Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 282 (2021) 481–540

Journal of Differential Equations

www.elsevier.com/locate/jde

Small linear perturbations of fractional Choquard equations with critical exponent

Xiaoming He [a], Vicenţiu D. Rădulescu [b,c,d,*]

[a] School of Science, Minzu University of China Beijing 100081, China

[b] Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

[c] Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

[d] Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Received 21 August 2020; revised 29 January 2021; accepted 10 February 2021

Abstract

We are concerned with the qualitative analysis of positive solutions to the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + a(x)u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & x \in \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N), \quad u(x) > 0, & x \in \mathbb{R}^N, \end{cases}$$

where $I_\alpha(x)$ is the Riesz potential, $s \in (0,1)$, $N > 2s$, $0 < \alpha < \min\{N, 4s\}$, and $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the fractional critical Hardy-Littlewood-Sobolev exponent. We first establish a nonlocal global compactness property in the framework of fractional Choquard equations. In the second part of this paper, we prove that the equation has at least one positive solution in the case of small perturbations of the potential that describes the linear term.

MSC: 35J20; 35A15; 35B33; 81Q05

Keywords: Fractional Choquard equation; Global compactness; Critical exponent; Concentration-compactness

* Corresponding author.

E-mail addresses: xmhe923@muc.edu.cn (X. He), radulescu@inf.ucv.ro (V.D. Rădulescu).

https://doi.org/10.1016/j.jde.2021.02.017

0022-0396/

1. Introduction

The Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3, \tag{1.1}$$

was first introduced in the pioneering work of Fröhlich [18] and Pekar [40] for the modeling of quantum polaron. This model corresponds to the study of free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarization that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (1.1) to describe an electron trapped in its own hole.

The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [17], Giulini and Großardt [22], Jones [24], and Schunck and Mielke [46]. Penrose [41,42] proposed equation (1.1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon. As pointed out by Lieb [28], Choquard used equation (1.1) to study steady states of the one component plasma approximation in the Hartree-Fock theory. Classification of solutions of (1.1) was first studied by Ma and Zhao [32]. For a broad survey of Choquard equations we refer to Moroz and van Schaftingen [37] and references therein. We also refer to Cassani and Zhang [9], Mingqi, Rădulescu and Zhang [34] and Seok [47] as recent relevant contributions to the study of Choquard-type problems. For more results on classical Choquard equations, we refer to [1,8,11,12,21,29,30,32,36] and the references therein.

In recent years, the fractional Laplace operator has attracted much attention. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, we refer to [4,20,26,27,43,44] and the monograph by Molica Bisci, Rădulescu and Servadei [35].

We note that the study of the existence, concentration and multiplicity of positive solutions for the fractional Choquard equations is not much in the literature. In [15,16], d'Avenia, Siciliano and Squassina studied the regularity, existence, nonexistence, symmetry as well as decays properties of weak solutions for the following fractional subcritical Choquard equation

$$(-\Delta)^s u + wu = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $w > 0$, a constant, $s \in (0,1), N > 2s, 0 < \alpha < \min\{N, 4s\}$, $\frac{2N-\alpha}{N} < p < 2^*_{\alpha,s}$, and $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the fractional Hardy-Littlewood-Sobolev critical exponent, $(-\Delta)^s$ denotes the fractional Laplacian operator. Later, in [48], the authors studied the existence of groundstates for the fractional Choquard equation

$$(-\Delta)^s u + u = (|x|^{-\alpha} * F(u)) f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where the nonlinearity is *subcritical* and satisfies the general Berestycki-Lions type conditions.

Let us consider the fractional critical Choquard equation

$$\begin{cases} (-\Delta)^s u + a(x)u = \left(\int\limits_{\Omega} \frac{|u|^{2^*_{\alpha,s}}}{|x-y|^{\alpha}} dx \right) |u|^{2^*_{\alpha,s}-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^N$, $a(x)$ is a potential function. When $\Omega = \mathbb{R}^N$ and $a(x) \equiv 0$, we have known that problem (1.4) has the positive solution

$$\frac{C_{N,\alpha,s}}{(1+|x|^2)^{\frac{N-2s}{2}}},$$

where $C_{N,\alpha,s}$ is the normalization constant depending only on N, α and s, and all positive solutions can be obtained by this one by translations and scale changes, see [25,38]. However, if $\Omega \neq \mathbb{R}^N$ or $a(x) \not\equiv 0$, the situation is not so simple. For example, if Ω is a bounded $C^{1,1}$ domain and $a(x) = \lambda$, constant, as a consequence of the Pohozaev identity [38,45], it follows

$$-s\lambda \int\limits_{\Omega} u^2 dx = \frac{\Gamma(s+1)^2}{2} \int\limits_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma,$$

where $\Gamma(\cdot)$ is the Gamma function. If Ω is a bounded starshaped domain, i.e., $(x \cdot \nu) > 0$ on $\partial\Omega$, then (1.4) has no nontrivial solution when $\lambda \geq 0$, and has a nontrivial solution while $\lambda < -\bar{\lambda}$ for some $\bar{\lambda} > 0$, see [38] for more details. In the case $\Omega = \mathbb{R}^N$ and $a(x) = \lambda \neq 0$, constant, a generalized version of the Pohozaev identity [15,36] associated to (1.4) reads as

$$\frac{N-2s}{2} \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{N\lambda}{2} \int\limits_{\mathbb{R}^N} |u|^2 dx = \frac{2N-\alpha}{22^*_{\alpha,s}} \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)u(y)|^{2^*_{\alpha,s}}}{|x-y|^{\alpha}} dxdy,$$

which implies that

$$s\lambda \int\limits_{\mathbb{R}^N} |u|^2 dx = 0$$

so there is no solution with $\lambda \neq 0$. If $\Omega = \mathbb{R}^N$ and $a(x) \not\equiv \lambda$, $\lambda \neq 0$, how to get nontrivial solutions for (P) becomes more complicated.

After a bibliography review we have found only a paper related to (P) that is due to Ma and Zhang [33]. In that paper Ma and Zhang considered the case $a(x) = \lambda V(x) - \beta$, and established the existence and multiplicity of weak solutions by variational methods if $\beta > 0$ is appropriately small and, $\gamma > 0$ large, and $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative function such that $\mathrm{int} V^{-1}(0)$ is a nonempty bounded set with smooth boundary. We notice that the energy of solution obtained in [33] (also in [25,38]) is *less than* the threshold value

$$c_{h,l} := \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

At this point a question arises naturally: If $\Omega = \mathbb{R}^N$ and $a(x) \not\equiv \lambda$, $\lambda \neq 0$, a constant, can we get a nontrivial solution of (1.4) with energy higher than $c_{h,l}$?

In this paper, we stress our attention to this case, and give an affirmative answer. We consider the following fractional Choquard equation involving pure critical nonlinearity

$$\begin{cases} (-\Delta)^s u + a(x)u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & x \in \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N), \quad u(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{P}$$

where $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^\alpha}, \quad \text{and} \quad A_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N-\alpha}{2})\pi^{N/2}2^{N-\alpha}},$$

and $s \in (0, 1)$, $N > 2s$, $0 < \alpha < \min\{N, 4s\}$ and $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$. The fractional Laplacian $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$(-\Delta)^s u(x) := C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where P.V. is a commonly used abbreviation for "in the principal value sense", and $C_{N,s}$ denotes the normalization constant. The work space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_s}(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy,$$

where $2^*_s = 2N/(N - 2s)$ is the fractional critical Sobolev constant. According to Propositions 3.4 and 3.6 of [39], we have that,

$$\|u\|^2 = |(-\Delta)^{\frac{s}{2}} u|_2^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy, \tag{1.5}$$

by omitting the normalization $C_{N,s}$. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces we refer the interested reader to [35], and to [5,6,23,35,52,53] and references therein for more results on fractional critical elliptic problems.

Before enunciating our theorem, we make the following assumptions on the function $a : \mathbb{R}^N \to \mathbb{R}^+$:

(a_1) The function a is positive on a set of positive measure.

(a_2) $a \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N-\alpha}{4s-\alpha} < p_2$ with $p_2 < \frac{N}{4s-N}$ if $2s < N < 4s$.

(a_3) The $L^{\frac{N}{2s}}$-norm of a satisfies the following inequality

$$|a|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} < \left(2^{\frac{4s-\alpha}{2N-\alpha}} - 1\right) S^{\frac{(2s-N)[(N-\alpha)(1-s)+2s]+(2N-\alpha)2s}{2s(N-\alpha+2s)}},$$

where

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}} \tag{1.6}$$

is the fractional best Sobolev exponent.

We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of problem (P), if $u \in D^{s,2}(\mathbb{R}^N)$ and for all $\varphi \in D^{s,2}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^N} a(x) u \varphi dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u \varphi dx.$$

In order to state the main result, we consider the C^1 class functional $I : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (P) given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} a(x) u^2 dx - \frac{1}{2 2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}} dx,$$

and for any $v \in D^{s,2}(\mathbb{R}^N)$,

$$I'(u)v = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^N} a(x) u v dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u v dx.$$

Using the above notations we are able to state our main result.

Theorem 1.1. *Assume that $(a_1) - (a_3)$ are satisfied, then problem (P) has a positive solution $u_0 \in D^{s,2}(\mathbb{R}^N)$ such that*

$$\frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} < I(u_0) < 2\frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}. \tag{1.7}$$

The proof of Theorem 1.1 is inspired from the seminal paper due to Benci and Cerami [7], where the authors investigated the existence of positive solution for the following classical Schrödinger equation with critical exponent

$$-\Delta u + a(x) u = |u|^{2^*-2} u, \quad x \in \mathbb{R}^N. \tag{1.8}$$

The authors used the properties of the solutions of the limit equation $-\Delta u = |u|^{2^*-2}u$ in combination with the variational methods and arguments of degree theory, to obtain positive solution to (1.8). However, we remark that in the proof of Theorem 1.1, the ideas of Benci and Cerami [7] cannot be immediately applicable to our situation due to the double nonlocal framework caused by the fractional Laplacian operator $(-\Delta)^s$ and the convolution term $I_\alpha * |u|^{2^*_{\alpha,s}}$. Some refined estimates for this problem are very necessary and delicate. For instance, see Lemmas 3.1, 4.11 and Theorem 3.2 below. After the publication of [7], some authors studied related problem to (1.8), see [2,3,10,13] and references therein.

We note that, when $s = 1$, the classical Laplacian case, Alves and Figueiredo [3] considered problem (P) and established a nonlocal version of global compactness lemma by applying a Cherrier type inequality (see Lemma 2.2 [3]). However, in the nonlocal case, $0 < s < 1$, the fractional type Cherrier inequality is not clear and not available in the literature, and so we can not make use of such kind of inequality to treat problem (P). Simultaneously, the nonlocality caused by the nonlocal operator $(-\Delta)^s$, makes the arguments explored in [3] unable to be carried out straightforwardly in our case because some estimates become more subtle to be established.

Another novelty is that, different from [25,33,38], where the energy of the solution is less than the level $c_{h,l}$, we show the existence of high energy solution of (P) satisfying (1.7). To this aim, we shall establish the version to $\mathbb{R}^N$ of Struwe's global compactness result [50] for problem (P) to study the behavior of Palais-Smale sequences, which may be useful also in other different context and has never appeared in the literature, to the best of our knowledge.

The paper is organized as follows. In Section 2 we give some preliminary results involving the limit problem. In Section 3, we establish a splitting theorem and show some compactness results involving the energy functional associated to (P). In Section 4, we prove some technical lemmas and the proof of Theorem 1.1 is completed in Section 5. In the Appendix, we give a preliminary lemma which is useful in proving the convergence of integral with nonlocal term.

Notation. Throughout this paper, $|\cdot|_q$ denotes for the norm in $L^q(\mathbb{R}^N)$; $B_r(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius r; The letters $C, C_i, i = 1, 2, \cdots$, denote various positive constants whose exact values are irrelevant, and $u^\pm = \max\{\pm u, 0\}$.

2. Limit problem

In this section we give some preliminary results involving the limit problem that will be useful in our approach. To begin with, we recall that the key point to apply variational method for problem (P) is the following Hardy-Littlewood-Sobolev inequality.

Proposition 2.1. *([29]) Let $t, r > 1$ and $0 < \alpha < n$ with $1/t + \alpha/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \alpha, r)$ independent of f, h such that*

$$\iint_{\mathbb{R}^{2n}} \frac{f(x)h(y)}{|x-y|^\alpha} dxdy \le C(t, n, \alpha, r)|f|_t |h|_r. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{\pi}{2} - \frac{\alpha}{2})}{\Gamma(N - \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)} \right\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (2.1) *if and only if* $f \equiv (constant)h$ *and*

$$h(x) = A(\gamma^2 + |x-a|^2)^{-\frac{2N-\alpha}{2}}$$

for some $A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$.

In general, let $f = h = |u|^q$, then by Hardy-Littlewood-Sobolev inequality we have,

$$\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)u(y)|^q}{|x-y|^\alpha} dxdy$$

is well defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\alpha}{N} = 2.$$

Therefore, for $u \in D^{s,2}(\mathbb{R}^N)$, by Sobolev embedding theorems, we must have

$$\frac{2N-\alpha}{N} \leq q \leq \frac{2N-\alpha}{N-2s}.$$

From this, for $u \in D^{s,2}(\mathbb{R}^N)$, we get

$$\left(\int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u(x)|^{2^*_{\alpha,s}} dx \right)^{\frac{1}{2^*_{\alpha,s}}} \leq [A_\alpha C(N,\alpha)]^{\frac{1}{2^*_{\alpha,s}}} |u|^2_{2^*_s} \triangleq \mathcal{D}_{N,\alpha}^{\frac{1}{2^*_{\alpha,s}}} |u|^2_{2^*_s}. \tag{2.2}$$

Let $S_{h,l}$ be the best constant

$$S_{h,l} = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} dx\right)^{\frac{1}{2^*_{\alpha,s}}}}, \tag{2.3}$$

and S be the best Sobolev constant

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*_s} dx\right)^{\frac{2}{2^*_s}}}, \tag{2.4}$$

where $2^*_s = 2N/(N-2s)$ is the fractional Sobolev critical exponent. It is well-known that $S_{h,l}$ and S are both achieved if and only if u is of the form

$$C\left(\frac{\varepsilon}{\varepsilon^2 + |x-x_0|^2}\right)^{\frac{N-2s}{2}}, \qquad x \in \mathbb{R}^N$$

for some $x_0 \in \mathbb{R}^N$, $C > 0$ and $\varepsilon > 0$ (see Theorem 2.15 of [16]). Moreover,

$$S_{h,l} = \frac{S}{\mathcal{D}_{N,\alpha}^{\frac{1}{2^*_{\alpha,s}}}}. \tag{2.5}$$

In particular, let

$$U(x) = \frac{\beta_1}{(1+|x|^2)^{(N-2s)/2}}, \quad \text{where} \quad \beta_1 = \left(\frac{S^{N/(2s)}\Gamma(N)}{\pi^{N/2}\Gamma(N/2)}\right)^{\frac{N-2s}{2N}}.$$

It follows from [14,52] that

$$S|U|^2_{2^*_s} = \|U\|^2, \tag{2.6}$$

and moreover, we have

$$\|U\|^2 = |U|^{2^*_s}_{2^*_s} = \int\limits_{\mathbb{R}^N} \frac{\beta_1^{2^*_s}}{(1+|x|^2)^N} dx = \beta_1^{\frac{2N}{N-2s}} \frac{\pi^{N/2}\Gamma(N/2)}{\Gamma(N)} = S^{\frac{N}{2s}}.$$

Put

$$\widetilde{U}(x) = S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} U(x),$$

then $\widetilde{U}(x)$ is the unique minimizer for $S_{h,l}$ and satisfies

$$(-\Delta)^s u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{2.7}$$

and

$$\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{U}|^2 dx = \int\limits_{\mathbb{R}^N} (I_\alpha * |\widetilde{U}|^{2^*_{\alpha,s}})|\widetilde{U}|^{2^*_{\alpha,s}} dx = S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}}. \tag{2.8}$$

