliptic operators modeled on the Euler-Lagrange equation of the functional (4.15). Specifically, in [43] its authors consider problems of the type

$$-\operatorname{div}(|Du|^{p-2}Du + |Du|^{p-2}\log(e+|Du|)) = \mu \tag{4.17}$$

coupled with homogeneous Dirichlet boundary conditions, where μ is a Borel measure with finite total mass, and prove existence results in optimal Marcinkiewicz spaces. In this setting there is still a lot to do. Considering problems with measure data and nonstandard growth conditions is an interesting direction of research that we expect to inspire the work of forthcoming researchers in next years.

A similar, but more general situation, has been considered in [39], where its authors examined generalized double phase functionals of the type

$$w \mapsto \int_\Omega [\Phi_1(|Dw|) + a(x)\Phi_2(|Dw|)]\;dx\,, \tag{4.18}$$

where $\Phi_1, \Phi_2\colon [0,\infty)\to[0,\infty)$ are two Young functions with suitable convexity and growth conditions (e.g., convex, non-decreasing). Note that both the functional in (4.3) and (4.15) belong to the class considered in (4.18).

Another borderline case, this time unlike the case of (4.18), has been considered in [89]:

$$w \mapsto \int_\Omega \left[|Dw|\log(1+|Dw|) + a(x)(1+|Dw|^2)^{q/2}\right]\;dx\;. \tag{4.19}$$

This is a sort of double phase interpolation between standard polynomial growth functionals and functionals with nearly linear growth. This time regularity results implied by gap bounds in the spirit of (4.13) can be obtained by assuming that $a(\cdot)$ belongs to a suitable Sobolev space. We refer to Subsection 4.6 and Section 8 for more on variational integrals with differentiable, Sobolev coefficients.

The next step after (4.19) is to consider of course functionals of the type

$$BV(\Omega)\ni w\mapsto |Dw|(\Omega) + \int_\Omega [a(x)|Dw|^2 + |w-f|^2]\,dx\;. \tag{4.20}$$

Here BV denotes the space of functions with bounded variation and $|Dw|(\Omega)$ denotes the total variation of w. These functionals have been studied for the first time in the paper [127] from this volume, where its authors have proved interesting Γ-convergence theorems via Sobolev functionals of the type

$$w \mapsto \int_\Omega [|Dw|^{1+\varepsilon} + a(x)|Dw|^2 + |w-f|^2]\,dx$$

and

$$w \mapsto |Dw|(\Omega) + \int_\Omega [(a(x)+\varepsilon)|Dw|^2 + |w-f|^2]\,dx\,,$$

where $\varepsilon > 0$. They are mainly motivated by image segmentation problems, and indeed the additional lower order term $|w-f|^2$ serves to make the minimizer stick to the original image f. It is worth noticing that nonautonomous functionals with nonstandard growth have already been employed to build image segmentation models, where functionals with a variable growth exponent are used. For this see [52].

Along another direction, different mixtures between functionals (4.2) and (4.3) have been considered in the literature, too. For instance, in [186] we find

$$w \mapsto \int_\Omega [|Dw|^{p(x)} + a(x)|Dw|^{q(x)}]\,dx\,. \tag{4.21}$$

In this case the regularity conditions are

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{\alpha}{n}\,, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)\,, \qquad \alpha \in (0,1]\,. \tag{4.22}$$

Moreover, $1 < p_0 \le p(\cdot) \le q(\cdot) < q_0$, are assumed to be Hölder continuous as well. The outcome is that gradients of minimizers are locally Hölder continuous. Later on, the strict inequality in (4.21) has been replaced by $\le$ in the setting of higher gradient integrability estimates [41] in this volume (see also [90] where the technique for reaching the delicate equality case has been first introduced).

The phenomenon displayed by the double phase functional, and concerning the delicate balance contained in bound (4.13), is not an isolated circumstance. For instance, multiphase problems can be considered as well, as first done in [93]. These involve functionals of the type

$$w \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q + b(x)|Dw|^s\right]\,dx\,, \qquad 1 < p \le q \le s\,. \tag{4.23}$$

In this situation we have that the local Hölder gradient continuity of the minimizers is ensured by

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}\,, \quad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)\,, \quad \frac{s}{p} \le 1 + \frac{\beta}{n}\,, \quad 0 \le b(\cdot) \in C^{0,\beta}(\Omega)\,, \quad \alpha, \beta \in (0,1]\,. \tag{4.24}$$

These assumptions are again seen to be optimal by the results in [110] and show that the regularity of coefficients distributes according to the growth with respect to the gradient. Similar structures have been considered in [101]. We note that the methods considered in [93] extend to more general functionals of the type

$$w \mapsto \int_\Omega [|Dw|^p + \sum_{k=1}^m a_k(x)|Dw|^{q_k}]\,dx\,, \qquad 1 < p < q_1 \le \cdots \le q_m\,. \tag{4.25}$$

Accordingly, the conditions allowing for regularity of minimizers become

$$\frac{q_k}{p} \le 1 + \frac{\alpha_k}{n}\,, \qquad 0 \le a_k(\cdot) \in C^{0,\alpha_k}(\Omega)\,, \qquad \alpha_k \in (0,1]\,. \tag{4.26}$$

Note that conditions (4.24) and (4.26) tell that, in order to rebalance larger exponents q_k, we need a larger rate α_k of Hölder continuity of the corresponding coefficient $a_k(\cdot)$.

4.2. Fully nonlinear structures

Very recently, anisotropic degenerate structures as in (4.2)-(4.3) have been treated in the setting of fully nonlinear differential equations. In particular, in [86] De Filippis has proved the local Hölder continuity of the gradient of viscosity solutions to degenerate equations of the type

$$[|Dw|^p + a(x)|Dw|^q]F(D^2u) \in L^\infty \,. \tag{4.27}$$

Here, the function $F(\cdot)$, which is defined on the set of $n \times n$-symmetric matrices, is assumed to be strongly elliptic (in the sense specified in [45], via maximal Pucci's operators). This extends basic work of Imbert & Silvestre [134] for the case $a(\cdot) \equiv 0$. Eventually, further results and developments concerning (4.27) have been obtained in the very interesting papers [81,82,104]. A model involving a variable exponent, i.e., $|Dw|^{p(x)}F(D^2u) \in L^\infty$, has been instead considered in [36].

4.3. Nonlinear Caldéron-Zygmund theory

Conditions of the type in (4.13) intervene also in the so-called Nonlinear Calderón-Zygmund theory. This is a far reaching extension of the classical linear Calderón-Zygmund theory based on singular integrals. To give an example, let us consider the following equation (or system):

