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Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations

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Abstract

In this paper, we study the following non-autonomous Choquard-Pekar equation:

$$\begin{cases} -\Delta u + V(x)u = (W * F(u))f(u), & x \in \mathbb{R}^N \ (N \geq 2), \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where the potential $V(x)$ is 1-periodic and 0 lies in a gap of the spectrum of the Schrödinger operator $-\Delta + V$. Under some general assumptions on the potential W and the nonlinearity f, we show the existence of ground state solutions. We also construct infinitely many geometrically distinct solutions by using the variational method and deformation arguments.

MSC: 35Q55; 35Q40; 35J20; 35J60; 46N50

Keywords: Choquard-Pekar equation; Ground state solution; Strongly indefinite problem; Geometrically distinct solutions

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1. Introduction

The Choquard-Pekar equation

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3, \tag{1.1}$$

was first introduced in the pioneering work of Fröhlich [19] and Pekar [35] for the modeling of quantum polaron. This model corresponds to the study of free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarization that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (1.1) to describe an electron trapped in its own hole.

The Choquard-Pekar equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [18], Giulini and Großardt [23], Jones [24], and Schunck and Mielke [40]. Penrose [36,37] proposed equation (1.1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon.

As pointed out by Lieb [27], Choquard used equation (1.1) to study steady states of the one component plasma approximation in the Hartree-Fock theory. Classification of solutions of (1.1) was first studied by Ma and Zhao [31]. Pointwise bounds and blow-up for Choquard-Pekar inequalities at isolated singularities have been studied by Ghergu and Taliaferro [21]. For the Choquard-type equation and related problems, we refer to [11,31,33,44] for the existence of solutions and multiplicity properties, to [13,22,47] for existence of sign-changing solutions, and to [12,32,45,48] for semiclassical solutions. See also [34] and references therein for a broad survey of Choquard equations.

This paper is concerned with the following non-autonomous Choquard-Pekar equation

$$\begin{cases} -\Delta u + V(x)u = (W * F(u))f(u), & x \in \mathbb{R}^N \ (N \geq 2) \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where V, W and f satisfy the following hypotheses:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is 1-periodic in x_i for $i = 1, 2, \ldots, N$, and

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta + V) \cap (0, \infty)];$$

(W1) $W(x)$ is an even function, and there exist $1 \leq r_1 \leq r_2 < \infty$ such that $W \in L^{r_1}(\mathbb{R}^N) + L^{r_2}(\mathbb{R}^N)$;

(W2) $W(x) \geq 0$ and on a neighborhood of 0 we have $W(x) > 0$;

(W3) there exists $C_0 > 0$ such that for all nonnegative $\varphi, \psi \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (W * \varphi)\psi \mathrm{d}x \le C_0 \sqrt{\int_{\mathbb{R}^N} (W * \varphi)\varphi \mathrm{d}x} \sqrt{\int_{\mathbb{R}^N} (W * \psi)\psi \mathrm{d}x};$$

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist $\mathcal{C}_0 > 0$ and $p_1, p_2 > 1$ with $(2r_2 - 1)/r_2 < p_1 \le p_2 < (2r_1 - 1)2^*/2r_1$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \le \mathcal{C}_0(|t|^{p_1-1} + |t|^{p_2-1}); \tag{1.3}$$

(F2) $F(t) \ge 0$ and $\lim_{|t|\to\infty} \frac{F(t)}{|t|} = \infty$;
(F5) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$.

When the particle interaction is attractive (that is, $W \ge 0$), problem (1.2) turns into (1.1) with $f(u) = u$, $N = 3$ and $V \equiv 1$, which also arises in the Hartree theory of bosonic systems (cf. [5,20]). The case when the particle interaction is repulsive (that is, $W \le 0$) concerns the Hartree equation for the Helium atom. In the sequel, we assume that W does not change sign, and we speak of the periodic case if the exterior potential V is periodic and nonconstant.

For the case when V is nonperiodic, by applying Strauss' lemma ([46, Corollary 1.26]) or using the fact the spectrum of $-\Delta + V$ is discrete at the bottom, problems like or similar to (1.2) have been widely investigated in the literature, see, e.g., [27–29,43] and the references therein.

In this paper we consider the periodic case. In this abstract setting there is a lack of compactness because of invariance by $\mathbb{Z}^N$-translation. The concentration-compactness argument is well employed to deal with this case. When $V = V_1 - \lambda$ is periodic and $\lambda \in \mathbb{R}$ is a free parameter, the eigenvalue problem (1.2) has been studied by Albanese [3] and Catto *et al.* [9] provided that $f(u) = u$; moreover, solutions with prescribed L^2 norm were obtained in this case. Depending on the location of 0 in the spectrum of $-\Delta + V$, there are many results about the existence and multiplicity solutions to equation (1.2). We recall some of these results in what follows.

i) For the positive definite case (that is, $V > 0$), the existence of nontrivial solutions can be obtained via the mountain pass theorem and weak sequential continuity of the Fréchet derivative of the energy functional associated to problem (1.2). Ackermann [1] studied (1.2) under assumptions (W1), (W2), (F1), (F5) and using the Ambrosetti-Rabinowitz type condition [4]:

(AR) there exists $\theta > 2$ such that for all $u \in \mathbb{R} \setminus \{0\}$

$$2f(u)u \ge \theta F(u) > 0.$$

Moreover, infinitely many geometrically distinct solutions of problem (1.2), that is, solutions which do not just differ by a translation, were constructed by using an abstract critical point theorem established by Bartsch and Ding (see [6, Theorem 4.2]).

ii) For the case when V changes sign such that (V1) holds, problem (1.2) is much more difficult to handle due to the fact that the operator $-\Delta + V$ has purely continuous spectrum, which consists of closed disjoint intervals (see [39, Theorem XIII.100]). In this case, problem (1.2) turns into a strongly indefinite problem. In [8], Buffoni *et al.* considered this case and proved the existence of a nontrivial solution for (1.2) by assuming that $f(u) = u$ and $W(x) = 1/|x|$. Next, by taking advantage of the fact that the Fourier transform of W is positive, they showed that the function Ψ defined later by (2.7) is convex, which plays a crucial role in their proof; see [8] for more details. As discussed in [5,20], for potentials W behaving like

$$W(x) \sim \frac{1}{|x|^6} + \frac{C}{|x|}, \quad \text{for } |x| \text{ large} \tag{1.4}$$

it is not clear how to modify this function near 0 such that the Fourier transform of W is non-negative. In order to avoid the discussion about the non-negativity of Fourier transform of W, Ackermann [1] introduced the Cauchy-Schwarz type inequality (W3) in order to show the boundedness of Palais-Smale ((PS) for short) sequences. Ackermann [1] also showed a variant of the Brezis-Lieb lemma in order to get a decomposition of the energy functional Φ (defined by (2.9)) corresponding to (1.2) along (PS) sequences. In this way it is obtained a nontrivial solution provided that hypotheses (V1), (W1), (W2), (W3), (F1) and (AR) hold, and there are constructed infinitely many geometrically distinct solutions when, additionally, condition (F5) is satisfied. Moreover, some examples and criteria for checking condition (W3) were also given there, see [1, relations (2.2) and (2.3)] and [2]. Thus, it is an interesting problem to find nontrivial solutions and infinitely many geometrically distinct solutions of (1.2) without assuming condition (W3) and the convexity of Ψ. As it is well known, the classical Ambrosetti-Rabinowitz condition (AR) plays an important role in establishing the mountain-pass or linking geometry and in verifying the Palais-Smale condition. That is why this technical condition is commonly used in the literature, see, e.g., [1,4,25,26,30]. However, there are many functions that do not satisfy condition (AR), see, for instance, $F(t) = |t| \ln(1 + t^2)$. So, it is quite natural to ask whether it is possible to obtain the results established in [1,8] without the classic condition (AR) or under a weaker condition.

In this paper we address the following question.

Question 1. Study problem (1.2) by getting rid of the Cauchy-Schwarz type inequality (W3), which is a crucial hypothesis of the proof developed by Ackermann [1]. In this case, how to obtain the existence of ground state solutions to problem (1.2) without using the classical condition (AR); see Remarks 1.4 and 1.6.

The analysis developed in this paper will bring some new difficulties both in the verification of the linking geometry and for proving the boundedness of Cerami sequences for the energy functional. Employing some new techniques and introducing some generic conditions on f, we obtain the existence of ground state solution for equation (1.2). Such an existence result, to the best of our knowledge, seems to be new, see Remark 1.3. We are also concerned with the existence of infinitely many geometrically distinct solutions of problem (1.2). Note that multiplicity results in [1,7,6,15] were constructed by using pseudo-index and the $(\mathrm{PS})_I$-attractor or $(\mathrm{C})_I$-attractor in order to obtain the deformations. Moreover, instead of dealing directly with the different exponents r_1, r_2, p_1, p_2 in (W1) and (F1), it needs to split W and F into a sum of functions, see [1]. One may ask whether is possible to consider the superquadratic part Ψ directly and to employ a direct approach to show the multiplicity results. In this paper we give an affirmative answer. First, we construct a profile decomposition of bounded sequences in $H^1(\mathbb{R}^N)$ (see Lemma 4.1) in order to analyse Cerami sequences. Next, applying the Lusternik-Schnirelmann theory and using deformation arguments, we obtain the existence of infinitely many geometrically distinct solutions of problem (1.2). The main results in this paper extend and complement the above mentioned results of [1,8]. More precisely, we will prove Theorems 1.1 and 1.2 below by using following conditions:

(F3) $\widehat{\mathcal{F}}(t) := f(t)t - F(t) \geq 0$, and there exist $c_1 > 0$, $\max\left\{1, \frac{(2r_1-1)N}{(N+2)r_1-N}\right\} < \kappa \leq \frac{p_2}{p_2-1}$ such that

$$|f(t)|^\kappa \leq c_1 \widehat{\mathcal{F}}(t), \quad \forall\, t \in \mathbb{R};$$

(F3′) $\widehat{\mathcal{F}}(t) := f(t)t - F(t) \geq 0$, and there exist $c_1 > 0$, $\max\left\{1, \frac{(2r_1-1)N}{(N+2)r_1-N}\right\} < \kappa_i \leq \frac{2r_2-1}{r_2-1}$, $i = 1, 2$ such that

$$|f(t)\chi_{[0,1)}(|t|)|^{\kappa_1} + |f(t)\chi_{[1,+\infty)}(|t|)|^{\kappa_2} \leq c_1 \widehat{\mathcal{F}}(t), \quad \forall\, t \in \mathbb{R};$$

(F4) $\lim_{t\to 0} \frac{f(t)}{t}$ exists.

Our main results read as follows.