Moreover, by the invariance of the scaling, we see that the functions

$$\begin{aligned}
\widetilde{U}_{\varepsilon,b}(x) = \varepsilon^{\frac{2s-N}{2}} \widetilde{U}\left(\frac{x-b}{\varepsilon}\right) &= S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} \varepsilon^{\frac{2s-N}{2}} U\left(\frac{x-b}{\varepsilon}\right) \\
&= S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} \frac{\beta_1 \varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x-b|^2)^{\frac{N-2s}{2}}} \\
&:= S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} U_{\varepsilon,b}(x) \\
&= \frac{b_{\alpha,s}\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x-b|^2)^{\frac{N-2s}{2}}}, \quad \forall\, \varepsilon > 0,\ \forall\, b \in \mathbb{R}^N,
\end{aligned} \tag{2.9}$$

solve equation (2.7) and satisfy (2.8), where

$$b_{\alpha,s} = S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} \beta_1. \tag{2.10}$$

Let us introduce the limit problem associated to problem (P),

$$\begin{cases} (-\Delta)^s u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & x \in \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N), \ \ u(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{P_∞}$$

whose the functional associated $I_\infty : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} dx.$$

Lemma 2.2. *Let (u_n) be a $(PS)_c$ sequence for I_∞. Then*

(i) *The sequence (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$.*
(ii) *If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^N)$, then $I'_\infty(u) = 0$.*
(iii) *If $c \in (-\infty, \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{(2N-\alpha)/N+2s-\alpha})$, then I_∞ satisfies the $(PS)_c$ condition, that is, up to subsequence, $u_n \to u$ in $D^{s,2}(\mathbb{R}^N)$.*

Proof. By hypothesis $I_\infty(u_n) \to c$ and $I'_\infty(u_n) \to 0$, then there exists $K > 0$ such that

$$K + \|u_n\| \geq I_\infty(u_n) - \frac{1}{22^*_{\alpha,s}} I'_\infty(u_n)u_n = \frac{N+2s-\alpha}{4N-2\alpha}\|u_n\|^2, \quad \forall n \in \mathbb{N},$$

proving (i). Since $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^N)$, up to a subsequence, we get

$$u_n \to L^q_{loc}(\mathbb{R}^N), \ \ q \in (2, 2^*_s)$$

and

$$u_n(x) \to u \ \text{ a.e. in } \ \mathbb{R}^N.$$

By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear map from $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. Thus

$$I_\alpha * |u_n|^{2^*_{\alpha,s}} \rightharpoonup I_\alpha * |u|^{2^*_{\alpha,s}} \ \text{ in } \ L^{\frac{2N}{\alpha}}(\mathbb{R}^N),$$

and for each $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$|u_n|^{2^*_{\alpha,s}-2}u_n\phi \to |u|^{2^*_{\alpha,s}-2}u\phi \ \text{ in } \ L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N).$$

The above limit implies that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n\phi dx \to \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\phi dx,$$

and so, $I'_\infty(u)\phi = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^N)$. Now (ii) follows from the density of $C_0^\infty(\mathbb{R}^N)$ in $D^{s,2}(\mathbb{R}^N)$.

In order to prove (iii), we consider $v_n = u_n - u$ and employing Brezis-Lieb lemma and Lemma 2.2 in [19] we get

$$\begin{aligned} o_n(1) = I'_\infty(u_n)u_n &= \|u_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} dx \\ &= \|v_n\|^2 + \|u\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} dx + o_n(1) \\ &= \|v_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx + o_n(1). \end{aligned} \tag{2.11}$$

Hence, up to a subsequence, we conclude

$$0 \le l = \lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx.$$

Suppose by contradiction that, $l > 0$. From the inequality

$$S_{h,l} \left[\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx \right]^{\frac{1}{2^*_{\alpha,s}}} \le \|v_n\|^2,$$

we obtain

$$l \ge [S_{\hbar,\ell}] l^{\frac{1}{2^*_{\alpha,s}}} \Rightarrow l \ge S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}. \tag{2.12}$$

In view of

$$I_\infty(u) = \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right)\|u\|^2 = \frac{N+2s-\alpha}{4N-2\alpha}\|u\|^2 \ge 0,$$

and

$$c = \frac{N+2s-\alpha}{4N-2\alpha}\|v_n\|^2 + I_\infty(u) + o_n(1), \tag{2.13}$$

we derive that

$$\begin{aligned} c &= \frac{N+2s-\alpha}{4N-2\alpha}\|v_n\|^2 + I_\infty(u) + o_n(1) \ge \frac{N+2s-\alpha}{4N-2\alpha}\|v_n\|^2 + o_n(1) \\ &= \frac{N+2s-\alpha}{4N-2\alpha} l \ge \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}, \end{aligned}$$

which is a contradiction. Hence $l = 0$ and

$$\|v_n\|^2 = \|u_n - u\|^2 \to 0.$$

The proof is now complete. □

Lemma 2.3. *If $u \in D^{s,2}(\mathbb{R}^N)$ is a nodal solution of $(P)_\infty$, then*

$$I_\infty(u) \geq 2^{\frac{4s-\alpha}{N+2s-\alpha}} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Proof. Arguing as in the proof of Proposition 3.2 [21], for all $t^+, t^- > 0$, we have

$$I_\infty(t^+u^+)^{\frac{N+2s-\alpha}{4s-\alpha}} + I_\infty(t^-u^-)^{\frac{N+2s-\alpha}{4s-\alpha}} \leq I_\infty(u)^{\frac{N+2s-\alpha}{4s-\alpha}}.$$

Fixing $t^+, t^- > 0$ such that $I'_\infty(t^\pm u^\pm)(t^\pm u^\pm) = 0$, it follows that

$$I_\infty(t^+u^+) \geq \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

The last two inequalities imply that

$$I_\infty(u) \geq 2^{\frac{4s-\alpha}{N+2s-\alpha}} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},$$

completing the proof. □

3. A nonlocal global compactness lemma

We start this section by establishing the following technical lemma for I_∞ which will be useful to prove our compactness theorem.

Lemma 3.1. *(Nonlocal global compactness property) Let (u_n) be a $(PS)_c$ sequence for the functional I_∞ with $u_n \rightharpoonup 0$ and $u_n \nrightarrow 0$. Then there exist a sequence $(R_n) \subset \mathbb{R}^+$ and a nontrivial solution $v_0 \in D^{s,2}(\mathbb{R}^N)$ of (P_∞) such that, up to a subsequence of (u_n), we have that*

$$w_n(x) = u_n(x) - R^{(N-2s)/2} v_0(R_n(x - x_n)) + o_n(1),$$

is a $(PS)_{c-I_\infty(v_0)}$ sequence for I_∞.

Proof. Let $(u_n) \subset D^{s,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I_∞, i.e.,

$$I_\infty(u_n) \to c \quad \text{and} \quad I'_\infty(u_n) \to 0. \tag{3.1}$$

From Lemma 2.2-(i) we know that (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, it follows from Lemma 2.2-(iii) that

$$c \geq \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Notice that

$$c + o_n(1) = I_\infty(u_n) - \frac{1}{22^*_{\alpha,s}} I'_\infty(u_n)u_n = \frac{N+2s-\alpha}{4N-2\alpha}\|u_n\|^2,$$

which implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \geq S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}. \tag{3.2}$$

Let L be a number such that $B_2(0)$ is covered by L balls of radius 1, $(R_n) \subset \mathbb{R}^+$, $(x_n) \subset \mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N} \int_{B_{R_n^{-1}}(y)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{B_{R_n^{-1}}(x_n)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \frac{S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}}{2L}$$

and

$$v_n(x) = R_n^{(2s-N)/2} u_n\left(\frac{x}{R_n} + x_n\right).$$

Using a change of variable, we can show that

$$\int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx = \frac{S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}}{2L} = \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx.$$

Now for each $\Phi \in D^{s,2}(\mathbb{R}^N)$, we define the function

$$\widetilde{\Phi}_n(x) := R_n^{(N-2s)/2} \Phi(R_n(x - x_n))$$

which satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \widetilde{\Phi}_n dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Phi dx, \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2} u_n \widetilde{\Phi}_n dx = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2} v_n \Phi dx. \tag{3.4}$$

These limits yield that

$$I_\infty(v_n) \to c \quad \text{and} \quad I'_\infty(v_n) \to 0.$$

From Lemma 2.2, there exists $v_0 \in D^{s,2}(\mathbb{R}^N)$ such that, up to a subsequence, $v_n \rightharpoonup v_0$ in $D^{s,2}(\mathbb{R}^N)$ and $I'_\infty(v_0) = 0$.

As a consequence of Lions's lemma [31], we have

$$\int_{\mathbb{R}^N} |v_n|^{2_s^*} \phi dx \to \int_{\mathbb{R}^N} |v_0|^{2_s^*} \phi dx + \sum_{j\in J} \phi(x_j)\nu_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N) \tag{3.5}$$

and

$$|(-\Delta)^{\frac{s}{2}} v_n|^2 \rightharpoonup \mu \geq |(-\Delta)^{\frac{s}{2}} v_0|^2 + \sum_{j\in J} \phi(x_j)\mu_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N),$$

for some $\{x_j\} \subset \mathbb{R}^N$ and for some $\{\nu_j\}_{j\in J}, \{\mu_j\}_{j\in J} \subset \mathbb{R}^+$ with $S\nu_j^{\frac{2}{2_s^*}} \leq \mu_j$, where J is at most a countable set. In fact, J is finite. To see this, consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$, for all $x \in \mathbb{R}^N$, $\phi(x) = 0$ for all $x \in B_2^c(0)$ and $\phi(x) = 1$ for all $x \in B_1(0)$. Now fix $x_j \in \mathbb{R}^N$, $j \in J$ and define $\psi_\rho(x) = \phi(\frac{x-x_j}{\rho})$, for each $\rho > 0$. Then $0 \leq \psi_\rho(x) \leq 1$, for all $x \in \mathbb{R}^N$, $\psi_\rho(x) = 0$ for all $x \in B_{2\rho}^c(x_j)$ and $\psi_\rho(x) = 1$ for all $x \in B_\rho(x_j)$. We have that $(v_n\psi_\rho)$ is bounded in $D^{s,2}(\mathbb{R}^N)$ and $I'_\infty(v_n)v_n\psi_\rho = o_n(1)$. Moreover, we have

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} (v_n\psi_\rho) dx = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2_{\alpha,s}^*})|v_n|^{2_{\alpha,s}^*} \psi_\rho dx + o_n(1). \tag{3.6}$$

Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} (v_n\psi_\rho) dx \\
&= \iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))^2\psi_\rho(y)}{|x-y|^{N+2s}} dxdy + \iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(\psi_\rho(x)-\psi_\rho(y))v_n(x)}{|x-y|^{N+2s}} dxdy.
\end{aligned} \tag{3.7}$$

It is easy to verify that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))^2\psi_\rho(y)}{|x-y|^{N+2s}} dxdy \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \psi_\rho(y) dy \to \int_{\mathbb{R}^N} \psi_\rho(y) d\mu \quad \text{as} \quad n \to +\infty
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} \psi_\rho(y)d\mu \to \mu(\{x_j\}) = \mu_j \quad \text{as} \quad \rho \to 0. \tag{3.8}$$

Moreover, using Hölder's inequality we obtain that

$$\begin{aligned} &\left| \iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(\psi_\rho(x)-\psi_\rho(y))v_n(x)}{|x-y|^{N+2s}} dxdy \right| \\ &\le \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|\cdot|\psi_\rho(x)-\psi_\rho(y)|\cdot|v_n(x)|}{|x-y|^{N+2s}} dxdy \\ &\le C\left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\rho(x)-\psi_\rho(y)|^2|v_n(x)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

By Lemma 3.6 in [53], we derive

$$\lim_{\rho\to 0}\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|\psi_\rho(x)-\psi_\rho(y)|^2|v_n(x)|^2}{|x-y|^{N+2s}} dxdy = 0. \tag{3.10}$$

On the other hand, using Proposition 2.1, we have, as $n \to +\infty$,

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\psi_\rho dx &\le C|v_n|_{2^*_s}^{2^*_{\alpha,s}} \left(\int_{\mathbb{R}^N} |v_n|^{2^*_s}\psi_\rho^{\frac{2^*_s}{2^*_{\alpha,s}}} dx \right)^{\frac{2N-\alpha}{2N}} \\ &\le C\left(\int_{\mathbb{R}^N} |v_n|^{2^*_s}\psi_\rho dx \right)^{\frac{2N-\alpha}{2N}} \to C\left(\int_{\mathbb{R}^N} \psi_\rho(y)d\nu \right)^{\frac{2^*_{\alpha,s}}{2^*_s}} \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \psi_\rho(y)d\nu \to \nu(\{x_j\}) = \nu_j \quad \text{as} \quad \rho \to 0. \tag{3.11}$$

Combining (3.7)-(3.11), we obtain

$$S\nu_j^{\frac{2}{2^*_s}} \le \mu_j \le C\nu_j^{\frac{2^*_{\alpha,s}}{2^*_s}}.$$

As $2^*_{\alpha,s} > 2$ and $\sum_{j\in J} \nu_j^{2/2^*_S} < \infty$, we see that ν_j does not converge to zero, which means that J is finite. From now on, de denote by $J = \{1, 2, \cdots, m\}$ and $\Gamma \subset \mathbb{R}^N$ the set given by

$$\Gamma = \{x_j \in \{x_j\}_{j\in J} : \ |x_j| > 1\}, \quad (x_j \ \text{ given by } \ (3.5)).$$

In the sequel, we are going to show that $v_0 \neq 0$. Suppose, by contradiction, that $v_0 = 0$. Then, by (3.5) we get

$$\int\limits_{\mathbb{R}^N} |v_n|^{2_s^*} \phi dx \to 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^N \backslash \{x_1, x_2, \cdots, x_m\}). \tag{3.12}$$

Since the sequence $\phi_n(x) = \phi(x)v_n(x)$ with $\phi \in C_c^\infty(\mathbb{R}^N \backslash \{x_1, x_2, \cdots, x_m\})$, is bounded, we have

$$I'_\infty(v_n)\phi_n = o_n(1),$$

that is

$$\begin{aligned}
& o_n(1) \\
&= \iint\limits_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(\phi(x)v_n(x) - \phi(y)v_n(y))}{|x-y|^{N+2s}} dxdy - \int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2} v_n(\phi v_n) dx \\
&= \iint\limits_{\mathbb{R}^{2N}} v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y)}{|x-y|^{N+2s}} dxdy + \iint\limits_{\mathbb{R}^{2N}} \phi(y) \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} dxdy \\
&\quad - \int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \phi dx.
\end{aligned} \tag{3.13}$$

Then,

$$\begin{aligned}
& \left| \iint\limits_{\mathbb{R}^{2N}} \phi(y) \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} dxdy \right| \\
&= \left| \int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \phi dx - \iint\limits_{\mathbb{R}^{2N}} v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}} dxdy + o_n(1) \right| \\
&\le \left| \int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \phi dx \right| + \left| \iint\limits_{\mathbb{R}^{2N}} v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}} dxdy \right| + o_n(1) \\
&\le \left| \int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \phi dx \right| + \|v_n\| \left[\iint\limits_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dxdy \right]^{\frac{1}{2}} + o_n(1).
\end{aligned} \tag{3.14}$$

Now, we show that

$$\iint_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy = o_n(1). \tag{3.15}$$

For this, let $r>0$ be such that $\mathrm{supp}(\phi) \subset B_r(0)$, and write $\mathbb{R}^{2N}$ as

$$\begin{aligned}\mathbb{R}^{2N} &= [(\mathbb{R}^N \backslash B_r(0)) \times (\mathbb{R}^N \backslash B_r(0))] \cup [(B_r(0)) \times (\mathbb{R}^N \backslash B_r(0))] \cup [(\mathbb{R}^N \backslash B_r(0)) \times (B_r(0))] \\ &= \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.\end{aligned}$$