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F)\,, \tag{4.28}$$

where $F \in L^p(\Omega;\mathbb{R}^n)$ is the assigned datum. We then have

$$F \in L^\gamma \Longrightarrow Du \in L^\gamma, \qquad \forall\, \gamma \geq p\,, \tag{4.29}$$

for any solution $u \in W^{1,p}$; see [95]. For $p = 2$, (4.29) is nothing but the classical Calderón-Zygmund theory. Both local and global versions of (4.29) can be proved, the last ones under suitable boundary conditions. There is a by now a large amount of literature on problems of type (4.28), both in the elliptic and in the parabolic case. When coming to non-uniformly elliptic problems and, in particular, under (p,q)-growth conditions, a complete parallel theory is still missing. Nevertheless, a perfect analog holds in the case special structures are considered. Operators stemming from (4.2) and (4.3) provide an example of this, where estimates can be proved under the same assumptions implying regularity of the minimizers. For instance, for equations and systems of the type

$$\operatorname{div}(|Du|^{p(x)-2}Du) = \operatorname{div}(|F|^{p(x)-2}F) \qquad \text{in } \Omega\,, \tag{4.30}$$

we have that

$$|F|^{p(x)} \in L^\gamma_{\rm loc}(\Omega) \Longrightarrow |Du|^{p(x)} \in L^\gamma_{\rm loc}(\Omega), \qquad \forall\, \gamma \geq 1\,, \tag{4.31}$$

provided condition (4.10) holds and that $|Du|^{p(x)} \in L^1_{\rm loc}(\Omega)$. See [173, Theorem 7.3] and related references, and [40] for a global result. The same happens in the case of double phase operators, where, for equations of the type

$$\operatorname{div}(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du) = \operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F) \quad \text{in } \Omega\,, \tag{4.32}$$

we have

$$|F|^p + a(x)|F|^q \in L^{\gamma}_{\rm loc}(\Omega) \Longrightarrow |Du|^p + a(x)|Du|^q \in L^{\gamma}_{\rm loc}(\Omega), \qquad \forall\, \gamma \geq 1\,, \tag{4.33}$$

provided (4.13) holds and that $|Du|^p + a(x)|Du|^q \in L^1_{\rm loc}(\Omega)$. For this see [90] and references, [38] for a boundary version, and [154] for estimates in Lorentz spaces. Both (4.31) and (4.33) are examples of how the mechanisms ruling regularity properties in the variational case then transfer to Calderón-Zygmund type theories. For instance, further results of Calderón-Zygmund theory type can be obtained using variable exponents in $p \equiv p(x)$ and $q \equiv q(x)$ in (4.32). These can be found in [41] of this volume. More delicate estimates and results of the type in (4.33) have been obtained in [200,201], and they make use of various kinds of fractional Maximal Operators. Very recently, Calderón-Zygmund estimates for multiphase problems - i.e., operators driven by the energy in (4.25) - have been proved in [9].

Estimates of the Calderón-Zygmund type can be extended to obstacle problems, too (see [42,154]); in that case various existence results are also available [213]. Operators with variable exponents, can be found in [154] of this volume.

We also note that the methods used to prove nonlinear Calderón-Zygmund type estimates for divergence form operators and in the nonuniformly elliptic setting are very general, and they extend to fully nonlinear equations, too, as, for instance, done in [153], again in this volume. It is worth remarking that analogs of variable exponent operators have also been considered in the fractional case, see for example [212] in this volume, and the related references.

Remark 5. There is actually a problematic point is defining weak solutions to both (4.30) and (4.32), in that it is not clear what kind of (distributional) solutions should be adopted and if there is a natural notion of energy solution according to Remark 1. This point will be addressed in Subsection 6.1.

4.4. Interpolative bounds in the nonautonomous case

Exactly as in the autonomous case − see Subsection 3.3 − the making of an assumptions of higher regularity on minimizers u allows to relax the bound in (4.13) in the case of the double phase functional (4.3). We report some results in [17]. The assumption that $u \in L^\infty$ leads to the relaxation of (4.13) in

$$q \leq p + \alpha\,, \tag{4.34}$$

from which local Hölder continuity of Du follows. Moreover, assuming that $u \in C^{0,\gamma}$ for $\gamma < 1$, we can use

$$q < p + \frac{\alpha}{1-\gamma} \tag{4.35}$$

to still obtain that Du is locally Hölder continuous. The sharpness of (4.34) has been proved in [102,110], where in fact counterexamples to continuity of minimizers have been found under condition (4.6). Finally, the sharpness of (4.35) is instead demonstrated in [10]. The bound in (4.34) is a nonautonomous counterpart of (3.12), whereas the bound in (4.35) shows the same asymptotic in (3.14). It is important to note that the counterexamples in [10,102,110] showing the sharpness of assumptions (4.13)-(4.35) are all based on the presence of the so called Lavrentiev phenomenon. For details see Section 5.

4.5. Nonautonomous integrals with fast growth

More dramatic examples of nonstandard growth conditions occur when nonautonomous integrands have fast growth. This is, for example, the case of

$$w \mapsto \int_\Omega \mathfrak{c}_1(x)\exp(\mathfrak{c}_2(x)|Dw|^p)\,dx\,, \tag{4.36}$$

where $p > 1$ and $0 < \nu \leq \mathfrak{c}_1(\cdot), \mathfrak{c}_2(\cdot) \leq L$. A main point here is that the integrand $F(\cdot)$ in (4.36) does not satisfy the so-called $\triangle_2$-condition, i.e., the control inequality

$$\tilde{F}(x, 2t) \lesssim \tilde{F}(x,t) \tag{4.37}$$

is not satisfied. As a consequence of (4.37), freezing coefficients at points x_0 does not preserve integrability

$$\exp(\mathfrak{c}_2(\cdot)|Dw|^p) \in L^1 \not\Longrightarrow \exp(\mathfrak{c}_2(x_0)|Dw|^p) \in L^1\,,$$

and standard techniques based on perturbation fail (a phenomenon that also occurs for integrands in (4.2) and (4.3)). Functionals with fast, non-polynomial growth conditions as in (4.36) are a classical topic. They have been treated for the first time by Duc & Eells [98], Lieberman [158] and Marcellini [167]. Evans [80] used them in the setting of weak KAM-theory. We shall revisit such functionals in Section 8.

4.6. General perturbation structures vs Sobolev coefficients

The model cases in (4.2) and (4.3) propose two meaningful examples where a complete regularity theory can be derived in the nonautonomous case. They both rely on the fact that, under the controlled oscillation conditions (4.9) and (4.13), some refined form of perturbation method is still possible (see [17]). Quite recently, a general approach aimed at unifying and extending these cases has been proposed in [125,131] for functionals of the type

$$w \mapsto \int_\Omega \tilde{F}(x, |Dw|)\,dx\,. \tag{4.38}$$

In these papers the convex function $t \mapsto \tilde{F}(x,t)$ satisfies a certain number of technical assumptions, amongst which the (p,q) growth conditions are formulated in the form

$$t \mapsto \frac{\partial_t \tilde{F}(x,t)}{t^{p-1}} \text{ is non-decreasing,} \qquad t \mapsto \frac{\partial_t \tilde{F}(x,t)}{t^{q-1}} \text{ is non-increasing} \tag{4.39}$$

for $1 < p \leq q$. The main assumption allowing for perturbation is then formulated by requiring that

$$\tilde{F}^+_B(t) \leq [1+\omega(\mathtt{r}(B))]\tilde{F}^-_B(t) \qquad \text{provided } \omega(\mathtt{r}(B)) \leq \tilde{F}^-_B(t) \leq \frac{1}{|B|} \tag{4.40}$$

for every $t > 0$, where $|B|$ denotes the measure of B and for a modulus of continuity $\omega(\cdot)$ (a non-decreasing, non-negative, concave function defined on non-negative real numbers), and where

$$\tilde{F}^+_B(t) := \sup_{x\in B} \tilde{F}(x,t) \qquad \text{and} \qquad \tilde{F}^-_B(t) := \inf_{x\in B} \tilde{F}(x,t)\,. \tag{4.41}$$

