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Concentration of solutions for fractional double-phase problems: critical and supercritical cases

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Abstract

This paper is concerned with concentration and multiplicity properties of solutions to the following fractional problem with unbalanced growth and critical or supercritical reaction:

$$\left\{\begin{array}{ll} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = h(u) + |u|^{r-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N),\ u > 0, & \text{in } \mathbb{R}^N, \end{array}\right\}$$

where ε is a positive parameter, $0 < s < 1$, $2 \leqslant p < q < N/s$, $(-\Delta)_t^s$ ($t \in \{p, q\}$) is the fractional t-Laplace operator, while $V : \mathbb{R}^N \mapsto \mathbb{R}$ and $h : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions. The analysis developed in this paper covers both critical and supercritical cases, that is, we assume that either $r = q_s^* := Nq/(N - sq)$ or $r > q_s^*$. The main results establish the existence of multiple positive solutions as well as related concentration properties. In the first case, due to the strong influence of the critical term, the result holds true for "high perturbations" of the subcritical nonlinearity. In the second framework, the result holds true for "low perturbations" of the supercritical nonlinearity. The concentration properties are achieved by combining topological and variational methods, provided that ε is small enough and in close relationship with the set where the potential V attains its minimum.

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1. Features of the paper and historical comments

In this paper, we are concerned with the study of concentration and multiplicity properties of solutions for a class of fractional double phase problems with critical or supercritical nonlinearity. The features of this paper are the following:

(i) the presence of several nonlocal operators with different growth, which generates a double phase associated energy;

(ii) the reaction combines the multiple effects generated by a subcritical term and a critical/supercritical nonlinearity;

(iii) the potential describing the absorption term satisfies a local condition and no information on the behavior of the potential at infinity is available;

(iv) the main concentration properties create a bridge between the global maximum point of the solution and the global minimum of the potential;

(v) our analysis combines the nonlocal nature of the fractional (p, q)-operator with the local perturbation in the absorption term.

Since the content of the paper is closely concerned with double phase problems, we start with a short description on the development this research. To the best of our knowledge, the first contributions to this field are due to J. Ball [10], in relationship with problems in nonlinear elasticity and composite materials. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. If $u : \Omega \to \mathbb{R}^N$ is the displacement and if Du is the $N \times N$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$I(u) = \int_\Omega f(x, Du(x))dx, \tag{1}$$

where the energy function $f = f(x, \xi) : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}$ is quasiconvex with respect to ξ, see Morrey [26]. A simple example considered by Ball is given by functions f of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geqslant c_1 |\xi|^p; \qquad \lim_{t\to+\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < N$. The condition $p \leqslant N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the variational integral (1) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus it is a continuous function if $p > N$. In accordance with these problems arising in nonlinear elasticity, Marcellini [22,23] considered continuous functions $f = f(x, u)$ with unbalanced growth that satisfy

$$c_1\,|u|^p \leqslant |f(x,u)| \leqslant c_2\,(1+|u|^q) \quad \text{for all } (x,u)\in\Omega\times\mathbb{R},$$

where c_1, c_2 are positive constants and $1\leqslant p\leqslant q$. Pioneering contributions to the study of various classes of problems with nonstandard growth are due to Mingione et al., see [1,11,18]. We also refer to Mingione and Rădulescu [24] for an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

2. Statement of the problem and the main results

In this paper we are first concerned with multiplicity and concentration properties of positive solutions to the following nonlinear fractional (p,q)-Laplacian problem with critical growth:

$$\left\{\begin{array}{ll} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \lambda f(u) + |u|^{q_s^*-2}u & \text{in } \mathbb{R}^N, \\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N),\ u>0, & \text{in } \mathbb{R}^N, \end{array}\right\} \qquad (P_\lambda)$$

where ε and λ are two positive parameters, $0<s<1$, $2\leqslant p<q<N/s$, $q_s^*=Nq/(N-sq)$ is the critical Sobolev exponent, $(-\Delta)_t^s$ (with $t\in\{p,q\}$) is the fractional t-Laplace operator, $V:\mathbb{R}^N\mapsto\mathbb{R}$ and $f:\mathbb{R}\mapsto\mathbb{R}$ are continuous functions.

The double-phase problem (P_λ) is motivated by numerous local and nonlocal models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [12, 13,16] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\mathrm{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-1/2} = 1+\frac{x}{2}+\frac{3}{2\cdot 2^2}x^2+\frac{5!!}{3!\cdot 2^3}x^3+\cdots+\frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1}+\cdots \text{ for } |x|<1.$$

Taking $x=2|\nabla u|^2$ and adopting the first order approximation, we obtain a particular case of the fractional problem (P_λ) for $p=2$ and $q=4$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

Now, we introduce the following hypotheses on the potential V and the nonlinearity f.

Let $V:\mathbb{R}^N\mapsto\mathbb{R}$ be a continuous function satisfying the following hypotheses:

(V_1) we have $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$;
(V_2) there exists an open bounded set $\Lambda\subset\mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M := \{x\in\Lambda : V(x)=V_0\} \neq \emptyset.$$

We point out that we do not assume any hypotheses about the upper boundedness or unboundedness of the potential V.

Meanwhile, $f \in C(\mathbb{R}, \mathbb{R})$ is supposed to verify the following assumptions:

(f_1) $\lim_{t\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0$;
(f_2) there exists $\nu \in (q, q_s^*)$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0;$$

(f_3) there is a constant $\theta \in (q, q_s^*)$ such that $0 < \theta F(t) := \theta \int_0^t f(\tau)d\tau \leqslant f(t)t$ for all $t > 0$;
(f_4) the map $t \mapsto \frac{f(t)}{t^{q-1}}$ is increasing for all $t \in (0, +\infty)$.

Since we are interested in finding positive solutions of problem (P_λ), we can assume that $f(t) = 0$ for all $t \leqslant 0$.

Recall that if A is a closed subset of a topological space Y, then we use $\mathrm{cat}_Y(A)$ to denote the Ljusternik-Schnirelmann category of A in Y, that is, the smallest number of closed and contractible sets in Y which cover A; see Willem [29] for more details.

The first major achievement of this work is reflected in the following "multiplicity and concentration" phenomena. This result corresponds to the critical case described in problem (P_λ). In this abstract setting, due to the strong influence of the critical term, the result holds true for "high perturbations" of the subcritical nonlinearity, that is, for large values of the positive parameter λ.

Theorem 1. *Assume that the nonlinearity f fulfills hypotheses (f_1)–(f_4) and the potential V verifies hypotheses (V_1)–(V_2). Then, there exists $\lambda^* > 0$ such that, for all $\lambda \in [\lambda^*, +\infty)$ and for every $\delta > 0$ with $M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, M) \leqslant \delta\} \subset \Lambda$, there exists $\varepsilon_{\delta,\lambda} > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_{\delta,\lambda})$, problem ($P_\lambda$) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is global maximum point of u_ε, then* $\lim_{\varepsilon\to 0} V(\varepsilon x_\varepsilon) = V_0$.

In the second part of this paper, we consider the supercritical case. In this case, we deal with the sum of two homogeneous nonlinearities and add a new positive parameter.

$$\left\{\begin{array}{ll} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = |u|^{\theta-2}u + \eta|u|^{r-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N),\ u > 0, & \text{in } \mathbb{R}^N, \end{array}\right\} \qquad (S_\eta)$$

where $\varepsilon, \eta > 0$, $0 < s < 1$, $sq < N$ and $2 \leqslant p < q < \theta < q_s^* < r$.

By combining the "modular-uniform" and the "L^∞-uniform" estimates of positive solutions with new truncation techniques, we can obtain the "multiplicity and concentration" of positive solutions to the supercritical problem (S_η). In this case, due to the strong influence of the supercritical term, the result holds true for "low perturbations", that is, for small values of the positive parameter η.

Theorem 2. *Assume that the potential V satisfies hypotheses (V_1)–(V_2). Then there exists $\eta_* > 0$ such that, for all $\eta \in (0, \eta_*]$ and for every $\delta > 0$ with $M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, M) \leqslant \delta\} \subset \Lambda$, there exists $\varepsilon_{\delta,\eta} > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_{\delta,\eta})$, problem ($S_\eta$) has at least*

$cat_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is global maximum point of u_ε, then $\lim_{\varepsilon\to 0} V(\varepsilon x_\varepsilon) = V_0$.

For related concentration and multiplicity properties of solutions, we refer to the recent paper by Alves, Ambrosio & Isernia [2], Alves & de Morais Filho [3], Ambrosio [6], Ambrosio, Isernia & Rădulescu [8], Ambrosio & Rădulescu [9], Gao, Tang & Chen [19], and Gu & Tang [20].

For the sake of simplicity, C, C_1, C_2,... in this paper denote positive constants whose exact values are unimportant and can be changed line by line, and the same C, C_1, C_2,... may represent different constants; $B_\rho(y)$ denotes the open ball centered at $y \in \mathbb{R}^N$ with radius $\rho > 0$ and $B^c_\rho(y)$ denotes the complement of $B_\rho(y)$ in $\mathbb{R}^N$. In particular, B_ρ and B^c_ρ denote $B_\rho(0)$ and $B^c_\rho(0)$, respectively.

3. Auxiliary results

Let $u : \mathbb{R}^N \mapsto \mathbb{R}$. For $0 < s < 1$ and $p > 1$, let us define $D^{s,p}(\mathbb{R}^N) = \overline{C^\infty_c(\mathbb{R}^N)}^{[\cdot]_{s,p}}$, where

$$[u]_{s,p} := \left[\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right]^{\frac{1}{p}} ;$$

By $W^{s,p}(\mathbb{R}^N)$ we denote the following fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{ u : |u|_p < +\infty,\ [u]_{s,p} < +\infty \right\}$$

equipped with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left([u]^p_{s,p} + |u|^p_p\right)^{\frac{1}{p}},$$

where $|u|^p_p := \int_{\mathbb{R}^N} |u|^p dx$.

For all $u, v \in W^{s,p}(\mathbb{R}^N)$, let us define

$$\langle u, v\rangle_{s,p} := \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy.$$

Now, let us recall the following embedding property; see the monograph by Molica Bisci, Rădulescu & Servadei [25] for more details.

Theorem 3. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$ satisfy $N > sp$. Then, there exists a constant $S_* := S_*(N, s, p) > 0$ such that*

$$|u|^p_{p^*_s} \leqslant S_*^{-1}[u]^p_{s,p} \text{ for all } u \in D^{s,p}(\mathbb{R}^N).$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [p, p^*_s]$ and compactly embedded in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for any $r \in [1, p^*_s)$.

In this work we need to introduce the following Banach space

$$X = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_X := \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

Notice that $W^{s,r}(\mathbb{R}^N)$ is a separable reflexive Banach space for all $r \in (1, +\infty)$, and so X is a separable reflexive Banach space.

For any fixed $\varepsilon \geqslant 0$, we also introduce the following Banach space

$$X_\varepsilon := \left\{ u \in X : \int\limits_{\mathbb{R}^N} V(\varepsilon x)\left(|u|^p + |u|^q\right) dx < +\infty \right\}$$

equipped with the norm

$$\|u\|_{X_\varepsilon} := \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

where $\|u\|_{V_\varepsilon,t}^t := [u]_{s,t}^t + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t dx$ for all $t > 1$.

4. The modified problem

In order to study problem (P_λ), we will modify the nonlinear term appropriately. When $\varepsilon > 0$ is small enough, we can use the variational method to obtain the solutions of the modified problem, which are indeed the solutions of the original problem. To be more precise, we shall adopt the penalization method proposed by del Pino & Felmer [15] to deal with problem (P_λ).

Without loss of generality, we may assume that

$$0 \in \Lambda \quad \text{and} \quad V(0) = V_0.$$

Let us choose $K > \lambda q/p > 0$ and take a unique number $a > 0$ such that

$$f(a) + \frac{a^{q_s^*-1}}{\lambda} = \frac{V_0}{K} a^{q-1}.$$

Then, we can define the following modified functions

$$\widetilde{f}(t) = \begin{cases} f(t) + \frac{(t^+)^{q_s^*-1}}{\lambda}, & \text{if } t \leqslant a, \\ \frac{V_0}{K} t^{q-1}, & \text{if } t > a \end{cases}$$

and

$$g(x,t)=\begin{cases}\chi_\Lambda(x)\left[f(t)+\frac{t^{q_s^*-1}}{\lambda}\right]+[1-\chi_\Lambda(x)]\,\widetilde{f}(t), & \text{if } t>0,\\ 0, & \text{if } t\leqslant 0,\end{cases}$$

where χ_Ω is the characteristic function on $\Omega\subset\mathbb{R}^N$. It is easy to check that the penalized nonlinearity g is a Carathéodory function and fulfills the following properties:

- (g_1) for each $\lambda>0$, $\lim_{t\to 0^+}\frac{g(x,t)}{t^{p-1}}=0$ uniformly for all $x\in\mathbb{R}^N$;
- (g_2) for each $\lambda>0$, $g(x,t)\leqslant f(t)+\frac{t^{q_s^*-1}}{\lambda}$ for all $x\in\mathbb{R}^N$ and $t>0$;
- $(g_3)_i$ for each $\lambda>0$, $0<\theta G(x,t):=\theta\int_0^t g(x,\tau)d\tau<g(x,t)t$ for all $x\in\Lambda$ and $t>0$;
- $(g_3)_{ii}$ for each $\lambda>0$, $0<qG(x,t)\leqslant g(x,t)t\leqslant\frac{V_0}{K}t^q$ for all $x\in\Lambda^c$ and $t>0$;
- (g_4) for each $\lambda>0$ and $x\in\mathbb{R}^N$, the map $t\mapsto\frac{g(x,t)}{t^{q-1}}$ is increasing in $(0,+\infty)$.

Remark 1. Let us introduce the following modified problem:

$$\left\{\begin{array}{l}(-\Delta)_p^s u+(-\Delta)_q^s u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=\lambda g(\varepsilon x,u),\\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N),\ u>0\end{array}\right\} \tag{2}$$

in $\mathbb{R}^N$. If u_ε is a solution of problem (2) satisfying

$$u_\varepsilon(x)\leqslant a \text{ for all } x\in\Lambda_\varepsilon^c,\ \text{ where } \Lambda_\varepsilon:=\left\{x\in\mathbb{R}^N:\varepsilon x\in\Lambda\right\},$$

then it is worth to pointing out that u_ε is also a solution of problem (P_λ).

Now, we define the functional $J_\varepsilon: X_\varepsilon\mapsto\mathbb{R}$ associated to problem (2), that is,

$$J_\varepsilon(u):=\frac{1}{p}\|u\|_{V_\varepsilon,p}^p+\frac{1}{q}\|u\|_{V_\varepsilon,q}^q-\lambda\int_{\mathbb{R}^N}G(\varepsilon x,u)dx \text{ for all } u\in X_\varepsilon.$$

Obviously, $J_\varepsilon\in C^1(X_\varepsilon,\mathbb{R})$ and its derivative can be expressed as follows

$$\begin{aligned}\langle J_\varepsilon'(u),v\rangle:=\langle u,v\rangle_{s,p}+\langle u,v\rangle_{s,q}+\int_{\mathbb{R}^N}V(\varepsilon x)\left(|u|^{p-2}u+|u|^{q-2}u\right)v dx\\ -\lambda\int_{\mathbb{R}^N}g(\varepsilon x,u)v dx \text{ for all } u,\ v\in X_\varepsilon.\end{aligned}$$

By $\mathcal{N}_\varepsilon$ we mean the Nehari manifold related to the functional J_ε, which is defined by

$$\mathcal{N}_\varepsilon:=\left\{u\in X_\varepsilon\setminus\{0\}:\langle J_\varepsilon'(u),u\rangle=0\right\}.$$

We set $c_\varepsilon:=\inf_{u\in\mathcal{N}_\varepsilon}J_\varepsilon(u)$. Let X_ε^+ denote the following open set

$$X_\varepsilon^+:=\left\{u\in X_\varepsilon:|\mathrm{supp}\,(u^+)\cap\Lambda_\varepsilon|>0\right\}$$

and set $S_\varepsilon^+ := S_\varepsilon \cap X_\varepsilon^+$, where $S_\varepsilon := \{u \in X_\varepsilon : \|u\|_{X_\varepsilon} = 1\}$.