Then we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy \\ &= \iint_{\Lambda_1} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy + \iint_{\Lambda_2} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy \\ &\quad + \iint_{\Lambda_3} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy.\end{aligned} \tag{3.16}$$

We are going to estimate each integral in (3.16). Since $\phi = 0$ in $\mathbb{R}^N \backslash B_r(0)$, we have

$$\iint_{\Lambda_1} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy = 0. \tag{3.17}$$

Using $|\phi| \le C_1$, $|\nabla\phi| \le C_2$ and the mean value theorem, we infer that

$$\begin{aligned}&\iint_{\Lambda_2} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy \\ &= \int_{B_r(0)} |v_n(x)|^2 dx \int_{\{y\in\mathbb{R}^N : |x-y|\le r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\ &\quad + \int_{B_r(0)} |v_n(x)|^2 dx \int_{\{y\in\mathbb{R}^N : |x-y|> r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\ &\le C|\nabla\phi|^2_{L^\infty(\mathbb{R}^N)} \int_{B_r(0)} |v_n(x)|^2 dx \int_{\{y\in\mathbb{R}^N : |x-y|\le r\}} \frac{1}{|x-y|^{N+2s-2}} dy \\ &\quad + C \int_{B_r(0)} |v_n(x)|^2 dx \int_{\{y\in\mathbb{R}^N : |x-y|> r\}} \frac{1}{|x-y|^{N+2s}} dy\end{aligned}$$

$$= Cr^{2-2s} \int\limits_{B_r(0)} |v_n(x)|^2 dx + Cr^{-2s} \int\limits_{B_r(0)} |v_n(x)|^2 dx = o_n(1). \tag{3.18}$$

Regarding the integral on Λ_3, we have

$$\begin{aligned}
&\iint\limits_{\Lambda_3} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&= \int\limits_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 dx \int\limits_{\{y\in B_r(0):\ |x-y|\le r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\quad + \int\limits_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 dx \int\limits_{\{y\in B_r(0):\ |x-y|> r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&= A_r + B_r.
\end{aligned} \tag{3.19}$$

Note that if $(x, y) \in (\mathbb{R}^N \setminus B_r(0)) \times B_r(0)$ and $|x-y| \le r$, then $|x| \le 2r$, we obtain

$$\begin{aligned}
A_r &\le \int\limits_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 dx \int\limits_{\{y\in B_r(0):\ |x-y|\le r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\le C|\nabla\phi|^2_{L^\infty(\mathbb{R}^N)} \int\limits_{B_{2r}(0)} |v_n(x)|^2 dx \int\limits_{\{z\in B_r(0):\ |z|\le r\}} \frac{1}{|z|^{N+2s-2}} dy \\
&= Cr^{2-2s} \int\limits_{B_{2r}(0)} |v_n(x)|^2 dx = o_n(1).
\end{aligned} \tag{3.20}$$

We observe that, there exists $K > 4$ such that

$$\Lambda_3 = (\mathbb{R}^N \setminus B_r(0)) \times (B_r(0)) \subset [B_{Kr}(0) \times B_r(0)] \cup [(\mathbb{R}^N \setminus B_{Kr}(0)) \times B_r(0)].$$

Then, we have the following estimates:

$$\begin{aligned}
&\int\limits_{B_{Kr}(0)} |v_n(x)|^2 dx \int\limits_{\{y\in B_r(0):\ |x-y|> r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\le C \int\limits_{B_{Kr}(0)} |v_n(x)|^2 dx \int\limits_{\{z\in \mathbb{R}^N:\ |z|> r\}} \frac{1}{|z|^{N+2s}} dz \\
&= Cr^{-2s} \int\limits_{B_{Kr}(0)} |v_n(x)|^2 dx = o_n(1).
\end{aligned} \tag{3.21}$$

We note that if $(x, y) \in (\mathbb{R}^N \setminus B_{Kr}(0)) \times B_r(0)$, then $|x-y| \ge |x| - |y| \ge \frac{|x|}{2} + \frac{K}{2}r - r > \frac{|x|}{2}$, and by using Hölder's inequality, we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} |v_n(x)|^2 dx \int\limits_{\{y\in B_r(0):\ |x-y|>r\}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\le C \int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} dx \int\limits_{\{y\in B_r(0):\ |x-y|>r\}} \frac{|v_n(x)|^2}{|x-y|^{N+2s}} dy \\
&\le Cr^N \int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} \frac{|v_n(x)|^2}{|x|^{N+2s}} dx \\
&\le Cr^N \left(\int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} |v_n(x)|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \left(\int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} |x|^{-(N+2s)\frac{2_s^*}{2_s^*-2}} dx\right)^{\frac{2_s^*-2}{2_s^*}} \\
&\le CK^{-N} \left(\int\limits_{\mathbb{R}^N\setminus B_{Kr}(0)} |v_n(x)|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \\
&\le C_1 K^{-N}.
\end{aligned} \tag{3.22}
$$

From (3.21) and (3.22), we obtain

$$
B_r \le C_1 K^{-N} + o_n(1). \tag{3.23}
$$

Combining (3.14)-(3.20) and (3.23), we derive

$$
\begin{aligned}
&\limsup_{n\to+\infty} \iint\limits_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\qquad = \lim_{K\to\infty} \limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dxdy = 0.
\end{aligned} \tag{3.24}
$$

By Proposition 2.1 and (3.12), we have

$$
\int\limits_{\mathbb{R}^N} (I_\alpha * |v_n|^{2_{\alpha,s}^*}) |v_n|^{2_{\alpha,s}^*} \phi dx \le C |v_n|_{2_s^*}^{2_{\alpha,s}^*} \left(\int\limits_{\mathbb{R}^N} |v_n|^{2_s^*} \phi dx\right)^{\frac{2_{\alpha,s}^*}{2_s^*}} = o_n(1). \tag{3.25}
$$

From (3.13), (3.24) and (3.25), we can conclude that

$$
\iint\limits_{\mathbb{R}^{2N}} \phi(y) \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}} dxdy \to 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N\setminus\{x_1, x_2, \cdots, x_m\}), \tag{3.26}
$$

which leads to

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \phi dx = o_n(1). \tag{3.27}$$

Let $\rho \in \mathbb{R}$ be a number such that $0 < \rho < \min\{\mathrm{dist}(\Gamma, \overline{B}_1(0)), 1\}$. We claim that

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \to 0. \tag{3.28}$$

Indeed, let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \phi(x) \le 1$ and $\phi(x) = 1$ if $x \in B_{1+\rho}(0)$. Set $\widetilde{\phi} = \phi|_{\mathbb{R}^N\setminus\{x_1,\cdots,x_m\}}$, by (3.27) we get

$$\begin{aligned} 0 &\le \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \le \int_{B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \\ &= \int_{B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \widetilde{\phi} dx \le \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \widetilde{\phi} dx \to 0, \end{aligned}$$

which implies that (3.28) is true.

Let $\Psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \Psi(x) \le 1$, $|\nabla\Psi(x)| \le 2$ for all $x \in \mathbb{R}^N$ and

$$\Psi(x) = \begin{cases} 1, & x \in B_{1+\frac{\rho}{3}}(0), \\ 0, & x \in B^c_{1+\frac{2\rho}{3}}(0), \end{cases}$$

and consider the sequence (Ψ_n) given by $\Psi_n(x) = \Psi(x)v_n(x)$.

Now, using (3.15), one has

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx \\ &\le 2 \int_{\mathbb{R}^N\setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} \frac{v_n(y)^2 |\Psi(x) - \Psi(y)|^2}{|x-y|^{N+2s}} dxdy \\ &\quad + 2 \int_{\mathbb{R}^N\setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} \frac{\Psi(x)^2 |v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} dxdy \\ &= o_n(1) + 2 \int_{\mathbb{R}^N\setminus B_{1+\rho}(0)} \Psi(x)^2 |(-\Delta)^{\frac{s}{2}} v_n(x)|^2 dx \\ &= o_n(1). \end{aligned} \tag{3.29}$$

Similarly, we can obtain the following estimate.

$$
\begin{aligned}
&\int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}}\Psi_n|^2 dx \\
&\leq 2 \int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \int\limits_{\mathbb{R}^N} \frac{v_n(x)^2|\Psi(x)-\Psi(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\quad + 2 \int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \int\limits_{\mathbb{R}^N} \frac{\Psi(y)^2|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\leq 2 \int\limits_{B_{1+\rho}(0)} \int\limits_{\mathbb{R}^N} \frac{v_n(x)^2|\Psi(x)-\Psi(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\quad + 2 \int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \int\limits_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}} dxdy \\
&= 2 \int\limits_{B_{1+\rho}(0)} \int\limits_{\mathbb{R}^N} \frac{v_n(x)^2|\Psi(x)-\Psi(y)|^2}{|x-y|^{N+2s}} dxdy + 2 \int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \\
&= o_n(1),
\end{aligned}
\tag{3.30}
$$

where in the last equality we have used the estimates (3.15) and (3.28).

Since (v_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, we derive that

$$
\begin{aligned}
&\int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\Psi_n dx + \int\limits_{B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\Psi_n dx \\
&\quad - \int\limits_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\Psi dx - \int\limits_{B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\Psi dx \\
&= - \int\limits_{\mathbb{R}^N\setminus B_{1+\rho}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\Psi_n dx + \int\limits_{\mathbb{R}^N\setminus B_{1+\rho}(0)} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\Psi dx + o_n(1) \\
&= - \int\limits_{\mathbb{R}^N\setminus B_{1+\rho}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\Psi_n dx + o_n(1)
\end{aligned}
\tag{3.31}
$$

Notice that from Hölder's inequality, (3.29) and (3.30), we get

$$
\int\limits_{\mathbb{R}^N\setminus B_{1+\rho}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}\Psi_n dx \to 0 \quad \text{as } n\to\infty, \tag{3.32}
$$

and

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.33}$$

Moreover, by Proposition 2.1 and (3.12) we get

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \Psi dx = o_n(1). \tag{3.34}$$

From (3.31)-(3.34) we obtain

$$\int_{B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx - \int_{B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx = o_n(1). \tag{3.35}$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx &= \int_{\mathbb{R}^N\setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + \int_{B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx \\
&= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + \int_{B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + o_n(1) \\
&= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx \\
&= \int_{B_{1+\rho}(0)} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx \\
&= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx + \int_{B_{1+\frac{\rho}{3}}(0)} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx.
\end{aligned} \tag{3.37}$$

From (3.34), (3.36)-(3.37), we derive that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx - \int_{\mathbb{R}^N} (I_\alpha * |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx = o_n(1).$$

Recall the definition of $S_{h,l}$, we have

$$\|\Psi_n\|^2\left[1-\frac{1}{S_{h,l}^{2^*_{\alpha,s}}}\|\Psi_n\|^{22^*_{\alpha,s}-2}\right]=\|\Psi_n\|^2-\frac{1}{S_{h,l}^{2^*_{\alpha,s}}}\|\Psi_n\|^{22^*_{\alpha,s}}$$
$$\leq\|\Psi_n\|^2-\int_{\mathbb{R}^N}(I_\alpha*|\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}}dx=o_n(1). \tag{3.38}$$

Since

$$\begin{aligned}\|\Psi_n\|^2&=\int_{B^c_{1+\rho}(0)}|(-\Delta)^{\frac{s}{2}}\Psi_n|^2dx+\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{\frac{s}{2}}\Psi_n|^2dx+\int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{\frac{s}{2}}\Psi_n|^2dx\\&=o_n(1)+\int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{\frac{s}{2}}\Psi_n|^2dx\\&\leq o_n(1)+\int_{B_2}|(-\Delta)^{\frac{s}{2}}v_n|^2dx,\end{aligned}$$

where in the last inequality we have used $\Psi_n=v_n$ in $B_{1+\frac{\rho}{3}(0)}$ and that $B_{1+\frac{\rho}{3}(0)}\subset B_2(0)$. Therefore, we arrive at

$$\begin{aligned}\|\Psi_n\|^2&\leq o_n(1)+\int_{\cup_{k=1}^L B_1(y_k)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx\\&\leq o_n(1)+\sum_{k=1}^L\int_{B_1(y_k)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx\\&\leq o_n(1)+L\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx\leq o_n(1)+\frac{S_{h,l}^{(2N-\alpha)/(N+2s-\alpha)}}{2},\end{aligned}$$

which gives that

$$\|\Psi_n\|\leq o_n(1)+\frac{S_{h,l}^{(2N-\alpha)/2(N+2s-\alpha)}}{2^{1/2}}\Rightarrow\|\Psi_n\|^{22^*_{\alpha,s}-2}\leq o_n(1)+\left(\frac{S_{h,l}^{(2N-\alpha)/2(N+2s-\alpha)}}{2^{1/2}}\right)^{22^*_{\alpha,s}-2}$$
$$\Rightarrow o_n(1)-\left(\frac{S_{h,l}^{(2N-\alpha)/2(N+2s-\alpha)}}{2^{1/2}}\right)^{22^*_{\alpha,s}-2}\leq-\|\Psi_n\|^{22^*_{\alpha,s}-2}. \tag{3.39}$$

It follows from (3.38), (3.39) that

$$
\begin{aligned}
&\|\Psi_n\|^2\left[1+o_n(1)-\frac{1}{S_{h,l}^{2^*_{\alpha,s}}}\left(\frac{S_{h,l}^{(2N-\alpha)/2(N+2s-\alpha)}}{2^{1/2}}\right)^{22^*_{\alpha,s}-2}\right] \\
&=\|\Psi_n\|^2\left[1+\frac{1}{S_{h,l}^{2^*_{\alpha,s}}}\left[o_n(1)-\left(\frac{S_{h,l}^{(2N-\alpha)/2(N+2s-\alpha)}}{2^{1/2}}\right)^{22^*_{\alpha,s}-2}\right]\right] \qquad (3.40)\\
&\le \|\Psi_n\|^2\left[1-\frac{1}{S_{h,l}^{2^*_{\alpha,s}}}\|\Psi_n\|^{22^*_{\alpha,s}-2}\right]\le o_n(1).
\end{aligned}
$$

Let us observe that

$$
\begin{aligned}
&\frac{2N-\alpha}{2(N+2s-\alpha)}(22^*_{\alpha,s}-2)-2^*_{\alpha,s} \\
&=\frac{2N-\alpha}{2(N+2s-\alpha)}\times\left[\frac{2(2N-\alpha)}{N-2s}-2\right]-\frac{2N-\alpha}{N-2s} \\
&=0,
\end{aligned}
$$

implies

$$
\|\Psi_n\|^2\left[1-\left(\frac{1}{2}\right)^{2^*_{\alpha,s}-1}\right]\le o_n(1),
$$

where we conclude that $\Psi_n\to 0$ in $D^{s,2}(\mathbb{R}^N)$. Since $v_n=\Psi_n$ in $B_1(0)$, we infer that

$$
0\le\int_{B_1(0)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx=\int_{B_1(0)}|(-\Delta)^{\frac{s}{2}}\Psi_n|^2dx\le\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v_n|^2dx,
$$

which yields to

$$
\int_{B_1(0)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx\to 0 \quad\text{as}\quad n\to\infty.
$$

But this last convergence contradicts to

$$
\int_{B_1(0)}|(-\Delta)^{\frac{s}{2}}v_n|^2dx=\frac{S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}}{2L},\quad \forall n\in\mathbb{N}.
$$

Hence, $v_0\neq 0$.