The modulus $\omega(\cdot)$ measures the oscillations of $x \mapsto F(x,z)$. See [131, Theorem 1.1] for the precise statement of the results. Perturbation is then made around minimizers of frozen, autonomous functionals $w \mapsto \int_B \bar{F}^-_B(|Dw|)\,dx$ whose integrand $\bar{F}^-_B$ is somehow comparable to $\tilde{F}^-_B$. This works thanks to the rebalancing condition (4.40). The assumption that

$$\omega(\cdot) \approx 1 \tag{4.42}$$

plus other less relevant technical ones, implies that minimizers are locally Hölder continuous for some exponent $\gamma \in (0,1]$. This result is available in [125]. In this case (4.42) accounts for measurable coefficients and actually no perturbation type approach can work. On the contrary, the proof proceeds via an analog of De Giorgi-Nash-Moser theory, still allowed by (4.40). Assuming that

$$\lim_{\mathtt{r}(B)\to 0} \omega(\mathtt{r}(B)) = 0\,, \tag{4.43}$$

one then obtains that minimizers are $C^{0,\gamma}$-regular, for every $\gamma < 1$. Eventually, assuming that

$$\omega(\mathtt{r}(B)) \lesssim [\mathtt{r}(B)]^{\beta} \qquad \text{for some } \beta \in (0,1] \tag{4.44}$$

implies the local Hölder continuity of the gradient of minimizers. Assumptions (4.42)-(4.44) parallel those in (4.9)-(4.11), and in fact coincide with them when $\tilde{F}(x,t) \equiv t^{p(x)}$. The above assumptions also allow to cover the double phase functional (4.3) and, applied to this case, reproduces (4.13).

Another interesting, dual approach has been proposed in [100], where general structures as in (4.38) have been considered under general (p,q)-growth, without assuming perturbation conditions as in (4.40). The approach in [100] also covers the vectorial case, while the one in [131] is at the moment still confined to the scalar one. Instead, Sobolev differentiability is assumed with respect to x. The idea is to assume that the partial map $x \mapsto \partial_t \tilde{F}(x,t)$ belongs to a suitable Sobolev space, in order to make it also Hölder continuous by embedding. This approach applied to model cases of the type

$$w \mapsto \int_{\Omega} \mathfrak{c}(x)\tilde{F}(|Dw|)\,dx\,, \qquad 0 < \nu \leq \mathfrak{c}(\cdot) \leq L\,, \tag{4.45}$$

corresponds to assume that $\mathfrak{c}(\cdot) \in W^{1,q}$ for some $d > n$. In turn, via the Sobolev-Morrey embedding, this implies the degree of Hölder continuity on $\mathfrak{c}(\cdot)$ that is necessary to get regularity properties according, for example, to what happens in the case (4.3) via conditions (4.13). For a detailed description of the results available using this approach in a wider setting, we refer to Subsection 8.1.

5. The Lavrentiev phenomena

This is a classical topic in the Calculus of Variations. It pertains to the occurrence of the fact that a functional $\mathcal{F}$, defined on a certain function space X, does not attain its infimum on a dense function subspace $Y \subset X$:

$$\inf_X \mathcal{F} < \inf_Y \mathcal{F}\,. \tag{5.1}$$

We have already mentioned it in Subsection 3.4. In the setting of functionals with (p,q)-growth it has been studied by Zhikov [214–217] and Marcellini [164], with important connections to Nonlinear Elasticity [13,111,112,164]. See also [1,37,103,109] for connections with relaxed functionals and more examples.

According to the classical definition, when considered in the setting of (p,q)-growth conditions, the Lavrentiev phenomenon (5.1) for functionals of the type (3.10), under conditions (1.8), occurs in the ball $B \Subset \Omega$ when

$$\inf_{w \in u_0 + W^{1,p}_0(B)} \mathcal{F}_0(w,B) < \inf_{w \in u_0 + W^{1,p}_0(B)\cap W^{1,q}_{\rm loc}(B)} \mathcal{F}_0(w,B)\,, \tag{5.2}$$

for a suitably regular boundary data u_0. This is by definition an obstruction to regularity of minimizers. An important catch, in our opinion, is that, in several cases, the same assumptions allowing for a priori regularity estimates for minimizers, can be used to prove the non-occurrence of the Lavrentiev phenomenon. In turn, as originally shown in [102], this fact can be reversed and the absence of Lavrentiev phenomenon can be eventually used to prove regularity estimates. We will elaborate on this approach in Subsection 5.1.

Related to the Lavrentiev phenomenon there lies the concept of approximation in energy. This takes place, for a map $w \in W^{1,p}(B;\mathbb{R}^N)$, when there exists a sequence of more regular maps $w_k \in W^{1,q}(B;\mathbb{R}^N)$ such that

$$\mathcal{F}_0(w_k, B) \to \mathcal{F}_0(w, B)\,. \tag{5.3}$$

As mentioned above, the same bounds allowing for regularity of minima, also work for the proof of approximation in energy. This fact goes back to the original work of Zhikov, who proved that, under condition (4.9),

$$\int_\Omega |Dw_k|^{p(x)}\,dx \to \int_\Omega |Dw|^{p(x)}\,dx \tag{5.4}$$

holds for a sequence of smooth maps $\{w_k\}$, whenever the right-hand side in the above display is finite. In a similar way, under condition (4.13), in [102] it has been proved that

$$\int_\Omega [|Dw_k|^p + a(x)|Dw_k|^q]\,dx \to \int_\Omega [|Dw|^p + a(x)|Dw|^q]\,dx \tag{5.5}$$

again for a sequence of smooth maps $\{w_k\}$. The fact extends even to the interpolation type bounds as in (4.34)-(4.35) and to general energies, as originally done in [102]. Indeed, let us consider a general Carathéodory integrand $F(\cdot)$ such that

$$|z|^p + a(x)|z|^q \lesssim F(x,z) \lesssim |z|^p + a(x)|z|^q + 1\,, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)\,, \tag{5.6}$$

holds for every $(x,z) \in \Omega \times \mathbb{R}^{N\times n}$. We then have the following.

Theorem 8 *(Baroni, Colombo & Mingione [17]). Let $u \in W^{1,p}(\Omega)$ be a minimizer of the functional $\mathcal{F}_0$ defined in (3.10), under assumption (5.6). Assume that either (4.34) or*

$$u \in C^{0,\gamma}(\Omega) \qquad \textit{and} \qquad q \le p + \frac{\alpha}{1-\gamma}\,, \qquad \gamma \in (0,1)$$

holds. Then, for every ball $B \Subset \Omega$, there exists a sequence $\{u_k\}$ of $W^{1,\infty}(B)$-regular functions such that $u_k \to u$ strongly in $W^{1,p}(B)$ and such that $F(u_k, B) \to \mathcal{F}(u, B)$.

Note that in Theorem 8 no convexity is assumed on $z \mapsto F(x,z)$; only (5.6) is needed. This result is sharp in view of the results from [10,110].

Issues like the absence of Lavrentiev phenomenon and the approximation in energy (5.3)-(5.5) do not only play a relevant role in regularity theory of minimizers, but are also linked to basic function theoretic properties of the Musielak-Orlicz spaces that are naturally associated with nonautonomous functionals such as (4.2) and (4.3). We shall elaborate on this in Subsection 6.1.