We observe that S_ε^+ is an incomplete $C^{1,1}$-manifold of codimension one. So, for all $u \in S_\varepsilon^+$ we have $X_\varepsilon = T_u S_\varepsilon^+ \bigoplus \mathbb{R}u$, where

$$T_u S_\varepsilon^+ := \left\{ v \in X_\varepsilon : \langle u, v\rangle_{s,p} + \langle u, v\rangle_{s,q} + \int\limits_{\mathbb{R}^N} V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) v dx = 0 \right\}.$$

Now, we show that the functional J_ε has a Mountain Pass Geometry (see Willem [29]).

Lemma 4. *For each fixed $\lambda > 0$, the following properties are fulfilled for the functional J_ε:*

(i) *there exist ϱ, $\rho > 0$ such that $J_\varepsilon(u) \geqslant \varrho$ with $\|u\|_{X_\varepsilon} = \rho$;*
(ii) *there exists $e \in X_\varepsilon^+$ satisfying $\|e\|_{X_\varepsilon} > \rho$ and $J_\varepsilon(e) < 0$.*

Proof. By (g_2), (f_1) and (f_2), we see that for all $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$|g(x,t)| \leqslant \sigma |t|^{p-1} + C_\sigma |t|^{q_s^*-1} \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Note that $\lambda > 0$ and $q > p > 1$. Thus, taking $\sigma = \frac{(q-p)V_0}{\lambda q}$, choosing $\|u\|_{X_\varepsilon} = \rho \in (0,1)$ and using Theorem 3, we have

$$J_\varepsilon(u) \geqslant \frac{1}{q}\left(\|u\|^q_{V_{\varepsilon,p}} + \|u\|^q_{V_{\varepsilon,q}}\right) - \frac{\lambda C_\sigma}{q_s^*}|u|^{q_s^*}_{q_s^*} \geqslant \frac{1}{2^{q-1}q}\|u\|^q_{X_\varepsilon} - C_1\|u\|^{q_s^*}_{X_\varepsilon},$$

for some constant $C_1 > 0$. Since $1 < q < q_s^*$, it is easy check that (i) is fulfilled.

(ii) According to (f_3), we infer that there exist two positive constants C_2, C_3 such that

$$F(x,t) \geqslant C_2 t^\theta - C_3 \text{ for all } t > 0.$$

The above inequality combined with $\lambda > 0$ and $1 < p < q < \theta$ implies that $J_\varepsilon(tu) \to -\infty$ as $t \to +\infty$ for all $u \in X_\varepsilon^+$. So, property (ii) also holds true. □

Since f is only assumed continuous, we need to establish some useful results in order to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S_ε^+.

Lemma 5. *Fix $\lambda > 0$ and assume that hypotheses (f_1)–(f_4) and (V_1)–(V_2) are fulfilled. Then, we have the following properties.*

(a) *For any fixed $u \in X_\varepsilon^+$, let the mapping $\ell_u : \mathbb{R}^+ \mapsto \mathbb{R}$ be defined by $\ell_u(t) := J_\varepsilon(tu)$. Then, there exists a unique $t_u > 0$ such that*

$$\ell_u'(t) > 0 \textit{ for all } t \in (0, t_u),$$
$$\ell_u'(t) < 0 \textit{ for all } t \in (t_u, +\infty).$$

(b) *There exists $\tau > 0$ independent of u such that $t_u \geqslant \tau$ for all $u \in S_\varepsilon^+$. Moreover, for each compact set $\mathcal{W} \subset S_\varepsilon^+$, there is a constant $C_{\mathcal{W}} > 0$ such that $t_u \leqslant C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

(c) *The mapping $\hat{m}_\varepsilon : X_\varepsilon^+ \mapsto \mathcal{N}_\varepsilon$ introduced by $\hat{m}_\varepsilon(u) := t_u u$ is continuous and $m_\varepsilon := \hat{m}_\varepsilon|_{S_\varepsilon^+}$ is a homeomorphism between S_ε^+ and $\mathcal{N}_\varepsilon$, and the inverse of m_ε is given by $m_\varepsilon^{-1}(u) := u/\|u\|_{X_\varepsilon}$.*

(d) *If there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset S_\varepsilon^+$ such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$ as $n \to \infty$, then $\|m_\varepsilon(u_n)\|_{X_\varepsilon} \to +\infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to +\infty$ as $n \to \infty$.*

Proof. (a) Similar to the proof of Lemma 4, we obtain $\ell_u(0) = 0$, $\ell_u(t) > 0$ for t sufficiently small and $\ell_u(t) < 0$ for t large enough. Thus, $\max_{t\geqslant 0} \ell_u(t)$ is achieved at some $t_u > 0$ satisfying $\ell_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

Next, we assert the uniqueness of $t_u > 0$. Otherwise, we may assume that there exist positive numbers t_1 and t_2 such that $\ell_u'(t_1) = \ell_u'(t_2) = 0$, that is,

$$t_i^{p-1}\|u\|_{V_\varepsilon,p}^p + t_i^{q-1}\|u\|_{V_\varepsilon,q}^q = \lambda \int_{\mathbb{R}^N} g(\varepsilon x, t_i u) u dx,\ i = 1, 2,$$

hence

$$\left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}}\right)\|u\|_{V_\varepsilon,p}^p = \lambda \int_{\mathbb{R}^N} \left[\frac{g(\varepsilon x, t_1 u^+)}{(t_1 u^+)^{q-1}} - \frac{g(\varepsilon x, t_2 u^+)}{(t_2 u^+)^{q-1}}\right](u^+)^q dx.$$

This equality, together with (g_4), $q > p > 1$ and $\lambda > 0$, implies that $t_1 = t_2$.

(b) For all $u \in S_\varepsilon^+$, we deduce from (a) that there exists $t_u > 0$ such that

$$t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q = \lambda \int_{\mathbb{R}^N} g(\varepsilon x, t_u u) u dx.$$

Using (g_2), (f_1), (f_2) and Theorem 3, for all $\sigma > 0$ we can take $C_\sigma > 0$ such that

$$t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q = \lambda \int_{\mathbb{R}^N} g(\varepsilon x, t_u u) u dx \leqslant \lambda\sigma t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + \lambda C_\sigma t_u^{q_s^*-1}\|u\|_{V_\varepsilon,q}^{q_s^*}.$$

Choosing $\sigma = 1/(2\lambda) > 0$ and recalling that $\|u\|_{X_\varepsilon} = 1$, we have

$$\frac{1}{2}t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \leqslant C t_u^{q_s^*-1}\|u\|_{V_\varepsilon,q}^{q_s^*} \leqslant C t_u^{q_s^*-1},$$

where $C := C(\lambda)$ is a positive constant.

Assume that $t_u \leqslant 1$. Then we have

$$\begin{aligned} C t_u^{q-1} &\leqslant \frac{1}{2}t_u^{q-1}\left(\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q\right) \\ &\qquad \text{for some constant } C > 0 \\ &\leqslant \frac{1}{2}t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \end{aligned}$$

$$
\begin{aligned}
&\qquad \text{(since } q > p,\ t_u \leqslant 1 \text{ and } 1 = \|u\|_{X_\varepsilon} \geqslant \|u\|_{V_\varepsilon,p})\\
&\leqslant C t_u^{q_s^*-1},\\
\Rightarrow\ & t_u \geqslant \tau \text{ for some constant } \tau > 0 \text{ (since } q_s^* > q),\\
&\qquad \text{where } \tau \text{ is dependent of } u.
\end{aligned}
$$

Assume that $t_u > 1$. Then we obtain

$$
\begin{aligned}
C t_u^{p-1} &\leqslant \frac{1}{2} t_u^{p-1} \left(\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q \right)\\
&\qquad \text{for some constant } C > 0\\
&\leqslant \frac{1}{2} t_u^{p-1} \|u\|_{V_\varepsilon,p}^p + t_u^{q-1} \|u\|_{V_\varepsilon,q}^q\\
&\qquad \text{(since } q > p,\ t_u > 1 \text{ and } 1 = \|u\|_{X_\varepsilon} \geqslant \|u\|_{V_\varepsilon,p})\\
&\leqslant C t_u^{q_s^*-1},\\
\Rightarrow\ & t_u \geqslant \tau \text{ for some constant } \tau > 0 \text{ (since } q_s^* > q > p),\\
&\qquad \text{where } \tau \text{ does not depend on } u.
\end{aligned}
$$

So, there exists $\tau > 0$ independent of u such that $t_u \geqslant \tau$ for all $u \in S_\varepsilon^+$.

Suppose that $\mathcal{W} \subset S_\varepsilon^+$ is a compact set. Arguing by contradiction, we may assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{W}$ such that $1 \leqslant t_n := t_{u_n} \to +\infty$ as $n \to \infty$. Since $\mathcal{W}$ is a compact set, there exists $u \in \mathcal{W}$ such that

$$
u_n \to u \text{ in } X_\varepsilon \text{ as } n \to \infty.
$$

Using the above facts and proceeding as in the proof of Lemma 4-(ii), we infer that

$$
J_\varepsilon(t_n u_n) \to -\infty \text{ as } n \to \infty. \tag{3}
$$

Moreover, for each $\varphi \in \mathcal{N}_\varepsilon$, we have $\langle J_\varepsilon'(\varphi), \varphi\rangle = 0$. From the above relation and $(g_3)_i$–$(g_3)_{ii}$, it follows that

$$
\begin{aligned}
J_\varepsilon(\varphi) &= J_\varepsilon(\varphi) - \frac{1}{\theta}\langle J_\varepsilon'(\varphi), \varphi\rangle\\
&\geqslant \frac{(\theta-q)(q-p)}{q^2\theta} \left(\|\varphi\|_{V_\varepsilon,p}^p + \|\varphi\|_{V_\varepsilon,q}^q \right)\\
&=: C \left(\|\varphi\|_{V_\varepsilon,p}^p + \|\varphi\|_{V_\varepsilon,q}^q \right),
\end{aligned}
$$

since $\theta > q > p$ and $K > \lambda q/p > 0$.

In the above inequality we choose $\varphi_n = t_n u_n \in \mathcal{N}_\varepsilon$, and then we obtain

$$
\begin{aligned}
J_\varepsilon(t_n u_n) &\geqslant C\left(t_n^p \|u_n\|_{V_\varepsilon,p}^p + t_n^q \|u_n\|_{V_\varepsilon,q}^q\right) \\
&\geqslant C\left(t_n^p \|u_n\|_{V_\varepsilon,p}^q + t_n^q \|u_n\|_{V_\varepsilon,q}^q\right) \\
&\quad \text{(since } q > p \text{ and } 1 = \|u_n\|_{X_\varepsilon} \geqslant \|u_n\|_{V_\varepsilon,p}) \\
&\geqslant C t_n^p \text{ for some constant } C > 0 \text{ (since } q > p,\ t_n \geqslant 1 \text{ and } \|u_n\|_{X_\varepsilon} = 1), \\
\Rightarrow \quad & -\infty \geqslant +\infty \text{ (see (3) and use the assumption } t_n \to +\infty \text{ as } n \to \infty),
\end{aligned}
$$

which is a contradiction.

(c) Clearly, the mappings $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. In fact, using (a), for any fixed $u \in X_\varepsilon^+$ it follows that there exists a unique $\hat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. In addition, if $u \in \mathcal{N}_\varepsilon$, then $u \in X_\varepsilon^+$. Otherwise, we obtain $|\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| = 0$. The above equality, hypothesis (V_1), the definition of g and $(g_3)_{ii}$ yield that

$$
\begin{aligned}
\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q &= \lambda \int_{\mathbb{R}^N} g(\varepsilon x, u) u dx \\
&= \lambda \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u^+) u^+ dx + \lambda \int_{\Lambda_\varepsilon} g(\varepsilon x, u^+) u^+ dx \\
&\leqslant \frac{\lambda}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x) |u|^q dx \\
&\leqslant \frac{p}{q} \int_{\Lambda_\varepsilon^c} V(\varepsilon x) |u|^q dx \ \left(\text{since } K > \frac{\lambda q}{p} > 0\right) \\
&\leqslant \frac{p}{q} \|u\|_{V_\varepsilon,q}^q, \\
\Rightarrow \quad & \left(1 - \frac{p}{q}\right)\left(\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q\right) \leqslant 0, \\
\Rightarrow \quad & u = 0 \text{ (since } q > p > 1),
\end{aligned}
$$

which contradicts the fact that $u \neq 0$. Hence, $m_\varepsilon^{-1}(u) = u/\|u\|_{X_\varepsilon} \in S_\varepsilon^+$ is well defined and continuous. For any $u \in S_\varepsilon^+$, it follows that

$$
m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{X_\varepsilon}} = \frac{u}{\|u\|_{X_\varepsilon}} = u.
$$

So, we conclude that m_ε is a bijection.

Next, we show that $\hat{m}_\varepsilon$ is continuous. To this end, let $\{u_n, u\}_{n\in\mathbb{N}} \subset X_\varepsilon^+$ be such that $u_n \to u$ in X_ε as $n \to \infty$. On account of the fact that $\hat{m}_\varepsilon(tu) = \hat{m}_\varepsilon(u)$ for all $t > 0$, we can assume that $\|u_n\|_{X_\varepsilon} = \|u\|_{X_\varepsilon} = 1$ for all $n \in \mathbb{N}$. Using (b), we see that there exists $t_n := t_{u_n} \to t_0 > 0$ (as $n \to \infty$) such that $t_n u_n \in \mathcal{N}_\varepsilon$, hence

$$t_n^p \|u_n\|_{V_\varepsilon,p}^p + t_n^q \|u_n\|_{V_\varepsilon,q}^q = \lambda \int_{\mathbb{R}^N} g(\varepsilon x, t_n u_n) t_n u_n dx.$$

In the above equality we pass to the limit as $n \to \infty$. Then we have

$$t_0^p \|u\|_{V_\varepsilon,p}^p + t_0^q \|u\|_{V_\varepsilon,q}^q = \lambda \int_{\mathbb{R}^N} g(\varepsilon x, t_0 u) t_0 u dx.$$

This implies that $t_0 u \in \mathcal{N}_\varepsilon$. From (a) it follows that $t_u = t_0$. Therefore, this leads to $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in X_ε^+ as $n \to \infty$. In conclusion, $\hat{m}_\varepsilon$ and m_ε are continuous mappings.

