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Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent

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Abstract

In this paper we are concerned with a new class of anisotropic quasilinear elliptic equations with a power-like variable reaction term. One of the main features of our work is that the differential operator involves partial derivatives with different variable exponents, so that the functional-analytic framework relies upon anisotropic Sobolev and Lebesgue spaces. Existence and nonexistence results are deeply influenced by the competition between the growth rates of the anisotropic coefficients. Our main results point out some striking phenomena related to the existence of a continuous spectrum in several distinct situations.

Keywords: Nonhomogeneous differential operators; Anisotropic Sobolev spaces; Continuous spectrum; Variational methods

1. Introduction

The purpose of this paper is to analyze the existence of solutions of the nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary, λ is a positive number, and p_i, q are continuous functions on $\overline{\Omega}$ such that $2 \leqslant p_i(x) < N$ and $q(x) > 1$ for any $x \in \overline{\Omega}$ and $i \in \{1, \ldots, N\}$.

In the particular case when $p_i = p$ for any $i \in \{1, \ldots, N\}$ the operator involved in (1) is the $p(\cdot)$-Laplace operator, i.e., $\Delta_{p(\cdot)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the isotropic p-Laplace

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operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(\cdot)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(\cdot)}$ is *not homogeneous*.

The study of nonlinear elliptic equations involving quasilinear homogeneous type operators like the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. In the case of nonhomogeneous $p(\cdot)$-Laplace operators the natural setting for this approach is the use of the variable exponent Sobolev spaces. The basic idea is to replace the Lebesgue spaces $L^p(\Omega)$ by more general spaces $L^{p(\cdot)}(\Omega)$, called *variable exponent Lebesgue spaces*. If the role played by $L^p(\Omega)$ in the definition of the Sobolev spaces $W^{m,p}(\Omega)$ is assigned instead to a variable Lebesgue space $L^{p(\cdot)}(\Omega)$ the resulting space is denoted by $W^{m,p(\cdot)}(\Omega)$ and called a *variable exponent Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Donaldson and Trudinger [9], and O'Neill [26] (see also Adams [2] for an excellent account of those works). The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ were thoroughly studied in the monograph by Musielak [25] and the papers by Edmunds et al. [10–12], Kovacik and Rákosník [21], Mihăilescu and Rădulescu [22–24], and Samko and Vakulov [34]. Variable Sobolev spaces have been used in the last decades to model various phenomena. Chen, Levine and Rao [7] proposed a framework for image restoration based on a variable exponent Laplacian. A second major application which uses nonhomogeneous Laplace operators is related to the modelling of electrorheological fluids (sometimes referred to as *smart fluids*). Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For a general account of the underlying physics consult Halsey [17] and for some technical applications Pfeiffer et al. [28]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids we refer to Acerbi and Mingione [1], Alves and Souto [3], Chabrowski and Fu [6], Diening [8], Fan et al. [14,15], Mihăilescu and Rădulescu [22–24], Rajagopal and Ruzicka [32], and Ruzicka [33].

In this paper, the operator involved in (1) is even more general than the $p(\cdot)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ is not adequate to study nonlinear problems of this type. This leads us to seek weak solutions for problem (1) in a more general variable exponent Sobolev space, which will be introduced in the next section of this paper.

As far as we are aware, nonlinear eigenvalue problems like (1) involving multiple anisotropic exponents have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction. Another major feature of this work is that, due to the "competition" between the growths of the functions p_i and q, some striking phenomena, not arising in the homogeneous case, characterize eigenvalue problems of this type.

2. Abstract framework

We recall in this section some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega})\colon \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u\colon\ u \text{ is a measurable real-valued function such that } \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty\right\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [21]. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \leqslant p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous [21, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$ [21, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2}$$

is valid.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:

$$|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1), \tag{3}$$

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \leqslant \rho_{p(\cdot)}(u) \leqslant |u|_{p(\cdot)}^{p^+}, \tag{4}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \leqslant \rho_{p(\cdot)}(u) \leqslant |u|_{p(\cdot)}^{p^-}, \tag{5}$$

$$|u_n - u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0, \tag{6}$$

since $p^+ < \infty$. For a proof of these facts see [21].