Theorem 1.1. *Assume that V, W and f satisfy* (V1), (W1), (W2), (F1), (F2) *and* (F3). *Then problem* (1.2) *has a solution $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{K}} \Phi > 0$, where $\mathcal{K} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \Phi'(u) = 0\}$. If, in addition, f satisfies* (F4) *and* (F5), *then problem* (1.2) *possesses infinitely many pairs of geometrically distinct solutions $\pm u$.*

Theorem 1.2. *Assume that V, W and f satisfy* (V1), (W1), (W2), (W3), (F1), (F2) *and* (F3′). *Then problem* (1.2) *has a solution $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{K}} \Phi > 0$. If, in addition, f satisfies* (F4) *and* (F5), *then problem* (1.2) *possesses infinitely many pairs of geometrically distinct solutions $\pm u$.*

Remark 1.3. For case $N = 3$ and $f(t) = t$, problem (1.2) reduces to the classical Choquard-Pekar equation:

$$-\Delta u + V(x)u = (W * u^2)u, \quad u \in H^1(\mathbb{R}^3).$$

Assuming that hypotheses (V1), (W1) and (W2) are satisfied, then existence and multiplicity results can be deduced from Theorem 1.1 directly. It is worth pointing out that the Cauchy-Schwarz type inequality associated with W, that is, condition (W3), is crucially needed in the argument of Ackermann [1] in order to establish the same results for the above Choquard-Pekar equation, see the remarks after Theorem 2.2 of [1].

Remark 1.4. Besides the power function $f(t) = a|t|^{p-2}t$ with $a > 0$ and $p \geq 2$, there are indeed many functions which satisfy (F1)-(F4). For example,

i). $F(t) = |t| \ln(1 + t^2)$. Then

$$f(t) = \operatorname{sign}(t) \ln(1 + t^2) + \frac{2|t|t}{1 + t^2}, \quad \widehat{\mathcal{F}}(t) = \frac{2|t|^3}{1 + t^2}.$$

Then f satisfies (F1)-(F4) with $1 \le r_1 \le r_2 < \infty$, $\frac{2r_2-1}{r_2} < p_1 = p_2 < \min\{3, \frac{(2r_1-1)2^*}{2r_1}\}$ and $\kappa = \frac{p_2}{p_2-1}$. But $f(t)$ does not satisfy (AR).

ii). $F(t) = \frac{|t|^p}{1+t^2}$ with $p > 3$. Then

$$f(t) = \frac{p|t|^{p-2}t + (p-2)|t|^p t}{(1+t^2)^2}, \quad \widehat{\mathcal{F}}(t) = \frac{(p-1)|t|^p + (p-3)|t|^{p+2}}{(1+t^2)^2}.$$

Then f satisfies (F1)-(F4) with $1 \le r_1 \le r_2 < \infty$, $\frac{2r_2-1}{r_2} < p_1 = p_2 = p-2 < \frac{(2r_1-1)2^*}{2r_1}$ and $\kappa = \frac{p-2}{p-3}$.

iii). $F(t) = \frac{a|t|^{p_1}+b|t|^{p_2}}{1+t^2}$ with $1 < p_2 - 2 < p_1 < p_2$ and $a, b > 0$. Then

$$f(t) = \frac{ap_1|t|^{p_1-2}t + bp_2|t|^{p_2-2}t + a(p_1-2)|t|^{p_1}t + b(p_2-2)|t|^{p_2}t}{(1+t^2)^2}$$

and

$$\widehat{\mathcal{F}}(t) = \frac{a(p_1-1)|t|^{p_1-2}t + b(p_2-1)|t|^{p_2-2}t + a(p_1-3)|t|^{p_1}t + b(p_2-3)|t|^{p_2}t}{(1+t^2)^2}.$$

Then f satisfies (F1)-(F4) with $1 \le r_1 \le r_2 < \infty$, $2 \le p_1 < p_2 < \frac{(2r_1-1)2^*}{2r_1}$ and $\kappa = \frac{p_2-2}{p_2-3}$.

As mentioned in [1, relation (2.3)], any nonnegative radial decreasing functions W satisfy (W3) and for W as in (1.4) we can use a simple regularization near 0 so that it satisfies (W1)-(W3).

Consider now the Choquard equation:

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.5}$$

where $N \ge 3$, $\alpha \in (0, N)$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}, \quad x \ne 0.$$

Then we have the following results.

Theorem 1.5. *Assume that V and f satisfy* (V1), (F1), (F2) *and* (F3′). *Then problem* (1.5) *has a solution $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{K}} \Phi > 0$. If, in addition, f satisfies* (F4) *and* (F5)*, then problem* (1.2) *possesses infinitely many pairs of geometrically distinct solutions $\pm u$.*

Remark 1.6.

i) Condition (F3′) is weaker than (F3) by the fact that $\frac{p_2}{p_2-1} < \frac{2r_2-1}{r_2-1}$, as it follows from (F1).

ii) Condition (F3′) weakens the Ambrosetti-Rabinowitz condition (AR). Indeed, if (AR) holds, then

$$1 - \frac{F(t)}{f(t)t} \ge 1 - \frac{2}{\mu} > 0, \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

Consider first the case $|t| < 1$. We choose $\kappa_1 = \frac{p_1}{p_1-1}$ and some $c_1 > 0$ such that

$$\frac{|f(t)t|^{\kappa_1-1}}{c_1|t|^{\kappa_1}} \leq 1 - \frac{2}{\mu} \leq 1 - \frac{F(t)}{f(t)t}, \quad \forall\, t \in \mathbb{R} \setminus \{0\},$$

which yields

$$|f(t)\chi_{[0,1)}(|t|)|^{\kappa_1} \leq c_1\widehat{\mathcal{F}}(t), \quad \forall\, t \in \mathbb{R};$$

For the case $|t| \geq 1$, we choose $\kappa_2 = \frac{p_2}{p_2-1}$ and some $c_1 > 0$ such that

$$\frac{|f(t)t|^{\kappa_2-1}}{c_1|t|^{\kappa_2}} \leq 1 - \frac{2}{\mu} \leq 1 - \frac{F(t)}{f(t)t}, \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

It follows that

$$|f(t)\chi_{[1,+\infty)}(|t|)|^{\kappa_2} \leq c_1\widehat{\mathcal{F}}(t), \quad \forall\, t \in \mathbb{R}.$$

Condition (F3$'$) holds in both cases.

Corollary 1.7. *Under the hypotheses of Theorems 1.1 and 1.2, the conclusions also hold for the following problem with repulsive particle interaction:*

$$-\Delta u + V(x)u = -(W * F(u))f(u), \quad u \in H^1(\mathbb{R}^N).$$

To complete this section, we sketch our proof. In order to obtain the existence of ground state solutions for problem (1.2), we first construct the linking structure of Φ. By using the linking theorem [26, Theorem 2.1], we will find the Cerami sequences for the energy functional Φ. Boundedness of the sequences will be proved by virtue of the technical conditions (F3) and (F3$'$). Next, by applying a concentration compactness argument, we find nontrivial solutions of problem (1.2). Finally, we constrain the functional Φ on the critical point set $\mathcal{K}$ and we show that the corresponding infimum is positive, then ground states of (1.2) are obtained by a standard argument. The proof of multiplicity results is carried out via the Lusternik-Schnirelmann theory and deformation arguments. In order to obtain the deformations we shall use the decomposition of Φ along the Cerami sequences and verify the discreteness of such sequences. This requires a deep analysis of the profile decomposition of bounded sequences in $H^1(\mathbb{R}^N)$ due to different exponents r_1, r_2, p_1, p_2 that appear in (W1) and (F1); see Lemma 4.1 for details. Next, arguing by contradiction, we succeed in establishing the existence of infinitely many geometrically distinct solutions for (1.2).

The present paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we establish the linking structure of Φ and show the existence of ground state solutions for problem (1.2). In Section 4, we construct a profile decomposition of bounded sequences in $H^1(\mathbb{R}^N)$ and we establish multiplicity results via deformation arguments.

2. Preliminaries

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. We say that a functional $\varphi \in C^1(X, \mathbb{R})$, φ is weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$; φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$. Recall that a sequence $\{u_n\} \subset X$ is called a Cerami sequence for φ at the level c ($(\mathrm{Ce})_c$-sequence for short) if $\varphi(u_n) \to c$ and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$.

Lemma 2.1. [25,26] *Let $(X, \|\cdot\|)$ be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X, \mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}\left(\|u^+\| - \|u^-\|\right) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(KS1) $\psi \in C^1(X, \mathbb{R})$ *is bounded from below and weakly sequentially lower semi-continuous;*
(KS2) ψ' *is weakly sequentially continuous;*
(KS3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \left\{u \in X^+ : \|u\| = \rho\right\}, \quad Q = \left\{v + se : v \in X^-,\ s \ge 0,\ \|v + se\| \le r\right\}.$$

Then there exist a constant $c \in [\kappa, \sup \varphi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Let $\mathcal{A} = -\Delta + V$. Then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^N)$ (see [17, Theorem 4.26]). Let $\{\mathcal{E}(\lambda) : -\infty \le \lambda \le +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [16, Theorem IV 3.3]). Let

$$E = \mathfrak{D}(|\mathcal{A}|^{1/2}), \quad E^- = \mathcal{E}(0)E, \quad E^+ = [id - \mathcal{E}(0)]E. \tag{2.1}$$

For any $u \in E$, we have $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0)u \in E^-, \quad u^+ := [id - \mathcal{E}(0)]u \in E^+ \tag{2.2}$$

and

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall\, u \in E \cap \mathfrak{D}(\mathcal{A}). \tag{2.3}$$

Define an inner product

$$(u, v) = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2}, \quad u, v \in E \tag{2.4}$$

and the corresponding norm

$$\|u\| = \left\||\mathcal{A}|^{1/2}u\right\|_2, \quad u \in E, \tag{2.5}$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$ and $\|\cdot\|_s$ is the norm of $L^s(\mathbb{R}^N)$. By (V1), $E = H^1(\mathbb{R}^N)$ with equivalent norms. Therefore, E embeds continuously in $L^s(\mathbb{R}^N)$ for all $2 \le s \le 2^*$. In addition, one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

Lemma 2.2. [1, Lemma 3.1] *Let $r > 1$ and s' be the conjugate exponent for $s = 2r/(2r-1)$, let $t \in [s, \infty)$, and let μ be given by $1/s' + 1/t = 1/\mu$. Assume that $U \in L^r(\mathbb{R}^N)$. Then the bilinear map $L^s \times L^t \to L^\mu$, sending (u, v) to $(U * u)v$, is well defined and continuous, with*

$$\|(U * u)v\|_\mu \le \|U * u\|_{s'}\|v\|_t \le \|U\|_r\|u\|_s\|v\|_t. \tag{2.6}$$

If $\{u_n\} \subset L^s$ and $\{v_n\} \subset L^t$ are bounded and either $u_n \to u$ in L^s and $v_n \to v$ in L^t_{loc} or $u_n \to u$ in L^s_{loc} and $v_n \to v$ in L^t, then $(U * u_n)v_n \to (U * u)v$ in L^μ.

We set

$$\Psi(u) = \int_{\mathbb{R}^N} (W * F(u))F(u)\mathrm{d}x, \quad \forall\, u \in E. \tag{2.7}$$

Employing Lemma 2.2 and using a standard argument, we obtain the following property.