(d) Assume that $\{u_n\}_{n\in\mathbb{N}} \subset S_\varepsilon^+$ is a sequence such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$ as $n \to \infty$. For any $\varphi \in \partial S_\varepsilon^+$ and $n \in \mathbb{N}$, then we obtain $|u_n^+| \leqslant |u_n - \varphi|$ a.e. in Λ_ε. Hence, from (V_1) and the Sobolev embedding theorem, for any $r \in [p, q_s^*]$ and $n \in \mathbb{N}$ it follows that

$$|u_n^+|_{L^r(\Lambda_\varepsilon)} \leqslant \inf_{\varphi\in\partial S_\varepsilon^+} |u_n - \varphi|_{L^r(\Lambda_\varepsilon)} \leqslant C_r \inf_{\varphi\in\partial S_\varepsilon^+} \|u_n - \varphi\|_{X_\varepsilon}.$$

Note that $q > p$. Then, for all $t > 0$ we can deduce from (V_1), (g_2), (f_1)–(f_2) and $(g_3)_{ii}$ that

$$\begin{aligned}
\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)dx &= \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n)dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)dx \\
&\leqslant \frac{V_0}{Kq} \int_{\Lambda_\varepsilon^c} t^q |u_n|^q dx + \int_{\Lambda_\varepsilon} \left[F(tu_n^+) + \frac{1}{\lambda q_s^*} (tu_n^+)^{q_s^*} \right] dx \\
&\leqslant \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q dx + C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p dx + C_2 t^{q_s^*} \int_{\Lambda_\varepsilon} (u_n^+)^{q_s^*} dx \\
&\leqslant \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q dx + \hat{C}_p t^p \operatorname{dist}(u_n, \partial S_\varepsilon^+)^p + \hat{C}_{q_s^*} t^{q_s^*} \operatorname{dist}(u_n, \partial S_\varepsilon^+)^{q_s^*},
\end{aligned}$$

where C_1, C_2, $\hat{C}_p$ and $\hat{C}_{q_s^*}$ are some positive constants. So, we have

$$\lambda \int_{\mathbb{R}^N} G(\varepsilon x, tu_n)dx \leqslant \frac{\lambda t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q dx + o_n(1) \text{ as } n \to \infty, \tag{4}$$

since $\lambda > 0$. Moreover, for any $t > 1$ we infer that

$$\begin{aligned}
&\frac{t^p}{p} \|u_n\|_{V_\varepsilon,p}^p + \frac{t^q}{q} \|u_n\|_{V_\varepsilon,q}^q - \frac{\lambda t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q dx \\
=&\frac{t^p}{p} \|u_n\|_{V_\varepsilon,p}^p + \frac{t^q}{q} [u_n]_{s,q}^q + \left(\frac{1}{q} - \frac{\lambda}{Kp} \right) t^q \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q dx
\end{aligned}$$

$$\begin{aligned}
&\geqslant C_1 t^p \|u_n\|^p_{V_\varepsilon,p} + C_2 t^q \|u_n\|^q_{V_\varepsilon,q} \quad \left(\text{since } K > \frac{\lambda q}{p} > 0\right) \\
&\geqslant C_1 t^p \|u_n\|^q_{V_\varepsilon,p} + C_2 t^q \|u_n\|^q_{V_\varepsilon,q} \text{ (due to } q > p \text{ and } 1 = \|u_n\|_{X_\varepsilon} \geqslant \|u_n\|_{V_\varepsilon,p}) \\
&\geqslant C_3 t^p \text{ (due to } q > p,\ t > 1 \text{ and } \|u_n\|_{X_\varepsilon} = 1),
\end{aligned} \tag{5}$$

where C_1, C_2 and C_3 are some positive constants. Recalling the definition of m_ε and invoking relations (4) and (5), for all $t > 1$ we can deduce that

$$\liminf_{n\to\infty} J_\varepsilon(m_\varepsilon(u_n)) \geqslant \liminf_{n\to\infty} J_\varepsilon(tu_n) \geqslant C_3 t^p.$$

The above inequality combined with the definition of J_ε and the arbitrariness of $t > 1$ means that

$$\liminf_{n\to\infty}\left[\frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p} + \frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}\right] \geqslant \liminf_{n\to\infty} J_\varepsilon(m_\varepsilon(u_n)) = +\infty,$$

and so $\|m_\varepsilon(u_n)\|_{X_\varepsilon} \to +\infty$ as $n \to \infty$. This proof is now complete. □

Now, we introduce the following functionals

$$\hat{\psi}_\varepsilon : X^+_\varepsilon \mapsto \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : S^+_\varepsilon \mapsto \mathbb{R}$$

defined by $\hat{\psi}_\varepsilon(u) := J_\varepsilon(\hat{m}_\varepsilon(u))$ for $u \in X^+_\varepsilon$ and $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{S^+_\varepsilon}$.

Using Lemma 5 and invoking Corollary 2.3 in Szulkin & Weth [28], we have the following result.

Lemma 6. *Fix $\lambda > 0$ and assume that hypotheses (f_1)–(f_4) and (V_1)–(V_2) are fulfilled. Then,*

(a) *$\hat{\psi}_\varepsilon \in C^1(X^+_\varepsilon, \mathbb{R})$ and $\langle \hat{\psi}'_\varepsilon(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{X_\varepsilon}}{\|u\|_{X_\varepsilon}} \langle J'_\varepsilon(\hat{m}_\varepsilon(u)), v\rangle$ for all $u \in X^+_\varepsilon$, all $v \in X_\varepsilon$;*
(b) *$\psi_\varepsilon \in C^1(S^+_\varepsilon, \mathbb{R})$ and $\langle \psi'_\varepsilon(u), v\rangle = \|m_\varepsilon(u)\|_{X_\varepsilon} \langle J'_\varepsilon(m_\varepsilon(u)), v\rangle$ for all $u \in S^+_\varepsilon$, all $v \in T_u S^+_\varepsilon$;*
(c) *if $\{u_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for ψ_ε, then $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for J_ε. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ is a bounded Palais-Smale sequence for J_ε, then $\{m^{-1}_\varepsilon(u_n)\}_{n\in\mathbb{N}} \subset S^+_\varepsilon$ is a Palais-Smale sequence for ψ_ε;*
(d) *$u \in S^+_\varepsilon$ is a critical point of ψ_ε if and only if $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$ is a critical point of J_ε. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in S^+_\varepsilon} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = c_\varepsilon.$$

Remark 2. Using the same ideas as in Szulkin & Weth [28], for each $\lambda > 0$ the following variational characterization of the infimum of the functional J_ε over $\mathcal{N}_\varepsilon$ is satisfied:

$$0 < c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in X^+_\varepsilon} \max_{t>0} J_\varepsilon(tu) = \inf_{u\in S^+_\varepsilon} \max_{t>0} J_\varepsilon(tu).$$

Furthermore, if

$$\hat{c}_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)), \text{ where } \Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\},$$

then arguing as in Willem [29], we can check that $c_\varepsilon = \hat{c}_\varepsilon$.

The main characteristic of the modified functional is that it satisfies a compactness condition. We start by proving the boundedness of Palais-Smale sequences.

Lemma 7. *For each $\lambda > 0$. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ is a $(PS)_c$ sequence for the functional J_ε at the level $c \in \mathbb{R}$. Then, the sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ is bounded.*

Proof. By a simple computation, for $n \in \mathbb{N}$ large enough we observe that

$$\begin{aligned}
C_1(1 + \|u_n\|_{X_\varepsilon}) &\geqslant J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p_{V_\varepsilon,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|^q_{V_\varepsilon,q} \\
&+ \frac{\lambda}{\theta}\int_{\Lambda_\varepsilon} [g(\varepsilon x, u_n)u_n - \theta G(\varepsilon x, u_n)]\,dx \\
&+ \frac{\lambda}{\theta}\int_{\Lambda^c_\varepsilon} [g(\varepsilon x, u_n)u_n - \theta G(\varepsilon x, u_n)]\,dx \\
&\geqslant \left(\frac{1}{q} - \frac{1}{\theta}\right)\left(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}\right) \\
&- \left(\frac{1}{q} - \frac{1}{\theta}\right)\frac{\lambda}{K}\int_{\Lambda^c_\varepsilon} V(\varepsilon x)\left(|u_n|^p + |u_n|^q\right)dx \\
&\quad (\text{using } \lambda > 0,\ \theta > q > p \text{ and } (g_3)_i,\ (g_3)_{ii}) \\
&\geqslant \left(\frac{1}{q} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{K}\right)\left(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}\right) \\
&\geqslant \left(\frac{1}{q} - \frac{1}{\theta}\right)\left(1 - \frac{p}{q}\right)\left(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}\right) \\
&\quad \left(\text{since } K > \frac{\lambda q}{p} > 0\right).
\end{aligned}$$

Using this inequality, we deduce that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ is bounded, and we omit the details here. Thus, we complete the proof of the lemma. □

Now, we show that the modified functional J_ε satisfies the Palais-Smale condition.

Lemma 8. *For each $\lambda > 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ be a $(PS)_c$ sequence for the modified functional J_ε at the level $c \in \mathbb{R}$. Then $u_n \to u \in X_\varepsilon$ as $n \to \infty$ for all $c \in (0, \frac{s}{N}S_*^{N/(sq)})$.*

Proof. From Lemma 7 it follows that $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ is bounded. Thus, up to a subsequence (still denoted by itself), we can assume that $u_n \xrightarrow{w} u$ in X_ε, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for all $r \in [1, q_s^*)$ as $n \to \infty$. A simple calculation can show that the weak limit u is actually the critical point of the modified functional J_ε. Therefore, $\langle J'_\varepsilon(u), u\rangle = 0$.

In order to prove that the Palais-Smale sequence satisfies the Palais-Smale condition, we need to establish the following asymptotic behavior with respect to large balls:

$$\limsup_{n\to\infty} \int_{B_R^c} \left\{ \int_{\mathbb{R}^N} \left[\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} \right] dy + V(\varepsilon x)\left(|u_n|^p + |u_n|^q\right) \right\} dx$$
$$\to 0 \text{ as } R \to +\infty. \tag{6}$$

For some fixed $R > 0$, let $\eta_R \in C^\infty(\mathbb{R}^N)$ be such that

$$0 \leqslant \eta_R \leqslant 1,\ \ \eta_R = 0 \text{ in } B_{\frac{R}{2}},\ \ \eta_R = 1 \text{ in } B_R^c$$

and

$$|\nabla \eta_R| \leqslant C/R$$

for some constant $C > 0$ (which is independent of R).

From the boundedness of $\{\eta_R u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$, it follows that $\langle J'_\varepsilon(u_n), \eta_R u_n\rangle \to 0$ as $n \to \infty$. So, for $n \in \mathbb{N}$ large enough we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p \eta_R(x)}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q \eta_R(x)}{|x-y|^{N+sq}} dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u_n|^p + |u_n|^q\right)\eta_R dx \\
&= o_n(1) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_R(x)-\eta_R(y))u_n(y)}{|x-y|^{N+sp}} dxdy \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\eta_R(x)-\eta_R(y))u_n(y)}{|x-y|^{N+sq}} dxdy \\
&+ \lambda \int_{\mathbb{R}^N} g(\varepsilon x, u_n) u_n \eta_R dx.
\end{aligned}$$

Let $R > 0$ large enough such that

$$\Lambda_\varepsilon \subset B_{\frac{R}{2}}.$$

Recalling the definitions of η_R and K, together with $(g_3)_{ii}$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p \eta_R(x)}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q \eta_R(x)}{|x-y|^{N+sq}} dxdy \\
&+ \left(1-\frac{p}{q}\right) \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u_n|^p + |u_n|^q\right)\eta_R dx \\
&\leqslant o_n(1) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_R(x)-\eta_R(y))u_n(y)}{|x-y|^{N+sp}} dxdy \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\eta_R(x)-\eta_R(y))u_n(y)}{|x-y|^{N+sq}} dxdy, \qquad (7)
\end{aligned}
$$

as $n \to \infty$.

Invoking the Hölder inequality and using the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$, we infer that there exists some constant $C > 0$ such that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_R(x)-\eta_R(y))u_n(y)}{|x-y|^{N+sp}} dxdy \right| \\
&\leqslant C \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\eta_R(x)-\eta_R(y)|^p |u_n(x)|^p}{|x-y|^{N+sp}} dxdy \right]^{\frac{1}{p}}. \qquad (8)
\end{aligned}
$$

On the other hand, by the definition of η_R, polar coordinates and the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\eta_R(x)-\eta_R(y)|^p |u_n(x)|^p}{|x-y|^{N+sp}} dxdy \\
&\leqslant \int_{\mathbb{R}^N}\int_{|x-y|>R} \frac{|\eta_R(x)-\eta_R(y)|^p |u_n(x)|^p}{|x-y|^{N+sp}} dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{|x-y|\leqslant R} \frac{|\eta_R(x)-\eta_R(y)|^p |u_n(x)|^p}{|x-y|^{N+sp}} dxdy \\
&\leqslant C \int_{\mathbb{R}^N}\int_{|z|>R} \frac{|u_n(x)|^p}{|z|^{N+sp}} dxdz + \frac{C}{R^p} \int_{\mathbb{R}^N}\int_{|z|\leqslant R} \frac{|u_n(x)|^p}{|z|^{N+sp-p}} dxdz \\
&\leqslant \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n|^p dx + \frac{C}{R^p} R^{-sp+p} \int_{\mathbb{R}^N} |u_n|^p dx \\
&\leqslant \frac{C}{R^{sp}}. \qquad (9)
\end{aligned}
$$

Using relations (8) and (9), we deduce that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))u_n(y)}{|x - y|^{N+sp}} dxdy \right| \leqslant \frac{C}{R^s}. \tag{10}$$

In a similar fashion, we also have

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))u_n(y)}{|x - y|^{N+sq}} dxdy \right| \leqslant \frac{C}{R^s}. \tag{11}$$

So, from (7), (10) and (11), we see that the claim (6) is fulfilled.

The claim (6), together with the locally compact embedding $X_\varepsilon \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N)$, implies that $u_n \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. Then, we deduce from the interpolation inequality that $u_n \to u$ in $L^r(\mathbb{R}^N)$ as $n \to \infty$ for all $r \in [p, q_s^*)$.

Because of the emergence of the critical nonlinear term, we need a more accurate analysis. Similar to the obtained relation (9), we also have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_R(x) - \eta_R(y)|^q |u_n(y)|^q}{|x - y|^{N+sp}} dxdy \leqslant \frac{C}{R^{sq}}. \tag{12}$$

Recalling the definition of η_R and the Sobolev inequality, and using relations (7), (10), (11), (12), we see that

$$\begin{aligned}
|u_n|^q_{L^{q_s^*}(B_R^c)} &\leqslant |u_n \eta_R|^q_{q_s^*} \leqslant C[u_n \eta_R]^q_{s,q} \\
&\leqslant C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q \eta_R(x)}{|x - y|^{N+sq}} dxdy \\
&\quad + C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^q |\eta_R(x) - \eta_R(y)|^q}{|x - y|^{N+sq}} dxdy \\
&\leqslant o_n(1) + \frac{C}{R^s} + \frac{C}{R^{sq}} \text{ as } n \to \infty.
\end{aligned}$$

Thus, we obtain

$$\lim_{R \to +\infty} \limsup_{n \to \infty} |u_n|^q_{L^{q_s^*}(B_R^c)} = 0. \tag{13}$$

Moreover, it is easy to see that

$$\lim_{R \to +\infty} \limsup_{n \to \infty} |u_n|^r_{L^r(B_R^c)} = 0 \text{ for any } r \in [p, q_s^*). \tag{14}$$

Taking into account of the growth of g and using relations (13), (14), for any $\sigma > 0$ we can find some $R := R(\sigma) > 0$ such that

$$\limsup_{n\to\infty} \int_{B_R^c} g(\varepsilon x, u_n) u_n dx \leqslant C\sigma. \tag{15}$$

Choosing $R > 0$ sufficiently large, we also have

$$\int_{B_R^c} g(\varepsilon x, u) u dx \leqslant \sigma. \tag{16}$$

By relations (15) and (16), for any $\sigma > 0$ it follows that

$$\limsup_{n\to\infty} \left| \int_{B_R^c} g(\varepsilon x, u_n) u_n dx - \int_{B_R^c} g(\varepsilon x, u) u dx \right| \leqslant C\sigma.$$

This estimate combined with the arbitrariness of $\sigma > 0$ means that

$$\lim_{n\to\infty} \int_{B_R^c} g(\varepsilon x, u_n) u_n dx = \int_{B_R^c} g(\varepsilon x, u) u dx. \tag{17}$$

The definition of g yields that

$$g(\varepsilon x, u_n) u_n \leqslant f(u_n) u_n + \frac{a^{q_s^*}}{\lambda} + \frac{V_0}{K} |u_n|^q \text{ for all } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

The above inequality, together with hypotheses (f_1)–(f_2), Theorem 3 and the Dominated Convergence Theorem, implies that

$$\lim_{n\to\infty} \int_{B_R \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)} g(\varepsilon x, u_n) u_n dx = \int_{B_R \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)} g(\varepsilon x, u) u dx. \tag{18}$$

Next, we prove that

$$\lim_{n\to\infty} \int_{B_R \cap \Lambda_\varepsilon} g(\varepsilon x, u_n) u_n dx = \int_{B_R \cap \Lambda_\varepsilon} g(\varepsilon x, u) u dx. \tag{19}$$

To this end, it is sufficient to prove that the following limit holds true:

$$\lim_{n\to\infty} \int_{\Lambda_\varepsilon} (u_n^+)^{q_s^*} dx = \int_{\Lambda_\varepsilon} (u^+)^{q_s^*} dx. \tag{20}$$

In the sequel, we will use the following notations:

$$|D^s u|^q(x) := \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} dy \text{ and } |D^s u_n|^q(x) := \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} dy \text{ for all } n \in \mathbb{N}.$$

Since the sequences $\{|D^s u_n|^q\}_{n\in\mathbb{N}}$ and $\left\{|u_n|^{q_s^*}\right\}_{n\in\mathbb{N}}$ are bounded in $L^1(\mathbb{R}^N)$, we deduce from Prokhorov's Theorem (up to a subsequence) that there exist two nonnegative bounded measures μ and υ such that

$$|D^s u_n|^q \xrightarrow{w} \mu \quad \text{ and } \quad |u_n|^{q_s^*} \xrightarrow{w} \upsilon \text{ as } n \to \infty \tag{21}$$

in the sense of measures. Invoking the concentration-compactness principle in Ambrosio [7], we know that there exist an at most countable index set I, sequences $\{x_i\}_{i\in \mathrm{I}} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in\mathrm{I}}$, $\{\upsilon_i\}_{i\in\mathrm{I}}$ in $(0,+\infty)$ such that

$$\upsilon = |u|^{q_s^*} + \sum_{i\in\mathrm{I}} \upsilon_i \delta_{x_i}, \ \mu \geqslant |D^s u|^q + \sum_{i\in\mathrm{I}} \mu_i \delta_{x_i}, \ S_* \upsilon_i^{\frac{q}{q_s^*}} \leqslant \mu_i \text{ for all } i \in \mathrm{I}. \tag{22}$$

Now, we show that $\{x_i\}_{i\in\mathrm{I}} \cap \Lambda_\varepsilon = \emptyset$. Otherwise, by contradiction, we assume that there exists some $i \in \mathrm{I}$ such that $x_i \in \Lambda_\varepsilon$. Let us define

$$\xi_\rho(x) := \xi\left(\frac{x-x_i}{\rho}\right) \text{ for any } \rho > 0,$$

where $\xi \in C_c^\infty(\mathbb{R}^N)$ is such that $0 \leqslant \xi \leqslant 1$, $\xi = 1$ in B_1, $\xi = 0$ in B_2^c and $|\nabla \xi|_{L^\infty(\mathbb{R}^N)} \leqslant 2$. We always assume that ρ is chosen so that the support of ξ_ρ is contained in Λ_ε.