Now, we are going to show that there is (w_n) in $D^{s,2}(\mathbb{R}^N)$ such that (w_n) is a $(PS)_{c-I_{v_0}}$ sequence for I_∞ satisfying

$$
w_n(x)=u_n(x)-R_n^{(N-2s)/2}v_0(R_n(x-x_n))+o_n(1),
$$

for some subsequence of (u_n) that still denoted by (u_n). For this, we consider $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$ for all $x \in \mathbb{R}^N$ and

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_1(0), \\ 0, & \text{if } x \in B_2^c(0), \end{cases}$$

and define a sequence by

$$w_n(x) = u_n(x) - R_n^{(N-2s)/2} v_0(R_n(x - x_n))\psi(\overline{R}_n(x - x_n)), \tag{3.41}$$

where $(\overline{R}_n)$ satisfies $\widetilde{R}_n = \frac{R_n}{\overline{R}_n} \to \infty$. From (3.41) we get

$$R_n^{(2s-N)/2} w_n(x) = R_n^{(2s-N)/2} u_n(x) - v_0(R_n(x - x_n))\psi(\overline{R}_n(x - x_n)).$$

Making change of variable, we get

$$R_n^{(2s-N)/2} w_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(2s-N)/2} u_n\left(\frac{z}{R_n} + x_n\right) - v_0(z)\psi\left(\frac{z}{\widetilde{R}_n}\right).$$

Denote by

$$\widetilde{w}_n(z) = R_n^{(2s-N)/2} w_n\left(\frac{z}{R_n} + x_n\right)$$

and since

$$v_n(x) = R_n^{(2s-N)/2} u_n\left(\frac{z}{R_n} + x_n\right),$$

we have,

$$\widetilde{w}_n(z) = v_n(z) - v_0(z)\psi\left(\frac{z}{\widetilde{R}_n}\right). \tag{3.42}$$

Set

$$\psi_n(z) := \psi\left(\frac{z}{\widetilde{R}_n}\right), \tag{3.43}$$

we see that

$$\psi_n(z) = \begin{cases} 1, & \text{if } z \in B_{\widetilde{R}_n}(0), \\ 0, & \text{if } z \in B^c_{2\widetilde{R}_n}(0). \end{cases}$$

By (3.42) and (3.43) we derive that

$$\widetilde{w}_n(z) = v_n(z) - v_0(z)\psi_n(z).$$

The proof is over if we can show that $v_0\psi_n \to v_0$ in $D^{s,2}(\mathbb{R}^N)$ and so (w_n) is a $(PS)_{c-I_\infty(v_0)}$ sequence for I_∞. To this aim, we note that

$$\begin{aligned}
&\|v_0\psi_n - v_0\|^2 \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(v_0\psi_n - v_0)|^2 dx \\
&= \iint_{\mathbb{R}^{2N}} \frac{|v_0(x)\psi_n(x) - v_0(x) - v_0(y)\psi_n(y) + v_0(y)|^2}{|x-y|^{N+2s}} dxdy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|v_0(x)[\psi_n(x) - \psi_n(y)] + [\psi_n(y) - 1][v_0(x) - v_0(y)]|^2}{|x-y|^{N+2s}} dxdy \\
&\le 2\iint_{\mathbb{R}^{2N}} \frac{|v_0(x)|^2[\psi_n(x) - \psi_n(y)]^2}{|x-y|^{N+2s}} dxdy + 2\iint_{\mathbb{R}^{2N}} \frac{[\psi_n(y) - 1]^2[v_0(x) - v_0(y)]^2}{|x-y|^{N+2s}} dxdy
\end{aligned} \tag{3.44}$$

As in the proof of (3.15), if we replace v_n by v_0, and ϕ by ψ_n, since $\operatorname{supp}(\psi_n) \subset B_{2\widetilde{R}_n}(0)$, it is easy to obtain the estimate

$$\iint_{\mathbb{R}^{2N}} \frac{|v_0(x)|^2[\psi_n(x) - \psi_n(y)]^2}{|x-y|^{N+2s}} dxdy = o_n(1).$$

On the other hand, taking into account that $|\psi_n(x) - 1| \le 2$, $|\psi_n(x) - 1| \to 0$ a.e. in $\mathbb{R}^N$ and $v_0 \in D^{s,2}(\mathbb{R}^N)$, the Dominated Convergence Theorem implies that

$$\iint_{\mathbb{R}^{2N}} \frac{[\psi_n(y) - 1]^2[v_0(x) - v_0(y)]^2}{|x-y|^{N+2s}} dxdy \to 0 \quad \text{as } n \to \infty.$$

Coming back to (3.44) we get

$$v_0\psi_n \to v_0 \quad \text{in } D^{s,2}(\mathbb{R}^N)$$

which implies that,

$$\widetilde{w}_n = v_n - v_0 + o_n(1).$$

Since $v_n \rightharpoonup v_0$ in $D^{s,2}(\mathbb{R}^N)$ and $v_n \to v_0$ a.e. in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_0|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(v_n - v_0)|^2 dx + o_n(1),$$

$$\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx = \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}} dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n - v_0|^{2^*_{\alpha,s}})|v_n - v_0|^{2^*_{\alpha,s}} dx + o_n(1),$$

which implies that

$$\begin{aligned}
I_\infty(w_n) &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}})|w_n|^{2^*_{\alpha,s}} dx \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{w}_n|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\widetilde{w}_n|^{2^*_{\alpha,s}})|\widetilde{w}_n|^{2^*_{\alpha,s}} dx \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx \\
&\quad - \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_0|^2 dx + \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}} dx + o_n(1) \\
&= I_\infty(v_n) - I_\infty(v_0) + o_n(1).
\end{aligned}$$

Therefore,

$$I_\infty(w_n) \to \widetilde{c} \quad \text{as} \quad n \to \infty,$$

where $\widetilde{c} = c - I_\infty(v_0)$. Now, we are going to show that

$$\|I'_\infty(\widetilde{w}_n) - I'_\infty(v_n) + I'_\infty(v_0)\|_{D^{-s,2}(\mathbb{R}^N)} \to 0. \tag{3.45}$$

Indeed, for all $\Phi \in D^{s,2}(\mathbb{R}^N)$ with $\|\Phi\| \le 1$, we have

$$\begin{aligned}
&\left|(I'_\infty(\widetilde{w}_n) - I'_\infty(v_n) + I'_\infty(v_0))\Phi\right| \\
&= \Bigg| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \widetilde{w}_n (-\Delta)^{\frac{s}{2}} \Phi dx - \int_{\mathbb{R}^N} (I_\alpha * |\widetilde{w}_n|^{2^*_{\alpha,s}})|\widetilde{w}_n|^{2^*_{\alpha,s}-2} \widetilde{w}_n \Phi dx \\
&\quad - \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Phi dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2} v_n \Phi dx \\
&\quad + \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_0 (-\Delta)^{\frac{s}{2}} \Phi dx - \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}-2} v_0 \Phi dx \Bigg| \\
&= \Bigg| \int_{\mathbb{R}^N} [(-\Delta)^{\frac{s}{2}} \widetilde{w}_n - (-\Delta)^{\frac{s}{2}} v_n + (-\Delta)^{\frac{s}{2}} v_0](-\Delta)^{\frac{s}{2}} \Phi dx
\end{aligned}$$

$$-\int_{\mathbb{R}^N}[(I_\alpha*|\widetilde{w}_n|^{2^*_{\alpha,s}})|\widetilde{w}_n|^{2^*_{\alpha,s}-2}\widetilde{w}_n-(I_\alpha*|v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n$$

$$+(I_\alpha*|v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}-2}v_0]\Phi dx\Bigg|$$

$$\leq\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\widetilde{w}_n-(-\Delta)^{\frac{s}{2}}v_n+(-\Delta)^{\frac{s}{2}}v_0|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\Phi|^2dx\right)^{\frac{1}{2}}$$

$$+\Bigg(\int_{\mathbb{R}^N}[(I_\alpha*|v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n-(I_\alpha*|\widetilde{w}_n|^{2^*_{\alpha,s}})|\widetilde{w}_n|^{2^*_{\alpha,s}-2}\widetilde{w}_n$$

$$-(I_\alpha*|v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}-2}v_0]^{\frac{2^*_s}{2^*_s-1}}dx\Bigg)^{\frac{2^*_s-1}{2^*_s}}\left(\int_{\mathbb{R}^N}|\Phi|^{2^*_s}dx\right)^{\frac{1}{2^*_s}}. \tag{3.46}$$

By the definition of $\widetilde{w}_n$, we have

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\widetilde{w}_n-(-\Delta)^{\frac{s}{2}}v_n+(-\Delta)^{\frac{s}{2}}v_0|^2dx=o_n(1). \tag{3.47}$$

Using Lemma A A, we get

$$\int_{\mathbb{R}^N}|(I_\alpha*|v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n-(I_\alpha*|\widetilde{w}_n|^{2^*_{\alpha,s}})|\widetilde{w}_n|^{2^*_{\alpha,s}-2}\widetilde{w}_n-(I_\alpha*|v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}-2}v_0|^{\frac{2^*_s}{2^*_s-1}}dx$$
$$=o_n(1).$$

Since v_0 is a nontrivial critical point of I_∞, we conclude that

$$I'_\infty(\widetilde{w}_n)=I'_\infty(v_n)+I'_\infty(v_0)+o_n(1)=I'_\infty(v_n)+o_n(1)=o_n(1).$$

As $\|I'_\infty(w_n)\|_{D^{-s,2}(\mathbb{R}^N)}\leq\|I'_\infty(\widetilde{w}_n)\|_{D^{-s,2}(\mathbb{R}^N)}$, it follows that $I'_\infty(w_n)\to 0$ and the proof of this lemma is over. $\square$

The next result is a version of nonlocal global compactness result for fractional Laplacian Choquard equation in $\mathbb{R}^N$ with critical growth.

Theorem 3.2. *(A splitting theorem) Let (u_n) be a $(PS)_c$ sequence for I with $u_n\rightharpoonup u_0$ in $D^{s,2}(\mathbb{R}^N)$. Then, up to a subsequence, (u_n) satisfies either,*

(a) *$u_n\to u_0$ in $D^{s,2}(\mathbb{R}^N)$ or,*

(b) *there exists* $k \in \mathbb{N}$ *and nontrivial solutions* $z_0^1, z_0^2, \cdots, z_0^k$ *for problem* $(P)_\infty$*, such that*

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2$$

and

$$I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j).$$

Proof. From the weak convergence, we have that u_0 is a critical point of I. Assume that $u_n \nrightarrow u_0$ in $D^{s,2}(\mathbb{R}^N)$ and let $(z_n^1) \subset D^{s,2}(\mathbb{R}^N)$ be the sequence given by $z_n^1 := u_n - u_0$. Then by hypothesis, we have $z_n^1 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ and $z_n^1 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$.

Using Lemma 3 [2] and Lemma A, we have

$$I_\infty(z_n^1) = I(u_n) - I(u_0) + o_n(1) \tag{3.48}$$

and

$$I'_\infty(z_n^1) = I'(u_n) - I'(u_0) + o_n(1). \tag{3.49}$$

By (3.48) and (3.49) we see that (z_n^1) is a $(PS)_{c_1}$ sequence for I_∞. Hence, by Lemma 3.1, there exist sequences $(R_{n,1}) \subset \mathbb{R}$, $(x_{n,1}) \subset \mathbb{R}^N$, $z_0^1 \in D^{s,2}(\mathbb{R}^N)$ nontrivial solution of (P_∞) and a $(PS)_{c_2}$ sequence (z_n^2) for I_∞ such that

$$z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-2s)/2} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1).$$

If we define

$$v_n^1(x) = R_{n,1}^{(2s-N)/2} z_n^1\left(\frac{x}{R_{n,1}} + x_{n,1}\right) \tag{3.50}$$

and

$$\widetilde{z}_n^2(x) = R_{n,1}^{(2s-N)/2} z_n^2\left(\frac{x}{R_{n,1}} + x_{n,1}\right),$$

we get

$$\widetilde{z}_n^2(x) = v_n^1(x) - z_0^1(x) + o_n(1) \tag{3.51}$$

and

$$\|v_n^1\| = \|z_n^1\| \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\alpha * |v_n^1|^{2^*_{\alpha,s}})|v_n^1|^{2^*_{\alpha,s}} dx = \int_{\mathbb{R}^N} (I_\alpha * |z_n^1|^{2^*_{\alpha,s}})|z_n^1|^{2^*_{\alpha,s}} dx. \tag{3.52}$$

Hence,

$$I_\infty(v_n^1) = I_\infty(z_n^1) \quad \text{and} \quad I'_\infty(v_n^1) \to 0 \quad \text{in} \quad D^{-s,2}(\mathbb{R}^N). \tag{3.53}$$

From (3.53) and Lemma 2.2, we have that (v_n^1) is a bounded sequence in $D^{s,2}(\mathbb{R}^N)$ and, up to a subsequence,

$$v_n^1 \rightharpoonup z_0^1 \quad \text{in} \quad D^{s,2}(\mathbb{R}^N). \tag{3.54}$$

As we arguing before, we conclude

$$I_\infty(\widetilde{z}_n^2) = I_\infty(v_n^1) - I_\infty(z_0^1) + o_n(1) = I_\infty(u_n) - I_\infty(u_0) - I_\infty(z_0^1) + o_n(1), \tag{3.55}$$

and

$$I'_\infty(\widetilde{z}_n^2) = I'_\infty(v_n^1) - I'_\infty(z_0^1) + o_n(1). \tag{3.56}$$

If $\widetilde{z}_n^2 \to 0$ in $D^{s,2}(\mathbb{R}^N)$, the proof is over for $k = 1$, since in this case, we get

$$\|u_n\|^2 \to \|u_0\|^2 + \|z_0^1\|^2.$$

If $\widetilde{z}_n^2 \nrightarrow 0$, using (3.51) and (3.54) that $\widetilde{z}_n^2 \rightharpoonup 0$, by (3.55) and (3.56) that $(\widetilde{z}_n^2)$ is a $(PS)_{c_2}$ sequence for I_∞. By Lemma 3.1, there exist sequences $(R_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N$, $z_0^2 \in D^{s,2}(\mathbb{R}^N)$ nontrivial solution of (P_∞) and a $(PS)_{c_3}$ sequence (z_n^3) for I_∞ such that

$$z_n^3(x) = \widetilde{z}_n^2(x) - R_{n,2}^{(N-2s)/2} z_0^2(R_{n,2}(x - x_{n,2})) + o_n(1).$$

If we define

$$v_n^2(x) = R_{n,2}^{(2s-N)/2} \widetilde{z}_n^2 \left(\frac{x}{R_{n,2}} + x_{n,2} \right) \tag{3.57}$$

and

$$\widetilde{z}_n^3(x) = R_{n,2}^{(2s-N)/2} z_n^3 \left(\frac{x}{R_{n,2}} + x_{n,2} \right),$$

we get

$$\widetilde{z}_n^3(x) = v_n^2(x) - z_0^2(x) + o_n(1). \tag{3.58}$$

Arguing of same way that was before, we have

$$\|\widetilde{z}_n^3\|^2 = \|u_n\|^2 - \|u_0\|^2 - \|z_0^1\|^2 - \|z_0^2\|^2 + o_n(1), \tag{3.59}$$

$$I_\infty(\widetilde{z}_n^3) = I(u_n) - I(u_0) - I_\infty(z_0^1) - I_\infty(z_0^2) + o_n(1), \tag{3.60}$$

and

$$I'_\infty(\widetilde{z}_n^3) = I'_\infty(v_n^2) - I'_\infty(z_0^2) + o_n(1). \tag{3.61}$$

If $\widetilde{z}_n^3 \to 0$ in $D^{s,2}(\mathbb{R}^N)$, the proof is over with $k = 2$, because $\|\widetilde{z}_n^3\|^2 \to 0$, and by (3.59), we obtain

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{2} \|z_0^j\|^2.$$

Moreover, from continuity of I_∞, we also have that $I_\infty(\widetilde{z}_n) \to 0$. Thus, by (3.60),

$$I_\infty(u_n) \to I(u_0) + \sum_{j=1}^{2} I_\infty(z_0^j) + o_n(1).$$

If $\widetilde{z}_n^3 \nrightarrow 0$, we can repeat the same arguments before and we shall find $z_0^1, z_0^2, \cdots, z_0^{k-1}$ nontrivial solutions of problem (P_∞) satisfying

$$\|\widetilde{z}_n^k\|^2 = \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|z_0^j\|^2 + o_n(1) \tag{3.62}$$

and

$$I_\infty(\widetilde{z}_n^k) = I(u_n) - I(u_0) - \sum_{j=1}^{k-1} I_\infty(z_0^j) + o_n(1). \tag{3.63}$$

By the definition of $S_{h,l}$,

$$S_{h,l}\left(\int_{\mathbb{R}^N} (I_\alpha * |z_0^j|^{2^*_{\alpha,s}})|z_0^j|^{2^*_{\alpha,s}}dx\right)^{\frac{1}{2^*_{\alpha,s}}} \leq \|z_0^j\|^2, \quad j = 1, 2, \cdots, k-1. \tag{3.64}$$

Since z_0^j is nontrivial solution of (P_∞), for all $j = 1, 2, \cdots, k-1$, we obtain

$$\|z_0^j\|^2 = \int_{\mathbb{R}^N} (I_\alpha * |z_0^j|^{2^*_{\alpha,s}})|z_0^j|^{2^*_{\alpha,s}}dx.$$

Therefore,

$$\|z_0^j\|^2 \geq S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}, \quad j = 1, 2, \cdots, k-1. \tag{3.65}$$

From (3.62) and (3.65),

$$\begin{aligned}\|\widetilde{z}_n^k\|^2 &= \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|z_0^j\|^2 + o_n(1) \\ &\leq \|u_n\|^2 - \|u_0\|^2 - (k-1) S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + o_n(1).\end{aligned} \tag{3.66}$$

Since (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, for k large enough, we conclude that $\widetilde{z}_n^k \to 0$ in $D^{s,2}(\mathbb{R}^N)$ and the proof is completed. □

An immediate consequence of the last theorem, are the next two conclusions.