If $p \in C_+(\overline{\Omega})$, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$, consisting of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(\cdot)}(\Omega)]^N$, endowed with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

is a separable and reflexive Banach space. As shown by Zhikov [39,40] the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$, but if the exponent variable p in $C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is

$$\bigl|p(x) - p(y)\bigr| \leqslant -\frac{M}{\log(|x-y|)} \quad \text{for all } x, y \in \Omega \text{ such that } |x-y| \leqslant 1/2, \tag{7}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$ and so the Sobolev space with zero boundary values, denoted by $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$, are meaningful, see [18,20]. Furthermore, if $p \in C_+(\overline{\Omega})$ satisfies (7), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, that is $H_0^{1,p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega)$ [19, Theorem 3.3]. Since Ω is an open bounded set and $p \in C_+(\overline{\Omega})$ satisfies (7), the $p(\cdot)$-*Poincaré inequality*

$$|u|_{p(\cdot)} \leqslant C|\nabla u|_{p(\cdot)}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where C depends on p, $|\Omega|$, $\operatorname{diam}(\Omega)$ and N [19, Theorem 4.3], and so

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$. Of course also the norm

$$\|u\|_{p(\cdot)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$. Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Note that when $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \geqslant N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous. Details, extensions and further references can be found in [18–21].

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study with sufficient accuracy problem (1). For this purpose, let us denote by $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\vec{p}=(p_1,\dots,p_N)$. We define $W_0^{1,\vec{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}(\cdot)}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}.$$

In the case when $p_i\in C_+(\overline{\Omega})$ are constant functions for any $i\in\{1,\dots,N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ is the constant vector $(p_1,\dots,p_N)$. The theory of such spaces was developed in [16,27,30,31,37,38]. It was proved that $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space for any $\vec{p}\in\mathbb{R}^N$ with $p_i>1$ for all $i\in\{1,\dots,N\}$. This result can be easily extended to $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Indeed, denoting by $X=L^{p_1(\cdot)}(\Omega)\times\cdots\times L^{p_N(\cdot)}(\Omega)$ and considering the operator $T:W_0^{1,\vec{p}(\cdot)}(\Omega)\to X$, defined by $T(u)=\nabla u$, it is clear that $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and X are isometric by T, since $\|Tu\|_X=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}=\|u\|_{\vec{p}(\cdot)}$. Thus, $T(W_0^{1,\vec{p}(\cdot)}(\Omega))$ is a closed subspace of X, which is a reflexive Banach space. By Proposition III.17 in [5] it follows that $T(W_0^{1,\vec{p}(\cdot)}(\Omega))$ is reflexive and consequently also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ we introduce $\vec{P}_+,\vec{P}_-\in\mathbb{R}^N$ as

$$\vec{P}_+=\big(p_1^+,\dots,p_N^+\big),\qquad \vec{P}_-=\big(p_1^-,\dots,p_N^-\big),$$

and $P_+^+,P_-^+,P_-^-\in\mathbb{R}^+$ as

$$P_+^+=\max\big\{p_1^+,\dots,p_N^+\big\},\qquad P_-^+=\max\big\{p_1^-,\dots,p_N^-\big\},\qquad P_-^-=\min\big\{p_1^-,\dots,p_N^-\big\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N\frac{1}{p_i^-}>1 \tag{8}$$

and define $P_-^\star\in\mathbb{R}^+$ and $P_{-,\infty}\in\mathbb{R}^+$ by

$$P_-^\star=\frac{N}{\sum_{i=1}^N 1/p_i^- -1},\qquad P_{-,\infty}=\max\{P_-^+,P_-^\star\}.$$

3. Main results

Our first main result extends Theorem 1 in [16] and states a compactness embedding between the spaces $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$.

Theorem 1. *Assume $\Omega\subset\mathbb{R}^N$ $(N\geqslant 3)$ is a bounded domain with smooth boundary. Assume relation* (8) *is fulfilled. For any $q\in C(\overline{\Omega})$ verifying*

$$1<q(x)<P_{-,\infty}\quad \text{for all } x\in\overline{\Omega}, \tag{9}$$

the embedding

$$W_0^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

Proof. Clearly $L^{p_i(\cdot)}(\Omega)$ is continuously embedded in $L^{p_i^-}(\Omega)$ for any $i\in\{1,\dots,N\}$, since $p_i^-\leqslant p_i(x)$ for all $x\in\overline{\Omega}$. Thus, for each $i\in\{1,\dots,N\}$ there exists a positive constant $C_i>0$ such that

$$|\varphi|_{p_i^-}\leqslant C_i|\varphi|_{p_i(\cdot)}\quad \text{for all } \varphi\in L^{p_i(\cdot)}(\Omega).$$