Lemma 2.3. *Assume that* (W2), (F1) *and* (F2) *hold. Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

Lemma 2.4. [10, Proposition 2.3] *Assume that* (V1) *hold and $q \in (2, \infty)$. Then*

$$\|\mathcal{E}(0)u\|_q \le c_2\|u\|_q, \quad \forall\, u \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N). \tag{2.8}$$

In view of Lemma 2.3, under assumptions (V1), (W1), (W2) and (F1), the solutions of problem (1.2) are the critical points of the following energy functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (W * F(u))F(u)\mathrm{d}x, \quad \forall\, u \in E. \tag{2.9}$$

Then Φ is of class $\mathcal{C}^1(E, \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^N} (W * F(u))f(u)v\mathrm{d}x, \quad \forall\, u, v \in E. \tag{2.10}$$

In view of (2.3) and (2.5), we have

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \frac{1}{2}\int_{\mathbb{R}^N} (W * F(u))F(u)\mathrm{d}x \tag{2.11}$$

and

$$\langle \Phi'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}^N} (W * F(u))f(u)u\mathrm{d}x, \quad \forall\, u = u^- + u^+ \in E. \tag{2.12}$$

3. The existence of ground state solutions

In this section, we show the existence of ground state solutions for problem (1.2).

Set $s_i = \frac{2r_i}{2r_i - 1}$, $i = 1, 2$, and define

$$\zeta(s_i) = \begin{cases} \frac{2^* s_i}{2^* - s_i}, & \text{if } s_i \geq \frac{N}{N-1} \text{ and } N \geq 3, \\ 2, & \text{if } s_i < \frac{N}{N-1} \text{ and } N \geq 3, \\ 2s_i, & \text{if } N = 2; \end{cases} \quad \eta(s_i) = \begin{cases} 2^*, & \text{if } s_i \geq \frac{N}{N-1} \text{ and } N \geq 3, \\ \frac{2s_i}{2 - s_i}, & \text{if } s_i < \frac{N}{N-1} \text{ and } N \geq 3, \\ 2s_i, & \text{if } N = 2. \end{cases} \tag{3.1}$$

It follows from (F1) that $s_i p_j \in (2, 2^*)$ for $i, j \in \{1, 2\}$. By (W2), there exists $r_0 > 0$ such that $W(x) > 0$ for $|x| \leq r_0$.

By Lemma 2.2, we have the following auxiliary property.

Lemma 3.1. *Assume that* (W1) *hold. Then*

i) *for $w_1, w_2 \in L^{s_i}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (W_i * |w_1|)|w_2|\mathrm{d}x \leq \mathcal{C}_i \|w_1\|_{s_i} \|w_2\|_{s_i}, \quad i = 1, 2; \tag{3.2}$$

ii) *for $w_1 \in L^{s_i}(\mathbb{R}^N)$, $w_2 \in L^{s_i p_j/(p_j - 1)}(\mathbb{R}^N)$ and $w_3 \in L^{s_i p_j}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (W_i * |w_1|)|w_2||w_3|\mathrm{d}x \leq \mathcal{C}_i \|w_1\|_{s_i} \|w_2\|_{s_i p_j/(p_j-1)} \|w_3\|_{s_i p_j}, \quad i, j = 1, 2; \tag{3.3}$$

iii) *for $w_1 \in L^{s_i}(\mathbb{R}^N)$, $w_2 \in L^{\zeta(s_i)}(\mathbb{R}^N)$ and $w_3 \in L^{\eta(s_i)}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (W_i * |w_1|)|w_2||w_3|\mathrm{d}x \leq \mathcal{C}_i \|w_1\|_{s_i} \|w_2\|_{\zeta(s_i)} \|w_3\|_{\eta(s_i)}, \quad i = 1, 2. \tag{3.4}$$

Lemma 3.2. *Assume that* (V1), (W1) *and* (F1) *hold. Then there exists* $\rho > 0$ *such that*

$$\hat{\kappa} := \inf\left\{\Phi(u) : u \in E^+, \|u\| = \rho\right\} > 0. \tag{3.5}$$

The proof of Lemma 3.2 is standard and hence is omitted.

Lemma 3.3. *Assume that* (V1), (W1), (W2), (F1) *and* (F2) *hold. Then for any* $e \in E^+$*, there is* $r > \rho$ *such that* $\sup \Phi(\partial Q) \le 0$*, where* ρ *is defined by Lemma 3.2 and*

$$Q = \left\{w + se^+ : w \in E^-,\ s \ge 0,\ \|w + se^+\| \le r\right\}. \tag{3.6}$$

Proof. It is sufficient to show that $\Phi(se + w) \to -\infty$ as $s \in \mathbb{R}$, $w \in E^-$, $\|se + w\| \to \infty$. Arguing indirectly, assume that for some sequence $\{s_n e + w_n\} \subset \mathbb{R}\ e \oplus E^-$ with $\|s_n e + w_n\| \to \infty$, there is $M > 0$ such that $\Phi(s_n e + w_n) \ge -M$ for all $n \in \mathbb{N}$. Set $v_n = (s_n e + w_n)/\|s_n e + w_n\| = t_n e + v_n^-$, then $\|t_n e + v_n^-\| = 1$. Passing to a subsequence, we may assume that $t_n \to \bar{t}$ and $v_n^- \rightharpoonup v^-$ in E, and so $v_n^- \to v^-$ a.e. on $\mathbb{R}^N$, and

$$\begin{aligned} -\frac{M}{\|s_n e + w_n\|^2} &\le \frac{\Phi(s_n e + w_n)}{\|s_n e + w_n\|^2} \\ &= \frac{t_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \frac{(W * F(s_n e + w_n))F(s_n e + w_n)}{\|s_n e + w_n\|^2}\mathrm{d}x. \end{aligned} \tag{3.7}$$

If $\bar{t} = 0$, it follows from (3.7) that

$$0 \le \frac{1}{2}\|v_n^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \frac{(W * F(s_n e + w_n))F(s_n e + w_n)}{\|s_n e + w_n\|^2}\mathrm{d}x \le \frac{t_n^2}{2}\|e\|^2 + \frac{M}{\|s_n e + w_n\|^2} \to 0,$$

which yields $\|v_n^-\| \to 0$, and so $1 = \|t_n e + v_n^-\| \to 0$, a contradiction.

If $\bar{t} \ne 0$, then $\bar{t}e + v^- \ne 0$. Hence, we can choose $x_0 \in \mathbb{R}^N$ and $r_1 \in (0, r_0/2)$ such that $|\bar{t}e + v^-| > 0$ for $x \in B_{r_1}(x_0)$. It follows from (W2), (F2) and (3.7) that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty}\left[\frac{t_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \frac{(W * F(s_n e + w_n))F(s_n e + w_n)}{\|s_n e + w_n\|^2}\mathrm{d}x\right] \\ &\le \frac{\bar{t}^2}{2}\|e\|^2 - \frac{1}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\frac{F((s_n e + w_n)(x))}{|(s_n e + w_n)(x)|}|(t_n e + v_n^-)(x)| \\ &\quad \times \frac{F((s_n e + w_n)(y))}{|(s_n e + w_n)(y)|}|(t_n e + v_n^-)(y)|\mathrm{d}x\mathrm{d}y \\ &\le \frac{\bar{t}^2}{2}\|e\|^2 - \frac{1}{2}\int_{B_{r_1}(x_0)}\int_{B_{r_1}(x_0)} W(x-y)\left[\liminf_{n\to\infty}\frac{F((s_n e + w_n)(x))}{|(s_n e + w_n)(x)|}|(t_n e + v_n^-)(x)|\right] \end{aligned}$$

$$\times \left[\liminf_{n\to\infty} \frac{F((s_n e + w_n)(y))}{|(s_n e + w_n)(y)|} |(t_n e + v_n^-)(y)| \right] \mathrm{d}x\mathrm{d}y$$
$$= -\infty,$$

a contradiction. □

By Lemmas 2.1, 2.3, 3.2 and 3.3, we have the following lemma.

Lemma 3.4. *Assume that* (V1), (W1), (W2), (F1) *and* (F2) *hold. Then there exist a constant* $\bar{c} > 0$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to \bar{c}, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.8}$$

Lemma 3.5. *Assume that* (V1), (W1), (W2), (F1), (F2) *and* (F3) *hold. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c \geq 0, \quad \langle \Phi'(u_n), u_n^{\pm} \rangle \to 0 \tag{3.9}$$

is bounded in E.

Proof. In view of (2.11), (2.12) and (3.9), one has

$$c + o(1) = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle = \frac{1}{2} \int_{\mathbb{R}^N} (W * F(u_n)) \widehat{\mathcal{F}}(u_n) \mathrm{d}x. \tag{3.10}$$

To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|$. If $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^2 \mathrm{d}x = 0$, then by Lions' concentration-compactness principle [46, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. By (F1) and (F3), we have

$$\max\left\{1, \frac{(2r_1 - 1)N}{(N+2)r_1 - N}\right\} < \kappa \leq \frac{p_2}{p_2 - 1} < \frac{2r_2 - 1}{r_2 - 1}. \tag{3.11}$$

Set $\kappa' = \kappa/(\kappa - 1)$. Then (3.11), together with the fact that $1 < p_1 \leq p_2$ and $s_1 \geq s_2 > 1$, implies

$$1 < p_1 \leq p_2 \leq \kappa', \quad 2 < s_2\kappa' \leq s_1\kappa' < 2^*. \tag{3.12}$$

By virtue of (W1), (W2), (F1), (F3), (2.8), (3.2), (3.10), (3.12) and the Hölder inequality, we find

$$\frac{1}{\|u_n\|} \left| \int_{\mathbb{R}^N} (W * F(u_n)) f(u_n) v_n^+ \mathrm{d}x \right|$$
$$\leq \frac{1}{\|u_n\|} \left[\int_{\mathbb{R}^N} (W * F(u_n)) |f(u_n)|^\kappa \mathrm{d}x \right]^{1/\kappa} \left[\int_{\mathbb{R}^N} (W * F(u_n)) |v_n^+|^{\kappa'} \mathrm{d}x \right]^{1/\kappa'}$$

$$
\begin{aligned}
&\leq \frac{C_1}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W*F(u_n))\widehat{\mathcal{F}}(u_n)\mathrm{d}x\right]^{1/\kappa}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}\\
&\leq \frac{C_2}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}\\
&= \frac{C_2}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W_1*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x+\int_{\mathbb{R}^N}(W_2*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}\\
&\leq \frac{C_2}{\|u_n\|}\left[\|W_1\|_{r_1}\|F(u_n)\|_{s_1}\|v_n^+\|_{s_1\kappa'}^{\kappa'}+\|W_2\|_{r_2}\|F(u_n)\|_{s_2}\|v_n^+\|_{s_2\kappa'}^{\kappa'}\right]^{1/\kappa'}\\
&\leq \frac{C_3}{\|u_n\|}\left[\left(\|u_n\|_{s_1p_1}^{p_1}+\|u_n\|_{s_1p_2}^{p_2}\right)\|v_n^+\|_{s_1\kappa'}^{\kappa'}+\left(\|u_n\|_{s_2p_1}^{p_1}+\|u_n\|_{s_2p_2}^{p_2}\right)\|v_n^+\|_{s_2\kappa'}^{\kappa'}\right]^{1/\kappa'}\\
&\leq \frac{C_4}{\|u_n\|}\left[\left(\|u_n\|^{p_1/\kappa'}+\|u_n\|^{p_2/\kappa'}\right)\left(\|v_n^+\|_{s_1\kappa'}+\|v_n^+\|_{s_2\kappa'}\right)\right]\\
&\leq C_5\left(\|v_n^+\|_{s_1\kappa'}+\|v_n^+\|_{s_2\kappa'}\right)\\
&\leq C_6\left(\|v_n\|_{s_1\kappa'}+\|v_n\|_{s_2\kappa'}\right)=o(1).
\end{aligned}
\tag{3.13}
$$

Similarly, we have

$$
\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^-\mathrm{d}x=o(1).
\tag{3.14}
$$

Hence, combining (3.13) with (3.14) and making use of (2.10) and (3.9), we have

$$
\begin{aligned}
1+o(1)&=\frac{\|u_n\|^2-\langle\Phi'(u_n),u_n^+-u_n^-\rangle}{\|u_n\|^2}\\
&=\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^+\mathrm{d}x-\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^-\mathrm{d}x\\
&\leq o(1).
\end{aligned}
$$

This contradiction shows that $\delta>0$.