Corollary 3.3. *Let (u_n) be a $(PS)_c$ sequence for I with $c \in \left(0, \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}\right)$. Then, up to a subsequence, (u_n) converges strongly in $D^{s,2}(\mathbb{R}^N)$.*

Proof. We have that (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, $u_n \rightharpoonup u_0$ in $D^{s,2}(\mathbb{R}^N)$ and $I'(u_0) = 0$. Suppose, by contradiction, that $u_n \nrightarrow u_0$ in $D^{s,2}(\mathbb{R}^N)$. From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $z_0^1, z_0^2, \cdots, z_0^k$ to problem (P_∞) such that,

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2$$

and

$$I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j).$$

Notice that

$$\begin{aligned}I(u_0) &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} a(x)|u_0|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2^*_{\alpha,s}})|u_0|^{2^*_{\alpha,s}} dx \\ &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\left(\int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2^*_{\alpha,s}})|u_0|^{2^*_{\alpha,s}} dx - \|u_0\|^2\right) - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2^*_{\alpha,s}})|u_0|^{2^*_{\alpha,s}} dx \\ &= \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,s}}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2^*_{\alpha,s}})|u_0|^{2^*_{\alpha,s}} dx \geq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}c &= I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j) \\ &\geq k \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} > \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},\end{aligned}$$

which is a contradiction with $c \in \left(0, \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}\right)$. □

Corollary 3.4. *The functional* $I : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ *satisfies the* $(PS)_c$ *condition with* $c \in \left(\frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{(2N-\alpha)/(N+2s-\alpha)}, 2^{\frac{4s-\alpha}{N+2s-\alpha}} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{(2N-\alpha)/(N+2s-\alpha)}\right)$.

Proof. Let (u_n) be a $(PS)_c$ sequence in $D^{s,2}(\mathbb{R}^N)$, that is,

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

Since (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, up to a subsequence, we have $u_n \rightharpoonup u_0$ in $D^{s,2}(\mathbb{R}^N)$ and $I'(u_0) = 0$. Suppose, by contradiction, that $u_n \nrightarrow u_0$ in $D^{s,2}(\mathbb{R}^N)$. Then from Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $z_0^1, z_0^2, \cdots, z_0^k$ to problem (P_∞) such that,

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2$$

and

$$I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j).$$

The above information ensures that $u_0 \neq 0$. Since $I(u_0) \geq 0$, then $k = 1$ and z_0^1 cannot change its sign, because otherwise, by Lemma 2.3,

$$I_\infty(z_0^1) \geq 2^{\frac{4s-\alpha}{N+2s-\alpha}} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},$$

which leads to a contradiction. Hence,

$$c = I(u_0) + I_\infty(z_0^1).$$

On the other hand, if z_0^1 has definite sign, by (2.8) and (2.9), we have

$$I_\infty(z_0^1) = \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

On the other hand, by a direct calculation,

$$I(u_0) \geq \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Hence,

$$c = I(u_0) + I_\infty(z_0^1) \geq 2\frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} > 2^{\frac{4-\alpha}{2N+4-2\alpha}} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},$$

which yields a contradiction. This completes the proof. □

In the sequel, we consider the functional $f : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$f(u) = \int\limits_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}|^2 + a(x)|u|^2 \right) dx$$

and the manifold $\mathcal{M} \subset D^{s,2}(\mathbb{R}^N)$ given by

$$\mathcal{M} = \left\{ u \in D^{s,2}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} dx = 1 \right\}.$$

The next results are direct consequence of the Corollaries above.

Lemma 3.5. *Let (u_n) be a sequence that satisfies*

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then, the sequence $(v_n) \subset D^{s,2}(\mathbb{R}^N)$ with $v_n = c^{(N-2s)/(2N+4s-2\alpha)} u_n$ satisfies the following limits.

$$I(v_n) \to \frac{N+2s-\alpha}{4N-2\alpha} c^{\frac{2N-\alpha}{N+2s-\alpha}} \quad \text{and} \quad I'(v_n) \to 0.$$

Lemma 3.6. *Suppose that there are a sequence $(u_n) \subset \mathcal{M}$ and*

$$c \in \left(S_{h,l}, 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l} \right)$$

such that

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then, up to a subsequence, $u_n \to u$ in $D^{s,2}(\mathbb{R}^N)$.

Corollary 3.7. *Suppose that there is a sequence $(u_n) \subset \mathcal{M}$ and*

$$c \in \left(S_{h,l}, 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l} \right)$$

such that

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.$$

Then I has a critical point $v_0 \in D^{s,2}(\mathbb{R}^N)$ with $I(v_0) = \frac{N+2s-\alpha}{4N-2\alpha} c^{(2N-\alpha)/(N+2s-\alpha)}$.

4. Technical lemmas

Throughout this section, we consider the function $\Psi_{\delta,b} \in D^{s,2}(\mathbb{R}^N)$ given by

$$\begin{aligned}\Psi_{\varepsilon,b}(x) &= S_{h,l}^{-\frac{N-2s}{2N+4s-2\alpha}} \widetilde{U}_{\varepsilon,b}(x)\\ &= S_{h,l}^{-\frac{N-2s}{2N+4s-2\alpha}} S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}} U_{\varepsilon,b}(x)\\ &= S_{h,l}^{-\frac{N-2s}{2N+4s-2}} b_{\alpha,s}\left(\frac{\varepsilon}{\varepsilon^2+|x-b|^2}\right)^{\frac{N-2s}{2}}\\ &:= B_{\alpha,s}\left(\frac{\varepsilon}{\varepsilon^2+|x-b|^2}\right)^{\frac{N-2s}{2}}, \quad x, b \in \mathbb{R}^N \quad \text{and} \quad \varepsilon > 0,\end{aligned} \tag{4.1}$$

where $U_{\varepsilon,b}(x)$ solves problem (P_∞) as we have pointed in Section 2. By a simple calculation, we get

$$\|\Psi_{\varepsilon,b}\|^2 = S_{h,l} \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\alpha * |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}})|\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}} dx = 1 \tag{4.2}$$

Now, we prove some properties on the function $\Psi_{\varepsilon,b}$ given by in (4.1). First of all, we recall

$$\Psi_{\varepsilon,b} \in \Sigma = \{u \in D^{s,2}(\mathbb{R}^N) : \ u \geq 0\} \tag{4.3}$$

and

$$\Psi_{\varepsilon,b} \in L^q(\mathbb{R}^N) \ \text{ for } \ q \in \left(\frac{N}{N-2s}, 2^*_{\alpha,s}\right], \ \ \forall \varepsilon > 0 \ \text{ and } \ \forall b \in \mathbb{R}^N. \tag{4.4}$$

Lemma 4.1. *For each $b \in \mathbb{R}^N$, we have*

(i) $\|\Psi_{\varepsilon,b}\|_{H^{1,\infty}(\mathbb{R}^N)} \to 0$ *as* $\varepsilon \to +\infty$,
(ii) $\|\Psi_{\varepsilon,b}\|_{H^{1,\infty}(\mathbb{R}^N)} \to +\infty$ *as* $\varepsilon \to 0$,
(iii) $|\Psi_{\varepsilon,b}|_q \to 0$ *when* $\varepsilon \to 0$, $\ \forall q \in \left(\frac{N}{N-2s}, 2^*_{\alpha,s}\right)$,
(iv) $|\Psi_{\varepsilon,b}|_q \to +\infty$ *when* $\varepsilon \to +\infty$, $\ \forall q \in \left(\frac{N}{N-2s}, 2^*_{\alpha,s}\right)$.

Proof. Using the definition of $\Psi_{\varepsilon,b}$ we have

$$|\nabla \Psi_{\varepsilon,b}(x)| = \frac{B_{\alpha,s}\varepsilon^{\frac{N-2s}{2}}|x-b|}{[\varepsilon^2+|x-b|^2]^{\frac{N-2s}{2}+1}}, \tag{4.5}$$

where $B_{\alpha,s} > 0$ is a constant given in (4.1), and then

$$\|\Psi_{\varepsilon,b}\|_{H^{1,\infty}(\mathbb{R}^N)} = B_{\alpha,s}\varepsilon^{-\frac{N+2-2s}{2}}$$

and consequently (i) and (ii) follow. Now

$$|\Psi_{\varepsilon,b}|_q^q = B_{\alpha,s}^q \varepsilon^{N-\frac{N-2s}{2}q} \int\limits_{\mathbb{R}^N} \left(\frac{1}{1+|z|^2}\right)^{\frac{N-2s}{2}q} dz, \tag{4.6}$$

and using the hypothesis that $\frac{N}{N-2s} < q < 2^*_{\alpha,s} < 2^*_s$, (iii) and (iv) follow immediately. □

Lemma 4.2. *For each $r > 0$, we have*

$$\int\limits_{\mathbb{R}^N\setminus B_r(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,0}|^2 dx \to 0 \quad \textit{when} \quad \varepsilon \to 0.$$

Proof. From Proposition 2.2 in [39], we have

$$\int\limits_{\mathbb{R}^N\setminus B_r(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,0}|^2 dx \le C \int\limits_{\mathbb{R}^N\setminus B_r(0)} |\nabla\Psi_{\varepsilon,0}|^2 dx. \tag{4.7}$$

Using (4.5) we get

$$\begin{aligned}\int\limits_{\mathbb{R}^N\setminus B_r(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,0}|^2 dx &\le C_1 \int\limits_{\mathbb{R}^N\setminus B_r(0)} \frac{\varepsilon^{N-2s}|x|^2}{[\varepsilon^2+|x|^2]^{N-2s+2}} \\ &= C_1\varepsilon^{2s-2} \int\limits_{\mathbb{R}^N\setminus B_{r/\varepsilon}(0)} \frac{|z|^2}{[1+|z|^2]^{N-2s+2}} dz \\ &\le C_2\varepsilon^{N-2s} r^{-N-2+4s} \to 0 \quad \text{as} \quad \varepsilon \to 0,\end{aligned} \tag{4.8}$$

which implies conclusion. □

Lemma 4.3. *Suppose that $a \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N-\alpha}{4s-\alpha} < p_2$ with $p_2 < \frac{N}{4s-N}$ if $2s < N < 4s$. Then for each $\tau > 0$, there are $\underline{\varepsilon} = \underline{\varepsilon}(\tau) > 0$ and $\overline{\varepsilon} = \overline{\varepsilon}(\tau) > 0$ such that*

$$\sup_{b\in\mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + \tau, \quad \varepsilon \in [0, \underline{\varepsilon}] \cup [\overline{\varepsilon}, \infty).$$

Proof. Let $y \in \mathbb{R}^N$, $q \in \left(\frac{2N-\alpha}{4s-\alpha}, p_2\right]$ and $t \in (1, \infty)$ with $\frac{1}{q} + \frac{1}{t} = 1$. By a simple calculus,

$$\frac{N}{N-2s} < 2t < 2^*_{\alpha,s}. \tag{4.9}$$

Since $\Psi_{\varepsilon,b} \in L^d(\mathbb{R}^N)$, $\forall d \in \left(\frac{N}{N-2s}, 2^*_{\alpha,s}\right)$, we get $|\Psi_{\varepsilon,b}|^2 \in L^t(\mathbb{R}^N)$. Then, using Hölder's inequality and change of variable, we obtain

$$\int_{\mathbb{R}^N} a(x)|\Psi_{\varepsilon,b}|^2 dx \le |a|_q \left(\int_{\mathbb{R}^N} \left| \frac{B_{\alpha,s}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|x-b|^2]^{\frac{N-2s}{2}}} \right|^{2t} dx \right)^{\frac{1}{t}}$$

$$= |a|_q \left(\int_{\mathbb{R}^N} \left| \frac{B_{\alpha,s}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|z|^2]^{\frac{N-2s}{2}}} \right|^{2t} dz \right)^{\frac{1}{t}}$$

$$= |a|_q \left(\int_{\mathbb{R}^N} |\Psi_{\varepsilon,0}|^{2t} dx \right)^{\frac{1}{t}} = |a|_q |\Psi_{\varepsilon,0}|_{2t}^2, \quad \forall b \in \mathbb{R}^N.$$

From Lemma 4.1-(iii), given $\tau > 0$, there exists $\underline{\varepsilon} = \underline{\varepsilon}(\tau) > 0$ such that

$$\sup_{b\in\mathbb{R}^N} f(\Psi_{\varepsilon,b}) \le S_{h,l} + \frac{\tau}{2} < S_{h,l} + \tau, \quad \forall \varepsilon \in (0, \underline{\varepsilon}].$$

Now, suppose that $q \in \left[p_1, \frac{2N-\alpha}{4s-\alpha}\right)$ with $t \in (0,\infty)$ and $\frac{1}{q} + \frac{1}{t} = 1$. Notice that $2t - 2^*_{\alpha,s} > 0$ and for $\varepsilon > 1$,

$$|\Psi_{\varepsilon,b}| \in L^\infty(\mathbb{R}^N) \tag{4.10}$$

and $|\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}} \in L^1(\mathbb{R}^N)$. Then $|\Psi_{\varepsilon,b}|^2 \in L^t(\mathbb{R}^N)$. Applying again Hölder's inequality with q and t, we get

$$\int_{\mathbb{R}^N} a(x)|\Psi_{\varepsilon,b}|^2 dx \le |a|_q \left(\int_{\mathbb{R}^N} |\Psi_{\varepsilon,0}|^{2t} dx \right)^{\frac{1}{t}}$$

$$= |a|_q \left(\int_{\mathbb{R}^N} |\Psi_{\varepsilon,0}|^{2t-2^*_{\alpha,s}} |\Psi_{\varepsilon,0}|^{2^*_{\alpha,s}} dx \right)^{\frac{1}{t}}$$

$$\le |a|_q |\Psi_{\varepsilon,0}|_{L^\infty(\mathbb{R}^N)}^{(2t-2^*_{\alpha,s})/t} \left(\int_{\mathbb{R}^N} |\Psi_{\varepsilon,0}|^{2^*_{\alpha,s}} dx \right)^{\frac{1}{t}}$$

$$\le C|a|_q \varepsilon^{\frac{2s-N}{2}\frac{2t-2^*_{\alpha,s}}{t}}, \quad \forall b \in \mathbb{R}^N.$$