If $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, then $\partial_{x_i} u \in L^{p_i(\cdot)}(\Omega)$ for each $i \in \{1, \ldots, N\}$. The above inequalities imply

$$\|u\|_{\vec{P}_-} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i^-} \leqslant C \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)} = C\|u\|_{\vec{p}(\cdot)},$$

where $C = \max\{C_1, \ldots, C_N\}$. Thus, we deduce that $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is continuously embedded in $W_0^{1,\vec{P}_-}(\Omega)$. On the other hand, since relation (9) holds true, we infer that $q^+ < P_{-,\infty}$. This fact combined with the result of Theorem 1 in [16] implies that $W_0^{1,\vec{P}_-}(\Omega)$ is compactly embedded in $L^{q^+}(\Omega)$. Finally, since $q(x) \leqslant q^+$ for each $x \in \overline{\Omega}$, we deduce that $L^{q^+}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. The above piece of information yields to the conclusion that $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$. The proof of Theorem 1 is complete. □

We give in what follows our main results regarding the existence of weak solutions for problem (1). By a *weak solution* for problem (1) we understand a function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\int_{\Omega} \left\{ \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} \varphi - \lambda |u|^{q(x)-2} u \varphi \right\} dx = 0$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Theorem 2. *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$P_+^+ < \min_{x \in \overline{\Omega}} q(x) \leqslant \max_{x \in \overline{\Omega}} q(x) < P_-^\star. \tag{10}$$

Then for any $\lambda > 0$ problem (1) *possesses a nontrivial weak solution.*

Theorem 3. *If $q \in C(\overline{\Omega})$ satisfies the inequalities*

$$1 < \min_{x \in \overline{\Omega}} q(x) \leqslant \max_{x \in \overline{\Omega}} q(x) < P_-^-, \tag{11}$$

then there exists $\lambda^{\star\star} > 0$ such that for any $\lambda > \lambda^{\star\star}$ problem (1) *possesses a nontrivial weak solution.*

Theorem 4. *If $q \in C(\overline{\Omega})$, with*

$$1 < \min_{x \in \overline{\Omega}} q(x) < P_-^- \quad \text{and} \quad \max_{x \in \overline{\Omega}} q(x) < P_{-,\infty}, \tag{12}$$

then there exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ problem (1) *possesses a nontrivial weak solution.*

Remark 1. If $q \in C(\overline{\Omega})$ verifies (11), then it satisfies (12). Consequently, the result of Theorem 3 can be completed with the conclusion of Theorem 4. More precisely, we deduce the following consequence.

Corollary 1. *Let $q \in C(\overline{\Omega})$ verify*

$$1 < \min_{x \in \overline{\Omega}} q(x) \leqslant \max_{x \in \overline{\Omega}} q(x) < P_-^-.$$

Then there exist $\lambda^\star > 0$ and $\lambda^{\star\star} > 0$ such that for any $\lambda \in (0, \lambda^\star)$ and $\lambda > \lambda^{\star\star}$ problem (1) *possesses a nontrivial weak solution.*

Remark 2. On the other hand, we point out that the result of Theorem 4 holds true in situations that extend relation (11), since in relation (12) we could have

$$1 < \min_{x \in \overline{\Omega}} q(x) < P_-^- < \max_{x \in \overline{\Omega}} q(x) < P_{-,\infty}.$$

4. Proof of Theorem 2

From now on E denotes the anisotropic variable exponent Orlicz–Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $\lambda > 0$ the energy functional corresponding to problem (1) is defined by $J_\lambda : E \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} - \frac{\lambda}{q(x)} |u|^{q(x)} \right\} dx. \tag{13}$$

Theorem 1 assures that $J_\lambda \in C^1(E, \mathbb{R})$ and the Fréchet derivative is given by

$$\langle J'_\lambda(u), v \rangle = \int_\Omega \left\{ \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v - \lambda |u|^{q(x)-2} uv \right\} dx$$

for all $u, v \in E$. Thus the weak solutions of (1) coincide with the critical points of J_λ.

In order to prove that the J_λ has a nontrivial critical point, our idea is to show that actually J_λ possesses a mountain pass geometry. It turns out that also the application of this, by now standard tools, is not straightforward, due to the fact that the *kinetic functional* J_λ is no longer homogeneous, as in the isotropic case.