Going if necessary to a subsequence, we may assume the existence of $k_n\in\mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)}|v_n|^2\mathrm{d}x>\frac{\delta}{2}$. Let $w_n(x)=v_n(x+k_n)$. Recall that $V(x)$ is 1-periodic in each of $x_1,x_2,\ldots,x_N$. Then $\|w_n\|=\|v_n\|=1$, and

$$
\int_{B_{1+\sqrt{N}}(0)}|w_n|^2\mathrm{d}x>\frac{\delta}{2}.
\tag{3.15}
$$

Passing to a subsequence, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$, $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (3.15) implies that $w \neq 0$.

Now we define $u_n^{k_n}(x) = u_n(x + k_n)$, then $u_n^{k_n}/\|u_n\| = w_n \to w$ a.e. on $\mathbb{R}^N$, $w \neq 0$. Hence, we can choose $x_0 \in \mathbb{R}^N$ and $r_2 \in (0, r_0/2)$ such that $|w(x)| > 0$ for a.e. $x \in B_{r_2}(x_0)$. For $x \in B_{r_2}(x_0)$, we have $\lim_{n\to\infty} |u_n^{k_n}(x)| = \infty$. Hence, it follows from (2.9), (3.9), (F2), (F3), (W2) and Fatou's lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 \right) - \frac{1}{2\|u_n\|^2} \int_{\mathbb{R}^N} (W * F(u_n)) F(u_n) \mathrm{d}x \right] \\
&\le \frac{1}{2} - \frac{1}{2} \liminf_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) \frac{F(u_n^{k_n}(x))}{|u_n^{k_n}(x)|} |w_n(x)| \frac{F(u_n^{k_n}(y))}{|u_n^{k_n}(y)|} |w_n(y)| \mathrm{d}x\mathrm{d}y \\
&\le \frac{1}{2} - \frac{1}{2} \int_{B_{r_2}(x_0)} \int_{B_{r_2}(x_0)} W(x-y) \left[\liminf_{n\to\infty} \frac{F(u_n^{k_n}(x))}{|u_n^{k_n}(x)|} |w_n(x)| \right] \\
&\qquad \times \left[\liminf_{n\to\infty} \frac{F(u_n^{k_n}(y))}{|u_n^{k_n}(y)|} |w_n(y)| \right] \mathrm{d}x\mathrm{d}y \\
&\le -\infty.
\end{aligned}
$$

This contradiction shows that $\{u_n\}$ is bounded. □

Lemma 3.6. *Assume that* (V1), (W1), (W2), (W3), (F1), (F2) *and* (F3′) *hold. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$
\Phi(u_n) \to c \ge 0, \quad \langle \Phi'(u_n), u_n^{\pm} \rangle \to 0 \tag{3.16}
$$

is bounded in E.

Proof. In view of (2.11), (2.12) and (3.16), one has (3.10) and

$$
c + o(1) = \Phi(u_n) = \frac{1}{2}\left(\|u_n^+\|^2 - \|u_n^-\|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N} (W * F(u_n)) F(u_n) \mathrm{d}x. \tag{3.17}
$$

To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|$. If $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^2 \mathrm{d}x = 0$, then by Lions' concentration-compactness principle [46, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set $\kappa_i' = \kappa_i/(\kappa_i - 1)$, $i = 1, 2$. Then (F1) and (F3′) imply

$$
2 \le s_2 \kappa_i' \le s_1 \kappa_i' < 2^*. \tag{3.18}
$$

From (W1), (3.2) and (3.18), we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (W * |v_n^+|^{\kappa_i'})|v_n^+|^{\kappa_i'} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} (W_1 * |v_n^+|^{\kappa_i'})|v_n^+|^{\kappa_i'} \mathrm{d}x + \int_{\mathbb{R}^N} (W_2 * |v_n^+|^{\kappa_i'})|v_n^+|^{\kappa_i'} \mathrm{d}x \\
&= \mathcal{C}_1 \|v_n^+\|_{s_1\kappa_i'}^{2\kappa_i'} + \mathcal{C}_2 \|v_n^+\|_{s_2\kappa_i'}^{2\kappa_i'}, \quad i = 1, 2.
\end{aligned} \tag{3.19}
$$

Set $f_1(t) = f(t)\chi_{[0,1)}(|t|)$ and $f_2(t) = f(t)\chi_{[1,+\infty)}(|t|)$. Then by virtue of (W1), (W2), (W3), (F1), (F3′), (3.10), (3.17), (3.19) and the Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{1}{\|u_n\|}\left|\int_{\mathbb{R}^N} (W * F(u_n)) f_i(u_n) v_n^+ \mathrm{d}x\right| \\
&\le \frac{1}{\|u_n\|}\left[\int_{\mathbb{R}^N} (W * F(u_n))|f_i(u_n)|^{\kappa_i} \mathrm{d}x\right]^{1/\kappa_i}\left[\int_{\mathbb{R}^N} (W * F(u_n))|v_n^+|^{\kappa_i'} \mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_1}{\|u_n\|}\left[\int_{\mathbb{R}^N} (W * F(u_n))\widehat{\mathcal{F}}(u_n) \mathrm{d}x\right]^{1/\kappa_i}\left[\int_{\mathbb{R}^N} (W * F(u_n))|v_n^+|^{\kappa_i'} \mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_2}{\|u_n\|}\left[\int_{\mathbb{R}^N} (W * F(u_n))|v_n^+|^{\kappa_i'} \mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_3}{\|u_n\|}\left[\int_{\mathbb{R}^N} (W * F(u_n))F(u_n) \mathrm{d}x\right]^{1/2\kappa_i'}\left[\int_{\mathbb{R}^N} (W * |v_n^+|^{\kappa_i'})|v_n^+|^{\kappa_i'} \mathrm{d}x\right]^{1/2\kappa_i'} \\
&\le \frac{C_4}{\|u_n\|}\|u_n\|^{1/\kappa_i'}\|v_n^+\| = o(1), \quad i = 1, 2.
\end{aligned} \tag{3.20}
$$

It follows that

$$
\frac{1}{\|u_n\|}\int_{\mathbb{R}^N} (W * F(u_n)) f(u_n) v_n^+ \mathrm{d}x = o(1). \tag{3.21}
$$

Similarly, we have

$$
\frac{1}{\|u_n\|}\int_{\mathbb{R}^N} (W * F(u_n)) f(u_n) v_n^- \mathrm{d}x = o(1). \tag{3.22}
$$

The rest of the proof is the same as for Lemma 3.5. □

Lemma 3.7. *Assume that V, W and f satisfy* (V1), (W1), (W2), (F1), (F2) *and* (F3). *Then* (1.2) *has a solution $\bar{u} \in E \setminus \{0\}$, that is,*

$$\mathcal{K} := \{u \in E \setminus \{0\} : \ \Phi'(u) = 0\} \neq \emptyset.$$

Proof. Combining Lemma 3.4 with Lemma 3.5, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (3.8). If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \mathrm{d}x = 0,$$

then by Lions' concentration-compactness principle [46, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. By virtue of (W1), (W2), (F1), (3.2), (3.10) and the Hölder inequality, we find

$$\begin{aligned}
\bar{c} + o(1) = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle &= \frac{1}{2}\int_{\mathbb{R}^N} (W * F(u_n))\widehat{\mathcal{F}}(u_n)\mathrm{d}x \\
&\leq C_1 \int_{\mathbb{R}^N} (W * F(u_n)) \left(|u_n|^{p_1} + |u_n|^{p_2}\right) \mathrm{d}x \\
&\leq C_2 \left[\|F(u_n)\|_{s_1} \left(\|u_n\|_{s_1 p_1}^{p_1} + \|u_n\|_{s_1 p_2}^{p_2}\right) + \|F(u_n)\|_{s_2} \left(\|u_n\|_{s_2 p_1}^{p_1} + \|u_n\|_{s_2 p_2}^{p_2}\right)\right] \\
&\leq C_3 \left[\left(\|u_n\|_{s_1 p_1}^{p_1} + \|u_n\|_{s_1 p_2}^{p_2}\right)^2 + \left(\|u_n\|_{s_2 p_1}^{p_1} + \|u_n\|_{s_2 p_2}^{p_2}\right)^2\right] \\
&= o(1).
\end{aligned} \tag{3.23}$$

This contradiction shows that $\delta > 0$. Using Lemma 2.3, by a standard argument, we can show that $\Phi'(\bar{u}) = 0$ for some $\bar{u} \in E \setminus \{0\}$. □

Similarly, we can prove the following lemma.

Lemma 3.8. *Assume that V, W and f satisfy* (V1), (W1), (W2), (W3), (F1), (F2) *and* (F3′). *Then* (1.2) *has a solution $\bar{u} \in E \setminus \{0\}$.*

Lemma 3.9. *Assume that* (V1), (W1), (W2), (F1), (F2) *and* (F3) *hold. Then*

i) $\vartheta_0 := \inf\{\|u\| : u \in \mathcal{K}\} > 0$;
ii) $c_0 := \inf\{\Phi(u) : u \in \mathcal{K}\} > 0$.