Then, given $\tau > 0$, there is $\overline{\varepsilon} = \overline{\varepsilon}(\tau) > 0$ such that

$$\varepsilon^{\frac{2s-N}{2}\frac{2t-2^*_{\alpha,s}}{t}} < \frac{\tau}{2C|a|_q}, \quad \forall \varepsilon \in [\overline{\varepsilon}, \infty),$$

which implies

$$\begin{aligned}
f(\Psi_{\varepsilon,b}) &= \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2 + a(x)|\Psi_{\varepsilon,b}|^2\right) dx \\
&\le S_{h,l} + \int_{\mathbb{R}^N} a(x)|\Psi_{\varepsilon,b}|^2 dx \\
&\le S_{h,l} + \sup_{b\in\mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Psi_{\varepsilon,b}|^2 dx \\
&\le S_{h,l} + \frac{\tau}{2} \le S_{h,l} + \tau, \quad \forall b \in \mathbb{R}^N \ \text{ and } \ \forall \varepsilon \in [\overline{\varepsilon}, \infty).
\end{aligned}$$

The proof is now complete. □

Lemma 4.4. *Suppose that* $|a|_{L^{N/(2s)}(\mathbb{R}^N)} < \left(2^{\frac{4s-\alpha}{2N-\alpha}} - 1\right) S^{\frac{(2s-N)[(N-\alpha)(1-s)+2s]+(2N-\alpha)2s}{2s(N-\alpha+2s)}}$. *Then*

$$\sup_{\varepsilon>0, b\in\mathbb{R}^N} f(\Psi_{\varepsilon,b}) < 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}.$$

Proof. From (2.8)-(2.10), (4.1)-(4.2) and (2.5), a direct calculation yields that

$$\begin{aligned}
|\Psi_{\varepsilon,b}|^2_{2_s^*} &= \left(S_{h,l}^{-\frac{N-2s}{2N+4s-2\alpha}} S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} \mathcal{D}_{N,\alpha}^{\frac{2s-N}{2(N-\alpha+2s)}}\right)^2 \left[\int_{\mathbb{R}^N} |U_{\varepsilon,b}(x)|^{2_s^*} dx\right]^{\frac{2}{2_s^*}} \\
&= S_{h,l}^{-\frac{N-2s}{N+2s-\alpha}} S^{\frac{(2s-N)(N-\alpha)}{2(N-\alpha+2s)}} D_{N,\alpha}^{\frac{2s-N}{N-\alpha+2s}} S^{\frac{N}{2s}\times\frac{2}{2_s^*}} \\
&= S_{h,l} S^{\frac{(N-2s)[(N-\alpha)(1-s)+2s]-(2N-\alpha)2s}{2s(N-\alpha+2s)}}.
\end{aligned}$$

Hence, by Hölder's inequality with $\frac{N}{2s}$ and $\frac{N}{N-2s}$, we obtain

$$\sup_{\delta>0, b\in\mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + |a|_{\frac{N}{2s}} |\Psi_{\varepsilon,b}|^2_{2_s^*} < S_{h,l} + \left(2^{\frac{4s-\alpha}{2N-\alpha}} - 1\right) S_{h,l} = 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}.$$

This completes the proof. □

In what follows, we consider the function

$$\xi(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ 1, & \text{if } |x| \ge 1 \end{cases}$$

and define $\alpha : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by

$$\alpha(u) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \xi(x)\right) |(-\Delta)^{\frac{s}{2}} u|^2 dx = (\beta(u), \gamma(u)),$$

where

$$\beta(u) = \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N} \frac{x}{|x|} |(-\Delta)^{\frac{s}{2}} u|^2 dx$$

and

$$\gamma(u) = \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N} \xi(x) |(-\Delta)^{\frac{s}{2}} u|^2 dx.$$

Lemma 4.5. *If $|b| \geq \frac{1}{2}$, then*

$$\beta(\Psi_{\varepsilon,b}) = \frac{b}{|b|} + o_\varepsilon(1) \quad when \ \ \varepsilon \to 0.$$

Proof. Given $\varepsilon > 0$, from Lemma 4.2, there is $\widehat{\varepsilon} > 0$ such that

$$\begin{aligned} \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N \setminus B_r(b)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}|^2 dx &= \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N \setminus B_r(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \\ &\leq C\widetilde{C}^2 \int\limits_{\mathbb{R}^N \setminus B_r(0)} \frac{\varepsilon^{N-2s}|x|^2}{[\varepsilon^2 + |x|^2]^{N-2s+2}} \\ &= C\widetilde{C}^2 \varepsilon^{2s-2} \int\limits_{\mathbb{R}^N \setminus B_{r/\varepsilon}(0)} \frac{|z|^2}{[1+|z|^2]^{N-2s+2}} dz \\ &\leq C_1 \varepsilon^{N-2s} r^{-N-2+4s} < \tau, \quad \forall \varepsilon \in (0, \widehat{\varepsilon}). \end{aligned} \tag{4.11}$$

Thus,

$$\begin{aligned} \left| \beta(\Psi_{\varepsilon,b}) - \frac{1}{S_{h,l}} \int\limits_{B_r(b)} \frac{x}{|x|} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}|^2 dx \right| &\leq \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N \setminus B_r(b)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}|^2 dx \\ &< \tau, \quad \forall \varepsilon \in (0, \widehat{\varepsilon}). \end{aligned} \tag{4.12}$$

Note that if ε is small enough, for $|b| \geq \frac{1}{2}$ and $x \in B_r(b)$,

$$\left| \frac{x}{|x|} - \frac{b}{|b|} \right| < 2\tau.$$

Then, by (4.11), (4.12), we have

$$\begin{aligned}
&\left|\frac{b}{|b|}-\frac{1}{S_{h,l}}\int\limits_{B_r(b)}\frac{x}{|x|}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2dx\right| \\
&=\left|\frac{1}{S_{h,l}}\int\limits_{B_r(b)}\left(\frac{b}{|b|}-\frac{x}{|x|}\right)|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2dx+\frac{1}{S_{h,l}}\int\limits_{\mathbb{R}^N\setminus B_r(b)}\frac{b}{|b|}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2dx\right| \\
&\le\frac{1}{S_{h,l}}\int\limits_{\mathbb{R}^N}2\tau|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2dx+\frac{1}{S_{h,l}}\int\limits_{\mathbb{R}^N\setminus B_r(b)}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2dx \\
&\le 2\tau+\tau=3\tau. \qquad (4.13)
\end{aligned}$$

From (4.12) and (4.13), we get

$$\left|\beta(\Psi_{\varepsilon,b})-\frac{b}{|b|}\right|<4\tau,$$

which completes the proof. □

Lemma 4.6. *Suppose that $a\in L^q(\mathbb{R}^N)$, $\forall q\in[p_1,p_2]$, where $1<p_1<\frac{2N-\alpha}{4s-\alpha}<p_2$ with $p_2<\frac{N}{4s-N}$ if $2s<N<4s$. Then for every $\varepsilon>0$, we have*

$$\lim_{|b|\to\infty}f(\Psi_{\varepsilon,b})=S_{h,l}.$$

Proof. Since

$$f(\Psi_{\varepsilon,b})=\int\limits_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}|^2+a(x)|\Psi_{\varepsilon,b}|^2\right)dx=S_{h,l}+\int\limits_{\mathbb{R}^N}a(x)|\Psi_{\varepsilon,b}|^2dx,$$

it is sufficient to prove that

$$\lim_{|b|\to\infty}\int\limits_{\mathbb{R}^N}a(x)|\Psi_{\varepsilon,b}|^2dx\to 0,\quad\forall\varepsilon>0. \qquad (4.14)$$

Notice that for given $\tau>0$, there exists $r_0>0$ such that

$$\left(\int\limits_{\mathbb{R}^N\setminus B_\rho(0)}|a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}}<\tau,\quad\forall\rho>r_0,$$

and

$$\left(\int\limits_{\mathbb{R}^N\setminus B_\rho(b)} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{1}{2^*_{\alpha,s}}} < \tau, \quad \forall \rho > r_0. \tag{4.15}$$

Let $r_0 < 2\rho < |b|$, and then $B_\rho(0)\cap B_\rho(b) = \emptyset$. By (4.6) and Hölder's inequality we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} a(x)|\Psi_{\varepsilon,b}|^2dx\\
&\leq \left(\int\limits_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(b))} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(b))} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&\quad+\left(\int\limits_{B_\rho(0)} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{(B_\rho(0)} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&\quad+\left(\int\limits_{B_\rho(b)} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{(B_\rho(b)} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&\leq \left(\int\limits_{\mathbb{R}^N\setminus B_\rho(0)} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{\mathbb{R}^N\setminus B_\rho(b)} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&\quad+\left(\int\limits_{\mathbb{R}^N} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{\mathbb{R}^N\setminus(B_\rho(b)} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&\quad+\left(\int\limits_{\mathbb{R}^N\setminus B_\rho(0)} |a(x)|^{\frac{2N-\alpha}{4s-\alpha}}dx\right)^{\frac{4s-\alpha}{2N-\alpha}} \left(\int\limits_{\mathbb{R}^N} |\Psi_{\varepsilon,b}|^{2^*_{\alpha,s}}dx\right)^{\frac{2}{2^*_{\alpha,s}}}\\
&< \tau\tau^2 + \tau^2|a|_{\frac{2N-\alpha}{4s-\alpha}} + \tau C\varepsilon^{\frac{\alpha}{2}}
\end{aligned}$$

which completes the proof. □

Now we define the set

$$\mathcal{Z} := \left\{u\in\mathcal{M}:\ \alpha(u) = \left(0,\frac{1}{2}\right)\right\}.$$

Lemma 4.7. *The set $\mathcal{Z}$ is not empty, that is, $\mathcal{Z}\neq\emptyset$.*

Proof. Since the function $\Psi_{\varepsilon,0}$ is symmetric about the origin, we have $\beta(\Psi_{\varepsilon,0}) = 0$. From Lemma 4.2 we see that

$$\gamma(\Psi_{\varepsilon,0}) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.16}$$

Note that,

$$\begin{aligned}
\gamma(\Psi_{\varepsilon,0}) &= \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \xi(x) |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \\
&= \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx + \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} (\xi(x) - 1) |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \\
&= 1 - \frac{1}{S_{h,l}} \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx.
\end{aligned} \tag{4.17}$$

From (4.5) and Proposition 2.2 [39], we have

$$\begin{aligned}
\int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx &\le C \int_{B_1(0)} |\nabla \Psi_{\varepsilon,0}|^2 dx \\
&\le C \int_{B_1(0)} \frac{\varepsilon^{N-2s} |x|^2}{[\varepsilon^2 + |x|^2]^{N-2s+2}} \\
&= C \varepsilon^{2s-2} \int_{B_1(0)} \frac{|z|^2}{[1 + |z|^2]^{N-2s+2}} dz \\
&\le C_1 \varepsilon^{2s-2} \to 0 \quad \text{as} \quad \varepsilon \to \infty.
\end{aligned} \tag{4.18}$$

Combining (4.17) and (4.18) we get

$$\gamma(\Psi_{\varepsilon,0}) \to 1 \quad \text{as} \quad \varepsilon \to \infty. \tag{4.19}$$

By (4.16) and (4.19) we see that there is a $\varepsilon_1 > 0$ such that $\Psi_{\varepsilon_1,0} \in \mathcal{Z}$. $\square$

Lemma 4.8. *The number $c_0 = \inf_{u \in \mathcal{Z}} f(u)$ satisfies the inequality $c_0 > S_{h,l}$.*

Proof. Since $\mathcal{Z} \subset \mathcal{M}$, we have

$$S_{h,l} \le c_0.$$

Suppose, by contradiction, that $S_{h,l} = c_0$. By Ekeland variational principle [51], there exists $(u_n) \subset D^{s,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} dx = 1, \quad \alpha(u_n) \to \left(0, \frac{1}{2}\right) \tag{4.20}$$

and

$$f(u_n) \to S_{h,l}, \quad f'|_{\mathcal{M}}(u_n) \to 0. \tag{4.21}$$

Thus, (u_n) is bounded in $D^{s,2}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u_0$ in $D^{s,2}(\mathbb{R}^N)$.

If $v_n = S_{h,l}^{(N-2s)/(2N+4s-2\alpha)} u_n$ and $v_0 = S_{h,l}^{(N-2s)/(2N+4s-2\alpha)} u_0$, we have that $v_n \rightharpoonup v_0$ in $D^{s,2}(\mathbb{R}^N)$. Furthermore, from (4.21) and Lemma 3.5 we obtain

$$I(v_n) \to \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} \quad \text{and} \quad I'(v_n) \to 0.$$

We are going to show that $v_0 \equiv 0$. First of all, note that

$$u_n \not\to u_0 \quad \text{in} \quad D^{s,2}(\mathbb{R}^N), \tag{4.22}$$

because otherwise, $u_0 \neq 0$ and

$$\begin{aligned} S_{h,l} &\le \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx \\ &< \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx + \int_{\mathbb{R}^N} a(x)|u_0|^2 dx = S_{h,l}, \end{aligned}$$

which is a contradiction. Thus, $v_n \not\to v_0$ in $D^{s,2}(\mathbb{R}^N)$ and since (v_n) is a $(PS)_c$ sequence for I, by Theorem 3.2 we have

$$I(v_n) \to I(v_0) + \sum_{j=1}^{k} I_\infty(z_0^j) = \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Using $I'_\infty(z_0^j) = 0$, we get

$$I(v_0) = 0, \quad k = 1, \quad z_0^1 > 0, \quad \text{and} \quad I_\infty(z_0^1) = \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}. \tag{4.23}$$

By virtue of v_0 being weak solution of (P), we have

$$I(v_0) = \frac{N+2s-\alpha}{4N-2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2^*_{\alpha,s}})|v_0|^{2^*_{\alpha,s}} dx = 0,$$

which implies that $v_0 \equiv 0$. Then, (v_n) is a $(PS)_c$ sequence for I such that $v_n \rightharpoonup 0$, $v_n \not\to 0$ and $\int_{\mathbb{R}^N} a(x)|v_n|^2 dx = o_n(1)$. Therefore,

$$\frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + o_n(1) = I(v_n) = I_\infty(v_n) + \int_{\mathbb{R}^N} a(x)|v_n|dx = I_\infty(v_n) + o_n(1) \tag{4.24}$$

and

$$\|I'_\infty(v_n)\|_{D^{-s,2}(\mathbb{R}^N)} \le \|I'(v_n)\|_{D^{-s,2}(\mathbb{R}^N)} + o_n(1). \tag{4.25}$$

From (4.24) and (4.25) we conclude that (v_n) is a $(PS)_c$ sequence for I_∞ and by Lemma 3.1, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$, z_0^1 nontrivial solution of (P_∞) and (w_n) a $(PS)_c$ sequence for I_∞ such that

$$v_n(x) = w_n(x) + R_n^{(N-2s)/2} z_0^1(R_n(x - x_n)) + o_n(1). \tag{4.26}$$

Setting

$$z_n(x) = R_n^{(N-2s)/2} z_0^1(R_n(x - x_n)),$$

and making change of variable, we have

$$I'_\infty(z_n)\varphi = I'_\infty(z_0^1)\varphi_n = 0, \quad \forall \varphi \in D^{s,2}(\mathbb{R}^N), \ \ \forall n \in \mathbb{N},$$

i.e., z_n is a solution of (P_∞), for all $n \in \mathbb{N}$. From z_0^1 solving (2.7) and (2.9) we know that there exist some $\varepsilon_n > 0$, $b_n \in \mathbb{R}^N$ such that

$$z_n(x) = b_{\alpha,s}\left(\frac{\varepsilon_n}{\varepsilon_n^2 + |x - b_n|^2}\right)^{\frac{N-2s}{2}}, \quad \forall x \in \mathbb{R}^N.$$

By (4.26), (4.1), we get

$$u_n(x) = \widetilde{w}_n(x) + \Psi_{\varepsilon_n,b_n}(x) + o_n(1),$$

where

$$\widetilde{w}_n(x) = S_{h,l}^{-\frac{N-2s}{2N+4s-2\alpha}} w_n(x), \quad \Psi_{\varepsilon_n,b_n}(x) = S_{h,l}^{-\frac{N-2s}{2N+4s-2\alpha}} z_n(x).$$

By (4.23) we derive that $w_n \to 0$, which implies that $\widetilde{w}_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$. Therefore, from (4.20), we have

$$\left(0, \frac{1}{2}\right) + o_n(1) = \alpha(u_n) = \alpha(\widetilde{w}_n + \Psi_{\varepsilon_n,b_n} + o_n(1)) = \alpha(\Psi_{\varepsilon_n,b_n})$$

which implies that

$$(i) \ \ \beta(\Psi_{\varepsilon_n,b_n}) \to 0$$

and

$$(ii)\ \ \gamma(\Psi_{\varepsilon_n,b_n}) \to \frac{1}{2}.$$

Passing to a subsequence, one of the following cases must occur.