We start with two auxiliary results.

Lemma 1. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \eta$.*

Proof. First, we point out that

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geqslant |u(x)|^{q(x)} \quad \text{for all } x \in \overline{\Omega}. \tag{14}$$

Using the above inequality and (13), we find

$$J_\lambda(u) \geqslant \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx - \frac{\lambda}{q^-} \left(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+} \right) \tag{15}$$

for any $u \in E$.

Since (10) holds, then E is continuously embedded both in $L^{q^-}(\Omega)$ and in $L^{q^+}(\Omega)$ by Theorem 1. It follows that there exist two positive constants B_1 and B_2 such that

$$B_1 \|u\|_{\vec{p}(\cdot)} \geqslant |u|_{q^+}, \qquad B_2 \|u\|_{\vec{p}(\cdot)} \geqslant |u|_{q^-} \quad \text{for all } u \in E. \tag{16}$$

Next, we focus our attention on the case when $u \in E$ and $\|u\|_{\vec{p}(\cdot)} < 1$. For such an element u we have $|\partial_{x_i} u|_{p_i(\cdot)} < 1$ and, by relation (5), we obtain

$$\frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{N^{P_+^+ - 1}} = N \left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}}{N} \right)^{P_+^+} \leqslant \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+} \leqslant \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{p_i^+} \leqslant \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx. \tag{17}$$

Relations (15)–(17) imply

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{P_+^+ N^{P_+^+ - 1}} \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \frac{\lambda}{q^-} \left[\left(B_1 \|u\|_{\vec{p}(\cdot)} \right)^{q^+} + \left(B_2 \|u\|_{\vec{p}(\cdot)} \right)^{q^-} \right] \\ &= \left(B_3 - B_4 \|u\|_{\vec{p}(\cdot)}^{q^+ - P_+^+} - B_5 \|u\|_{\vec{p}(\cdot)}^{q^- - P_+^+} \right) \|u\|_{\vec{p}(\cdot)}^{P_+^+} \end{aligned}$$

for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} < 1$, where B_3, B_4 and B_5 are positive constants.

Since the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = B_3 - B_4 t^{q^+ - P_+^+} - B_5 t^{q^- - P_+^+}$$

is positive in a neighborhood of the origin, the conclusion of the lemma follows at once. $\square$

Lemma 2. *There exists $e \in E$ with $\|e\|_{\vec{p}(\cdot)} > \eta$ (where η is given in Lemma 1) such that $J_\lambda(e) < 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$ and $\psi \not\equiv 0$, be fixed and let $t > 1$. By (13)

$$\begin{aligned} J_\lambda(t\psi) &= \int_\Omega \left\{ \sum_{i=1}^N \frac{t^{p_i(x)}}{p_i(x)} |\partial_{x_i}\psi|^{p_i(x)} - \lambda \frac{t^{q(x)}}{q(x)} |\psi|^{q(x)} \right\} dx \\ &\leqslant \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i}\psi|^{p_i(x)}\, dx - \frac{\lambda t^{q^-}}{q^+} \int_\Omega |\psi|^{q(x)}\, dx. \end{aligned}$$

Since $q^- > P_+^+$ by (10), it is clear that $\lim_{t\to\infty} J_\lambda(t\psi) = -\infty$. Then for $t > 1$ large enough we can take $e = t\psi$ such that $\|e\|_{\vec{p}(\cdot)} > \eta$ and $J_\lambda(e) < 0$. This completes the proof. □

Proof of Theorem 2. By Lemmas 1 and 2 and the mountain pass theorem of Ambrosetti and Rabinowitz [4] (see also Pucci and Serrin [29] for the case of mountains of zero altitude), we deduce the existence of a sequence $(u_n) \subset E$ such that

$$J_\lambda(u_n) \to \bar{c} > 0 \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad (\text{in } E^\star) \text{ as } n \to \infty. \tag{18}$$

We prove that (u_n) is bounded in E. In order to do that, we assume by contradiction that passing eventually to a subsequence, still denoted by (u_n), we have $\|u_n\|_{\vec{p}(\cdot)} \to \infty$ and $\|u_n\|_{\vec{p}(\cdot)} > 1$ for all n.