5.1. Regularity via a Lavrentiev gap

Here we discuss an approach pursued in [89,92,102]. In order to measure the failure of (5.3), it is possible to introduce suitable gap functionals, already considered in [1,20,102]. Let us take a functional of the type in (3.10) under (p, q)-growth conditions (1.8), which is convex with respect to the gradient variable. With $B \Subset \Omega$ denoting a ball, for every $w \in W^{1,1}(B;\mathbb{R}^N)$ we define (see also [109,164])

$$\mathcal{F}_0^q(w, B) := \inf_{\{w_k\}\subset W^{1,q}(B;\mathbb{R}^N)} \left\{ \liminf_k \mathcal{F}_0(w_k, B) : \ w_k \to w \text{ in } L^1(B;\mathbb{R}^N) \right\}$$

and, from there

$$\mathcal{L}_0^q(w, B) := \mathcal{F}_0^q(w, B) - \mathcal{F}_0(w, B)$$

this last one whenever $\mathcal{F}_0(w, B)$ is finite; we set $\mathcal{L}_0^q(w, B) = 0$ otherwise. Note that if $w \in W^{1,1}(B;\mathbb{R}^N)$ is a function such that $\mathcal{F}_0(w, B) < \infty$, then $\mathcal{L}^q(w, B) = 0$ iff there exists a sequence $\{w_k\} \subset W^{1,q}(B;\mathbb{R}^N)$ such that $w_k \rightharpoonup w$ weakly in $W^{1,1}(B;\mathbb{R}^N)$ and $\mathcal{F}(w_k, B) \to \mathcal{F}(w, B)$. We then assume that the integrand $F\colon \Omega \times \mathbb{R}^{N\times n} \to [0, \infty)$ is such that $z \to F(\cdot, z)$ is locally C^1-regular and satisfies

$$\begin{cases} \nu|z|^p \le F(x, z) \le L(1 + |z|^q)\,, \\ \nu\left(\lambda^2 + |z_1|^2 + |z_2|^2\right)^{(p-2)/2} |z_1 - z_2|^2 \le (\partial_z F(x, z_1) - \partial_z F(x, z_2)) \cdot (z_1 - z_2)\,, \\ |\partial_z F(x, z) - \partial_z F(y, z)| \le L|x - y|^\alpha (1 + |z|^{q-1})\,, \end{cases} \tag{5.7}$$

whenever $x, y \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^{N\times n}$, where $1 < p \le q$, $\lambda \in [0, 1]$, $\alpha \in (0, 1]$ and $0 < \nu \le 1 \le L$ are fixed constants. We have the following.

Theorem 9 *(Esposito, Leonetti & Mingione [102]). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional $\mathcal{F}_0$ in (3.10) under assumptions (5.7) and*

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}\,. \tag{5.8}$$

If

$$\mathcal{L}_0^q(u, B) = 0 \tag{5.9}$$

holds for a ball $B \Subset \Omega$, then $u \in W^{1,q}_{\mathrm{loc}}(B;\mathbb{R}^N)$.

In other words, the bound (5.8) found for the double phase functional now works for any nonautonomous functional, as assumption (5.9) allows us to avoid to assume a specific double phase structure. An additional assumption as in (5.6) then guarantees that (5.9) holds, and therefore we can always conclude with the higher integrability $u \in W^{1,q}_{\mathrm{loc}}(B;\mathbb{R}^N)$ in case of a double sided bound. We remark that a global version of the higher integrability result of Theorem 9 has been recently obtained by L. Koch [142].

This approach can be extended to the case of bounded minimizers, and connects to the results of Subsection 4.4. For this we define, for every $w \in W^{1,1}(B;\mathbb{R}^N)$:

$$\mathcal{F}_b^q(w, B) := \inf_{\{w_k\}\subset W^{1,q}(B;\mathbb{R}^n)\cap L^\infty(B;\mathbb{R}^N)} \left\{ \liminf_k \mathcal{F}_0(w_k, B) : \ w_k \to w \text{ in } L^\infty(B;\mathbb{R}^N) \right\}$$

and, whenever $\mathcal{F}(w, B)$ is finite, we set

$$\mathcal{L}_b^q(w, B) := \mathcal{F}_b^q(w, B) - \mathcal{F}(w, B)\,.$$

We set $\mathcal{L}_b^q(w, B) = 0$ if $\mathcal{F}(w, B) = \infty$.

Theorem 10 *(De Filippis & Mingione [89]). Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N) \cap L^{\infty}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional $\mathcal{F}_0$ in (3.10) $p \geq 2$, and assume that $q < p + \alpha$ and (5.7) hold. Assume also that*

$$\mathcal{L}^q_b(u, B_R) = 0 \tag{5.10}$$

holds for a ball $B \Subset \Omega$. If $\tilde{p}$ is such that $q < \tilde{p} < p + \alpha$, then $u \in W^{1,\tilde{p}}_{\rm loc}(B;\mathbb{R}^N)$.

Remark 6. A few comments are now in order.

- By Theorem 8 we get that an additional assumption such as (5.6) guarantees (5.10). As a consequence, in both Theorems 9 and 10, we can replace (5.9) and (5.10) by a double sided control as in (5.6). The same happens when a double side control of the type

$$b(x)G(z) \lesssim F(x,z) \lesssim b(x)G(z) + 1\,, \qquad 0 \leq b(\cdot), 1/b(\cdot) \in L^{\infty}(\Omega) \tag{5.11}$$

  holds and $G\colon \mathbb{R}^{N\times n} \to [0,\infty)$ is a convex function (see [102, Lemma 12] or [92, Corollary 1] for details).
- The Lavrentiev gap also intervenes in the setting of obstacle problems when considering nonautonomous integrals. In this case one minimizes a functional such as (3.10) under a constraint. Specifically, given a measurable function $\psi\colon \Omega \to \mathbb{R}$ and the convex set $\mathcal{K}_\psi(\Omega) := \{w \in W^{1,1}_{\rm loc}(\Omega)\colon w(x) \geq \psi(x)$ for a.e. $x \in \Omega\}$, a function $u \in W^{1,1}_{\rm loc}(\Omega) \cap \mathcal{K}_\psi(\Omega)$ is a constrained minimizer of $\mathcal{F}_0$ if, for every open subset $\tilde{\Omega} \Subset \Omega$, we have $\mathcal{F}_0(u;\tilde{\Omega}) < \infty$ and if $\mathcal{F}_0(u;\tilde{\Omega}) \leq \mathcal{F}_0(w;\tilde{\Omega})$ holds for every competitor $w \in u + W^{1,1}_0(\tilde{\Omega})$ such that $w \in \mathcal{K}_\psi(\Omega)$. In this regard, the first regularity result using the Lavrentiev gap in the setting of obstacle problems, is due to De Filippis [85] in this volume. In the more specific situation of Musielak-Orlicz spaces, obstacle problems have been also considered in [140]. Sharp results on regularity of minimizers with respect to the regularity of ψ can be found in [91].
- Lavrentiev gap functionals can be used in finding further interpolative gap bounds when dealing with general nonautonomous functionals with (p,q)-growth. In this case bounds of the type in (3.13) and (3.14) are effective to prove that minimizers are locally $W^{1,q}$-regular provided a suitable Lavrentiev gap functional is used; for this we refer to [92].
- When the Lavrentiev gap does not vanish one can still prove regularity results. Specifically, one can prove that minimizers of the relaxed functional are more regular. This, in turn, implies that original minimizer are regular when the Lavrentiev gap vanishes. Results as those in Theorem 9 follow therefore as corollaries. Let us give an example in that setting. A minimizer of the functional $\mathcal{F}^q_0(\cdot, B)$ is defined as a map $u \in W^{1,1}(B;\mathbb{R}^N)$ such that $\mathcal{F}^q_0(u,B) < \infty$ and such that $\mathcal{F}^q_0(u,B) \leq \mathcal{F}^q_0(w,B)$ holds for every $w \in u + W^{1,1}_0(B;\mathbb{R}^N)$. Now, if $u \in W^{1,1}_{\rm loc}(\Omega)$ is a minimizer of the original functional $\mathcal{F}_0$ in (3.10) such that $\mathcal{L}^q_0(u,B) = 0$, then