(a) $\varepsilon_n \to +\infty$ when $n \to +\infty$;
(b) $\varepsilon_n \to \overline{\varepsilon} \neq 0$ when $n \to +\infty$;
(c) $\varepsilon_n \to 0$ and $b_n \to \overline{b}$ when $n \to +\infty$ with $|\overline{b}| < \frac{1}{2}$;
(d) $\varepsilon_n \to 0$ when $n \to +\infty$ and $|b_n| \geq \frac{1}{2}$ for n sufficiently large.

Suppose that (a) is true. Then

$$\gamma(\Psi_{\varepsilon_n,b_n}) = 1 - \frac{1}{S_{h,l}} \int\limits_{B_1(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx$$

together with Lemma 4.1 gives

$$|\gamma(\Psi_{\varepsilon_n,b_n}) - 1| = \frac{1}{S_{h,l}} \int\limits_{B_1(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx = o_n(1),$$

which contradicts (ii).

Suppose that (b) is true. In this case we may suppose that $|b_n| \to +\infty$ because if $b_n \to \overline{b}$, we can prove that

$$\Psi_{\varepsilon_n,b_n} \to \Psi_{\overline{\varepsilon},\overline{b}} \quad \text{in} \quad D^{s,2}(\mathbb{R}^N).$$

Since $\widetilde{w}_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$ and $u_n = \widetilde{w}_n + \Psi_{\varepsilon_n,b_n} + o_n(1)$, we see that (u_n) converges in $D^{s,2}(\mathbb{R}^N)$ but this is a contradiction with (4.22). Hence,

$$\begin{aligned}
\gamma(\Psi_{\varepsilon_n,b_n}) &= \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N} \xi(x)|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx \\
&= \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N\setminus B_1(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx \\
&= 1 - \frac{1}{S_{h,l}} \int\limits_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,0}|^2 dx.
\end{aligned} \tag{4.27}$$

Applying Lebesgue's theorem we can show that

$$\int\limits_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,0}|^2 dx \to 0 \quad \text{as} \quad n \to +\infty$$

and from (4.27) we obtain

$$\gamma(\Psi_{\varepsilon_n,b_n}) \to 1 \quad \text{when} \;\; n \to +\infty,$$

which contradicts to (ii).

Suppose that (c) is true. Note that

$$\begin{aligned}
\gamma(\Psi_{\varepsilon_n,b_n}) &= \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N} \xi(x)|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx \\
&= \frac{1}{S_{h,l}} \int\limits_{\mathbb{R}^N\setminus B_1(0)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,b_n}|^2 dx \\
&= 1 - \frac{1}{S_{h,l}} \int\limits_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,0}|^2 dx.
\end{aligned} \tag{4.28}$$

Therefore, using again the Lebesgue theorem, we find

$$\lim_{n\to+\infty} \int\limits_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon_n,0}|^2 dx = S_{h,l}.$$

From (4.28) we get

$$\gamma(\Psi_{\varepsilon_n,b_n}) \to 0,$$

which is a contradiction with (ii).

Suppose that (d) is true. Since $|b_n| \geq \frac{1}{2}$ for n large, we have that $b_n \nrightarrow 0$ in $\mathbb{R}^N$. From Lemma 4.5 we have

$$\beta(\Psi_{\delta_n,b_n}) = \frac{b_n}{|b_n|} + o_n(1).$$

Thus,

$$\beta(\Psi_{\varepsilon_n,b_n}) \nrightarrow 0$$

which is a contradiction with (i). So, $S_{h,l} < c_0$ and the proof is completed. □

Lemma 4.9. *There is* $\varepsilon_1 \in (0, 1/2)$ *such that*

(a) $f(\Psi_{\varepsilon_1,b}) < \frac{S_{h,l}+c_0}{2}, \quad \forall b \in \mathbb{R}^N$;
(b) $\gamma(\Psi_{\varepsilon_1,b}) < \frac{1}{2}, \quad \forall b \in \mathbb{R}^N$ *such that* $|b| < \frac{1}{2}$;
(c) $\left|\beta(\Psi_{\varepsilon_1,b}) - \frac{b}{|b|}\right| < \frac{1}{4}, \quad \forall b \in \mathbb{R}^N$ *such that* $|b| \geq \frac{1}{2}$.

Proof. Using Lemma 4.3, with $\tau = \frac{c_0 - S_{h,l}}{2} > 0$ and $\varepsilon_1 < \min\{\underline{\varepsilon}, 1/2\}$, we conclude that

$$f(\Psi_{\varepsilon,b}) \leq \sup_{b\in\mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + \frac{c_0 - S_{h,l}}{2} = \frac{S_{h,l}+c_0}{2}, \quad \forall b \in \mathbb{R}^N, \tag{4.29}$$

showing that (a) holds true. Now, be the definition of ξ,

$$\gamma(\Psi_{\varepsilon,b}) = 1 - \frac{1}{S_{h,l}} \int\limits_{B_1(-b)} |(-\Delta)^{\frac{s}{2}} \Psi_{\delta,0}|^2 dx.$$

By Lebesgue's theorem,

$$\lim_{\varepsilon \to 0} \int\limits_{B_1(-b)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx = S_{h,l}$$

proving (b).

Notice that from Lemma 4.5, we derive that

$$\beta(\Psi_{\varepsilon,b}) = \frac{b}{|b|} + o_\varepsilon(1) \quad \text{when } \varepsilon \to 0, \;\; \forall b \in \mathbb{R}^N; \;\; |b| \geq \frac{1}{2}$$

and the proof is over. $\square$

Lemma 4.10. *There is $\varepsilon_2 > 1$ such that*

(a) $f(\Psi_{\varepsilon_2,y}) < \frac{S_{h,l}+c_0}{2}$, $\forall b \in \mathbb{R}^N$;
(b) $\gamma(\Psi_{\varepsilon_2,b}) > \frac{1}{2}$, $\forall b \in \mathbb{R}^N$.

Proof. By Lemma 4.3, we can take $\tau = \frac{c_0 - S_{h,l}}{2} > 0$ and $\varepsilon_2 > \max\{\overline{\varepsilon}, 1\}$ such that

$$f(\Psi_{\varepsilon,b}) \leq \sup_{b \in \mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + \frac{c_0 - S_{h,l}}{2} = \frac{S_{h,l} + c_0}{2}, \quad \forall b \in \mathbb{R}^N. \tag{4.30}$$

Moreover, by the definition of ξ and Lemma 4.1, we see that

$$\gamma(\Psi_{\varepsilon,b}) \to 1 \quad \text{when} \quad \varepsilon \to +\infty$$

and the proof is over. $\square$

Lemma 4.11. *There exists $R > 0$ such that*

(a) $f(\Psi_{\varepsilon,b}) < \frac{S_{h,l}+c_0}{2}$, $\forall b$; $|b| \geq R$ *and* $\varepsilon \in [\varepsilon_1, \varepsilon_2]$;
(b) $(\beta(\Psi_{\varepsilon,b})|b)_{\mathbb{R}^N} > 0$, $\forall b$; $|b| \geq R$ *and* $\varepsilon \in [\varepsilon_1, \varepsilon_2]$.

Proof. Since ε varies in the compact set $[\varepsilon_1, \varepsilon_2]$ by Lemma 4.6 we can find an $R_1 > 0$, big enough, by choosing $\tau = \frac{c_0 - S_{h,l}}{2} > 0$ so that

$$f(\Psi_{\varepsilon,b}) < S_{h,l} + \tau = \frac{S_{h,l} + c_0}{2}, \quad \forall b; \;\; |b| \geq R_1 \;\text{ and }\; \varepsilon \in [\varepsilon_1, \varepsilon_2],$$

and item (a) follows. Now we prove item (b). Note that, for each $y\in\mathbb{R}^N$ we consider the sets $(\mathbb{R}^N)_b^+=\{x\in\mathbb{R}^N:\ (x|b)_{\mathbb{R}^N}>0\}$, $(\mathbb{R}^N)_b^-=\mathbb{R}^N\backslash(\mathbb{R}^N)_b^+$. Since ε varies in the compact set $[\varepsilon_1,\varepsilon_2]$, we can prove there is $R_2>0$ big enough and $r\in(0,1/4)$ such that the following things are true if $|b|\geq R_2$ and $|b-\widetilde{b}|=1/2$,

$$B_r(\widetilde{b})=\{x\in\mathbb{R}^N:\ \ |x-\widetilde{b}|<r\}\subset(\mathbb{R}^N)_b^+.$$

Note that for each $x\in B_r(\widetilde{b})$, by means of equality (1.5) we have

$$\begin{aligned}
|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2&=\int\limits_{\mathbb{R}^N}\left|\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|x-b|^2]^{\frac{N-2s}{2}}}-\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|z-b|^2]^{\frac{N-2s}{2}}}\right|^2\frac{1}{|x-z|^{N+2s}}dz\\
&\geq\int\limits_{B_{r/2}(b)\backslash B_{r/4}(b)}\left|\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|x-b|^2]^{\frac{N-2s}{2}}}-\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+|z-b|^2]^{\frac{N-2s}{2}}}\right|^2\frac{1}{|x-z|^{N+2s}}dz\\
&\geq\int\limits_{B_{r/2}(b)\backslash B_{r/4}(b)}\left|\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+(3/4)^2]^{\frac{N-2s}{2}}}-\frac{\widetilde{C}\varepsilon^{\frac{N-2s}{2}}}{[\varepsilon^2+(1/8)^2]^{\frac{N-2s}{2}}}\right|^2\frac{1}{(7/8)^{N+2s}}dz\\
&\geq\frac{\widetilde{C}^2\varepsilon_1^{N-2s}}{(7/8)^{N+2s}}\int\limits_{B_{r/2}(b)\backslash B_{r/4}(b)}\left|\frac{1}{[\varepsilon_2^2+(3/4)^2]^{\frac{N-2s}{2}}}-\frac{1}{[\varepsilon_2^2+(1/8)^2]^{\frac{N-2s}{2}}}\right|^2dz\\
&=H_1>0.
\end{aligned}\tag{4.31}$$

Therefore,

$$\begin{aligned}
(\beta(\Psi_{\varepsilon,b})|b)_{\mathbb{R}^N}&\geq\frac{1}{S_{h,l}}\int\limits_{B_r(\widetilde{b})}\frac{(x|b)_{\mathbb{R}^N}}{|x|}H_1dx+\frac{1}{S_{h,l}}\int\limits_{(\mathbb{R}^N)_b^-}\frac{(x|b)_{\mathbb{R}^N}}{|x|}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx\\
&=\frac{|b|}{S_{h,l}}\int\limits_{B_r(\widetilde{b})}\frac{(x|b)_{\mathbb{R}^N}}{|x||b|}H_1dx-\frac{|b|}{S_{h,l}}\int\limits_{(\mathbb{R}^N)_b^-}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx.
\end{aligned}\tag{4.32}$$

Moreover, we have

$$\frac{1}{S_{h,l}}\int\limits_{B_r(\widetilde{b})}\frac{(x|b)_{\mathbb{R}^N}}{|x||b|}H_1dx\geq\frac{C}{S_{h,l}}\int\limits_{B_r(\widetilde{b})}\frac{H_1}{|x|}dx=H_2>0.\tag{4.33}$$

From Proposition 2.2 [39], (4.5) and (4.8), we can derive that

$$
\begin{aligned}
&\int\limits_{(\mathbb{R}^N)_b^-} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx \leq \int\limits_{B_{R_2}^c(b)} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx\\
&\leq C\int\limits_{B_{R_2}^c(b)} |\nabla\Psi_{\varepsilon,b}|^2dx = C\int\limits_{B_{R_2}^c(0)} |\nabla\Psi_{\varepsilon,0}|^2dx\\
&\leq C\varepsilon^{2s-2}\int\limits_{B_{(R_2)/\varepsilon}^c(0)} \frac{|z|^2}{(1+|z|^2)^{N-2s+2}}dz = C_1\varepsilon^{N-2s}R_2^{-N-2+4s}.
\end{aligned}
\tag{4.34}
$$

It is possible to choose $R_2 > 0$ large, such that for all $b \in \mathbb{R}^N$ with $|b| > R_2$, we have

$$
\frac{1}{S_{h,l}}\int\limits_{(R^N)_b^-} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx < \frac{H_2}{2}. \tag{4.35}
$$

From (4.32)-(4.35), we get

$$
\begin{aligned}
(\beta(\Psi_{\varepsilon,b})|b)_{\mathbb{R}^N} &\geq |b|\left\{\frac{1}{S_{h,l}}\int\limits_{B_r(\widetilde{b})} \frac{(x|b)_{\mathbb{R}^N}}{|x||b|}H_1dx - \frac{1}{S_{h,l}}\int\limits_{(R^N)_b^-} |(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,b}(x)|^2dx\right\}\\
&\geq |b|\left(H_2 - \frac{H_2}{2}\right) > 0,
\end{aligned}
\tag{4.36}
$$

for all $b \in \mathbb{R}^N$ with $|b| > R = \max\{R_1, R_2\} > 0$ and for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, and the proof is over. □

5. Proof of the main theorem

In this section we prove Theorem 1.1. For this aim, we first fix some notations and give some more technical lemmas. Let us consider now the sets

$$
V = \{(b,\varepsilon) \in \mathbb{R}^N \times (0,\infty) : \ |b| < R \ \text{ and } \ \varepsilon \in (\varepsilon_1, \varepsilon_2)\},
$$

where $\varepsilon_1, \varepsilon_2$ and R are given in Lemmas 4.9, 4.10 and 4.11, respectively. Moreover, we define the continuous function $Q : \mathbb{R}^N \times (0,\infty) \to D^{s,2}(\mathbb{R}^N)$ given by

$$
Q(b,\varepsilon) = \Psi_{\varepsilon,b}.
$$

Under the above notations, we also denote the sets

$$
\Theta = \{Q(b,\varepsilon) : \ (b,\varepsilon) \in \overline{V}\},
$$

$$
\mathcal{H} = \left\{h \in C(\Sigma\cap\mathcal{M}, \Sigma\cap\mathcal{M}) : \ h(u) = u, \ \forall u \in (\Sigma\cap\mathcal{M}) : \ f(u) < \frac{S_{h,l}+c_0}{2}\right\},
$$

and

$$\Gamma = \{\mathcal{A} \subset (\Sigma \cap \mathcal{M}) : \mathcal{A} = h(\Theta), h \in \mathcal{H}\}.$$

Clearly, $\Theta \subset (\Sigma \cap \mathcal{M})$, $\Theta = Q(\overline{V})$ is compact and $\mathcal{H} \neq \emptyset$, because the identity function is in $\mathcal{H}$.