Relation (18) and the above considerations imply that for n large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\|_{\vec{p}(\cdot)} &\geqslant J_\lambda(u) - \frac{1}{q^-}\langle J'_\lambda(u_n), u_n\rangle \\ &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx. \end{aligned}$$

For each $i \in \{1, \ldots, N\}$ and n we define

$$\alpha_{i,n} = \begin{cases} P_+^+ & \text{if } |\partial_{x_i} u_n|_{p_i(\cdot)} < 1, \\ P_-^- & \text{if } |\partial_{x_i} u_n|_{p_i(\cdot)} > 1. \end{cases}$$

Using relations (4) and (5) we infer that for n large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\|_{\vec{p}(\cdot)} &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{\alpha_{i,n}} \\ &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \sum_{\{i;\alpha_{i,n}=P_+^+\}} \left(|\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_+^+}\right) \\ &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-}\right) \frac{1}{N^{P_-^-}} \|u_n\|_{\vec{p}(\cdot)}^{P_-^-} - N\left(\frac{1}{P_+^+} - \frac{1}{q^-}\right). \end{aligned} \tag{19}$$

Dividing by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ in the above inequality and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that (u_n) is bounded in E. This information, combined with the fact that E is reflexive, implies that there exist a subsequence, still denoted by (u_n), and $u_0 \in E$ such that (u_n) converges weakly to u_0 in E. Since, by Theorem 1, the space E is compactly embedded in $L^{q(\cdot)}(\Omega)$, it follows that (u_n) converges strongly to u_0 in $L^{q(\cdot)}(\Omega)$. Then by inequality (2) we deduce

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q(x)-2} u_n (u_n - u_0)\, dx = 0.$$

This fact and relation (18) yield

$$\lim_{n\to\infty}\langle J_\lambda^{'}(u_n), u_n-u_0\rangle = 0.$$

Thus, we deduce that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n(\partial_{x_i}u_n-\partial_{x_i}u_0)\,dx = 0. \tag{20}$$

Since (u_n) converge weakly to u_0 in E, by relation (20) we find

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega \big(|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n-|\partial_{x_i}u_0|^{p_i(x)-2}\partial_{x_i}u_0\big)(\partial_{x_i}u_n-\partial_{x_i}u_0)\,dx = 0. \tag{21}$$

Next, we apply the following inequality (see Simon [35, formula (2.2)])

$$\big(|\xi|^{r-2}\xi-|\psi|^{r-2}\psi\big)\cdot(\xi-\psi)\geqslant 2^{-r}|\xi-\psi|^r, \quad \xi,\psi\in\mathbb{R}^N, \tag{22}$$

valid for all $r\geqslant 2$. Relations (21) and (22) show that actually (u_n) converges strongly to u_0 in E. Then by relation (18) we have

$$J_\lambda(u_0)=\bar{c}>0 \quad\text{and}\quad J_\lambda'(u_0)=0,$$

that is, u_0 is a nontrivial weak solution of Eq. (1). □

5. Proof of Theorem 3

For any $\lambda>0$ let J_λ be defined as in (13). This time our idea is to show that J_λ possesses a nontrivial global minimum point in E. We start with the following auxiliary result.

Lemma 3. *The functional J_λ is coercive on E.*

Proof. By relations (15) and (16) we deduce that for all $u\in E$,

$$J_\lambda(u)\geqslant\frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}\,dx-\frac{\lambda}{q^-}\Big[\big(B_1\|u\|_{\vec{p}(\cdot)}\big)^{q^+}+\big(B_2\|u\|_{\vec{p}(\cdot)}\big)^{q^-}\Big]. \tag{23}$$

Now, we focus our attention on the elements $u\in E$ with $\|u\|_{\vec{p}(\cdot)}>1$. Using the same techniques as in the proof of (19) combined with relation (23), we find that

$$J_\lambda(u)\geqslant\frac{1}{P_+^+N^{P_-^-}}\|u\|_{\vec{p}(\cdot)}^{P_-^-}-\frac{N}{P_+^+}-\frac{\lambda}{q^-}\Big[\big(B_1\|u\|_{\vec{p}(\cdot)}\big)^{q^+}+\big(B_2\|u\|_{\vec{p}(\cdot)}\big)^{q^-}\Big]$$

for any $u\in E$ with $\|u\|_{\vec{p}(\cdot)}>1$. Since by relation (11) we have $P_-^->q^+\geqslant q^-$ we infer that $J_\lambda(u)\to\infty$ as $\|u\|_{\vec{p}(\cdot)}\to\infty$. In other words, J_λ is coercive in E, completing the proof. □