$$\mathcal{F}^q_0(u,B) = \mathcal{F}_0(u,B) + \mathcal{L}^q_0(u,B) = \mathcal{F}_0(u,B) \leq \mathcal{F}_0(w,B) \leq \mathcal{F}^q_0(w,B)$$

  holds whenever $w \in u + W^{1,1}_0(B)$. Therefore u also minimizes the relaxed functional $\mathcal{F}^q_0(\cdot,B)$ and its regularity in the ball B can be inferred by the available results on minimizers of $\mathcal{F}^q_0(\cdot,B)$. For this approach we refer to [102,142,165].

6. Connections

6.1. Function spaces theory, Harmonic analysis

The integrands considered in (4.2)-(4.3) belong to a larger class of integrands related to so-called Musielak-Orlicz spaces. Let us briefly recall the setting. We consider a function $\Phi\colon \Omega \times [0,\infty) \to [0,\infty)$, i.e., such that

$x \mapsto \Phi(x,t)$ is measurable for every $t \geq 0$ and $t \mapsto \Phi(x,t)$ is convex and non-decreasing for almost every $x \in \Omega$. We assume that

$$0 = \Phi(x,0) = \lim_{t\to 0+} \Phi(x,t) \qquad \text{and} \qquad \lim_{t\to\infty} \Phi(x,t) = \infty$$

again for every $x \in \Omega$. The Musielak-Orlicz space $L^{\Phi}(\Omega)$ is then defined as the space of measurable maps $f\colon \Omega \to \mathbb{R}^k$ such that there exists $h > 0$ such that

$$\int_{\Omega} \Phi\left(x, h|f|\right) dx < \infty .$$

It is equipped with the following Luxemburg type norm:

$$\|f\|_{L^{\Phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(x, \frac{|f|}{\lambda}\right) dx \leq 1 \right\} < \infty .$$

The local version $L^{\Phi}_{\rm loc}(\Omega)$ can be defined in the usual fashion. Such spaces are a significant generalization of the classical Orlicz spaces, given by the case Φ depends only on t. Examples are given by

$$\Phi_1(x,t) = t^{p(x)} \qquad \text{and} \qquad \Phi_2(x,t) = t^p + a(x)t^q, \qquad 1 \leq p \leq q\,. \tag{6.1}$$

Where $1 < p(x)$ and $0 \leq a(\cdot)$ are continuous and bounded functions. These two examples are obviously related to functionals in (4.2)-(4.3). Now, the crucial point is that the assumptions on the function Φ that are necessary in order to get regularity of minimizers of integrands of the type

$$w \mapsto \int_{\Omega} \Phi(x, |Dw|)\, dx\,,$$

often coincide, modulo minor technicalities, with those admitting fundamental good properties of the function spaces in question. The typical, common circle of ideas, as first highlighted by Zhikov [214–217], is related to

- Regularity of minimizers
- Absence of Lavrentiev phenomenon
- Density of smooth functions
- Boundedness of Maximal operators
- Boundedness of Singular and Fractional Integral Operators
- Possibility of defining a suitable notion of energy solution for critical points of functionals and general equations (see Remark 1).

For instance, when considering Φ_1 in (6.1), assumption (4.9) guaranteeing continuity of minimizers of (4.2), also make it possible to prove density of smooth functions in L^{Φ_1} and boundedness of various maximal and integral operators (see [96,185]). Moreover, optimal Sobolev and Sobolev-Morrey embedding have been established. We note that more specific embeddings can be proved when considering special geometries of the domain, with suitable bounds on $p(\cdot)$; an example is provided in [4] in this volume (see also the related paper [211] in this volume).

Similarly, assumption (4.13), already guaranteeing the regularity of minimizers of (4.3), also enables to prove density of smooth functions in L^{Φ_2} [102] (and approximation in energy (5.5)) and boundedness of maximal and integral operators (see [17] for references).

Recently, in connection to the approach already described in Subsection 4.6, a certain number of properties of the generating function Φ have been singled out in order to develop a general function theoretic approach to Musielak-Orlicz spaces L^{Φ}, which also have connections to Harmonic Analysis. For this we refer to the recent book [126]. For instance, in [130] Hästo has proved the boundedness of the classical Hardy-Littlewood Maximal operator

$$M(f)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy\,,$$

(which is defined for any locally integrable $f\colon \mathbb{R}^n \to \mathbb{R}$) in Musielak-Orlicz spaces

$$\|M(f)\|_{L^{\Phi}(\Omega)} \lesssim \|f\|_{L^{\Phi}(\Omega)}\,. \tag{6.2}$$

The crucial fact is that the main assumption on Φ (that must be superlinear in t) in order to get (6.2) is of the type (4.40) with $\omega(\cdot)$ as in (4.42). Indeed, this roughly requires that

$$\Phi_B^+(t) \lesssim \Phi_B^-(t) \qquad \text{holds provided } 1 \lesssim \Phi_B^-(t) \leq \frac{1}{|B|} \tag{6.3}$$

for every ball $B \subset \Omega$, where Φ_B^+, Φ_B^- are defined as in (4.41). After this, under similar assumptions, one can also prove interpolation and extrapolation theorems, as shown in [68]. Further boundedness results for maximal operators in the double phase case are contained in this volume [175]. Accordingly, associated results on the validity of Hardy inequality are also available here in [176].

The occurrence of (6.2) and (6.3) has important function theoretic implications. It allows, for example, to deduce the density of smooth functions in L^{Φ}. When connecting this fact to functionals as in (1.1), this has two important consequences:

- It implies the possibility of approximation in energy in the sense of (5.3), and therefore the absence of the Lavrentiev phenomenon. This, according to Subsection 5.1, can be used to prove the regularity of minima. Moreover, as noticed above in connection to (6.3), the assumptions guaranteeing (6.2) are often the same of those used to prove regularity.
- In several cases, it allows to identify the natural class of test functions admissible in the weak formulation of the Euler-Lagrange equation (1.3), beyond minimality. This means we can consider critical points of the functional $\mathcal{F}$, i.e., distributional solutions to (1.3)

$$\int_{\Omega} \partial_z F(x, Du) \cdot D\varphi\, dx = \int_{\Omega} f \cdot \varphi\, dx$$

  and use them as test functions φ maps that are proportional to u. In other words, it is possible to define the proper notion of energy solution u as a distributional solution to (1.3) such that $Du \in L^{\Phi}(\Omega)$. This approach has been already described in the pioneering work of Zhikov [214–217] and we also refer to [96] for details. Let us give two examples, that also help to clarify the contents of Remarks 1 and 5. When considering the weak form of (4.30), we can use as a test function any φ that satisfies $|D\varphi|^{p(x)} \in L_{\rm loc}(\Omega)$, provided we start with an energy solution u, that is, in this case, $|Du|^{p(x)} \in L^1_{\rm loc}(\Omega)$ is satisfied, namely, $|Du| \in L^{\Phi_1}_{\rm loc}(\Omega)$ and Φ_1 is in (6.1). This follows by a standard density argument provided (4.10) holds, which is at this stage also an assumption for regularity, that is, for getting (4.31). (Note that in this case, to get the only density argument, the weaker assumption (4.9) suffices). Similarly, when considering the weak form of (4.32), any function φ can be used to test the weak formulation if $|D\varphi|^p + a(x)|D\varphi|^q \in L^1_{\rm loc}(\Omega)$ and $|Du|^p + a(x)|Du|^q \in L^1_{\rm loc}(\Omega)$, that is, u is an energy solution, $|Du| \in L^{\Phi_2}_{\rm loc}(\Omega)$. Again, this follows by a density argument provided condition (4.13) holds. Also in this case, this condition works both for regularity and helps to make the density argument work. An example of this approach is, for instance, provided in [90].