We first deduce from the Hölder inequality, the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subset X_\varepsilon$ and Lemma 2.2 in Ambrosio [7] that

$$\begin{aligned} &\lim_{\rho\to 0}\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))(\xi_\rho(x)-\xi_\rho(y))u_n(y)}{|x-y|^{N+st}} dxdy \right| \\ &\leqslant C \lim_{\rho\to 0}\limsup_{n\to\infty} \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\xi_\rho(x)-\xi_\rho(y)|^t |u_n(x)|^t}{|x-y|^{N+st}} dxdy \right]^{\frac{1}{t}} = 0 \end{aligned} \tag{23}$$

for $t \in \{p, q\}$.

Clearly, we have

$$\lim_{\rho\to 0}\limsup_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^t \xi_\rho dx = 0 \text{ for } t \in \{p, q\}. \tag{24}$$

Taking into account the fact that f has a subcritical growth and recalling the definition of ξ_ρ, we obtain

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)u_n\xi_\rho dx = 0. \tag{25}$$

Since we have $\langle J_\varepsilon'(u_n), \xi_\rho u_n\rangle \to 0$ as $n\to\infty$, for $n\in\mathbb{N}$ large enough we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p\xi_\rho(x)}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q\xi_\rho(x)}{|x-y|^{N+sq}}dxdy\\
&+\int_{\mathbb{R}^N} V(\varepsilon x)\left(|u_n|^p+|u_n|^q\right)\xi_\rho dx\\
&= o_n(1) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\xi_\rho(x)-\xi_\rho(y))u_n(y)}{|x-y|^{N+sp}}dxdy\\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\xi_\rho(x)-\xi_\rho(y))u_n(y)}{|x-y|^{N+sq}}dxdy\\
&+\lambda\int_{\mathbb{R}^N} f(u_n)u_n\xi_\rho dx + \int_{\mathbb{R}^N} |u_n|^{q_s^*}\xi_\rho dx.
\end{aligned} \tag{26}$$

Putting together (21), (23), (24), (25) and (26), we deduce that $\upsilon_i \geqslant \mu_i$.

Note that $\langle J_\varepsilon'(u_n), u_n^-\rangle = o_n(1)$ as $n\to\infty$, where $u_n^- := \min\{u_n, 0\}$, and $g(x,t)=0$ for $t\leqslant 0$. So, we can deduce that

$$\|u_n^-\|_{V_\varepsilon,p}^p + \|u_n^-\|_{V_\varepsilon,q}^q \leqslant o_n(1) \text{ as } n\to\infty,$$

which implies that $u_n^-\to 0$ in X_ε as $n\to\infty$. Consequently, from $(g_3)_i$–$(g_3)_{ii}$ and $p<q$, it follows that

$$\begin{aligned}
c &= J_\varepsilon(u_n) - \frac{1}{q}\langle J_\varepsilon'(u_n), u_n\rangle + o_n(1)\\
&= \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_{V_\varepsilon,p}^p + \lambda\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left[\frac{1}{q}g(\varepsilon x,u_n)u_n - G(\varepsilon x,u_n)\right]dx\\
&+\lambda\int_{\Lambda_\varepsilon}\left[\frac{1}{q}f(u_n)u_n - F(u_n)\right]dx + \left(\frac{1}{q}-\frac{1}{q_s^*}\right)\int_{\Lambda_\varepsilon}|u_n^+|^{q_s^*}dx + o_n(1)\\
&\geqslant \frac{s}{N}\int_{\Lambda_\varepsilon}\left(|u_n|^{q_s^*} - |u_n^-|^{q_s^*}\right)dx + o_n(1) \text{ (using } u_n^-\to 0 \text{ in } X_\varepsilon \text{ as } n\to\infty \text{ and Theorem 3)}\\
&\geqslant \frac{s}{N}\int_{\Lambda_\varepsilon}|u_n|^{q_s^*}dx + o_n(1)
\end{aligned}$$

$$\geqslant \frac{s}{N} \int\limits_{\Lambda_\varepsilon} |u_n|^{q_s^*} \xi_\rho dx + o_n(1),$$

as $n \to \infty$.

In the above inequality, we pass to the limit as $\rho \to 0$. Then, we deduce from relation (22) and $\upsilon_i \geqslant \mu_i$ that

$$c > \frac{s}{N} \sum_{\{i \in \mathrm{I}\,:\, x_i \in \Lambda_\varepsilon\}} \xi_\rho(x_i)\upsilon_i \geqslant \frac{s}{N}\upsilon_i \geqslant \frac{s}{N} S_*^{\frac{N}{sq}},$$

which contradicts $c < \frac{s}{N} S_*^{\frac{N}{sq}}$. So, relation (20) holds true.

By combining relation (20) with (g_2), (f_1), (f_2), Theorem 3 and the Dominated Convergence Theorem, we deduce that relation (19) holds true. Thus, by (17), (18) and (19) it follows that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} g(\varepsilon x, u_n)u_n dx = \int\limits_{\mathbb{R}^N} g(\varepsilon x, u)u dx. \tag{27}$$

On the other hand, we can use $\langle J_\varepsilon'(u), u\rangle = 0$ and $\langle J_\varepsilon'(u_n), u_n\rangle = o_n(1)$ (as $n \to \infty$) to infer that

$$\|u_n\|_{V_\varepsilon,p}^p + \|u_n\|_{V_\varepsilon,q}^q = \lambda \int\limits_{\mathbb{R}^N} g(\varepsilon x, u_n)u_n dx + o_n(1) \text{ as } n \to \infty$$

and

$$\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q = \lambda \int\limits_{\mathbb{R}^N} g(\varepsilon x, u)u dx.$$

Consequently, by above two relations, together with (27), we obtain

$$\|u_n\|_{V_\varepsilon,p}^p + \|u_n\|_{V_\varepsilon,q}^q = \|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q + o_n(1) \text{ as } n \to \infty.$$

Therefore, we can deduce from the above relation and the Brezis-Lieb lemma [14] that

$$\|u_n - u\|_{V_\varepsilon,p}^p + \|u_n - u\|_{V_\varepsilon,q}^q = o_n(1) \text{ as } n \to \infty,$$

which means that $u_n \to u$ in X_ε as $n \to \infty$. This proof is now complete. □

Corollary 9. *The modified functional J_ε fulfills the $(PS)_c$ condition on S_ε^+ at any level $c \in (0, \frac{s}{N} S_*^{N/(sq)})$.*

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset S_\varepsilon^+$ is a Palais-Samle sequence for the functional ψ_ε at the level c, that is,

$$\psi_\varepsilon(u_n) \to c \text{ in } \mathbb{R} \quad \text{ and } \quad \psi_\varepsilon'(u_n) \to 0 \text{ in } (T_{u_n} S_\varepsilon^+)' \text{ as } n \to \infty.$$

From Lemma 6-(c), it follows that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}} \subset X_\varepsilon$ is also a Palais-Samle sequence for the functional J_ε at the level c. Thus, we can derive from Lemma 8 that the functional J_ε satisfies the $(PS)_c$ condition. Hence, we pass to a subsequence and we can find some $u \in S_\varepsilon^+$ such that $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in X_ε as $n \to \infty$. This fact combined with Lemma 5-(c) implies that $u_n \to u$ in S_ε^+ as $n \to \infty$. The proof of the corollary is now complete. □

5. The limit problem

In what follows, we need to consider the limit problem associated with problem (P_λ), that is,

$$\left\{\begin{array}{ll} (-\Delta)_p^s u + (-\Delta)_q^s u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \lambda f(u) + |u|^{q_s^*-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N),\ u > 0,\ \lambda > 0 & \text{in } \mathbb{R}^N. \end{array}\right\} \qquad (Q_\lambda)$$

The energy functional $I_{V_0} : X_0 \mapsto \mathbb{R}$ related to problem (Q_λ) is defined by

$$I_{V_0}(u) := \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q + V_0\left(\frac{1}{p}|u|_p^p + \frac{1}{q}|u|_q^q\right) - \lambda \int_{\mathbb{R}^N} F(u)dx - \frac{1}{q_s^*}\int_{\mathbb{R}^N} (u^+)^{q_s^*}dx$$

for all $u \in X_0$. By standard arguments, we know that the functional I_{V_0} is well-defined and belongs to C^1, and it holds

$$\begin{aligned} \langle I'_{V_0}(u), v\rangle = \langle u, v\rangle_{s,p} + \langle u, v\rangle_{s,q} + \int_{\mathbb{R}^N} V_0\left(|u|^{p-2}u + |u|^{q-2}u\right) v dx \\ - \lambda \int_{\mathbb{R}^N} f(u) v dx - \int_{\mathbb{R}^N} (u^+)^{q_s^*-1} v dx, \end{aligned}$$

for all $u, v \in X_0$.

Let us consider the Nehari manifold associated with the functional I_{V_0}, that is,

$$\mathcal{N}_0 := \left\{u \in X_0 \setminus \{0\} : \langle I'_{V_0}(u), u\rangle = 0\right\}.$$

Moreover, we set $c_{V_0} := \inf_{u\in\mathcal{N}_0} I_{V_0}(u)$.

Next, we define the following sets

$$X_0^+ := \left\{u \in X_0 : |\mathrm{supp}\,(u^+)| > 0\right\} \quad \text{and} \quad S_0^+ = S_0 \cap X_0^+,$$

where S_0 is the unit sphere of X_0.

As in section 4, S_0^+ is also an incomplete $C^{1,1}$-manifold of codimension one and contained in X_0^+. Hence, $X_0 = T_u S_0^+ \bigoplus \mathbb{R}u$ for each $u \in S_0^+$, where

$$T_u S_0^+ := \left\{ v \in X_0 : \langle u, v\rangle_{s,p} + \langle u, v\rangle_{s,q} + \int_{\mathbb{R}^N} V_0\left(|u|^{p-2}u + |u|^{q-2}u\right) v dx = 0 \right\}.$$

Arguing as in the proof of Lemma 5, we can show that the following results are fulfilled.

Lemma 10. *For any fixed $\lambda > 0$, assume that (f_1)–(f_4) and (V_1)–(V_2) are fulfilled. Then, we have the following properties*:

(a) *for any fixed $u \in X_0^+$, let the mapping $\ell_u : \mathbb{R}^+ \mapsto \mathbb{R}$ be defined by $\ell_u(t) := I_{V_0}(tu)$. Then, there is a unique $t_u > 0$ such that*

$$\ell_u'(t) > 0 \text{ for all } t \in (0, t_u),$$
$$\ell_u'(t) < 0 \text{ for all } t \in (t_u, +\infty);$$

(b) *there exists $\tau > 0$ independent of u such that $t_u \geqslant \tau$ for all $u \in S_0^+$. Moreover, for each compact set $\mathcal{W} \subset S_0^+$, there is a constant $C_{\mathcal{W}} > 0$ such that $t_u \leqslant C_{\mathcal{W}}$ for all $u \in \mathcal{W}$;*
(c) *the mapping $\hat{m}_0 : X_0^+ \mapsto \mathcal{N}_0$ introduced by $\hat{m}_0(u) := t_u u$ is continuous and $m_0 := \hat{m}_0|_{S_0^+}$ is a homeomorphism between S_0^+ and $\mathcal{N}_0$, and the inverse of m_0 is given by $m_0^{-1}(u) := u/\|u\|_{X_0}$;*
(d) *if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset S_0^+$ such that $\operatorname{dist}(u_n, \partial S_0^+) \to 0$ as $n \to \infty$, then $\|m_0(u_n)\|_{X_0} \to +\infty$ and $I_{V_0}(m_0(u_n)) \to +\infty$ as $n \to \infty$.*

Consider the mappings

$$\hat{\psi}_{V_0} : X_0^+ \mapsto \mathbb{R} \quad \text{and} \quad \psi_{V_0} : S_0^+ \mapsto \mathbb{R}$$

defined by $\hat{\psi}_{V_0}(u) := I_{V_0}(\hat{m}_0(u))$ for all $u \in X_0^+$ and $\psi_{V_0} := \hat{\psi}_{V_0}|_{S_0^+}$.

Lemma 11. *For any fixed $\lambda > 0$, assume that (f_1)–(f_4) and (V_1)–(V_2) are fulfilled. Then,*

(a) *$\hat{\psi}_{V_0} \in C^1(X_0^+, \mathbb{R})$ and $\langle \hat{\psi}_{V_0}'(u), v\rangle = \frac{\|\hat{m}_0(u)\|_{X_0}}{\|u\|_{X_0}} \langle I_{V_0}'(\hat{m}_0(u)), v\rangle$ for all $u \in X_0^+$, all $v \in X_0$;*
(b) *$\psi_{V_0} \in C^1(S_0^+, \mathbb{R})$ and $\langle \psi_{V_0}'(u), v\rangle = \|m_0(u)\|_{X_0} \langle I_{V_0}'(m_0(u)), v\rangle$ for all $u \in S_0^+$, all $v \in T_u S_0^+$;*
(c) *if $\{u_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for ψ_{V_0}, then $\{m_0(u_n)\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for I_{V_0}. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ is bounded Palais-Smale sequence for I_{V_0}, then $\{m_0^{-1}(u_n)\}_{n\in\mathbb{N}} \subset S_0^+$ is a Palais-Smale sequence for ψ_{V_0};*
(d) *$u \in S_0^+$ is a critical point of ψ_{V_0} if and only if $m_0(u) \in \mathcal{N}_0$ is a critical point of I_{V_0}. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in S_0^+} \psi_{V_0}(u) = \inf_{u\in\mathcal{N}_0} I_{V_0}(u) = c_{V_0}.$$

Arguing as in Lemma 4, we can show that the functional I_{V_0} has a Mountain Pass Geometry (see Willem [29]).

Lemma 12. *For each fixed $\lambda > 0$, the following properties are fulfilled for the functional I_{V_0}:*

(i) *there exist $\hat{\varrho}$, $\hat{\rho} > 0$ such that $I_{V_0}(u) \geqslant \hat{\varrho}$ with $\|u\|_{X_0} = \hat{\rho}$;*
(ii) *there exists $\hat{e} \in X_0^+$ satisfying $\|\hat{e}\|_{X_0} > \hat{\rho}$ and $I_{V_0}(\hat{e}) < 0$.*

Remark 3. Arguing as in Section 4, for each $\lambda > 0$ we have the following variational characterization for c_{V_0}:

$$0 < c_{V_0} = \inf_{u \in \mathcal{N}_0} I_{V_0}(u) = \inf_{u \in X_0^+} \max_{t>0} I_{V_0}(tu) = \inf_{u \in S_0^+} \max_{t>0} I_{V_0}(tu).$$

Moreover, if

$$\hat{c}_{V_0} := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} I_{V_0}(\gamma(t)), \text{ where } \Gamma_0 := \left\{\gamma \in C([0,1], X_0) : \gamma(0) = 0 \text{ and } I_{V_0}(\gamma(1)) < 0\right\},$$

proceeding as in Willem [29], we can check that $c_{V_0} = \hat{c}_{V_0}$.