Proof. We only consider the case where $N \geq 3$, since the case where $N = 2$ can be dealt with similar arguments. Lemma 3.7 shows that $\mathcal{K} \neq \emptyset$. Let $\{u_n\} \subset \mathcal{K}$ such that $\|u_n\| \to \vartheta_0$. From (2.10), one has

$$\|u_n\|^2 = \int_{\mathbb{R}^N} (W * F(u_n)) f(u_n)(u_n^+ - u_n^-)\mathrm{d}x. \tag{3.24}$$

Hence, from (F1), (3.3) and (3.24), we have

$$\begin{aligned}
\|u_n\|^2 &= \int_{\mathbb{R}^N} (W * F(u_n)) f(u_n)(u_n^+ - u_n^-)\mathrm{d}x \\
&\le C_1 \int_{\mathbb{R}^N} (W * |F(u_n)|) \left(|u_n|^{p_1-1} + |u_n|^{p_2-1}\right) |u_n^+ - u_n^-|\mathrm{d}x \\
&\le C_2 \|F(u_n)\|_{s_1} \|u_n\|_{s_1p_1}^{p_1-1} \|u_n^+ - u_n^-\|_{s_1p_1} \\
&\quad + C_2 \|F(u_n)\|_{s_2} \|u_n\|_{s_2p_1}^{p_1-1} \|u_n^+ - u_n^-\|_{s_2p_1} \\
&\quad + C_2 \|F(u_n)\|_{s_1} \|u_n\|_{s_1p_2}^{p_2-1} \|u_n^+ - u_n^-\|_{s_1p_2} \\
&\quad + C_2 \|F(u_n)\|_{s_2} \|u_n\|_{s_2p_2}^{p_2-1} \|u_n^+ - u_n^-\|_{s_2p_2} \\
&\le C_3 \left(\|u_n\|_{s_1p_1}^{p_1} + \|u_n\|_{s_1p_2}^{p_2}\right) \Big[\|u_n\|_{s_1p_1}^{p_1-1} \|u_n^+ - u_n^-\|_{s_1p_1} \\
&\quad + \|u_n\|_{s_1p_2}^{p_2-1} \|u_n^+ - u_n^-\|_{s_1p_2}\Big] \\
&\quad + C_3 \left(\|u_n\|_{s_2p_1}^{p_1} + \|u_n\|_{s_2p_2}^{p_2}\right) \Big[\|u_n\|_{s_2p_1}^{p_1-1} \|u_n^+ - u_n^-\|_{s_2p_1} \\
&\quad + \|u_n\|_{s_2p_2}^{p_2-1} \|u_n^+ - u_n^-\|_{s_2p_2}\Big] \\
&\le C_4 \left(\|u_n\|^{p_1} + \|u_n\|^{p_2}\right) \left(\|u_n\|^{p_1} + \|u_n\|^{p_2}\right) \\
&\le \frac{1}{2}\|u_n\|^2 + C_5\|u_n\|^{2p_2}.
\end{aligned}$$

Therefore

$$\vartheta_0 + o(1) = \|u_n\| \ge (2C_5)^{-1/(2p_2-2)} > 0. \tag{3.25}$$

This shows that i) holds. Next, we prove that ii) also holds. Let $\{u_n\} \subset \mathcal{K}$ such that $\Phi(u_n) \to c_0$. Then $\langle \Phi'(u_n), v\rangle = 0$ for any $v \in E$. From (2.11) and (2.12), one has

$$c_0 + o(1) = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \frac{1}{2}\int_{\mathbb{R}^N} (W * F(u_n))\widehat{\mathcal{F}}(u_n)\mathrm{d}x. \tag{3.26}$$

By i), we can define $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|^2$. Set $\kappa' = \kappa/(\kappa - 1)$. By virtue of (F3), (3.26) and the Hölder inequality, we have that

$$\begin{aligned}
&\frac{1}{\|u_n\|}\int_{\mathbb{R}^N} (W * F(u_n)) f(u_n) v_n^+ \mathrm{d}x \\
&\le \frac{1}{\|u_n\|}\left[\int_{\mathbb{R}^N} (W * F(u_n))|f(u_n)|^{\kappa}\mathrm{d}x\right]^{1/\kappa} \left[\int_{\mathbb{R}^N} (W * F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}
\end{aligned}$$

$$\leq \frac{C_6}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W*F(u_n))\widehat{\mathcal{F}}(u_n)\mathrm{d}x\right]^{1/\kappa}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}$$

$$= \frac{C_7}{\|u_n\|}[c_0+o(1)]^{1/\kappa}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa'}\mathrm{d}x\right]^{1/\kappa'}$$

$$\leq \frac{C_8}{\|u_n\|}\left[\left(\|u_n\|^{p_1/\kappa'}+\|u_n\|^{p_2/\kappa'}\right)\left(\|v_n^+\|_{s_1\kappa'}+\|v_n^+\|_{s_2\kappa'}\right)\right][c_0+o(1)]^{1/\kappa}$$

$$\leq C_9\,[c_0+o(1)]^{1/\kappa}. \tag{3.27}$$

Similarly, we have

$$\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^-\mathrm{d}x \leq C_{10}\,[c_0+o(1)]^{1/\kappa}. \tag{3.28}$$

Hence, combining (3.27) with (3.28) and making use of (2.10), we deduce that

$$\begin{aligned} 1 &= \frac{\|u_n\|^2-\langle\Phi'(u_n),u_n^+-u_n^-\rangle}{\|u_n\|^2} \\ &= \frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^+\mathrm{d}x+\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^-\mathrm{d}x \\ &\leq C_{11}\,[c_0+o(1)]^{1/\kappa}. \end{aligned}$$

We conclude that $c_0>0$. □

Lemma 3.10. *Assume that* (V1), (W1), (W2), (W3), (F1), (F2) *and* (F3′) *hold. Then*

i) $\vartheta_0 := \inf\{\|u\| : u\in\mathcal{K}\}>0$;
ii) $c_0 := \inf\{\Phi(u) : u\in\mathcal{K}\}>0$.

Proof. The proof of i) is the same as of Lemma 3.9.

Next, we prove that ii) also holds. We only consider the case where $N\geq 3$, since the case where $N=2$ can be dealt similarly. Let $\{u_n\}\subset\mathcal{K}$ such that $\Phi(u_n)\to c_0$. Then $\langle\Phi'(u_n),v\rangle=0$ for any $v\in E$. From (2.11) and (2.12), we obtain (3.26) and

$$c_0+o(1)=\Phi(u_n)=\frac{1}{2}\left[\|u_n^+\|^2-\|u_n^-\|^2-\int_{\mathbb{R}^N}(W*F(u_n))F(u_n)\mathrm{d}x\right]. \tag{3.29}$$

Let $v_n=u_n/\|u_n\|$. Then $1=\|v_n\|$. Set $f_1(t)$ and $f_2(t)$ as in Lemma 3.6 and set $\kappa_i'=\kappa_i/(\kappa_i-1)$, $i=1,2$. By virtue of (W3), (F3′), (3.19), (3.26) and the Hölder inequality, we have

$$
\begin{aligned}
&\frac{1}{\|u_n\|}\left|\int_{\mathbb{R}^N}(W*F(u_n))f_i(u_n)v_n^+\mathrm{d}x\right| \\
&\le \frac{1}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W*F(u_n))|f_i(u_n)|^{\kappa_i}\mathrm{d}x\right]^{1/\kappa_i}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa_i'}\mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_1}{\|u_n\|}\left[\int_{\mathbb{R}^N}(W*F(u_n))\widehat{\mathcal{F}}(u_n)\mathrm{d}x\right]^{1/\kappa_i}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa_i'}\mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_2}{\|u_n\|}[c_0+o(1)]^{1/\kappa_i}\left[\int_{\mathbb{R}^N}(W*F(u_n))|v_n^+|^{\kappa_i'}\mathrm{d}x\right]^{1/\kappa_i'} \\
&\le \frac{C_3}{\|u_n\|}[c_0+o(1)]^{1/\kappa_i}\left[\int_{\mathbb{R}^N}(W*F(u_n))F(u_n)\mathrm{d}x\right]^{1/2\kappa_i'} \\
&\quad\times\left[\int_{\mathbb{R}^N}(W*|v_n^+|^{\kappa_i'})|v_n^+|^{\kappa_i'}\mathrm{d}x\right]^{1/2\kappa_i'} \\
&\le \frac{C_4}{\|u_n\|}[c_0+o(1)]^{1/\kappa_i}\,\|u_n\|^{1/\kappa_i'}\|v_n^+\| \\
&\le C_5\,[c_0+o(1)]^{1/\kappa_i}, \quad i=1,2.
\end{aligned}
\tag{3.30}
$$

It follows that

$$
\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^+\mathrm{d}x \le C_6\left\{[c_0+o(1)]^{1/\kappa_1}+[c_0+o(1)]^{1/\kappa_2}\right\}. \tag{3.31}
$$

Similarly, we have

$$
\frac{1}{\|u_n\|}\int_{\mathbb{R}^N}(W*F(u_n))f(u_n)v_n^-\mathrm{d}x \le C_7\left\{[c_0+o(1)]^{1/\kappa_1}+[c_0+o(1)]^{1/\kappa_2}\right\}. \tag{3.32}
$$

The rest of the proof is the same as the one of Lemma 3.9. $\square$

Proof of the first part in Theorem 1.1. By Lemmas 3.7 and 3.9, $\mathcal{K}\neq\emptyset$ and $c_0:=\inf\{\Phi(u): u\in\mathcal{K}\}>0$. Let $\{u_n\}\subset\mathcal{K}$ such that $\Phi(u_n)\to c_0$. By virtue of Lemma 3.5, the sequence $\{u_n\}$ is bounded in E. Hence, by a standard argument, we deduce that there exists $\bar{u}\in E\setminus\{0\}$ such that $\Phi'(\bar{u})=0$ and $\Phi(\bar{u})=c_0>0$. $\square$

Proof of the first part in Theorem 1.2. By Lemmas 3.8 and 3.10, $\mathcal{K} \neq \emptyset$ and $c_0 := \inf\{\Phi(u) : u \in \mathcal{K}\} > 0$. Let $\{u_n\} \subset \mathcal{K}$ such that $\Phi(u_n) \to c_0$. In view of Lemma 3.6, the sequence $\{u_n\}$ is bounded in E. Hence, by a standard argument, we deduce that there exists $\bar{u} \in E \setminus \{0\}$ such that $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) = c_0 > 0$. □

4. The existence of infinitely many solutions

In this section, we give the proofs of the second part in Theorem 1.1 and Theorem 1.2. To this end, we need some notations. For $d_2 \geq d_1 > -\infty$, we set

$$\Phi^{d_2} := \{u \in E : \Phi(u) \leq d_2\}, \quad \Phi_{d_1} := \{u \in E : \Phi(u) \geq d_1\}, \quad \Phi_{d_1}^{d_2} := \Phi_{d_1} \cap \Phi^{d_2},$$

and

$$\mathcal{K}_c := \{u \in \mathcal{K} : \Phi(u) = c\}.$$

Lemma 4.1. *Assume that* (W1), (W2), (F1) *and* (F4) *hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$, then for $v_n = u_n - \bar{u}$, up to a subsequence of $\{u_n\}$, we have*

$$\sup_{\varphi \in H^1(\mathbb{R}^N), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^N} [(W * F(u_n)) f(u_n) - (W * F(v_n)) f(v_n) - (W * F(\bar{u})) f(\bar{u})] \varphi \mathrm{d}x \right| = o(1). \tag{4.1}$$

Proof. From $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$, we deduce, up to a subsequence that $u_n - \bar{u} = v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, $v_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $v_n \to 0$ a.e. $x \in \mathbb{R}^N$. For any $a > 0$, we set

$$A_n^a = \{x \in \mathbb{R}^N : |v_n(x)| \leq a\}, \quad B_n^a = \mathbb{R}^N \setminus A_n^a.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, then there exists a constant $C_1 > 0$ such that

$$|B_n^a| \leq \frac{1}{a^2} \int_{B_n^a} |v_n|^2 \mathrm{d}x \leq \frac{C_1}{a^2} \to 0 \quad \text{as } a \to \infty. \tag{4.2}$$

Define $g(t) = \lim_{s \to t} f(s)/s$ for $t \in \mathbb{R}$. Then (F1) and (F4) imply $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exists $C_2 > 0$ such that

$$|g(t)| \leq C_2(|t|^{p_1 - 2} + |t|^{p_2 - 2}), \quad \forall\, t \in \mathbb{R}. \tag{4.3}$$