We lastly note that abstract functional theoretic properties of the spaces linked to nonuniformly elliptic operators are important in order to settle down the right framework for solvability of nonlinear problems. There the anisotropicity of the operators considered makes new phenomena appear. This is the case of the study of eigenvalues [66,174], see also a related papers in this volume [4,8,14,53,181].

6.2. Anisotropic capacities, Hausdorff measures, removability of singularities

In this section we want to report two recent tools playing an important role in the analysis of fine properties of solutions to problems with nonstandard growth. In elliptic theory it is standard to associate, to each problem, a function space with related capacity and Hausdorff measure. In the case of problems with p-growth we have the familiar p-capacity. Upon passing to Orlicz spaces, different notions of capacities can be defined accordingly, using the associated Young functions; we refer to the classical books [132,171] and to Frehse's paper [114] for an overview. In the case of a general Musielak-Orlicz spaces, generated by a function $\Phi(\cdot)$, an approach has been recently proposed in [18,88]. Specifically, following [18], we shall consider a function $\Phi\colon \Omega\times[0,\infty)\to[0,\infty)$ as in Subsection 6.1, with the property that there exists $\beta\in(0,1)$ such that, for every $x\in\Omega$

$$\Phi(x,\beta)\leq 1 \qquad \text{and} \qquad \Phi\left(x,1/\beta\right)\geq 1 \tag{6.4}$$

hold, and moreover we assume that

$$\frac{\Phi(x,s)}{s}\lesssim\frac{\Phi(x,t)}{t} \qquad \text{whenever} \quad 0<s\leq t\,. \tag{6.5}$$

For a compact subset $K\subset\Omega$, we denote

$$Cap^*_\Phi(K)\equiv Cap^*_\Phi(K,\Omega):=\inf_{f\in\mathcal{C}(K)}\int_\Omega \Phi(x,|Df|)\,dx \tag{6.6}$$

where $\mathcal{C}(K):=\left\{f\in W^{1,\Phi}(\Omega)\cap C_0(\Omega)\colon f\geq 1 \,\text{in}\; K,\ f\geq 0\right\}$. Then, first for open subsets $U\subset\Omega$, and next for general sets $E\subset\Omega$, the capacity Cap_Φ is defined as

$$\begin{cases} Cap_\Phi(U):=\displaystyle\sup_{K\subset U,\ K \text{ is compact}} Cap^*_\Phi(K)\\ Cap_\Phi(E):=\displaystyle\inf_{E\subset\tilde U\subset\Omega,\ \tilde U \text{ is open}} Cap_\Phi(\tilde U)\,. \end{cases} \tag{6.7}$$

As shown in [18], the set function Cap_Φ is a Choquet capacity in the sense that $Cap^*_\Phi(K)=Cap_\Phi(K)$ holds for every compact subset set $K\subset\Omega$.

Beside the intrinsic capacity Cap_Φ, we also have an intrinsic Hausdorff measure $\mathcal{H}_\Phi$, as introduced for the first time in [88]. We assume that

$$\Phi(x,t)\lesssim m(x)t^n\,, \qquad \text{for all } t\geq 1\,,\quad \text{a.e. } x\in\Omega\,,\quad \text{where } 0\leq m(\cdot)\in L^1_{\rm loc}(\Omega)\,. \tag{6.8}$$

Then define the basic set function

$$h_\Phi(B) = \int_B \Phi\left(x, 1/\mathtt{r}(B)\right) \, dx \, . \tag{6.9}$$

The use of Carathéodory's construction then gives an outer measure. Specifically, let $E \subset \Omega$ be any subset, and

$$\mathcal{H}_{\Phi,\kappa}(E) = \inf_{\mathcal{C}_E^\kappa} \sum_j h_\Phi(B_j) \, , \tag{6.10}$$

where

$$\mathcal{C}_E^\kappa = \left\{ \{B_j\}_{j\in\mathbb{N}} \text{ is a collection of balls } B_j \subset \Omega \text{ covering } E \text{ and such that } \mathtt{r}(B_j) \leq \kappa \right\} .$$

The definitions in (6.9) and (6.10) imply that if $0 < \kappa_1 < \kappa_2 < \infty$, then $\mathcal{C}_E^{\kappa_1} \subset \mathcal{C}_E^{\kappa_2}$, so that $\mathcal{H}_{\Phi,\kappa_1}(E) \geq \mathcal{H}_{\Phi,\kappa_2}(E)$ and therefore the limit

$$\mathcal{H}_\Phi(E) := \lim_{\kappa\to 0} \mathcal{H}_{\Phi,\kappa}(E) = \sup_{\kappa>0} \mathcal{H}_{\Phi,\kappa}(E) \tag{6.11}$$

exists. This turns out to be a Borel regular measure [88], that in the case $\Phi(x,t) \equiv t^p$ for $p \leq n$, is equivalent (up to constants) to the usual $(n-p)$-dimensional spherical Hausdorff measure. When $\Phi(x,t) \equiv w(x)t^p$ for $p \leq n$ and $w(\cdot)$ is a non-negative and measurable function, $\mathcal{H}_\Phi$ is of the type of the weighted Hausdorff measures in [207]. We refer to [88] for further properties of $\mathcal{H}_\Phi$ and, in particular, for its relations with the capacity in (6.7).

These measures, introduced in [88], reveal to be an essential tool to prove estimates on singular sets (see Subsection 6.3) and sharp theorems for the removability singularities in problems with nonstandard growth conditions. For instance, as for removability of singularities, they have shown to play for double phase problems the same role as standard p-Capacities in the case of the p-Laplacian operator. This has been first shown in [56]. Eventually, this approach has been extended to more general operators in [57] in this volume. We expect that Hausdorff type measures defined in (6.11) will have permanent value in the analysis of problems with nonstandard growth.

The analysis of partial differential equations in Musielak-Orlicz spaces has undergone a tumultuous development in the last years, see for instance [5,6,44,58–60,79,105,123,128]. For an overview of results we mention and the recent survey [54]. For an approach pioneering the intrinsic ones used in Musielak-Orlcz spaces, see [19].