The next result shows that we can compare c_{V_0} with a suitable constant which involves S_*.

Lemma 13. *There exists $\lambda_* > 0$ such that $c_{V_0} \in (0, \frac{s}{N} S_*^{N/(sq)})$ for each $\lambda \geqslant \lambda_*$.*

Proof. Clearly, there exists $t_\lambda > 0$ such that $I_{V_0}(t_\lambda \hat{e}) = \max_{t \geqslant 0} I_{V_0}(t\hat{e})$, where $\hat{e}$ is given by Lemma 12. As a consequence of $\langle I'_{V_0}(t_\lambda \hat{e}), t_\lambda \hat{e}\rangle = 0$, it holds

$$t_\lambda^p \|\hat{e}\|_{V_0,p}^p + t_\lambda^q \|\hat{e}\|_{V_0,q}^q = \lambda \int_{\mathbb{R}^N} f(t_\lambda \hat{e}) t_\lambda \hat{e} dx + t_\lambda^{q_s^*} |\hat{e}|_{q_s^*}^{q_s^*}. \tag{28}$$

The above relation combined with (f_3) implies that

$$t_\lambda^p \|\hat{e}\|_{V_0,p}^p + t_\lambda^q \|\hat{e}\|_{V_0,q}^q \geqslant t_\lambda^{q_s^*} |\hat{e}|_{q_s^*}^{q_s^*}.$$

Since $p < q < q_s^*$, we conclude that $\{t_\lambda\}_{\lambda>0} \subset \mathbb{R}$ is bounded, and so there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ ($\lambda_n \to +\infty$ as $n \to \infty$) such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. Arguing by contradiction and using (28), we can infer that $t_0 = 0$.

Next, we define $\gamma(t) = t\hat{e}$ with $t \in [0, 1]$. So, $\gamma \in \Gamma_0$ and we have

$$0 < c_{V_0} \leqslant \max_{t\in[0,1]} I_{V_0}(t\hat{e}) = I_{V_0}(t_\lambda \hat{e}) \leqslant t_\lambda^p \|\hat{e}\|_{V_0,p}^p + t_\lambda^q \|\hat{e}\|_{V_0,q}^q. \tag{29}$$

Then we can take $\lambda > 0$ large enough to ensure that

$$t_\lambda^p \|\hat{e}\|_{V_0,p}^p + t_\lambda^q \|\hat{e}\|_{V_0,q}^q < \frac{s}{N} S_*^{\frac{N}{sq}},$$

which means that $0 < c_{V_0} < \frac{s}{N} S_*^{\frac{N}{sq}}$ for $\lambda > 0$ sufficiently large. In particular, the fact that $t_\lambda \to 0$ as $\lambda \to +\infty$, together with (29), implies that $c_{V_0} \to 0$ as $\lambda \to +\infty$. $\square$

Lemma 14. *For each $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ is a $(PS)_{c_{V_0}}$ sequence of the functional I_{V_0} at the level c_{V_0}. Then $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded and there exist a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and some constants R, $\alpha > 0$ such that*

$$\liminf_{n\to\infty} \int\limits_{B_R(y_n)} |u_n|^q dx \geqslant \alpha.$$

Proof. Similar to the argument of Lemma 7, we are able to deduce that $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded. Proceeding by contradiction, we may suppose that for all $R > 0$ the following limit is fulfilled:

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^q dx = 0.$$

By Lemma 2.2 in Alves, Ambrosio & Isernia [2], it follows that

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^N) \text{ for all } r \in (p, q_s^*) \text{ as } n \to \infty. \tag{30}$$

In particular, we deduce from (f_1)–(f_2) and (30) that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} F(u_n)dx = \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} f(u_n)u_n dx = 0.$$

According to the fact that $\langle I'_{V_0}(u_n), u_n\rangle = o_n(1)$ as $n \to \infty$, we obtain

$$\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} = \lambda \int\limits_{\mathbb{R}^N} f(u_n)u_n dx + |u_n^+|^{q_s^*}_{q_s^*} + o_n(1) \text{ as } n \to \infty.$$

Then, we can pass to a subsequence and we assume that

$$\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} \to b \geqslant 0 \quad \text{and} \quad |u_n^+|^{q_s^*}_{q_s^*} \to b \geqslant 0 \text{ as } n \to \infty.$$

Next, for each $\lambda \geqslant \lambda_*$ we show that $b = 0$. Otherwise, $b > 0$. Clearly, we observe that

$$c_{V_0} \geqslant \frac{s}{N} b.$$

On the other hand, from Theorem 3 it follows that

$$\begin{aligned} &\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} \geqslant [u_n]^q_{s,q} \geqslant S_* |u_n^+|^q_{q_s^*}, \\ \Rightarrow\ & b \geqslant S_*^{\frac{N}{sq}}. \end{aligned}$$

So, we have

$$c_{V_0} \geqslant \frac{s}{N} b \geqslant \frac{s}{N} S_*^{\frac{N}{sq}}.$$

But, for each $\lambda \geqslant \lambda_*$, from Lemma 13 we know that

$$c_{V_0} < \frac{s}{N} S_*^{\frac{N}{sq}}.$$

Now, we get a contradiction for each $\lambda \geqslant \lambda_*$. Thus, for each $\lambda \geqslant \lambda_*$ we have $b = 0$. But, this leads to $\|u_n\|_{X_0} \to 0$ as $n \to \infty$, and then we infer that $I_{V_0}(u_n) \to 0$ as $n \to \infty$. This is also a contradiction since $I_{V_0}(u_n) \to c_{V_0} > 0$ as $n \to \infty$. The proof of the lemma is now complete. □

Remark 4. Fix arbitrarily $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. If u is the weak limit of a $(PS)_{c_{V_0}}$ sequence for the functional I_{V_0}, then we can assume that $u \neq 0$. Indeed, $u_n \xrightarrow{w} 0$, and if $u_n \nrightarrow 0$ in X_0 as $n \to \infty$, then we can use Lemma 14 to conclude that there exist a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and some positive constants R, α such that

$$\liminf_{n\to\infty} \int\limits_{B_R(y_n)} |u_n|^q dx \geqslant \alpha.$$

Let us define $v_n(x) := u_n(x + y_n)$. According to the invariance of $\mathbb{R}^N$ by translation, we infer that $\{v_n\}_{n\in\mathbb{N}} \subset X_0$ is a bounded $(PS)_{c_{V_0}}$ sequence for the functional I_{V_0}. Thus, there exists $0 \neq v \in X_0$ such that $v_n \xrightarrow{w} v$ as $n \to \infty$.

Theorem 15. *For each $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13, problem (Q_λ) has a positive ground state solution.*

Proof. By a variant of the Mountain Pass Theorem without the Palais-Smale condition (see Willem [29]) and applying Lemma 12, we know that there is a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ of the functional I_{V_0} at the level c_{V_0}. As in the proof of Lemma 7, we can infer that $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded. Now, we can pass to a subsequence (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) and suppose that there exists $u \in X_0$ such that $u_n \xrightarrow{w} u$ in X_0 and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for all $r \in [1, q_s^*)$ as $n \to \infty$.

With the same ideas as in the proof of Lemma 8, we can show that $I'_{V_0}(u) = 0$. By Remark 4 we can directly assume that $u \neq 0$. In addition, (f_3) combined with Fatou's Lemma yields

$$c_{V_0} \leqslant I_{V_0}(u) = I_{V_0}(u) - \frac{1}{q}\langle I'_{V_0}(u), u\rangle \leqslant \liminf_{n\to\infty}\left[I_{V_0}(u_n) - \frac{1}{q}\langle I'_{V_0}(u_n), u_n\rangle \right] = c_{V_0}.$$

So, we get $I_{V_0}(u) = c_{V_0}$.

Eventually, we show that $u > 0$ in $\mathbb{R}^N$. Since $\min\{u, 0\} =: u^- \in X_0$, and recalling that $f(t) = 0$ for $t \leqslant 0$ and using $\langle I'_{V_0}(u), u^-\rangle = 0$, we can obtain

$$\|u^-\|^p_{V_0,p} + \|u^-\|^q_{V_0,q} \leqslant 0,$$

which means that $u^- = 0$, and so $u \geqslant 0$ in $\mathbb{R}^N$. Consequently, $u \geqslant 0$ and $u \not\equiv 0$. Arguing as in the proof of Lemma 24, we deduce that $u \in L^\infty(\mathbb{R}^N)$. By Corollary 2.1 in Ambrosio & Rădulescu [9], we see that $u \in C^\sigma(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$. Proceeding as the proof of Theorem 1.1-(ii) in Jarohs [21], we infer that u is positive in $\mathbb{R}^N$. This proof is now complete. □

Firstly, we introduce a compactness result for the autonomous problem, which will be very useful in the sequel.

Lemma 16. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ is a sequence such that $I_{V_0}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$, then $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ admits a convergent subsequence.*

Proof. Note that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ and $I_{V_0}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$. Then we deduce from Lemma 10-(c), Lemma 11-(d) and the definition of c_{V_0} that

$$w_n := m_0^{-1}(u_n) \in S_0^+ \text{ for all } n \in \mathbb{N}$$

and

$$\psi_{V_0}(w_n) = I_{V_0}(u_n) \to c_{V_0} = \inf_{w\in S_0^+} \psi_{V_0}(w) \text{ in } \mathbb{R} \text{ as } n \to \infty.$$

Consider the mapping

$$\mathcal{E}(u) = \begin{cases} \psi_{V_0}(u) & \text{if } u \in S_0^+, \\ +\infty & \text{if } u \in \partial S_0^+. \end{cases}$$

Now, we are able to present the following properties:

(i) $(\overline{S_0^+}, \delta_{V_0})$, where $\delta_{V_0}(u, v) = \|u - v\|_{X_0}$, is a complete metric space;
(ii) $\mathcal{E} \in C(\overline{S_0^+}, \mathbb{R} \cup \{+\infty\})$, by Lemma 10-(d);
(iii) $\mathcal{E}$ is bounded below, by Lemma 11-(d).

So, we can apply the Ekeland variational principle (see Ekeland [17]) to the functional $\mathcal{E}$, and then we find a sequence $\{\hat{w}_n\}_{n\in\mathbb{N}} \subset S_0^+$ such that $\{\hat{w}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{c_{V_0}}$ for the functional ψ_{V_0} and $\|\hat{w}_n - w_n\|_{X_0} = o_n(1)$ as $n \to \infty$. This means that $\psi_{V_0}(\hat{w}_n) \to c_{V_0}$ in $\mathbb{R}$ and $\psi'_{V_0}(\hat{w}_n) \to 0$ in $(T_{\hat{w}_n} S_0^+)'$ as $n \to \infty$. Using Lemma 11, Theorem 15 and proceeding as in the proof of Corollary 9, the proof of the remainder of the lemma can be completed. □

Next, we establish the following useful relationship between the minimax levels c_ε and c_{V_0}.

Lemma 17. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Then*

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0} < \frac{s}{N} S_*^{\frac{N}{sq}}.$$

Proof. For each $\lambda \geqslant \lambda_*$, we assume that ω is a positive ground state solution of problem (Q_λ). Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ satisfy $0 \leqslant \zeta \leqslant 1$, $\zeta(x) = 1$ in B_r and $\operatorname{supp}(\zeta) \subset B_{2r} \subset \Lambda$ for some $r > 0$. Now, we define $\omega_\varepsilon(x) := \zeta_\varepsilon(x)\omega(x)$, where $\zeta_\varepsilon(x) = \zeta(\varepsilon x)$ for $\varepsilon > 0$. Applying the Dominated Convergence Theorem and arguing as in the proof of Lemma 2.2 in Ambrosio [5], we can deduce that

$$\omega_\varepsilon \to \omega \text{ in } X_0 \quad \text{ and } \quad I_{V_0}(\omega_\varepsilon) \to I_{V_0}(\omega) = c_{V_0} \text{ as } \varepsilon \to 0. \tag{31}$$

Next, for each $\varepsilon > 0$ we take a unique number $t_\varepsilon > 0$ satisfying

$$J_\varepsilon(t_\varepsilon \omega_\varepsilon) = \max_{t \geqslant 0} J_\varepsilon(t\omega_\varepsilon).$$

From $\operatorname{supp}(\zeta) \subset \Lambda$ and the definition of g, for $\varepsilon > 0$ small enough, it follows that

$$t_\varepsilon^p \|\omega_\varepsilon\|_{V_\varepsilon,p}^p + t_\varepsilon^q \|\omega_\varepsilon\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} \left[\lambda f(t_\varepsilon\omega_\varepsilon)t_\varepsilon\omega_\varepsilon + (t_\varepsilon\omega_\varepsilon)^{q_s^*}\right] dx,$$

$$\Rightarrow t_\varepsilon^{p-q} \|\omega_\varepsilon\|_{V_\varepsilon,p}^p + \|\omega_\varepsilon\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} \frac{\lambda f(t_\varepsilon\omega_\varepsilon) + (t_\varepsilon\omega_\varepsilon)^{q_s^*-1}}{(t_\varepsilon\omega_\varepsilon)^{q-1}} \omega_\varepsilon^q \, dx. \tag{32}$$

If $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, relations (31)–(32), together with $q > p$ and (f_3) yield that $+\infty > \|\omega\|_{V_0,q}^q = +\infty$. This is impossible.

Up to a subsequence, we can now suppose that $t_\varepsilon \to t_0 \in [0, +\infty)$ as $\varepsilon \to 0$. Indeed, $t_0 > 0$. Otherwise, we are able to infer that $\|\omega\|_{V_0,p} = 0$, which implies that $\omega \equiv 0$. Since $\omega > 0$ in $\mathbb{R}^N$, we reach a contradiction. Then we pass to the limit as $\varepsilon \to 0$ in the relation (32), and using relation (31) we have

$$t_0^{p-q} \|\omega\|_{V_0,p}^p + \|\omega\|_{V_0,q}^q = \int_{\mathbb{R}^N} \frac{\lambda f(t_0\omega) + (t_0\omega)^{q_s^*-1}}{(t_0\omega)^{q-1}} \omega^q \, dx.$$

Then, we can use the above equality, (f_4) and $\omega \in \mathcal{N}_0$ to get the fact that $t_0 = 1$. Eventually, we deduce from the Dominated Convergence Theorem, $t_0 = 1$ and $\omega \in \mathcal{N}_0$ that

$$\lim_{\varepsilon \to 0} J_\varepsilon(t_\varepsilon\omega_\varepsilon) = I_{V_0}(\omega) = c_{V_0}.$$

The above relation, together with the definition of t_ε, implies that $c_{V_0} \geqslant \limsup_{\varepsilon \to 0} c_\varepsilon$. On the other hand, from (V_1) we see that $\liminf_{\varepsilon \to 0} c_\varepsilon \geqslant c_{V_0}$. By Lemma 13, we have

$$\lim_{\varepsilon \to 0} c_\varepsilon = c_{V_0} < \frac{s}{N} S_*^{\frac{N}{sq}} \text{ for each } \lambda \geqslant \lambda_*.$$

This proof is now complete. □

We complete this section with the following existence property.

Theorem 18. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Assume that hypotheses (f_1)–(f_4) and (V_1)–(V_4) are fulfilled. Then, for $\varepsilon > 0$ sufficiently small, problem (2) admits a nontrivial nonnegative solution.*

Proof. According to Lemma 4, Remark 2, Lemma 17 and Lemma 8, we can use the Mountain Pass Theorem of Ambrosetti & Rabinowitz [4] to conclude that, for each $\lambda \geqslant \lambda_*$ and if $\varepsilon > 0$ is small enough, there exists a critical point $u_\varepsilon \in X_\varepsilon \setminus \{0\}$ for the functional J_ε. It is easy to check that $u_\varepsilon \geqslant 0$ in $\mathbb{R}^N$ and $u_\varepsilon \not\equiv 0$. This proof is now complete. □

6. The barycenter map

In this section, we provide some technical results, which will be used to prove the multiplicity of solutions for the modified problem (2).