Moreover, we have

$$f(v_n + \bar{u}) - f(v_n) = g(v_n + \bar{u})\bar{u} + [g(v_n + \bar{u}) - g(v_n)]v_n. \tag{4.4}$$

From (F1) and (F4), we obtain $2 \leq p_1 \leq p_2 < (2r_1 - 1)2^*/(2r_1)$, which, together with (3.1), implies

$$2 \le \zeta(s_2) \le \zeta(s_1) \le \zeta(s_1)(p_2 - 1) < 2^*. \tag{4.5}$$

By (4.3) and (4.5), the Hölder inequality and the Sobolev inequality, we find

$$\begin{aligned}
&\int_{B_n^a} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&\le C_3 \int_{B_n^a} \left(|v_n|^{p_1-2} + |\bar{u}|^{p_1-2} + |v_n|^{p_2-2} + |\bar{u}|^{p_2-2}\right)^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&\le C_4 \int_{B_n^a} \Big(|v_n|^{\zeta(s_i)(p_1-1)} + |v_n|^{\zeta(s_i)(p_2-1)} \\
&\qquad + |\bar{u}|^{\zeta(s_i)(p_1-2)} |v_n|^{\zeta(s_i)} + |\bar{u}|^{\zeta(s_i)(p_2-2)} |v_n|^{\zeta(s_i)}\Big) \mathrm{d}x \\
&\le C_4 \left\{ \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{\zeta(s_i)(p_1-1)}{2^*}} \left[\|v_n\|_{2^*}^{\zeta(s_i)(p_1-1)} + \|\bar{u}\|_{2^*}^{\zeta(s_i)(p_1-2)} \|v_n\|_{2^*}^{\zeta(s_i)}\right] \right. \\
&\qquad \left. + \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{\zeta(s_i)(p_2-1)}{2^*}} \left[\|v_n\|_{2^*}^{\zeta(s_i)(p_2-1)} + \|\bar{u}\|_{2^*}^{\zeta(s_i)(p_2-2)} \|v_n\|_{2^*}^{\zeta(s_i)}\right] \right\} \\
&\le C_5 \left\{ \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{\zeta(s_i)(p_1-1)}{2^*}} + \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{\zeta(s_i)(p_2-1)}{2^*}} \right\} \quad \text{for } N \ge 3
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\int_{B_n^a} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&= \int_{B_n^a} |g(v_n + \bar{u}) - g(v_n)|^{2s_i} |v_n|^{2s_i} \mathrm{d}x \\
&\le C_6 \int_{B_n^a} \left(|v_n|^{2s_i(p_1-1)} + |v_n|^{2s_i(p_2-1)} + |\bar{u}|^{2s_i(p_1-2)} |v_n|^{2s_i} + |\bar{u}|^{2s_i(p_2-2)} |v_n|^{2s_i}\right) \mathrm{d}x \\
&\le C_7 \left\{ \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{s_i(p_1-1)}{4p_2}} \left[\|v_n\|_{8p_2}^{2s_i(p_1-1)} + \|\bar{u}\|_{8p_2}^{2s_i(p_1-2)} \|v_n\|_{8p_2}^{2s_i}\right] \right. \\
&\qquad \left. + \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{s_i(p_2-1)}{4p_2}} \left[\|v_n\|_{8p_2}^{2s_i(p_2-1)} + \|\bar{u}\|_{8p_2}^{2s_i(p_2-2)} \|v_n\|_{8p_2}^{2s_i}\right] \right\} \\
&\le C_8 \left\{ \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{s_i(p_1-1)}{4p_2}} + \left[\mathrm{meas}(B_n^a)\right]^{1-\frac{s_i(p_2-1)}{4p_2}} \right\} \quad \text{for } N = 2.
\end{aligned} \tag{4.7}$$

For any $\varepsilon > 0$, relations (4.2), (4.6) and (4.7) imply that there exists $\hat{a} > 0$ such that

$$\int_{B_n^{\hat{a}}} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \le \varepsilon, \quad \forall\, n \in \mathbb{N}. \tag{4.8}$$

By the uniformly continuity of g on $[-\hat{a}, \hat{a}]$, there exists $\delta > 0$ such that

$$|g(t+h) - g(t)| < \varepsilon \quad \text{for all } (t,h) \in [-\hat{a}, \hat{a}] \times [-\delta, \delta]. \tag{4.9}$$

Set

$$G^\delta = \{x \in \mathbb{R}^N : |\bar{u}(x)| \le \delta\}, \quad D^\delta = \mathbb{R}^N \setminus G^\delta.$$

Clearly,

$$\text{meas}(D^\delta) \le \frac{1}{\delta^2} \int_{D^\delta} |\bar{u}|^2 \mathrm{d}x \le \frac{\|\bar{u}\|_2^2}{\delta^2}. \tag{4.10}$$

From (4.3), (4.5), (4.9), (4.10), the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\int_{A_n^{\hat{a}}} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&= \int_{A_n^{\hat{a}} \cap G^\delta} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&\quad + \int_{A_n^{\hat{a}} \cap D^\delta} |g(v_n + \bar{u}) - g(v_n)|^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&\le \varepsilon^{\zeta(s_i)} \int_{A_n^{\hat{a}} \cap G^\delta} |v_n|^{\zeta(s_i)} \mathrm{d}x + C_9 \int_{A_n^{\hat{a}} \cap D^\delta} \Big(|v_n|^{p_1-2} + |\bar{u}|^{p_1-2} \\
&\quad + |v_n|^{p_2-2} + |\bar{u}|^{p_2-2} \Big)^{\zeta(s_i)} |v_n|^{\zeta(s_i)} \mathrm{d}x \\
&\le \varepsilon^{\zeta(s_i)} \|v_n\|_{\zeta(s_i)}^{\zeta(s_i)} + C_{10} \int_{D^\delta} \Big(|v_n|^{\zeta(s_i)(p_1-1)} + |v_n|^{\zeta(s_i)(p_2-1)} \\
&\quad + |\bar{u}|^{\zeta(s_i)(p_1-2)} |v_n|^{\zeta(s_i)} + |\bar{u}|^{\zeta(s_i)(p_2-2)} |v_n|^{\zeta(s_i)} \Big) \mathrm{d}x \\
&\le C_{11}\varepsilon + C_{10} \int_{D^\delta} \Big(|v_n|^{\zeta(s_i)(p_1-1)} + |v_n|^{\zeta(s_i)(p_2-1)} \Big) \mathrm{d}x \\
&\quad + C_{10} \left(\int_{D^\delta} |v_n|^{\zeta(s_i)(p_1-1)} \mathrm{d}x \right)^{\frac{1}{p_1-1}} \|\bar{u}\|_{\zeta(s_i)(p_1-1)}^{\zeta(s_i)(p_1-2)} \\
&\quad + C_{10} \left(\int_{D^\delta} |v_n|^{\zeta(s_i)(p_2-1)} \mathrm{d}x \right)^{\frac{1}{p_2-1}} \|\bar{u}\|_{\zeta(s_i)(p_2-1)}^{\zeta(s_i)(p_2-2)}
\end{aligned}$$

$$\leq C_{11}\varepsilon + o(1). \tag{4.11}$$

Here we have used $v_n \rightharpoonup 0$ and $\mathrm{meas}(D^\delta) \leq \frac{\|\bar{u}\|_2^2}{\delta^2}$. By a standard argument, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |g(v_n + \bar{u})|^{\zeta(s_i)} |\bar{u}|^{\zeta(s_i)} \mathrm{d}x \leq \varepsilon, \quad i = 1, 2 \tag{4.12}$$

and

$$\int_{\mathbb{R}^N \setminus B_R(0)} |f(\bar{u})|^{\zeta(s_i)} \mathrm{d}x \leq \varepsilon, \quad i = 1, 2. \tag{4.13}$$

Since $u_n \to \bar{u}$ in $L^2(B_R(0)) \cap L^{\zeta(s_1)(p_2-1)}(B_R(0))$, then [46, Lemma A.1] implies that there exists $w \in L^2(B_R(0)) \cap L^{\zeta(s_1)(p_2-1)}(B_R(0))$ such that, up to a subsequence,

$$|\bar{u}(x)|,\ |u_n(x)| \leq w(x), \quad \text{a.e. } x \in B_R(0). \tag{4.14}$$

Then by (F1) and (4.14), we have

$$\begin{aligned}
&|f(u_n) - f(v_n) - f(\bar{u})|^{\zeta(s_i)} \\
&\leq C_{12} \left(|u_n|^{p_1-1} + |\bar{u}|^{p_1-1} + |u_n|^{p_2-1} + |\bar{u}|^{p_2-1} \right)^{\zeta(s_i)} \\
&\leq C_{13} \left[|u_n|^{\zeta(s_i)(p_1-1)} + |\bar{u}|^{\zeta(s_i)(p_1-1)} + |u_n|^{\zeta(s_i)(p_2-1)} + |\bar{u}|^{\zeta(s_i)(p_2-1)} \right] \\
&\leq 2C_{13} \left[|w|^{\zeta(s_i)(p_1-1)} + |w|^{\zeta(s_i)(p_2-1)} \right], \quad \text{a.e. } x \in B_R(0).
\end{aligned} \tag{4.15}$$

Since $|w|^{\zeta(s_i)(p_1-1)} + |w|^{\zeta(s_i)(p_2-1)} \in L^1(B_R(0))$ and

$$f(u_n) - f(v_n) - f(\bar{u}) \to 0, \quad \text{a.e. } x \in B_R(0),$$

then it follows from (4.15) and the Lebesgue dominated convergence theorem that

$$\int_{B_R(0)} |f(u_n) - f(v_n) - f(\bar{u})|^{\zeta(s_i)} \, \mathrm{d}x = o(1). \tag{4.16}$$

From (4.8), (4.11), (4.12), (4.13) and (4.16), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(u_n) - f(v_n) - f(\bar{u})|^{\zeta(s_i)} \, \mathrm{d}x \\
&\leq \int_{B_R(0)} |f(u_n) - f(v_n) - f(\bar{u})|^{\zeta(s_i)} \, \mathrm{d}x + C_{14} \int_{\mathbb{R}^N \setminus B_R(0)} |f(\bar{u})|^{\zeta(s_i)} \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+C_{14}\int_{\mathbb{R}^N\setminus B_R(0)} |f(u_n)-f(v_n)|^{\zeta(s_i)}\,\mathrm{d}x\\
&\le \int_{B_R(0)} |f(u_n)-f(v_n)-f(\bar{u})|^{\zeta(s_i)}\,\mathrm{d}x + C_{14}\int_{\mathbb{R}^N\setminus B_R(0)} |f(\bar{u})|^{\zeta(s_i)}\,\mathrm{d}x\\
&\quad +C_{15}\int_{A_n^{\hat{a}}} |g(v_n+\bar{u})-g(v_n)|^{\zeta(s_i)}\,|v_n|^{\zeta(s_i)}\mathrm{d}x\\
&\quad +C_{15}\int_{B_n^{\hat{a}}} |g(v_n+\bar{u})-g(v_n)|^{\zeta(s_i)}\,|v_n|^{\zeta(s_i)}\mathrm{d}x\\
&\quad +C_{15}\int_{\mathbb{R}^N\setminus B_R(0)} |g(v_n+\bar{u})|^{\zeta(s_i)}|\bar{u}|^{\zeta(s_i)}\mathrm{d}x\\
&\le C_{16}\varepsilon+o(1).
\end{aligned}
\tag{4.17}
$$