Proof of Theorem 3. The same arguments as in the proof of Lemma 3.4 of [22] can be used in order to show that J_λ is weakly lower semicontinuous on E. By Lemma 3, the functional J_λ is also coercive on E. These two facts enable us to apply Theorem 1.2 in [36] in order to find that there exists $u_\lambda\in E$ a global minimizer of J_λ and thus a weak solution of problem (1).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0>1$ be a fixed real number and Ω_1 be an open subset of Ω with $|\Omega_1|>0$, we deduce that there exists $v_0\in C_0^\infty(\Omega)\subset E$ such that $v_0(x)=t_0$ for any $x\in\overline{\Omega}_1$ and $0\leqslant v_0(x)\leqslant t_0$ in $\Omega\setminus\Omega_1$. We have

$$J_\lambda(v_0)=\int_\Omega\left\{\sum_{i=1}^{N}\frac{1}{p_i(x)}|\partial_{x_i}v_0|^{p_i(x)}-\frac{\lambda}{q(x)}|v_0|^{q(x)}\right\}dx\leqslant L-\frac{\lambda}{q^+}\int_{\Omega_1}|v_0|^{q(x)}\,dx\leqslant L-\frac{\lambda}{q^+}t_0^{q^-}|\Omega_1|,$$

where L is a positive constant. Thus, there exists $\lambda^{\star\star} > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^{\star\star}, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geqslant \lambda^{\star\star}$ and thus u_λ is a nontrivial weak solution of problem (1) for λ large enough. The proof of Theorem 3 is complete. □

6. Proof of Theorem 4

For any $\lambda > 0$, let J_λ be defined as in (13). Applying Ekeland's variational principle [13], we show that assumption (12) implies that the functional J_λ has a nontrivial critical point. We start with two auxiliary results.

Lemma 4. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there are ρ, $a > 0$ such that $J_\lambda(u) \geqslant a > 0$ for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Proof. Since (12) holds, it follows by Theorem 1 that E is continuously embedded in $L^{q(\cdot)}(\Omega)$. Thus, there exists a positive constant c_1 such that

$$|u|_{q(\cdot)} \leqslant c_1 \|u\|_{\vec{p}(\cdot)} \quad \text{for all } u \in E. \tag{24}$$

We fix $\rho \in (0, 1)$ such that $\rho < 1/c_1$. Then relation (24) implies

$$|u|_{q(\cdot)} < 1 \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{p}(\cdot)} = \rho.$$

Furthermore, relation (5) yields

$$\int_\Omega |u|^{q(x)}\,dx \leqslant |u|_{q(\cdot)}^{q^-} \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{p}(\cdot)} = \rho. \tag{25}$$

Relations (24) and (25) imply

$$\int_\Omega |u|^{q(x)}\,dx \leqslant c_1^{q^-} \|u\|_{\vec{p}(\cdot)}^{q^-} \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{p}(\cdot)} = \rho. \tag{26}$$

Taking into account relations (17) and (26), we deduce that for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$ the following inequalities hold true:

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{P_+^+ N^{P_+^+ - 1}} \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)}\,dx \geqslant \frac{1}{P_+^+ N^{P_+^+ - 1}} \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \frac{\lambda c_1^{q^-}}{q^-} \|u\|_{\vec{p}(\cdot)}^{q^-} \\ &= \rho^{q^-} \left(\frac{1}{P_+^+ N^{P_+^+ - 1}} \rho^{P_+^+ - q^-} - \frac{\lambda c_1^{q^-}}{q^-} \right). \end{aligned}$$