Let $\delta > 0$ be such that

$$M_\delta := \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leqslant \delta\right\} \subset \Lambda \tag{33}$$

and $\eta \in C^\infty([0, +\infty), [0, 1])$ be a non-increasing cut-off function verifying $\eta(t) = 1$ for $t \in (0, \delta/2)$, $\eta(t) = 0$ for $t \in [\delta, +\infty)$ and $|\eta'(t)| \leqslant C$ for some constant $C > 0$. For each $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13, we may assume that ω is a positive ground state solution to the limit problem (Q_λ). For any $y \in M$, we introduce the following function

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

with the unique number $t_\varepsilon > 0$ satisfying

$$\max_{t \geqslant 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}),$$

and we consider the mapping $\Phi_\varepsilon : M \mapsto \mathcal{N}_\varepsilon$ defined by

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

Lemma 19. *For each $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13, the mapping Φ_ε has the following property:*

$$\lim_{\varepsilon \to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M.$$

Proof. Arguing by contradiction, we may assume that there exist $\delta_0 > 0$, $\{y_n\}_{n \in \mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ in $\mathbb{R}$ as $n \to \infty$ such that

$$\left|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}\right| \geqslant \delta_0. \tag{34}$$

We first note that for each $n \in \mathbb{N}$ and for all $z \in B_{\frac{\delta}{\varepsilon_n}}$, $\varepsilon_n z \in B_\delta$, hence $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$.

Applying the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$ and recalling that $G(x, t) = F(x, t) + \frac{1}{\lambda q_s^*} t^{q_s^*}$ in $\Lambda \times [0, +\infty)$ and $\eta(t) = 0$ for $t \geqslant \delta$, we get

$$
\begin{aligned}
J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{t_{\varepsilon_n}^q}{q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \\
&\quad - \lambda \int_{\mathbb{R}^N} G(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})dx \\
&= \frac{t_{\varepsilon_n}^p}{p}\left[[\eta(|\varepsilon_n \cdot|)\omega]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)\omega(z))^p dz \right] \\
&\quad + \frac{t_{\varepsilon_n}^q}{q}\left[[\eta(|\varepsilon_n \cdot|)\omega]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)\omega(z))^q dz \right] \\
&\quad - \int_{\mathbb{R}^N} \left[\lambda F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)\omega(z)) + \frac{1}{q_s^*}(t_{\varepsilon_n}\eta(|\varepsilon_n z|)\omega(z))^{q_s^*} \right] dz.
\end{aligned} \tag{35}
$$

Next, we prove that the sequence $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ verifies $t_{\varepsilon_n} \to 1$ in $\mathbb{R}$ as $n \to \infty$. Using the definition of t_{ε_n}, we see that $t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$, that is,

$$
\begin{aligned}
& t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \\
&= \int_{\mathbb{R}^N} \frac{\lambda f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\Psi_{\varepsilon_n,y_n} + t_{\varepsilon_n}^{q_s^*-1}(\Psi_{\varepsilon_n,y_n})^{q_s^*}}{t_{\varepsilon_n}^{q-1}} dx \\
& \quad (\text{since } g = f \text{ on } \Lambda \times [0, +\infty)) \\
&= \int_{\mathbb{R}^N} \frac{\lambda f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)\omega(z)) + (t_{\varepsilon_n}\eta(|\varepsilon_n z|)\omega(z))^{q_s^*-1}}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)\omega(z))^{q-1}} (\eta(|\varepsilon_n z|)\omega(z))^q dz.
\end{aligned} \tag{36}
$$

Clearly, $\eta(|x|) = 1$ for $x \in B_{\frac{\delta}{2}}$ and $B_{\frac{\delta}{2}} \subset B_{\frac{\delta}{\varepsilon_n}}$ for $n \in \mathbb{N}$ sufficiently large. Then, we conclude from (36) that

$$
t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \geqslant \int_{B_{\frac{\delta}{2}}} \frac{\lambda f(t_{\varepsilon_n}\omega(z)) + (t_{\varepsilon_n}\omega(z))^{q_s^*-1}}{(t_{\varepsilon_n}\omega(z))^{q-1}} (\omega(z))^q dz.
$$

In addition, ω is a continuous and positive function in $\mathbb{R}^N$, so there exists $\bar{z} \in \mathbb{R}^N$ such that

$$
\omega(\bar{z}) = \min_{z\in \overline{B}_{\frac{\delta}{2}}} \omega(z) > 0.
$$

Consequently, we infer from (f_4) that

$$
t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \geqslant \left[\frac{\lambda f(t_{\varepsilon_n}\omega(\bar{z}))}{(t_{\varepsilon_n}\omega(\bar{z}))^{q-1}}(\omega(\bar{z}))^q + t_{\varepsilon_n}^{q_s^*-q}(\omega(\bar{z}))^{q_s^*} \right] \left| B_{\frac{\delta}{2}} \right|. \tag{37}
$$

Arguing as in the proof of Lemma 2.2 in Ambrosio [5], we can obtain

$$\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},t} \to \|\omega\|_{V_0,t} \in (0,+\infty) \text{ in } \mathbb{R} \text{ as } n\to\infty \text{ for } t\in\{p,q\}. \tag{38}$$

Hence, if $t_{\varepsilon_n} \to +\infty$, we see that

$$\lim_{n\to\infty}\left(t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q\right) = \|\omega\|_{V_0,q}^q, \tag{39}$$

since $q>p$. Additionally, from (f_3) it follows that

$$\lim_{n\to\infty}\frac{f(t_{\varepsilon_n}\omega(\bar z))}{(t_{\varepsilon_n}\omega(\bar z))^{q-1}} = +\infty. \tag{40}$$

Since $q_s^* > q$, from (37), (39) and (40), we get a contradiction. Therefore, we pass to a subsequence and assume that there exists t_0 such that $t_{\varepsilon_n} \to t_0 \geqslant 0$ as $n\to\infty$. Applying (36), (38) and using (f_1)–(f_2), we obtain that $t_0>0$.

In (36) we pass to the limit as $n\to\infty$, then we can use (38) and the Dominated Convergence Theorem to conclude that

$$t_0^{p-q}\|\omega\|_{V_0,p}^p + \|\omega\|_{V_0,q}^q = \int_{\mathbb{R}^N}\frac{\lambda f(t_0\omega) + (t_0\omega)^{q_s^*-1}}{(t_0\omega)^{q-1}}\omega^q\,dx. \tag{41}$$

Since $\omega\in\mathcal{N}_0$, we see that

$$\|\omega\|_{V_0,p}^p + \|\omega\|_{V_0,q}^q = \int_{\mathbb{R}^N}\left[\lambda f(\omega)\omega + \omega^{q_s^*}\right]dx. \tag{42}$$

Consequently, from (41), (42) and (f_4), it follows that $t_0=1$.

In (35) we pass to the limit as $n\to\infty$ and we have

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(\omega) = c_{V_0}.$$

This contradicts relation (34). The proof is now complete. □

For each $\lambda\geqslant\lambda_*$, where λ_* is given in Lemma 13, let us define the function $e:\mathbb{R}^+\mapsto\mathbb{R}^+$ such that $e(\varepsilon) := \sup_{y\in M}\left|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}\right|$ for all $\varepsilon>0$. Then, we introduce the following subset of $\mathcal{N}_\varepsilon$:

$$\hat{\mathcal{N}}_\varepsilon := \left\{u\in\mathcal{N}_\varepsilon : J_\varepsilon(u)\leqslant c_{V_0}+e(\varepsilon)\right\}.$$

From Lemma 19 it follows that $e(\varepsilon)\to 0$ as $\varepsilon\to 0$. Additionally, we deduce from the definition of the function e that $\Phi_\varepsilon(y)\in\hat{\mathcal{N}}_\varepsilon$ for any $y\in M$ and $\varepsilon>0$, and so $\hat{\mathcal{N}}_\varepsilon\neq\emptyset$.

For any $\delta>0$ given by (33), let us choose $\rho:=\rho(\delta)>0$ such that $M_\delta\subset B_\rho$. Consider the map $\hat\zeta:\mathbb{R}^N\mapsto\mathbb{R}^N$ defined by

$$\hat{\xi}(x) := \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geqslant \rho. \end{cases}$$

Now, we introduce the following barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \mapsto \mathbb{R}^N$ defined by

$$\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^N} \hat{\xi}(\varepsilon x)\,(|u|^p + |u|^q)\,dx}{\int_{\mathbb{R}^N} (|u|^p + |u|^q)\,dx}$$

for all $u \in \mathcal{N}_\varepsilon$.

Lemma 20. *For each $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13, the map β_ε has the following property:*

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \textit{ uniformly in } y \in M.$$

Proof. Arguing by contradiction, there exist $\delta_0 > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\left|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n\right| \geqslant \delta_0. \tag{43}$$

By the definitions of Φ_{ε_n}, β_{ε_n}, $\hat{\xi}$ and using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can conclude that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} \left[\hat{\xi}(\varepsilon_n z + y_n) - y_n\right] \left[|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q\right] dz}{\int_{\mathbb{R}^N} [|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q]\,dz}.$$

Thanks to $\{y_n\}_{n\in\mathbb{N}} \subset M \subset M_\delta$, by the Dominated Convergence Theorem, we can derive that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = 0,$$

which contradicts relation (43). This proof is now complete. □

Lemma 21. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Assume that the sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ satisfy $\varepsilon_n \to 0$ in $\mathbb{R}$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$, then there is a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that the sequence $\left\{\hat{u}_n(x) := u_n(x + \hat{y}_n)\right\}_{n\in\mathbb{N}}$ admits a subsequence which converges in X_0. Furthermore, the sequence $\left\{y_n := \varepsilon_n \hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ has a subsequence $\{y_n\}_{n\in\mathbb{N}}$ (still denoted by itself) such that $y_n \to y_0 \in M$ as $n \to \infty$.*

Proof. It is worth to pointing out that $\lambda \geqslant \lambda_*$. In the fashion of the proof of Lemma 7, we see that $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded. Then, arguing as in the proof of Lemma 14 and Remark 4, we know that there are a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and two positive constants $R, \alpha > 0$ such that

$$\liminf_{n\to\infty} \int\limits_{B_R(\hat{y}_n)} |u_n|^q\,dx \geqslant \alpha.$$

Set

$$\hat{u}_n(x) := u_n(x + \hat{y}_n).$$

Thus, $\{\hat{u}_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded. So, passing to a subsequence, there exists $0 \neq \hat{u} \in X_0$ such that

$$\hat{u}_n \xrightarrow{w} \hat{u} \text{ in } X_0 \text{ as } n \to \infty.$$

Let $t_n > 0$ be such that

$$\hat{v}_n := t_n \hat{u}_n \in \mathcal{N}_0$$

and set

$$y_n := \varepsilon_n \hat{y}_n.$$

Therefore, we conclude that

$$\begin{aligned}
c_{V_0} &\leqslant I_{V_0}(\hat{v}_n) \text{ (from the definition of } c_{V_0}) \\
&\leqslant \frac{1}{p}[\hat{v}_n]_{s,p}^p + \frac{1}{q}[\hat{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\hat{v}_n|^p + \frac{1}{q}|\hat{v}_n|^q\right)dx \\
&- \int_{\mathbb{R}^N} \left[\lambda F(\hat{v}_n) + \frac{1}{q_s^*}(\hat{v}_n^+)^{q_s^*}\right]dx \\
&\leqslant \frac{t_n^p}{p}[u_n]_{s,p}^p + \frac{t_n^q}{q}[u_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x)\left(\frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q\right)dx \\
&- \lambda \int_{\mathbb{R}^N} G(\varepsilon x, t_n u_n)dx \text{ (by } (g_2)) \\
&= J_{\varepsilon_n}(t_n u_n) \leqslant J_{\varepsilon_n}(u_n) \text{ (since } u_n \in \mathcal{N}_{\varepsilon_n}) \\
&= c_{V_0} + o_n(1) \text{ as } n \to \infty.
\end{aligned}$$

This implies that

$$I_{V_0}(\hat{v}_n) \to c_{V_0} \text{ in } \mathbb{R} \text{ as } n \to \infty \text{ and } \hat{v}_n \in \mathcal{N}_0.$$

Clearly, the sequence $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset X_0$ is bounded. Thus, up to a subsequence if necessary, still denoted by itself, we may assume that there is an element $\hat{v} \in X_0$ such that $\hat{v}_n \xrightarrow{w} \hat{v}$ in X_0 as $n \to \infty$. It is easy to see that the sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded and it holds that $t_n \to t_0 \geqslant 0$ as $n \to \infty$.

We claim that $t_0 > 0$. Otherwise, $t_0 = 0$, so, we infer from the boundedness of $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset X_0$ that $\|\hat{v}_n\|_{X_0} = t_n\|\hat{u}_n\|_{X_0} \to 0$ in $\mathbb{R}$ as $n \to \infty$, and so $I_{V_0}(\hat{v}_n) \to 0$ in $\mathbb{R}$ as $n \to \infty$, but this is impossible, since $c_{V_0} > 0$. Thus, $t_0 > 0$. We deduce from the uniqueness of the weak limit that $\hat{v} = t_0\hat{u}$ and $\hat{u} \neq 0$. Then, from Lemma 16 it follows that

$$\hat{v}_n \to \hat{v} \text{ in } X_0 \text{ as } n \to \infty, \tag{44}$$

and so $\hat{u}_n \to \hat{u}$ in X_0 as $n \to \infty$. Moreover, $I_{V_0}(\hat{v}) = c_{V_0}$ and $\langle I'_{V_0}(\hat{v}), \hat{v}\rangle = 0$.

Next, we shall prove that the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ has a subsequence, still denoted by itself, such that $y_n \to y_0 \in M$ as $n \to \infty$. We first show the boundedness of $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$. Otherwise, the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ is not bounded. So, we may assume that there exists a subsequence, still denoted by itself, such that $|y_n| \to +\infty$ in $\mathbb{R}$ as $n \to \infty$. Then, we choose $R > 0$ large enough such that $\Lambda \subset B_R$ and we may suppose that $|y_n| > 2R$ for $n \in \mathbb{N}$ sufficiently large, and so for all $x \in B_{R/\varepsilon_n}$ we have

$$|\varepsilon_n x + y_n| \geqslant |y_n| - |\varepsilon_n x| > R.$$

Therefore, recalling that the definition of g, for $n \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned}
\|\hat{u}_n\|^p_{V_0,p} + \|\hat{u}_n\|^q_{V_0,q} &\leqslant \lambda \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \hat{u}_n)\hat{u}_n dx \\
&\leqslant \lambda \int_{B_{R/\varepsilon_n}} \widetilde{f}(\hat{u}_n)\hat{u}_n dx + \int_{B^c_{R/\varepsilon_n}} \left[\lambda f(\hat{u}_n)\hat{u}_n + (\hat{u}_n^+)^{q_s^*}\right] dx \\
&\leqslant \lambda \int_{B_{R/\varepsilon_n}} \frac{V_0}{K}\left(|\hat{u}_n|^p + |\hat{u}_n|^q\right) dx + \int_{B^c_{R/\varepsilon_n}} \left[\lambda f(\hat{u}_n)\hat{u}_n + (\hat{u}_n^+)^{q_s^*}\right] dx, \\
&\quad \left(\text{since } \widetilde{f}(\hat{u}_n)\hat{u}_n \leqslant \frac{V_0}{K}\left(|\hat{u}_n|^p + |\hat{u}_n|^q\right) \text{ on } B_{R/\varepsilon_n}\right) \\
&\leqslant \frac{p}{q} \int_{B_{R/\varepsilon_n}} V_0\left(|\hat{u}_n|^p + |\hat{u}_n|^q\right) dx + \int_{B^c_{R/\varepsilon_n}} \left[\lambda f(\hat{u}_n)\hat{u}_n + (\hat{u}_n^+)^{q_s^*}\right] dx \\
&\quad (\text{ since } K > \lambda q/p > 0) \\
&\leqslant \frac{p}{q}\left(\|\hat{u}_n\|^p_{V_0,p} + \|\hat{u}_n\|^q_{V_0,q}\right) + \int_{B^c_{R/\varepsilon_n}} \left[\lambda f(\hat{u}_n)\hat{u}_n + (\hat{u}_n^+)^{q_s^*}\right] dx.
\end{aligned}$$

Since $\hat{u}_n \to \hat{u}$ in X_0 as $n \to \infty$ and using the Dominated Convergence Theorem, we obtain

$$\int_{B^c_{R/\varepsilon_n}} \left[\lambda f(\hat{u}_n)\hat{u}_n + (\hat{u}_n^+)^{q_s^*}\right] dx = o_n(1) \text{ as } n \to \infty.$$

So, we have

$$\left(1 - \frac{p}{q}\right)\left(\|\hat{u}_n\|^p_{V_0,p} + \|\hat{u}_n\|^q_{V_0,q}\right) \leqslant o_n(1) \text{ as } n \to \infty.$$

Using $q > p$ and $\hat{u}_n \to \hat{u} \neq 0$ in X_0 as $n \to \infty$ again, we achieve a contradiction.