Since $\varepsilon>0$ is arbitrary, then it follows from (4.17) that

$$
\int_{\mathbb{R}^N} |f(u_n)-f(v_n)-f(\bar{u})|^{\zeta(s_i)}\,\mathrm{d}x = o(1), \quad i=1,2. \tag{4.18}
$$

Since $F(t)=\int_0^t f(s)\mathrm{d}s$, by a standard argument, we can prove that

$$
\int_{\mathbb{R}^N} |F(u_n)-F(v_n)-F(\bar{u})|^{s_i}\,\mathrm{d}x = o(1), \quad i=1,2. \tag{4.19}
$$

Note that for any $\varphi\in E$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} [(W*F(u_n))f(u_n)-(W*F(v_n))f(v_n)-(W*F(\bar{u}))f(\bar{u})]\varphi\mathrm{d}x\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,[F(u_n(y))f(u_n(x))-F(v_n(y))f(v_n(x))\\
&\quad -F(\bar{u}(y))f(\bar{u}(x))]\,\varphi(x)\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,[F(u_n(y))-F(v_n(y))-F(\bar{u}(y))]\\
&\quad \times[f(u_n(x))-f(v_n(x))]\,\varphi(x)\mathrm{d}x\mathrm{d}y\\
&\quad +\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)F(\bar{u}(y))\,[f(u_n(x))-f(v_n(x))-f(\bar{u}(x))]\,\varphi(x)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,[F(u_n(y))-F(v_n(y))]\,f(v_n(x))\varphi(x)\mathrm{d}x\mathrm{d}y\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)F(v_n(y))\,[f(u_n(x))-f(v_n(x))]\,\varphi(x)\mathrm{d}x\mathrm{d}y\\
:=\;& I_1+I_2+I_3+I_4.
\end{aligned}
\tag{4.20}
$$

From (F1), (3.4), (4.5), (4.18) and (4.19), we have

$$
\begin{aligned}
&|I_1|\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,|F(u_n(y))-F(v_n(y))-F(\bar{u}(y))|\\
&\qquad \times |f(u_n(x))-f(v_n(x))|\,|\varphi(x)|\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N} (W*|F(u_n)-F(v_n)-F(\bar{u})|)\,|f(u_n)-f(v_n)|\,|\varphi|\mathrm{d}x\\
&\le \mathcal{C}_1\,\|F(u_n)-F(v_n)-F(\bar{u})\|_{s_1}\,\|f(u_n)-f(v_n)\|_{\zeta(s_1)}\,\|\varphi\|_{\eta(s_1)}\\
&\qquad +\mathcal{C}_2\,\|F(u_n)-F(v_n)-F(\bar{u})\|_{s_2}\,\|f(u_n)-f(v_n)\|_{\zeta(s_2)}\,\|\varphi\|_{\eta(s_2)}\\
&\le o(1)\|\varphi\|
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
|I_2|&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,|F(\bar{u}(y))|\,|f(u_n(x))-f(v_n(x))-f(\bar{u}(x))|\,|\varphi(x)|\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N} (W*|F(\bar{u})|)\,|f(u_n)-f(v_n)-f(\bar{u})|\,|\varphi|\mathrm{d}x\mathrm{d}y\\
&\le \mathcal{C}_1\,\|F(\bar{u})\|_{s_1}\,\|f(u_n)-f(v_n)-f(\bar{u})\|_{\zeta(s_1)}\,\|\varphi\|_{\eta(s_1)}\\
&\qquad +\mathcal{C}_2\,\|F(\bar{u})\|_{s_2}\,\|f(u_n)-f(v_n)-f(\bar{u})\|_{\zeta(s_2)}\,\|\varphi\|_{\eta(s_2)}\\
&\le o(1)\|\varphi\|.
\end{aligned}
\tag{4.22}
$$

Since $F(\bar{u})\in L^{s_1}(\mathbb{R}^N)\cap L^{s_2}(\mathbb{R}^N)$, then for any $\varepsilon>0$, there exists $R_1>0$ such that

$$
\left(\int_{\mathbb{R}^N\setminus B_{R_1}(0)} |F(\bar{u}(y))|^{s_i}\,\mathrm{d}y\right)^{\frac{1}{s_i}}<\varepsilon,\quad i=1,2. \tag{4.23}
$$

Let $\Omega_\varepsilon:=\{x\in\mathbb{R}^N\setminus B_1(0): W(x)\ge\varepsilon\}$ and $\Omega_\varepsilon^c:=\{x\in\mathbb{R}^N\setminus B_1(0): W(x)<\varepsilon\}$. Then (W1) implies that $\mathrm{meas}(\Omega_\varepsilon)<\infty$. Hence, it follows from the Hölder inequality and $v_n\to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2\le s<2^*$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{R_1+1}(0)} |f(v_n(x))||\varphi(x)| \left[\int_{B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))|\,\mathrm{d}y \right] \mathrm{d}x \\
&= \int_{B_{R_1}(0)} |F(\bar{u}(y))| \left[\int_{\mathbb{R}^N \setminus B_{R_1+1}(0)} W(x-y)|f(v_n(x))||\varphi(x)|\mathrm{d}x \right] \mathrm{d}y \\
&\le \left(\sup_{(\mathbb{R}^N \setminus B_1(0)) \cap \Omega_\varepsilon^c} W \right) \left(\int_{B_{R_1}(0)} |F(\bar{u}(y))|\,\mathrm{d}y \right) \left(\int_{\mathbb{R}^N \setminus B_{R_1+1}(0)} |f(v_n(x))|^2 \mathrm{d}x \right)^{\frac{1}{2}} \|\varphi\|_2 \\
&\quad + \int_{B_{R_1}(0)} |F(\bar{u}(y))| \left[\int_{\Omega_\varepsilon} W(z)|f(v_n(z+y))||\varphi(z+y)|\mathrm{d}z \right] \mathrm{d}y \\
&\le (C_{17}\varepsilon + o(1))\|\varphi\|.
\end{aligned} \tag{4.24}
$$

From (3.2), (4.23) and $v_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le s < 2^*$, we have

$$
\begin{aligned}
&\int_{B_{R_1+1}(0)} |f(v_n(x))||\varphi(x)| \left[\int_{B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))|\,\mathrm{d}y \right] \mathrm{d}x \\
&\le \mathcal{C}_1 \left(\int_{B_{R_1}(0)} |F(\bar{u}(y))|^{s_1}\,\mathrm{d}y \right)^{\frac{1}{s_1}} \left(\int_{B_{R_1+1}(0)} |f(v_n)|^{\zeta(s_1)} \mathrm{d}x \right)^{\frac{1}{\zeta(s_1)}} \|\varphi\|_{\eta(s_1)} \\
&\quad + \mathcal{C}_2 \left(\int_{B_{R_1}(0)} |F(\bar{u}(y))|^{s_2}\,\mathrm{d}y \right)^{\frac{1}{s_2}} \left(\int_{B_{R_1+1}(0)} |f(v_n)|^{\zeta(s_2)} \mathrm{d}x \right)^{\frac{1}{\zeta(s_2)}} \|\varphi\|_{\eta(s_2)} \\
&\le o(1)\|\varphi\|
\end{aligned} \tag{4.25}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |f(v_n(x))||\varphi(x)| \left[\int_{\mathbb{R}^N \setminus B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))|\,\mathrm{d}y \right] \mathrm{d}x \\
&\le \mathcal{C}_1 \left(\int_{\mathbb{R}^N \setminus B_{R_1}(0)} |F(\bar{u}(y))|^{s_1}\,\mathrm{d}y \right)^{\frac{1}{s_1}} \|f(v_n)\|_{\zeta(s_1)} \|\varphi\|_{\eta(s_1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{C}_2 \left(\int\limits_{\mathbb{R}^N \setminus B_{R_1}(0)} |F(\bar{u}(y))|^{s_2} \, dy \right)^{\frac{1}{s_2}} \|f(v_n)\|_{\zeta(s_2)} \|\varphi\|_{\eta(s_2)} \\
&\le C_{18}\varepsilon \|\varphi\|. \qquad (4.26)
\end{aligned}
$$

From (4.24), (4.25) and (4.26), we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} W(x-y)\,|F(\bar{u}(y))|\,|f(v_n(x))||\varphi(x)| dx dy \\
&= \int\limits_{B_{R_1+1}(0)} |f(v_n(x))||\varphi(x)| \left[\int\limits_{B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))| \, dy \right] dx \\
&\quad + \int\limits_{\mathbb{R}^N \setminus B_{R_1+1}(0)} |f(v_n(x))||\varphi(x)| \left[\int\limits_{B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))| \, dy \right] dx \\
&\quad + \int\limits_{\mathbb{R}^N} |f(v_n(x))||\varphi(x)| \left[\int\limits_{\mathbb{R}^N \setminus B_{R_1}(0)} W(x-y)\,|F(\bar{u}(y))| \, dy \right] dx \\
&\le [C_{19}\varepsilon + o(1)]\|\varphi\|. \qquad (4.27)
\end{aligned}
$$

Hence, it follows from (3.4), (4.19) and (4.27) that

$$
\begin{aligned}
|I_3| &\le \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} W(x-y)\,|F(u_n(y)) - F(v_n(y))|\,|f(v_n(x))||\varphi(x)| dx dy \\
&\le \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} W(x-y)\,|F(u_n(y)) - F(v_n(y)) - F(\bar{u}(y))|\,|f(v_n(x))||\varphi(x)| dx dy \\
&\quad + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} W(x-y)\,|F(\bar{u}(y))|\,|f(v_n(x))||\varphi(x)| dx dy \\
&\le \mathcal{C}_1 \|F(u_n) - F(v_n) - F(\bar{u})\|_{s_1} \|f(v_n)\|_{\zeta(s_1)} \|\varphi\|_{\eta(s_1)} \\
&\quad + \mathcal{C}_2 \|F(u_n) - F(v_n) - F(\bar{u})\|_{s_2} \|f(v_n)\|_{\zeta(s_2)} \|\varphi\|_{\eta(s_2)} + [C_{19}\varepsilon + o(1)]\|\varphi\| \\
&\le [C_{19}\varepsilon + o(1)]\|\varphi\|. \qquad (4.28)
\end{aligned}
$$