6.3. Manifold constrained problems; boundary regularity

A classical field of research in Geometric Analysis deals with regularity properties of Harmonic mappings. A model example occurs when minimizing the Dirichlet integral

$$w \mapsto \int_\Omega |Dw|^2 \, dx$$

amongst all the competitors $w \in W^{1,2}(\Omega;\mathbb{R}^N)$ such that $|w| = 1$, i.e., with values contained within the sphere. The sphere can be of course replaced by more general manifolds. The theory has been initiated by Schoen & Uhlenbeck [189], and has been developed at length; we refer to [179] for a comprehensive presentation and a list of references. It is a partial regularity theory in the sense of Subsection 3.9, i.e., minimizers are found to be regular in an open subset of the domain whose complement is negligible and has Hausdorff dimension less than $n-2$. Very recently the constrained minimization problems for functionals in (4.2) and (4.3) have been studied in [83] and [88], respectively. In the latter mentioned paper the size of the singular set of constrained minimizers has been estimated using the Hausdorff type measures $\mathcal{H}_{\Phi_2}$ introduced in (6.11) (recall Φ_2 is defined in (6.1)).

Another interesting direction, which is at the moment mostly untouched even for model cases, is the one of boundary regularity of solutions. Some results have been obtained in [199] in this volume.

7. Parabolic problems

The regularity theory of problems with nonstandard growth in the parabolic case appears to be much less developed, although it already gained a certain number of contributions. In the standard, nondegenerate situations, a main reference is of course the classical treatise [147]. The regularity theory of the evolutionary p-Laplacean equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$$

instead goes back to fundamental papers of DiBenedetto (see [73]), DiBenedetto & Friedman (see [74,75]), and Chen & DiBenedetto (see [51]). Note that the papers [74,75] deal with the vector valued case and therefore extend to parabolic systems the classical results of Uhlenbeck [208].

As for the nonstandard case, and especially the gradient flow of integral functionals of the type in (3.5), i.e.,

$$u_t - \operatorname{div} \partial F(Du) = 0,$$

a satisfying existence theory has been developed only recently. As for regularity, some results appeared, sporadically, in the more traditional setting of nonuniformly parabolic equations and very often in the form of a priori estimates for more regular solutions (see for instance [47,155,159,160]). More recently, a series of papers by Bögelein, Duzaar & Marcellini (see [29–32]) has offered a systematic approach to parabolic problems, considering different kinds of solutions. Further results in this direction can be found in [84,117,194–196]. Despite this, the regularity theory is still to be developed in many respects. Almost all such results deal with the case of (p,q)-growth conditions. Very little is at the moment known concerning the case of non-polynomial growth conditions. In another direction, we mention the recent works [58–60], where parabolic problems are studied in the context of Musielak-Orlicz spaces and existence results are obtained, also using the concept of Lavrentiev phenomenon.

8. Sensitivity on data

When considering uniformly elliptic problems, the following regularity question is classical.

Problem. Find minimal regularity assumptions on f and $x \mapsto F(x,\cdot), x \mapsto a(x,\cdot)$, such that they guarantee local Lipschitz continuity of minimizers of the functional $\mathcal{F}$ in (1.1) and weak solutions to (1.2), respectively, provided this type of regularity holds when $f \equiv 0$ and no x-dependence occurs, i.e., $F(x,z) \equiv F(z)$ and $a(x,z) \equiv a(z)$.

In this respect, for the classical model case

$$-\operatorname{div}(\mathfrak{c}(x)|Du|^{p-2}Du) = f\,, \quad p > 1\,, \quad 0 < \nu \le \mathfrak{c}(\cdot) \le L\,, \tag{8.1}$$

we have the following.

Theorem 11 *(Nonlinear Stein Theorem [144]). Let $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a solution to (8.1). If $f \in L(n,1)(\Omega;\mathbb{R}^N)$, and $\mathfrak{c}(\cdot)$ is Dini continuous, then Du is continuous.*

Observe that f belongs to the Lorentz space $L(n,1)(\Omega;\mathbb{R}^N)$ iff

$$\|f\|_{L(n,1)(\Omega)} := \int_0^\infty |\{x\in\Omega : |f(x)|>\lambda\}|^{1/n}\, d\lambda < \infty\,. \tag{8.2}$$

This is a borderline space as

$$L^q \subset L(n,1) \subset L^n \tag{8.3}$$

holds for every $q > n$. Here, the Dini continuity property of $\mathfrak{c}(\cdot)$ is fixed by

$$\int_0 \omega(\varrho)\,\frac{d\varrho}{\varrho} < \infty\,, \tag{8.4}$$

where $\omega(\cdot)$ denotes the modulus of continuity of $\mathfrak{c}(\cdot)$. Note that, for $\mathfrak{c}(\cdot) \equiv 1$ and $p = 2$, Theorem 11 is a classical result of Stein [198]. Theorem 11 is optimal both with respect to condition (8.2), as shown by Cianchi [62], and with respect to (8.4), and this has been shown by Jin, Maz'ya & Van Schaftingen [139]. Global analogs concerning gradient boundedness of the Nonlinear Stein Theorem have been obtained by Cianchi & Maz'ya [63,64]; see [99] for local Lipschitz estimates in the autonomous case.

We now wonder what happens when passing from (8.1) to operators and functionals with nonstandard growth conditions. For this, we will consider special structure functionals of Uhlenbeck type

$$w \mapsto \int_\Omega [\tilde F(x,|Dw|) - f\cdot w]\,dx = \int_\Omega \left[\tilde F(x,Dw) - f\cdot w\right]\,dx \tag{8.5}$$

and we will deal with essentially three significant cases

- Functionals with (p,q)-growth
- Functionals with fast growth as in (4.36)
- General uniformly elliptic operators as in (2.6)-(2.7).

The main outcome is then twofold:

- The condition assumed on f is independent of the operator considered; when $n > 2$, it amounts to assuming that $f \in L(n,1)$. In other words, this last one is a universal condition for Lipschitz regularity, a matter uniformly or nonuniformly elliptic problems are connected with.
- The degree of assumed regularity of $x \mapsto F(x,z)$ depends instead on the growth conditions of the integrand $F(\cdot)$ with respect to z. This is totally consistent with what we have seen in Section 4.

To proceed, we fix the relevant function space that f is assumed to belong to, that is,

$$|f| \in \mathfrak{X}(\Omega) = \begin{cases} L(n,1)(\Omega) & \text{if } n>2,\\ L^2(\log L)^{\mathfrak{a}}(\Omega)\,,\ \mathfrak{a}>2 & \text{if } n=2\,. \end{cases} \tag{8.6}$$

We recall that the space $L^2(\log L)^{\mathfrak{a}}(\Omega;\mathbb{R}^N)$ with $\mathfrak{a}\geq 0$, consists of all the measurable maps $f\colon \Omega\to\mathbb{R}^N$ such that

$$f \in L^2(\log L)^{\mathfrak{a}}(\Omega;\mathbb{R}^N) \Longleftrightarrow \int_\Omega |f|^2 \log^{\mathfrak{a}}(\mathrm{e}+|f|)\,dx < \infty\,.$$

This is another "borderline space' around L^2, as, similarly to (8.3)

$$L^q \subset L^2(\log L)^{\mathfrak{a}} \subset L^2$$

holds for every $q > 2$. Therefore, $L^2(\log L)^{\mathfrak{a}}$ can be used as a replacement of $L(2,1)$ for $n = 2$.