Now, we get the boundedness of $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$. Passing to a subsequence (still denoted by $\{y_n\}_{n\in\mathbb{N}}$), we may assume that there exists $y_0 \in \mathbb{R}^N$ such that $y_n \to y_0 \in \overline{\Lambda}$ as $n \to \infty$. In fact, if $y_0 \notin \overline{\Lambda}$, we can find a positive number $r > 0$ such that $y_n \in B_{r/2}(y_0) \subset \overline{\Lambda}^c$. Arguing as before, we can reach a contradiction. Hence, $y_0 \in \overline{\Lambda}$.

It remains to show that $V(y_0) = V_0$. Arguing by contradiction again we may assume that $V(y_0) > V_0$. From (44), together with Fatou's Lemma and the invariance by translations of $\mathbb{R}^N$ it follows that

$$
\begin{aligned}
c_{V_0} &= I_{V_0}(\hat{v}) \\
&< \liminf_{n\to\infty} \left\{ \frac{1}{p}[\hat{v}_n]_{s,p}^p + \frac{1}{q}[\hat{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left(\frac{1}{p}|\hat{v}_n|^p + \frac{1}{q}|\hat{v}_n|^q \right) dx \right. \\
&\quad \left. - \int_{\mathbb{R}^N} \left[\lambda F(\hat{v}_n) + \frac{1}{q_s^*}(\hat{v}_n^+)^{q_s^*} \right] dx \right\} \\
&\leqslant \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \leqslant \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) = c_{V_0}.
\end{aligned}
$$

This leads to a contradiction. Therefore, by hypothesis (V_2), we know that $y_0 \in M$. This proof is now complete. □

Lemma 22. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Then, for any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0} \sup_{u\in\hat{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0$ in $\mathbb{R}$ as $n \to \infty$, then we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\hat{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1) \text{ as } n \to \infty.$$

Taking into account the fact that $\{u_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we infer that

$$c_{V_0} \leqslant c_{\varepsilon_n} \leqslant J_{\varepsilon_n}(u_n) \leqslant c_{V_0} + e(\varepsilon_n),$$

and so

$$\lim_{n\to\infty} J_{\varepsilon_n}(u_n) = c_{V_0}.$$

Then, we deduce from Lemma 21 that for $n \in \mathbb{N}$ large enough there exists $\{\hat{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \hat{y}_n \in M_\delta$. So, we have

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} \left[\hat{\zeta}(\varepsilon_n z + y_n) - y_n \right] \left[|u_n(z+\hat{y}_n)|^p + |u_n(z+\hat{y}_n)|^q \right] dz}{\int_{\mathbb{R}^N} \left[|u_n(z+\hat{y}_n)|^p + |u_n(z+\hat{y}_n)|^q \right] dz}.$$

On account of the facts that $\{\hat{u}_n(\cdot+\hat{y}_n)\}_{n\in\mathbb{N}} \subset X_0$ admits a convergent subsequence and $\varepsilon_n z + y_n \to y_0 \in M$ as $n\to\infty$, we can infer that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$ in $\mathbb{R}^N$ as $n\to\infty$. Thus, there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Now, we complete the proof of the lemma. □

7. Multiple solutions for problem (2)

In this section we will focus on establishing a relationship between the topology of M and the number of solutions to problem (2). Since $\mathcal{N}_\varepsilon$ is not a C^1 submanifold of X_ε and S_ε^+ is not a complete metric space, we cannot use directly the standard Ljusternik-Schnirelmann theory, but we can circumvent this difficulty by applying the abstract results in Szulkin & Weth [28].

Theorem 23. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Assume that (f_1)–(f_4) and (V_1)–(V_2) are valid, then for any $\delta > 0$ satisfying $M_\delta \subset \Lambda$, there exists $\hat{\varepsilon}_{\delta,\lambda} > 0$ such that, for any $\varepsilon \in (0, \hat{\varepsilon}_{\delta,\lambda})$, problem (2) admits at least $cat_{M_\delta}(M)$ positive solutions.*

Proof. For any fixed $\varepsilon > 0$ we consider the mapping $\alpha_\varepsilon : M \mapsto S_\varepsilon^+$ defined by

$$\alpha_\varepsilon(y) := m_\varepsilon^{-1}(\Phi_\varepsilon(y)) \text{ for all } y \in M.$$

Then, we infer from Lemma 19 that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M. \tag{45}$$

Define the function

$$\bar{e}(\varepsilon) := \sup_{y\in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) - c_{V_0}|.$$

So, relation (45) implies that $\bar{e}(\varepsilon) \to 0$ in $\mathbb{R}$ as $\varepsilon \to 0$. Also, we introduce the following set:

$$\hat{S}_\varepsilon^+ := \left\{\omega \in S_\varepsilon^+ : \psi_\varepsilon(\omega) \leqslant c_{V_0} + \bar{e}(\varepsilon)\right\}.$$

Clearly, for all $y \in M$ and $\varepsilon > 0$, $\psi_\varepsilon(\alpha_\varepsilon(y)) \in \hat{S}_\varepsilon^+$, that is, $\hat{S}_\varepsilon^+ \neq \emptyset$.

Using the above information and invoking Lemma 19, Lemma 10-(c), Lemma 20 and Lemma 22, we derive that there exists $\hat{\varepsilon} = \hat{\varepsilon}_{\delta,\lambda} > 0$ such that, for all $\varepsilon \in (0, \hat{\varepsilon}_{\delta,\lambda})$, the following diagram is well-defined:

$$M \xmapsto{\Phi_\varepsilon} \Phi_\varepsilon(M) \xmapsto{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xmapsto{m_\varepsilon} \Phi_\varepsilon(M) \xmapsto{\beta_\varepsilon} M_\delta.$$

By Lemma 20 and decreasing $\hat{\varepsilon}$ if necessary, then, for all $y \in M$ we obtain $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + l(\varepsilon, y)$, where $|l(\varepsilon, y)| \leqslant \delta/2$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \hat{\varepsilon})$. Hence, $H(t, y) :=$

$y+(1-t)l(\varepsilon, y)$ for $(t, y) \in [0,1] \times M$ is homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \left(m_\varepsilon^{-1} \circ \Phi_\varepsilon\right)$ and the inclusion map $id : M \mapsto M_\delta$. This fact means that

$$cat_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geqslant cat_{M_\delta}(M). \tag{46}$$

Note that $\lambda \geqslant \lambda_*$. From Corollary 9, Lemma 17 and Theorem 27 in Szulkin & Weth [28], with $c = c_\varepsilon \leqslant c_{V_0} + \bar{e}(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, it follows that ψ_ε admits at least $cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\hat{S}_\varepsilon^+$. Finally, applying Lemma 6-(d) and (46), we see that the functional J_ε has at least $cat_{M_\delta}(M)$ critical points in $\hat{\mathcal{N}}_\varepsilon$. This proof is now complete. □

8. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. The most important thing is to show that the solutions obtained in Theorem 23 satisfy the following estimate:

$$\text{for } \varepsilon > 0 \text{ sufficiently small, } u_\varepsilon(x) \leqslant a \text{ for all } x \in \Lambda_\varepsilon^c.$$

Then we can exploit this fact and recall the definitions of g, G, and deduce that these solutions are indeed solutions of the original problem (P_λ). For this purpose, we shall consider the regularity of nonnegative solutions of problem (2). Inspired by Moser [27] and Ambrosio & Rădulescu [9], we start with the following lemma which plays an important role in the study of behavior of the maximum points of solutions to problem (P_λ).

Lemma 24. *Fix $\lambda \geqslant \lambda_*$, where λ_* is given in Lemma 13. Let $\varepsilon_n \to 0$ in $\mathbb{R}$ as $n \to \infty$ and $u_n \in \hat{\mathcal{N}}_{\varepsilon_n}$ be a solution to problem (2). Then, $J_{\varepsilon_n}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$, and there is some sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^\infty(\mathbb{R}^N)$ and $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$, for some constant $C > 0$. Furthermore,*

$$\hat{u}_n(x) \to 0 \text{ as } |x| \to +\infty \text{ uniformly in } n \in \mathbb{N}. \tag{47}$$

Proof. Note that $\lambda \geqslant \lambda_*$ and $u_n \in \hat{\mathcal{N}}_{\varepsilon_n}$. Proceeding as in the proof of Lemma 22, we know that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$. Then, we can use Lemma 21 to deduce that there is a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \to \hat{u}(\cdot) \in X_0$ and $y_n := \varepsilon_n \hat{y}_n \to y_0 \in M$ as $n \to \infty$. For any $L > 0$ and $\beta > 1$ we introduce the function

$$\psi(\hat{u}_n) := \hat{u}_n \hat{u}_{n,L}^{q(\beta-1)} \in X_{\varepsilon_n}, \text{ where } \hat{u}_{n,L} := \min\left\{\hat{u}_n, L\right\}.$$

Choosing $\psi(\hat{u}_n)$ as test function, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{p-2}(\hat{u}_n(x) - \hat{u}_n(y))(\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x-y|^{N+sp}} dxdy$$

$$+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{q-2}(\hat{u}_n(x) - \hat{u}_n(y))(\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x-y|^{N+sq}} dxdy$$

$$+\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|\hat{u}_n|^{p-2}\hat{u}_n\psi(\hat{u}_n)dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|\hat{u}_n|^{q-2}\hat{u}_n\psi(\hat{u}_n)dx$$

$$= \lambda \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \hat{u}_n)\psi(\hat{u}_n)dx.$$

According to the growth of g, we see that for any $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|g(x,t)| \leqslant \sigma|t|^{p-1} + C_\sigma|t|^{q_s^*-1} \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Using (V_1) and taking $\sigma \in (0, V_0/\lambda)$, together with the above relations, we can conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{p-2}(\hat{u}_n(x) - \hat{u}_n(y))(\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x-y|^{N+sp}}dxdy$$
$$+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{q-2}(\hat{u}_n(x) - \hat{u}_n(y))(\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x-y|^{N+sq}}dxdy$$
$$\leqslant C\int_{\mathbb{R}^N} |\hat{u}_n|^{q_s^*}\hat{u}_{n,L}^{q(\beta-1)}dx \tag{48}$$

for some constant $C > 0$.

Let us introduce the following functions

$$\varphi(t) := \frac{|t|^q}{q} \quad \text{and} \quad \Upsilon(t) := \int_0^t (\psi'(\tau))^{\frac{1}{q}}d\tau.$$

We first observe that ψ is an increasing function, thus it holds

$$(a-b)(\psi(a) - \psi(b)) \geqslant 0 \quad \text{for all } a,\ b \in \mathbb{R}. \tag{49}$$

Then we can use (49) and the Jensen inequality to obtain that

$$\varphi'(a-b)(\psi(a) - \psi(b)) \geqslant |\Upsilon(a) - \Upsilon(b)|^q \text{ for all } a,\ b \in \mathbb{R}. \tag{50}$$

Obviously, we have

$$\Upsilon(\hat{u}_n) \geqslant \frac{1}{\beta}\hat{u}_n\hat{u}_{n,L}^{\beta-1}. \tag{51}$$

Thus, by (48), (49), (50) and (51) and using Theorem 3, we can find some constant $C > 0$ such that

$$|\hat{u}_n\hat{u}_{n,L}^{\beta-1}|_{q_s^*}^q \leqslant C\beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q_s^*}\hat{u}_{n,L}^{q(\beta-1)}dx. \tag{52}$$

Choose $\beta = \frac{q_s^*}{q}$ and let $R > 0$ large enough. Combining $\hat{u}_n \to \hat{u}$ in X_0 as $n \to \infty$ with the Hölder inequality, we can infer that there exists some constant $C > 0$ such that

$$\left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}}\right)^{q_s^*} dx\right]^{\frac{q}{q_s^*}} \leqslant C\beta^q \int_{\mathbb{R}^N} R^{q_s^*-q} \hat{u}_n^{q_s^*} dx + C\epsilon \left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}}\right)^{q_s^*} dx\right]^{\frac{q}{q_s^*}}.$$

Then, we choose a fixed $\epsilon \in (0, 1/C)$ and infer that

$$\left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}}\right)^{q_s^*} dx\right]^{\frac{q}{q_s^*}} \leqslant C\beta^q \int R^{q_s^*-q} \hat{u}_n^{q_s^*} dx < +\infty.$$

In the above inequality we pass to the limit as $L \to +\infty$ and we have $\hat{u}_n \in L^{\frac{q_s^{*2}}{q}}(\mathbb{R}^N)$.

Due to $0 \leqslant \hat{u}_{n,L} \leqslant \hat{u}_n$, then in (52) we pass to the limit as $L \to +\infty$ and we get

$$|\hat{u}_n|_{\beta q_s^*}^{\beta q} \leqslant C\beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q_s^*+q(\beta-1)} dx.$$

This fact means that

$$\left(\int_{\mathbb{R}^N} \hat{u}_n^{\beta q_s^*} dx\right)^{\frac{1}{q_s^*(\beta-1)}} \leqslant (C^{1/q}\beta)^{\frac{1}{\beta-1}} \left[\int_{\mathbb{R}^N} \hat{u}_n^{q_s^*+q(\beta-1)} dx\right]^{\frac{1}{q(\beta-1)}}.$$

Now, we consider the sequence $\{\beta_m\}_{m\geqslant 1} \subset \mathbb{R}$ ($m \in \mathbb{N}$) which satisfies the following recursive relation:

$$q_s^* + q(\beta_{m+1} - 1) = \beta_m q_s^* \quad \text{and} \quad \beta_1 = \frac{q_s^*}{q}.$$

It follows that

$$\beta_{m+1} = \beta_1^m(\beta_1 - 1) + 1,$$

and so

$$\lim_{m\to\infty} \beta_m = +\infty.$$

Let us define

$$T_m := \left(\int_{\mathbb{R}^N} \hat{u}_n^{\beta_m q_s^*} dx\right)^{\frac{1}{q_s^*(\beta_m-1)}}.$$

So, we have

$$T_{m+1} \leqslant (C^{1/q}\beta_{m+1})^{\frac{1}{\beta_{m+1}-1}} T_m.$$

Clearly, using a standard iteration argument we have

$$T_{m+1} \leqslant \prod_{k=1}^{m} (C^{1/q}\beta_{k+1})^{\frac{1}{\beta_{k+1}-1}} T_1 \leqslant \overline{C} T_1, \text{ where } \overline{C} \text{ is independent of } m.$$

In the above inequality we pass to the limit as $m \to \infty$ and then we conclude that $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leqslant C$ uniformly in $n \in \mathbb{N}$.