Similarly to (4.27), we can show that

$$
\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} W(x-y)\,|F(v_n(y))|\,|f(\bar{u}(x))||\varphi(x)| dx dy
$$

$$
\begin{aligned}
&= \int_{B_{R_1+1}(0)} |F(v_n(y))| \left[\int_{B_{R_1}(0)} W(x-y)|f(\bar{u}(x))||\varphi(x)|\mathrm{d}x \right] \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N \setminus B_{R_1+1}(0)} |F(v_n(y))| \left[\int_{B_{R_1}(0)} W(x-y)|f(\bar{u}(x))||\varphi(x)|\mathrm{d}x \right] \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N} |F(v_n(y))| \left[\int_{\mathbb{R}^N \setminus B_{R_1}(0)} W(x-y)|f(\bar{u}(x))||\varphi(x)|\mathrm{d}x \right] \mathrm{d}y \\
&\leq [C_{20}\varepsilon + o(1)]\|\varphi\|, \qquad (4.29)
\end{aligned}
$$

which, together with (3.4) and (4.18), implies that

$$
\begin{aligned}
|I_4| &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,|F(v_n(y))|\,|f(u_n(x)) - f(v_n(x))||\varphi(x)|\mathrm{d}x\mathrm{d}y \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,|F(v_n(y))|\,|f(u_n(x)) - f(v_n(x)) - f(\bar{u}(x))||\varphi(x)|\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,|F(v_n(y))|\,|f(\bar{u}(x))||\varphi(x)|\mathrm{d}x\mathrm{d}y \\
&\leq \mathcal{C}_1 \,\|F(v_n)\|_{s_1}\,\|f(u_n) - f(v_n) - f(\bar{u})\|_{\zeta(s_1)}\,\|\varphi\|_{\eta(s_1)} \\
&\quad + \mathcal{C}_2 \,\|F(v_n)\|_{s_2}\,\|f(u_n) - f(v_n) - f(\bar{u})\|_{\zeta(s_2)}\,\|\varphi\|_{\eta(s_2)} + [C_{20}\varepsilon + o(1)]\|\varphi\| \\
&\leq [C_{20}\varepsilon + o(1)]\|\varphi\|. \qquad (4.30)
\end{aligned}
$$

Then it follows from (4.20), (4.21), (4.22), (4.28) and (4.30) that

$$
\begin{aligned}
&\sup_{\varphi\in E, \|\varphi\|\leq 1} \left| \int_{\mathbb{R}^N} [(W * F(u_n))f(u_n) - (W * F(v_n))f(v_n) - (W * F(\bar{u}))f(\bar{u})]\,\varphi \mathrm{d}x \right| \\
&\leq (C_{19} + C_{20})\varepsilon + o(1).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows from the above inequality. □

Applying Lemma 4.1, we can prove the following lemma by standard arguments (see [15, Lemma 4.2]).

Lemma 4.2. *Assume that* (V1), (W1), (W2), (F1) *and* (F4) *hold. If* $u_n \rightharpoonup \bar{u}$ *in* E*, then*

$$\Phi(u_n) = \Phi(\bar{u}) + \Phi(u_n - \bar{u}) + o(1), \qquad (4.31)$$

$$\Phi'(u_n) = \Phi'(\bar{u}) + \Phi'(u_n - \bar{u}) + o(1). \qquad (4.32)$$

By the existence results established in Theorem 1.1 and Theorem 1.2, problem (1.2) has a nontrivial solution $\bar{u} \in E$ satisfying $\Phi(\bar{u}) = c_0 := \inf_{\mathcal{K}} \Phi > 0$ under (V1), (W1), (W2) and (F1)-(F3). Therefore, $\mathcal{K} \supseteq \mathcal{K}_{c_0} \neq \emptyset$. As in [14], [15] and [42], we choose a subset $\mathcal{F}$ of $\mathcal{K}$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset \mathcal{K}$ has a unique representative in $\mathcal{F}$. It suffices to show that the set $\mathcal{F}$ is infinite. Employing the ideas used in [44], we can get the desired result. Here we sketch the proofs for reader's convenience. We argue by contradiction and assume that

$$\mathcal{F} \text{ is a finite set.} \tag{4.33}$$

Let $[a]$ denote the integer part of $a \in \mathbb{R}$. As a consequence of Lemma 4.2 and Lemmas 3.5, 3.9 (or Lemmas 3.6, 3.10), we have the following result (see [14], [15]).

Lemma 4.3. *Assume that* (V1), (W1), (W2), (F1), (F2) *and* (F3)-(F4) (*or* (F3$'$) *and* (W3)) *hold. Let* $\{u_n\}$ *be a* $(\mathrm{Ce})_c$*-sequence for* Φ *in* E*. Then either*

i) $u_n \to 0$ *in* E (*and hence* $c = 0$)*; or*
ii) $c \geq c_0$ *and there exist a positive integer* $l \leq [c/c_0]$*, points* $\bar{u}_1, ..., \bar{u}_l \in \mathcal{K}$*, a subsequence denoted again by* $\{u_n\}$*, and sequences* $\{a_n^i\} \subset \mathbb{Z}^N$ *such that*

$$\left\| u_n - \sum_{i=1}^{l} a_n^i * \bar{u}_i \right\| \to 0 \quad \text{as } n \to \infty,$$

$$\left| a_n^i - a_n^j \right| \to \infty \quad \text{for } i \neq j \text{ as } n \to \infty$$

and

$$\sum_{i=1}^{l} \Phi(\bar{u}_i) = c.$$

For any $c \geq c_0$, as in [14] and [15], we let

$$\mathcal{F}_c := \left\{ \sum_{i=1}^{j} (a_i * u_i) : 1 \leq j \leq \left[\frac{c}{c_0} \right],\ a_i \in \mathbb{Z}^N,\ u_i \in \mathcal{F} \right\}.$$

It follows that $\mathcal{F}_{c'} \subseteq \mathcal{F}_c$ for any $c \geq c' \geq c_0$. Following an argument of [14], we obtain the following property.

Lemma 4.4. *Let* $c \geq c_0$*. Then* $\kappa_c := \inf\{\|u_1 - u_2\| : u_1, u_2 \in \mathcal{F}_c, u^1 \neq u^2\} > 0$.

Lemma 4.5. (Discreteness of (Ce)-sequences). *Let* $c \geq c_0$*. If* $\{u_n^1\}, \{u_n^2\} \subset \Phi_{c_0}^c$ *are two* (Ce)*-sequences for* Φ*, then either* $\lim_{n\to\infty} \|u_n^1 - u_n^2\| = 0$ *or* $\limsup_{n\to\infty} \|u_n^1 - u_n^2\| \geq \kappa_c$.

Proof. By virtue of Lemma 3.5 or Lemma 3.6, $\{u_n^1\}$ and $\{u_n^2\}$ are two bounded (Ce)-sequences for Φ. Next, by Lemma 4.3, there exist two sequences $\{w_n^1\}, \{w_n^2\} \subset \mathcal{F}_c$ such that

$$\|u_n^i - w_n^i\| \to 0, \quad i = 1, 2. \tag{4.34}$$

By Lemma 4.4, we have that either $\lim_{n\to\infty} \|w_n^1 - w_n^2\| = 0$ or $\limsup_{n\to\infty} \|w_n^1 - w_n^2\| \geq \kappa_c$. Hence, the conclusion follows from (4.34). □

In the following, for a subset $A \subset E$ and $\delta > 0$, we denote $U_\delta(A) := \{v \in E : \mathrm{dist}(v, A) < \delta\}$. For a closed and symmetric subset $A \subset E \setminus \{0\}$ (that is, $A = -A = \bar{A}$), denote by $\gamma(A)$ the Krasnoselskii genus of A (see [38,41]).

Let $c > c_0$ and $\delta \in (0, \kappa_c/4)$. It is easy to show that there exist $\alpha > 0$ and $b_0 \in (0, (c - c_0)/2]$ such that

$$(1 + \|u\|)\|\Phi'(u)\| \geq \alpha \ \text{ for all } \ u \in \Phi_{c-2b_0}^{c+2b_0} \setminus U_\delta(\mathcal{F}_c). \tag{4.35}$$

For each $u \in E \setminus (\mathcal{K} \cup \{0\})$, consider the Cauchy problem:

$$\begin{cases} \dfrac{d\eta}{dt} = -g(\eta), \\ \eta(0, u) = u, \end{cases} \tag{4.36}$$

where

$$g(u) = \frac{(1 + \|u\|)W(u)}{\|\Phi'(u)\|}, \quad u \in E \setminus (\mathcal{K} \cup \{0\}), \tag{4.37}$$

and $W : E \setminus (\mathcal{K} \cup \{0\}) \to E$ is an odd locally Lipschitz continuous map such that (see [41, Lemma II.3.9])

$$\begin{cases} \|W(u)\| \leq 2\|\Phi'(u)\|, \\ \langle \Phi'(u), W(u)\rangle \geq \|\Phi'(u)\|^2. \end{cases} \tag{4.38}$$

It follows from (4.37) and (4.38) that

$$\|g(u)\| = \frac{(1 + \|u\|)\|W(u)\|}{\|\Phi'(u)\|} \leq 2(1 + \|u\|), \quad \forall\, u \in E \setminus (\mathcal{K} \cup \{0\}). \tag{4.39}$$

Then by the existence-uniqueness theorem for ordinary differential equations, we get that for each $u \in E \setminus (\mathcal{K} \cup \{0\})$, problem (4.36) has a unique solution $\eta(t, u)$ defined on $[0, \infty)$, and $\eta(t, u)$ is odd with respect to $u \in E$.

Using the same arguments as in the proofs of [44, Lemmas 4.7, 4.8, 4.9 and 4.10], we have the following lemmas.

Lemma 4.6. *Let $c > c_0$, $b \in (0, b_0]$ and $u \in E \setminus (\mathcal{K} \cup \{0\})$ be such that $c - b \leq \Phi(\eta(t, u)) \leq c + b$ for all $t \in [0, \infty)$. Then $u_\infty := \lim_{t\to\infty} \eta(t, u)$ exists and $u_\infty \in \Phi_{c-b}^{c+b} \cap \mathcal{K}$.*

Lemma 4.7. *Let $c > c_0$. If $\mathcal{K}_c = \emptyset$, then there exists $\varepsilon > 0$ such that $\lim_{t\to\infty} \Phi(\eta(t, u)) < c - \varepsilon$ for $u \in \Phi^{c+\varepsilon}$.*

Lemma 4.8. *Let $c > c_0$. Then for every $\delta \in (0, \kappa_c/4)$, there exist $\varepsilon = \varepsilon(c, \delta) > 0$ and an odd and continuous map $\varphi : \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{F}_c) \to \Phi^{c-\varepsilon}$.*

Lemma 4.9. *Let $c \geq c_0$. Then for every $\delta \in (0, \kappa_c/4)$, $\gamma(\overline{U_\delta(\mathcal{F}_c)}) = 1$.*

Proof of the second parts in Theorems 1.1 and 1.2. For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $A \subset E \setminus \{0\}$ with $\gamma(A) \geq j$. Moreover, we consider the nondecreasing sequence of Lusternik-Schnirelmann values for Φ defined by

$$c_k := \inf\{c \geq c_0 : \gamma(\Phi^c) \geq k\}, \quad k \in \mathbb{N}. \tag{4.40}$$

Taking the advantage of Lemmas 4.7, 4.8 and 4.9, as in [44], we show that

$$\mathcal{K}_{c_k} \neq \emptyset \text{ and } c_k < c_{k+1}, \quad k \in \mathbb{N}. \tag{4.41}$$

Then it follows that there is an infinite sequence $\{u_k\}$ of pairs of geometrically distinct critical points of Φ with $\Phi(u_k) = c_k$, contrary to (4.33). The proof is now complete. □

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