8.1. Functionals with (p, q)-growth

We consider a sample result from [91], regarding general functionals as in (1.1) in the vectorial case $N \geq 1$. A scalar analog in the autonomous case can be found in [20]. We assume that

$$\begin{cases} \nu(|z|^2+\mu^2)^{p/2} \leq F(x,z) \leq L(|z|^2+\mu^2)^{q/2} + L(|z|^2+\mu^2)^{p/2}, \\ (|z|^2+\mu^2)|\partial_{zz}F(x,z)| \leq L(|z|^2+\mu^2)^{q/2} + L(|z|^2+\mu^2)^{p/2}, \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \leq \partial_{zz}F(x,z)\xi\cdot\xi\,, \end{cases} \tag{8.7}$$

hold for $z, \xi \in \mathbb{R}^n$ such that $|z| \neq 0$. Here $0 < \nu \leq 1 \leq L$ and $\mu \in [0,1]$ are fixed constants, and $t \mapsto \tilde{F}(x,t) \in C^1_{\rm loc}[0,\infty) \cap C^2_{\rm loc}(0,\infty)$ for $x \in \Omega$. Of course, $1 < p \leq q$ is required. We, moreover, assume that

$$t \mapsto \frac{\tilde{F}'(x,t)}{(t^2+\mu^2)^{(p-2)/2}t} \quad \text{is non-decreasing} \tag{8.8}$$

for every $x \in \Omega$. As for the crucial dependence on x, we assume that for every $t \geq 0$, $x \mapsto \tilde{F}'(x,t) \in W^{1,d}(\Omega)$ and that

$$|\partial_x \tilde{F}'(x,t)| \leq h(x)\left[(t^2+\mu^2)^{(q-1)/2} + (t^2+\mu^2)^{(p-1)/2}\right], \quad h(\cdot) \in L^d(\Omega)\,,\ d > n \tag{8.9}$$

holds for $x \in \Omega$ and $t \geq 0$. The following holds.

Theorem 12 *(De Filippis & Mingione [91]). Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional in (8.5), under assumptions (8.6)-(8.9) with $p \geq 2$ and*

$$\frac{q}{p} \leq 1 + \frac{1}{n} - \frac{1}{d}\,. \tag{8.10}$$

Then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$.

Remark 7 *(Models and Sharpness).* We note that

- When $f \equiv 0$, Theorem 12 has been proved for the first time in [100]. Note that, when considering product type integrands as (4.45), Theorem 12 says that minimizers are locally Lipschitz continuous provided $\mathfrak{c}(\cdot) \in W^{1,d}(\Omega)$ for some $d > n$ and (8.10) is satisfied. For the non-homogeneous double phase functional in (4.14) condition (8.9) again amounts to assuming that $0 \leq a(\cdot) \in W^{1,d}(\Omega)$ with (4.13) (in addition to the requirement that $q < p^2$ when $n = 2$).
- The sharpness of assumptions (8.9)-(8.10) can be tested by comparison with (8.10) with (4.13). Sobolev-Morrey embedding yields $a(\cdot) \in C^{0,\alpha}$ with $\alpha = 1 - n/d$. By this we find that conditions (4.13) and (8.10) coincide. In turn, (4.13) is sharp by the counterexamples in [102,110]. Therefore, (8.9) is the

sharp differentiable version of (4.13), which is stronger than (4.13). This assumption is weaker than the Lipschitz continuity of $a(\cdot)$, as it is usually considered in the literature.

8.2. Functionals with fast growth

When coming to functionals with fast growth, as in Subsection 3.8, the theory becomes (only apparently) unexpectedly closer to the one for uniformly elliptic problems. We take sequences of exponents functions $\{p_k(\cdot)\}$ and coefficients $\{\mathfrak{c}_k(\cdot)\}$, all defined on the open subset $\Omega \subset \mathbb{R}^n$, such that

$$\begin{cases} 1 < p_{\mathfrak{m}} \leq p_0(\cdot) \leq p_M\,, \quad 0 < p_m \leq p_k(\cdot) \leq p_M\,, \quad \text{for } k \geq 1 \\ 0 < \nu \leq \mathfrak{c}_k(\cdot) \leq L\,, \quad p_k(\cdot), \mathfrak{c}_k(\cdot) \in W^{1,d}(\Omega)\,, \quad d > n\,, \quad \text{for } k \geq 0\,. \end{cases} \tag{8.11}$$

We then inductively define, for every $k \in \mathbb{N}$, the functions $\mathbf{e}_k \colon \Omega \times [0,\infty) \to \mathbb{R}$ as

$$\begin{cases} \mathbf{e}_{k+1}(x,t) & := & \exp\left(\mathfrak{c}_{k+1}(x)\,[\mathbf{e}_k(t)]^{p_{k+1}(x)}\right) \\ \mathbf{e}_0(x,t) & := & \exp\left(\mathfrak{c}_0(x) t^{p_0(x)}\right)\,, \end{cases}$$

and consider functionals defined by

$$w \mapsto \int_\Omega \left[\mathbf{e}_k(x,|Dw|) - fw\right]\, dx\,. \tag{8.12}$$

We then have the following.

Theorem 13 *(De Filippis & Mingione [91]). Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional in (8.12) for some $k \in \mathbb{N}$, under assumptions (8.11) and such that f satisfies (8.6). Then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$.*

In contrast to the case of (p,q)-growth functionals considered in Theorem 12, here any exponent $d > n$ is sufficient for regularity. No quantitative bound of the type in (8.10) is required. This apparently counterintuitive fact can be justified by observing that, although the functionals in (8.12) have faster growth than polynomials, their rates of nonuniform ellipticity decreases with respect to the growth of the energy density. In other words, considering for simplicity the autonomous case, and recalling (3.19), we find

$$\lim_{|z|\to\infty} \frac{[\mathcal{R}_{\partial_z F}(z)]^\delta}{F(z)} = 0$$

for any possible composition of exponentials and $\delta > 0$. This is not the case for functionals with (p,q)-growth. More details and technical explanations can be found in [20,91].

8.3. General uniformly elliptic operators

For operators in (2.6)-(2.7) we have

Theorem 14 *(De Filippis & Mingione [91]). Let $u \in W^{1,A(\cdot)}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a weak solution to*

$$-\operatorname{div}(\mathfrak{c}(x)\tilde{a}(|Du|)Du) = f\,, \qquad 0 < \nu \leq \mathfrak{c}(\cdot) \leq L\,,$$

where $A(\cdot)$ is defined in (2.8), under the assumptions in (2.7). If $|f|, |D\mathfrak{c}| \in \mathfrak{X}(\Omega)$, then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$. Moreover, there exists a positive radius $R_* \equiv R_*(n,N,i_{\rm a},s_{\rm a},\mathfrak{c}(\cdot)) \leq 1$ such that if $B \Subset \Omega$ is a ball with radius $\mathcal{r}(B) \leq R_*$, then

$$\|A(|Du|)\|_{L^\infty(sB)} \leq \frac{c}{(1-s)^n[\mathfrak{r}(B)]^n}\|A(|Du|)\|_{L^1(B)} + c\|f\|_{\mathfrak{X}(B)}^{\frac{i_a+2}{i_a+1}} + c$$

holds for every $s \in (0,1)$*, where* $c \equiv c(n, N, \nu, L, i_a, s_a)$*.*

As a particular case, this applies to (8.1) by taking $\tilde{a}(t) \equiv \mathfrak{c}(x)t^{p-2}$. Even in the homogeneous case $f \equiv 0$, this provides a new regularity criterion, which goes beyond the known and classical one in (8.4). Indeed, $D\mathfrak{c} \in L(n,1)$ implies that $\mathfrak{c}(\cdot)$ is continuous [198], but not necessarily with a modulus of continuity $\omega(\cdot)$ satisfying (8.4). We refer to [91] for results concerning the more general systems of the type $-\operatorname{div}(\tilde{a}(x,|Du|)Du) = f$.

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