Next, let us define

$$\kappa_n := -V(\varepsilon_n x + y_n)\left(\hat{u}_n^{p-1} + \hat{u}_n^{q-1}\right) + \lambda g(\varepsilon_n x + y_n, \hat{u}_n).$$

We observe that $\hat{u}_n$ fulfills the following equation:

$$(-\Delta)_p^s \hat{u}_n + (-\Delta)_q^s \hat{u}_n = \kappa_n \quad \text{ in } \mathbb{R}^N.$$

By the growth hypotheses on g, Corollary 2.1 in Ambrosio & Rădulescu [9], $\hat{u}_n \to \hat{u}$ in X_0 as $n \to \infty$ and the uniformly boundedness of $\left\{\hat{u}_n\right\}_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^N) \cap X_0$, we can infer that $\hat{u}_n(x) \to 0$ in $\mathbb{R}$ as $|x| \to +\infty$ uniformly with respect to $n \in \mathbb{N}$. The proof of this lemma is now finished. $\square$

Proof of Theorem 1 completed. Let us take $\delta > 0$ small enough such that $M_\delta \subset \Lambda$. We claim that there exists $\bar{\varepsilon}_{\delta,\lambda} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_{\delta,\lambda})$ and any solution $u_\varepsilon \in \hat{\mathcal{N}}_\varepsilon$ of problem (2), we have

$$|u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a. \tag{53}$$

Otherwise, we may assume that there is a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\varepsilon_n \to 0$ as $n \to \infty$, $u_{\varepsilon_n} \in \hat{\mathcal{N}}_{\varepsilon_n}$ such that $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geqslant a. \tag{54}$$

Clearly, we know that $J_{\varepsilon_n}(u_{\varepsilon_n}) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$, and then we can use Lemma 21 to deduce that there is a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\hat{u}_n(\cdot) := u_{\varepsilon_n}(\cdot + \hat{y}_n) \to \hat{u}(\cdot)$ in X_0 and $\varepsilon_n \hat{y}_n \to y_0 \in M$ as $n \to \infty$.

Let r be a positive real number such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$, and so $B_{\frac{r}{\varepsilon_n}}\left(\frac{y_0}{\varepsilon_n}\right) \subset \Lambda_{\varepsilon_n}$. Additionally, for n large enough we can conclude that $\Lambda_{\varepsilon_n}^c \subset B^c_{\frac{r}{\varepsilon_n}}(\hat{y}_n)$. Using (47), we see that $\hat{u}_n(x) \to 0$ as $|x| \to +\infty$ uniformly in $n \in \mathbb{N}$. Therefore, we can find $R > 0$ such that $\hat{u}_n(x) < a$ for any $|x| \geqslant R$, $n \in \mathbb{N}$. Consequently, $u_{\varepsilon_n}(x) < a$ for any $x \in B_R^c(\hat{y}_n)$, $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$ large enough, we obtain that

$$\Lambda_{\varepsilon_n}^c \subset B^c_{\frac{r}{\varepsilon_n}}(\hat{y}_n) \subset B_R^c(\hat{y}_n).$$

Hence, we derive that $u_{\varepsilon_n}(x) < a$ for any $x \in \Lambda^c_{\varepsilon_n}$ and for all $n \in \mathbb{N}$ large enough, which contradicts relation (54).

Fix $\varepsilon \in (0, \varepsilon_{\delta,\lambda})$, where $\varepsilon_{\delta,\lambda} := \min\{\hat{\varepsilon}_{\delta,\lambda}, \bar{\varepsilon}_{\delta,\lambda}\}$. It follows from Theorem 23 that problem (2) has at least $cat_{M_\delta}(M)$ nontrivial solutions for each $\lambda \geqslant \lambda_*$. If u_ε represents one of these solutions, we have that $u_\varepsilon \in \hat{\mathcal{N}}_\varepsilon$, and then we can apply relation (53) and review the definitions of the modified nonlinearity g to infer that u_ε is also a solution to problem (P_λ) for each $\lambda \geqslant \lambda_*$. Thus, problem ($P_\lambda$) admits at least $cat_{M_\delta}(M)$ nontrivial solutions for each $\lambda \geqslant \lambda_*$.

Eventually, we show the behavior of the maximum points of solutions of problem (P_λ) for each $\lambda \geqslant \lambda_*$. Let us choose $\varepsilon_n \to 0$ (as $n \to \infty$) and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_{\varepsilon_n}$ of solutions for problem (P_λ) as before. From (g_1) it follows that there exists a positive constant $\iota < a$ such that

$$g(\varepsilon x, t)t \leqslant \frac{V_0}{K}\left(t^p + t^q\right) \text{ for any } x \in \mathbb{R}^N,\ t \in [0, \iota]. \tag{55}$$

Proceeding as before, there exists $R > 0$ such that

$$|u_n|_{L^\infty(B^c_R(\hat{y}_n))} < \iota. \tag{56}$$

In addition, we can extract a subsequence $\{u_n\}_{n\in\mathbb{N}} \subset X_{\varepsilon_n}$ (still denoted by itself) and assume that it has

$$|u_n|_{L^\infty(B_R(\hat{y}_n))} \geqslant \iota. \tag{57}$$

Otherwise, if relation (57) does not hold, we deduce from (56) that $|u_n|_{L^\infty(\mathbb{R}^N)} < \iota$. Taking into account $u_n \in \mathcal{N}_{\varepsilon_n}$ again, (55) and recalling that $K > \lambda q/p > 0$, we get

$$\begin{aligned} \|u_n\|^p_{V_{\varepsilon_n},p} + \|u_n\|^q_{V_{\varepsilon_n},q} &\leqslant \lambda \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n)u_n dx, \\ &\leqslant \frac{pV_0}{q} \int_{\mathbb{R}^N} \left(|u_n|^p + |u_n|^q\right) dx. \end{aligned}$$

This fact means that $\|u_n\|_{X_{\varepsilon_n}} = 0$, which is absurd. Thus, relation (57) is fulfilled. Using relations (56) and (57), we see that if p_n is a global maximum point of u_n and $p_n = \hat{y}_n + q_n$ for some $q_n \in B_R$. Therefore, $\varepsilon_n p_n \to y_0 \in M$ as $n \to \infty$. This fact combined the continuity of the potential V yields that $V(\varepsilon_n p_n) \to V(y_0) = V_0$ in $\mathbb{R}$ as $n \to \infty$. This proof is now complete. □

9. The supercritical case

In the last section we study problem (S_η). Since problem (S_η) has a supercritical nonlinear term, we first truncate the nonlinearity $f_\eta(u) := |u|^{\theta-2}u + \eta|u|^{r-2}u$ in an appropriate way. Let $b > 0$ be a real number, and its value will be fixed later. Now, we introduce the following truncation function:

$$f_\eta(t)=\begin{cases}0 & \text{if } t<0,\\ t^{\theta-1}+\eta t^{r-1} & \text{if } 0\leqslant t<b,\\ (1+\eta b^{r-\theta})t^{\theta-1} & \text{if } t\geqslant b.\end{cases}$$

It is easy to see that f_η is a continuous function and verifies the following properties:

(f_1') $\lim_{t\to 0}\frac{f_\eta(t)}{|t|^{p-1}}=0$;
(f_2') there exists $\nu\in(q,q_s^*)$ such that

$$\lim_{|t|\to+\infty}\frac{f_\eta(t)}{|t|^{\nu-1}}=0;$$

(f_3') $0<\theta F_\eta(t):=\theta\int_0^t f_\eta(\tau)d\tau\leqslant f_\eta(t)t$ for all $t>0$;
(f_4') the map $t\mapsto\frac{f_\eta(t)}{t^{q-1}}$ is increasing for all $t\in(0,+\infty)$,
(f_η) $f_\eta(t)\leqslant(1+\eta b^{r-\theta})t^{\theta-1}$ for all $t\geqslant 0$.

Next, we consider the following truncation problem:

$$\left\{\begin{array}{ll}(-\Delta)_p^s u+(-\Delta)_q^s u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=f_\eta(u) & \text{in } \mathbb{R}^N,\\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N),\ u>0,\ \eta>0 & \text{in } \mathbb{R}^N.\end{array}\right\} \tag{T_η}$$

The functional $J_{\varepsilon,\eta}:X_\varepsilon\mapsto\mathbb{R}$ associated to problem (T_η) is defined by

$$J_{\varepsilon,\eta}(u):=\frac{1}{p}\|u\|_{V_\varepsilon,p}^p+\frac{1}{q}\|u\|_{V_\varepsilon,q}^q-\int\limits_{\mathbb{R}^N}F_\eta(u)dx \text{ for all } u\in X_\varepsilon.$$

Also, we introduce the functional of the limit problem of problem (T_η) as follows

$$J_{0,\eta}(u):=\frac{1}{p}[u]_{s,p}^p+\frac{1}{q}[u]_{s,q}^q+V_0\left(\frac{1}{p}|u|_p^p+\frac{1}{q}|u|_q^q\right)-\int\limits_{\mathbb{R}^N}F_\eta(u)dx \text{ for all } u\in X_0.$$

Arguing as in the proof of Theorem 1 (see also Ambrosio & Rădulescu [9, Theorem 1.1]), we infer that for each $\eta\geqslant 0$ and $\delta>0$, there exists $\hat{\varepsilon}(\delta,\eta)>0$ such that, for any $\varepsilon\in(0,\hat{\varepsilon}(\delta,\eta))$, problem ($T_\eta$) has at least $cat_{M_\delta}(M)$ positive solutions. If $u_{\varepsilon,\eta}$ denotes one of these solutions, we can show that the $W^{s,q}$-norm of $u_{\varepsilon,\eta}$ can be uniformly estimated with respect to $\eta\geqslant 0$. This statement can be expressed as follows:

Lemma 25. *Fix $\eta\geqslant 0$. Then there is a constant $C>0$ such that, for any $\varepsilon>0$ small enough,*

$$\|u_{\varepsilon,\eta}\|_{V_\varepsilon,q}\leqslant C,$$

where C is independent of η and ε.

Proof. By a simple examination of the proof of Theorem 1, we know that for each solution $u_{\varepsilon,\eta}$ of problem (T_η) fulfills the following relation:

$$J_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \leqslant c_{0,\eta} + e_\eta(\varepsilon),$$

where $c_{0,\eta}$ is the mountain pass level with respect to the functional $J_{0,\eta}$ and $e_\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Decreasing $\hat{\varepsilon}(\delta, \eta)$ if necessary, we may assume that for any $\varepsilon \in (0, \hat{\varepsilon}(\delta, \eta))$,

$$J_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \leqslant c_{0,\eta} + 1.$$

On account of the fact that $c_{0,0} \geqslant c_{0,\eta}$ for all $\eta \geqslant 0$, for any $\varepsilon \in (0, \hat{\varepsilon}(\delta, \eta))$ we infer that

$$J_{\varepsilon,\eta}(u_{\varepsilon,\eta}) \leqslant c_{0,0} + 1. \tag{58}$$

In addition, we have

$$\begin{aligned} J_{\varepsilon,\eta}(u_{\varepsilon,\eta}) &= J_{\varepsilon,\eta}(u_{\varepsilon,\eta}) - \frac{1}{\theta}\langle J'_{\varepsilon,\eta}(u_{\varepsilon,\eta}), u_{\varepsilon,\eta}\rangle \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_{\varepsilon,\eta}\|^p_{V_\varepsilon,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|u_{\varepsilon,\eta}\|^q_{V_\varepsilon,q} \\ &+ \int\limits_{\mathbb{R}^N} \left[\frac{1}{\theta} f_\eta(u_{\varepsilon,\eta})u_{\varepsilon,\eta} - F_\eta(u_{\varepsilon,\eta})\right] dx \\ &\geqslant \left(\frac{1}{q} - \frac{1}{\theta}\right) \|u_{\varepsilon,\eta}\|^q_{V_\varepsilon,q} \text{ (using } (f_3')). \end{aligned} \tag{59}$$

Combining (58) and (59), we derive that there exists some constant $C > 0$, independent of ε and η, such that

$$\|u_{\varepsilon,\eta}\|_{V_\varepsilon,q} \leqslant C,$$

for any $\varepsilon \in (0, \hat{\varepsilon}(\delta, \eta))$. The proof is now complete. □

Finally, our aim is to prove that $u_{\varepsilon,\eta}$ is the solution of the original problem (S_η) if η is small enough. For this goal, we shall develop a suitable Moser iteration technique. For simplicity, let $u = u_{\varepsilon,\eta}$ and set $u_L := \min\{u, L\}$. Choosing $\psi(u) := uu_L^{q(\beta-1)}$ ($\beta > 1$ will be chosen later) as the test function in problem (T_η), we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\psi(u(x)) - \psi(u(y)))}{|x - y|^{N+sp}} dxdy \\ &+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))(\psi(u(x)) - \psi(u(y)))}{|x - y|^{N+sq}} dxdy \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\psi(u)dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{q-2}u\psi(u)dx \\
&= \int_{\mathbb{R}^N} f_\eta(u)\psi(u)dx.
\end{aligned}
$$

Then, we can argue as in the proof of relation (52) and use (f_η) to conclude that

$$
\begin{aligned}
&|uu_L^{\beta-1}|_{q_s^*}^q \leqslant C_1\beta^q\left(1+\eta b^{r-\theta}\right)\int_{\mathbb{R}^N} u^\theta u_L^{q(\beta-1)}dx, \\
\Rightarrow\ &|uu_L^{\beta-1}|_{q_s^*}^q \leqslant C_1\beta^q\left(1+\eta b^{r-\theta}\right)|u|_{q_s^*}^{\theta-q}|uu_L^{\beta-1}|_{\kappa^*}^q \\
&\text{(using the Hölder inequality, where } \kappa^* = qq_s^*/(q_s^*-\theta+q)\text{)},
\end{aligned}
\tag{60}
$$

where C_1 is a positive constant and independent of ε and η.

Now, we can deduce from Lemma 25, Theorem 3 and relation (60) that there exist $C_2 > 0$, independent of ε and η, such that

$$
|uu_L^{\beta-1}|_{q_s^*}^q \leqslant C_2\beta^q\left(1+\eta b^{r-\theta}\right)|uu_L^{\beta-1}|_{\kappa^*}^q. \tag{61}
$$

If $u^\beta \in L^{\kappa^*}(\mathbb{R}^N)$ then, from the fact that $u_L \leqslant u$ and relation (61), it follows that

$$
|uu_L^{\beta-1}|_{q_s^*}^q \leqslant C_2\beta^q\left(1+\eta b^{r-\theta}\right)|u|_{\beta\kappa^*}^{q\beta} < +\infty \text{ (using the assumption } u^\beta \in L^{\kappa^*}(\mathbb{R}^N)\text{)}. \tag{62}
$$

In relation (62) we pass to the limit as $L \to +\infty$ and use the Fatou's Lemma to deduce that

$$
|u|_{q_s^*\beta} \leqslant (C_2+\eta C_2 b^{r-\theta})^{\frac{1}{q\beta}}\beta^{\frac{1}{\beta}}|u|_{\beta\kappa^*} \tag{63}
$$

for $u^{\beta\kappa^*} \in L^1(\mathbb{R}^N)$.

Set $\beta = \frac{q_s^*}{\kappa^*} > 1$. Due to $u \in L^{q_s^*}(\mathbb{R}^N)$, relation (63) holds true for this choice of β. Via a standard iterative step, for each $1 \leqslant m \in \mathbb{N}$, we deduce that

$$
\begin{aligned}
&|u|_{q_s^*\beta^m} \leqslant (C_2+\eta C_2 b^{r-\theta})^{\sum_{k=1}^m q^{-1}\beta^{-k}}\beta^{\sum_{k=1}^m k\beta^{-k}}|u|_{q_s^*}, \\
\Rightarrow\ &|u|_{L^\infty(\mathbb{R}^N)} \leqslant (C_2+\eta C_2 b^{r-\theta})^{\sum_{k=1}^\infty q^{-1}\beta^{-k}}\beta^{\sum_{k=1}^\infty k\beta^{-k}}C_3 \\
&\text{(letting } m \to \infty \text{ and using Theorem 3, Lemma 25)},
\end{aligned}
\tag{64}
$$

where C_3 is a positive constant and independent of ε and η.

Take

$$
b := 2C_2^{\sum_{k=1}^\infty q^{-1}\beta^{-k}}\beta^{\sum_{k=1}^\infty k\beta^{-k}}C_3 \quad \text{and} \quad \eta_* := \left(2^{\frac{1}{\sum_{k=1}^\infty q^{-1}\beta^{-k}}-1} - \frac{1}{2}\right)b^{\theta-r}.
$$

Then, from (64) it follows that

$$|u|_{L^\infty(\mathbb{R}^N)} < b \text{ for any } \eta \in [0, \eta_*],$$

which implies that $u = u_{\varepsilon,\eta}$ is a solution of problem (S_η). This proof of Theorem 2 is now complete. □

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