Hence if we define

$$\lambda^\star = \frac{q^-}{2 c_1^{q^-} P_+^+ N^{P_+^+ - 1}} \rho^{P_+^+ - q^-}, \tag{27}$$

then for any $\lambda \in (0, \lambda^\star)$ and $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$ the number $a = \rho^{P_+^+}/2 P_+^+ N^{P_+^+ - 1}$ is such that

$$J_\lambda(u) \geqslant a > 0.$$

This completes the proof. □

Lemma 5. *There exists $\varphi \in E$ such that $\varphi \geqslant 0$, $\varphi \not\equiv 0$ and $J_\lambda(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. Assumption (12) implies that $q^- < P_-^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < P_-^-$. On the other hand, since $q \in C(\overline{\Omega})$, it follows that there exists an open set $\Omega_2 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_2$. Thus, we conclude that $q(x) \leqslant q^- + \epsilon_0 < P_-^-$ for all $x \in \Omega_2$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\varphi) \supset \overline{\Omega}_2$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_2$ and $0 \leqslant \varphi \leqslant 1$ in Ω. Then using the above information and (13), for any $t \in (0,1)$, we have

$$\begin{aligned}
J_\lambda(t\varphi) &= \int_\Omega \left\{ \sum_{i=1}^N \frac{t^{p_i(x)}}{p_i(x)} |\partial_{x_i}\varphi|^{p_i(x)} - \lambda \frac{t^{q(x)}}{q(x)} |\varphi|^{q(x)} \right\} dx \\
&\leqslant \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i}\varphi|^{p_i(x)}\,dx - \frac{\lambda}{q^+} \int_\Omega t^{q(x)} |\varphi|^{q(x)}\,dx \\
&\leqslant \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i}\varphi|^{p_i(x)}\,dx - \frac{\lambda}{q^+} \int_{\Omega_2} t^{q(x)} |\varphi|^{q(x)}\,dx \\
&\leqslant \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i}\varphi|^{p_i(x)}\,dx - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} \int_{\Omega_2} |\varphi|^{q(x)}\,dx.
\end{aligned}$$

Therefore

$$J_\lambda(t\varphi) < 0$$

for $t < \delta^{1/(P_-^- - q^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda P_-^-}{q^+} \int_{\Omega_2} |\varphi|^{q(x)}\,dx \Big/ \sum_{i=1}^N \int_\Omega |\partial_{x_i}\varphi|^{p_i(x)}\,dx \right\}.$$

This is possible since we claim that $\sum_{i=1}^N \int_\Omega |\partial_{x_i}\varphi|^{p_i(x)}\,dx > 0$. Indeed, it is clear that

$$\int_{\Omega_2} |\varphi|^{q(x)}\,dx \leqslant \int_\Omega |\varphi|^{q(x)}\,dx \leqslant \int_\Omega |\varphi|^{q^-}\,dx.$$

On the other hand, $E = W_0^{1,\vec{p}(x)}(\Omega)$ is continuously embedded in $L^{q^-}(\Omega)$ and thus, there exists a positive constant c_2 such that

$$|\varphi|_{q^-} \leqslant c_2 \|\varphi\|_{\vec{p}(\cdot)}.$$

The last two inequalities imply that

$$\|\varphi\|_{\vec{p}(\cdot)} > 0$$

and combining this fact with relations (4) or (5) the claim follows at once. The proof of the lemma is now completed. □

Proof of Theorem 4. Let $\lambda^\star > 0$ be defined as in (27) and $\lambda \in (0, \lambda^\star)$. By Lemma 4 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{28}$$

On the other hand, by Lemma 5, there exists $\varphi \in E$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, relations (17), (26) and (5) imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geqslant \frac{1}{P_+^+ N^{P_+^+ - 1}} \|u\|_{\vec{p}(\cdot)}^{P_+^+} - \frac{\lambda c_1^{q^-}}{q^-} \|u\|_{\vec{p}(\cdot)}^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle (see [13]) to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon,$$
$$J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\vec{p}(\cdot)}, \quad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \leqslant \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leqslant \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\|_{\vec{p}(\cdot)}$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} \geqslant 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} + \epsilon\|v\|_{\vec{p}(\cdot)} \geqslant 0.$$

Letting $t \to 0$ it follows that $\langle J'_\lambda(u_\epsilon), v\rangle + \epsilon\|v\|_{\vec{p}(\cdot)} > 0$ and we infer that $\|J'_\lambda(u_\epsilon)\| \leqslant \epsilon$.

We deduce that there exists a sequence $(w_n) \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J'_\lambda(w_n) \to 0. \tag{29}$$

It is clear that (w_n) is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, (w_n) converges weakly to w in E. Actually, with similar arguments as those used in the end of Theorem 2 we can show that (w_n) converges strongly to w in E. Thus, by (29)

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J'_\lambda(w) = 0, \tag{30}$$

that is, w is a nontrivial weak solution for problem (1). This completes the proof. □

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