Lemma 5.1. *Let $\mathcal{F} : \overline{V} \to \mathbb{R}^{N+1}$ be a function given by*

$$\mathcal{F} = (\alpha \circ Q)(b, \varepsilon) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \xi(x) \right) |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}|^2 dx.$$

Then

$$deg(\mathcal{F}, V, (0, 1/2)) = 1.$$

Proof. Let $G : [0, 1] \times \overline{V} \to \mathbb{R}^{N+1}$ be the homotopy given by

$$G(t, (b, \varepsilon)) = t\mathcal{F}(b, \varepsilon) + (1 - t)I_{\overline{V}}(b, \varepsilon),$$

where $I_{\overline{V}}(b, \varepsilon)$ is the identity. We are going to show that $(0, 1/2) \notin G([0, 1] \times (\partial V))$, i.e.,

$$t\beta(\Psi_{\varepsilon,b}) + (1 - t)b \neq 0, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall (b, \varepsilon) \in \partial V \tag{5.1}$$

or

$$t\gamma(\Psi_{\varepsilon,b}) + (1 - t)\varepsilon \neq \frac{1}{2}, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall (b, \varepsilon) \in \partial V. \tag{5.2}$$

Notice that $\partial V = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 = \{(b, \varepsilon_1) : |b| < 1/2\},$$
$$\Gamma_2 = \{(b, \varepsilon_1) : 1/2 \le |b| \le R\},$$
$$\Gamma_3 = \{(b, \varepsilon_2) : |b| \le R\},$$

and

$$\Gamma_4 = \{(b, \varepsilon) : |b| = R, \quad \text{and} \quad \varepsilon \in [\varepsilon_1, \varepsilon_2]\}.$$

If $(b, \varepsilon) \in \Gamma_1$, then $(b, \varepsilon) = (b, \varepsilon_1)$. From Lemma 4.9, $\varepsilon_1 < \frac{1}{2}$ and $\gamma(\Psi_{\varepsilon_1,b}) < \frac{1}{2}$. Hence,

$$\begin{aligned} t\gamma(\Psi_{\varepsilon,b}) + (1 - t)\varepsilon &= t\gamma(\Psi_{\varepsilon_1,b}) + (1 - t)\varepsilon_1 \\ &< \frac{t}{2} + \frac{1 - t}{2} = \frac{1}{2}, \quad \forall t \in [0, 1] \text{ and } \forall (b, \varepsilon) \in \Gamma_1 \end{aligned}$$

showing that (5.2) is true and $(0, 1/2) \notin G([0, 1] \times \Gamma_1)$.

If $(b, \varepsilon) \in \Gamma_2$, then $(b, \varepsilon) = (b, \varepsilon_1)$ and $|b| \ge \frac{1}{2}$. Using Triangle inequality and by Lemma 4.9-(c) we derive

$$
\begin{aligned}
|t\beta(\Psi_{\varepsilon,b})-(1-t)b| &\geq \left|(1-t)b+\frac{tb}{|b|}\right|-\left|\frac{tb}{|b|}-t\beta(\Psi_{\varepsilon_1,b})\right| \\
&= \frac{|(1-t)|b|+t|}{|b|}|b|-t\left|\frac{b}{|b|}-\beta(\Psi_{\varepsilon_1,b})\right| \\
&= |(1-t)|b|+t|-t\left|\frac{b}{|b|}-\beta(\Psi_{\varepsilon_1,b})\right| \\
&= (1-t)|b|+t-t\left|\frac{b}{|b|}-\beta(\Psi_{\varepsilon_1,b})\right| \\
&\geq (1-t)|b|+t-\frac{t}{4}\geq \frac{1}{2}+\frac{t}{4}>0 \ \ \forall t\in[0,1] \ \text{ and } \ \forall (b,\varepsilon)\in\Gamma_2,
\end{aligned}
$$

showing that (5.1) is true and $(0,1/2)\notin G([0,1]\times\Gamma_2)$.

If $(\varepsilon,\varepsilon)\in\Gamma_3$, then $(b,\varepsilon)=(b,\varepsilon_2)$. By Lemma 4.10, $\varepsilon_2>\frac{1}{2}$ and $\gamma(\Psi_{\varepsilon_2,b})>\frac{1}{2}$, and so,

$$
t\gamma(\Psi_{\varepsilon,b})+(1-t)\varepsilon=t\gamma(\Psi_{\varepsilon_2,b})+(1-t)\varepsilon_2>\frac{t}{2}+\frac{1-t}{2}=\frac{1}{2}, \quad \forall t\in[0,1] \ \text{ and } \ \forall (b,\varepsilon)\in\Gamma_3,
$$

showing that (5.2) is true and $(0,1/2)\notin G([0,1]\times\Gamma_3)$.

If $(b,\varepsilon)\in\Gamma_4$, we must have $|b|=R$. Thus, by Lemma 4.11-(b), $(\beta(\Psi_{\varepsilon,b}|b)_{\mathbb{R}^N}>0$. From this,

$$
(t\beta(\Psi_{\varepsilon,b})+(1-t)b|b)_{\mathbb{R}^N}=t(\beta(\Psi_{\varepsilon,b})|b)_{\mathbb{R}^N}+(1-t)(b|b)_{\mathbb{R}^N}>0, \quad \forall t\in[0,1].
$$

Therefore (5.1) is true and $(0,1/2)\notin G([0,1]\times\Gamma_4)$.

The previous analysis ensures that $(0,1/2)\notin G([0,1]\times\partial V)$, then by properties of the topological degree

$$
\deg(\mathcal{F},V,(0,1/2))=\deg(I_{\overline{V}},V,(0,1/2)).
$$

Since $(0,1/2)\in V$, we deduce that

$$
\deg(\mathcal{F},V,(0,1/2))=\deg(I_{\overline{V}},V,(0,1/2))=1. \quad \square
$$

Lemma 5.2. *If $\mathcal{A}\in\Gamma$, then $\mathcal{A}\cap\mathcal{Z}\neq\emptyset$.*

Proof. It is sufficient to prove that for all $h\in\mathcal{H}$, there exists $(b_0,\varepsilon_0)\in\overline{V}$ such that

$$
(\alpha\circ h\circ Q)(b_0,\varepsilon_0)=\left(0,\frac{1}{2}\right).
$$

Given $h\in\mathcal{H}$, let

$$
\mathcal{F}_h:\overline{V}\to\mathbb{R}^{N+1}
$$

be the continuous function given by

$$\mathcal{F}_h(b,\varepsilon) = (\alpha \circ h \circ Q)(b,\varepsilon).$$

We are going to prove that $\mathcal{F}_h = \mathcal{F}$ in ∂V. Note that

$$\partial V = L_1 \cup L_2 \cup L_3, \tag{5.3}$$

where

$$L_1 = \{(b,\varepsilon_1) : |b| \le R\}, \quad L_2 = \{(b,\varepsilon_2) : |b| \le R\}$$

and

$$L_3 = \{(b,\varepsilon) : |b| = R \text{ and } \varepsilon \in [\varepsilon_1,\varepsilon_2]\}.$$

If $(b,\varepsilon) \in L_1$, then $(b,\varepsilon) = (b,\varepsilon_1)$ and by Lemma 4.9-(a), we get

$$f(Q(b,\varepsilon)) = f(Q(b,\varepsilon_1) = f(\Psi_{\varepsilon_1,b}) < \frac{S_{h,l}+c_0}{2}, \quad \forall (b,\varepsilon) \in L_1. \tag{5.4}$$

If $(b,\varepsilon) \in L_2$, then $(b,\varepsilon) = (b,\varepsilon_2)$ and by Lemma 4.10-(a), we get

$$f(Q(b,\varepsilon)) = f(Q(b,\varepsilon_2) = f(\Psi_{\varepsilon_2,b}) < \frac{S_{h,l}+c_0}{2}, \quad \forall (b,\varepsilon) \in L_2. \tag{5.5}$$

If $(b,\varepsilon) \in L_3$, then $|b| = R$ and by Lemma 4.11-(a), we have

$$f(Q(b,\varepsilon)) = f(\Psi_{\varepsilon,b}) < \frac{S_{h,l}+c_0}{2}, \quad \forall (b,\varepsilon) \in L_3. \tag{5.6}$$

By (5.4)-(5.6), we derive that

$$f(Q(b,\varepsilon)) < \frac{S_{h,l}+c_0}{2}, \quad \forall (b,\varepsilon) \in \partial V.$$

Thus,

$$\begin{aligned}
\mathcal{F}_h(b,\varepsilon) &= (\alpha \circ h \circ Q)(b,\varepsilon) = (\alpha \circ h)Q(b,\varepsilon) \\
&= \alpha(h(Q(b,\varepsilon))) \\
&= (\alpha \circ Q)(b,\varepsilon) = \mathcal{F}(b,\varepsilon), \quad \forall (b,\varepsilon) \in \partial V.
\end{aligned}$$

In view of $(0,1/2) \notin \mathcal{F}(\partial V)$, we have

$$\deg(\mathcal{F}, V, (0,1/2)) = \deg(\mathcal{F}_h, V, (0,1/2)).$$

From Lemma 5.1, we get

$$\deg(\mathcal{F}_h, V, (0,1/2)) = \deg(\mathcal{F}, V, (0,1/2)) = 1,$$

and there exists $(b_0, \varepsilon_0) \in V$ such that

$$\mathcal{F}_h(b_0, \varepsilon_0) = (\alpha \circ h \circ Q)(b_0, \varepsilon_0) = \left(0, \frac{1}{2}\right)$$

and the proof is over. $\square$

Proof of Theorem 1.1. Let us define the number

$$c = \inf_{\mathcal{A} \in \Gamma} \sup_{u \in \mathcal{A}} f(u).$$

Our first step is to prove that

$$S_{h,l} < c < 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}. \tag{5.7}$$

Note that, by Lemma 4.4,

$$c = \inf_{\mathcal{A} \in \Gamma} \sup_{u \in \mathcal{A}} f(u) \le \max_{u \in \Theta} f(u) \le \sup_{y \in \mathbb{R}^N, \delta \in (0,\infty)} f(\Psi_{\varepsilon,b}) < 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}.$$

On the other hand, from Lemma 5.2 and Lemma 4.8 we get

$$S_{h,l} < c_0 = \inf_{u \in \mathcal{Z}} f(u) \le c = \inf_{\mathcal{A} \in \Gamma} \sup_{u \in \mathcal{A}} f(u) < 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}, \tag{5.8}$$

from where it follows (5.7).

Using the definition of c, there exists $(u_n) \subset (\Sigma \cap \mathcal{M})$ such that

$$f(u_n) \to c. \tag{5.9}$$

Suppose, by contradiction, that

$$f'|_{\mathcal{M}}(u_n) \nrightarrow 0.$$

Then, there exists $(u_{n_j}) \subset (u_n)$ such that

$$\|f'|_{\mathcal{M}}(u_{n_j})\|_* \ge l > 0, \quad \forall j \in \mathbb{N}.$$

By a Deformation Lemma [51], there exists a continuous application $\eta : [0, 1] \times (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$, $\varepsilon_0 > 0$ such that

(1) $\eta(0, u) = u$;
(2) $\eta(t, u) = u, \forall u \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \backslash f^{c+\varepsilon_0}\}, \ \forall t \in [0, 1]$;
(3) $\eta(1, f^{c+\varepsilon_0/2} \subset f^{c-\varepsilon_0/2}$.

From the definition of c, there exists $\widehat{\mathcal{A}} \in \Gamma$ such that

$$c \leq \max_{u \in \widehat{\mathcal{A}}} f(u) < c + \frac{\varepsilon_0}{2},$$

where

$$\widehat{\mathcal{A}} \subset f^{c+\frac{\varepsilon_0}{2}}. \tag{5.10}$$

Since $\widehat{\mathcal{A}} \in \Gamma$, we have $\widehat{\mathcal{A}} \subset (\Sigma \cap \mathcal{M})$ and there exists $\widetilde{h} \in \mathcal{H}$ such that

$$\widetilde{h}(\Theta) = \widehat{\mathcal{A}}. \tag{5.11}$$

From definition of η, we have

$$\eta(1, \widehat{\mathcal{A}}) \subset (\Sigma \cap \mathcal{M}). \tag{5.12}$$

Let $h^* : (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$ be the function given by $h^*(u) = \eta(1, \widetilde{h}(u))$ and note that $h^* \in C(\Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M})$. We are going to show that

$$f^{c+\varepsilon_0} \backslash f^{c-\varepsilon_0} \subset f^{2^{(4s-\alpha)(2N-\alpha)} S_{h,l}} \backslash f^{(S_{h,l}+c_0)/2}. \tag{5.13}$$

Indeed, given $u \in f^{c+\varepsilon_0} \backslash f^{c-\varepsilon_0}$, we have

$$c - \varepsilon_0 < f(u) \leq c + \varepsilon_0$$

and by (5.8), for ε_0 sufficiently small, we get

$$c - \varepsilon_0 < f(u) \leq c + \varepsilon_0 < 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l}. \tag{5.14}$$

On the other hand, Lemma 4.8 together with (5.8) gives

$$\frac{S_{h,l} + c_0}{2} < c_0 - \varepsilon_0 \leq c - \varepsilon_0 < f(u). \tag{5.15}$$

Now, (5.13) follows from (5.14) and (5.15).

Consider $u \in (\Sigma \cap \mathcal{M})$ with

$$f(u) < \frac{S_{h,l} + c_0}{2}. \tag{5.16}$$

Then,

$$\widetilde{h}(u) = u$$

and from (5.16), we have that $u \notin f^{2^{(4s-\alpha)(2N-\alpha)} S_{h,l}} \backslash f^{(S_{h,l}+c_0)/2}$ and by (5.13), we have

$$u \notin f^{c+\varepsilon_0} \backslash f^{c-\varepsilon_0}.$$

Thus,

$$u \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \backslash f^{c+\varepsilon_0}\}$$

and by Deformation Lemma, we obtain

$$\eta(1, u) = u.$$

Therefore,

$$h^*(u) = \eta(1, \widetilde{h}(u)) = \eta(1, u) = u,$$

which shows that $h^* \in \mathcal{H}$, and so,

$$h^*(\Theta) = \eta(1, \widetilde{h}(\Theta)) = \eta(1, \widehat{\mathcal{A}}) \in \Gamma. \tag{5.17}$$

Hence,

$$c = \inf_{\mathcal{A} \in \Gamma} \max_{u \in \mathcal{A}} f(u) \leq \max_{u \in \eta(1, \widehat{\mathcal{A}})} f(u). \tag{5.18}$$

On the other hand, from Deformation Lemma and (5.10), we get

$$\eta(1, \widehat{\mathcal{A}}) \subset \eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}.$$

That is,

$$f(u) \leq c - \frac{\varepsilon_0}{2}, \quad \forall u \in \eta(1, \widehat{\mathcal{A}}),$$

which implies that

$$\max_{u \in \eta(1, \widehat{\mathcal{A}})} f(u) \leq c - \frac{\varepsilon_0}{2}$$

and using (5.18), we conclude that

$$c \leq \max_{u \in \eta(1, \bar{\mathcal{A}})} f(u) \leq c - \frac{\varepsilon_0}{2},$$

which is a contradiction. Therefore, we must have

$$f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0$$

and from Lemma 3.6, up to a subsequence, we have $u_n \to \widetilde{u}_0$ in $D^{s,2}(\mathbb{R}^N)$, and $\widetilde{u}_0$ satisfies that

$$f(\widetilde{u}_0) = c \quad \text{and} \quad f'|_{\mathcal{M}}(\widetilde{u}_0) = 0,$$

and from (5.7), we get $S_{h,l} < f(\widetilde{u}_0) < 2^{(4s-\alpha)(2N-\alpha)} S_{h,l}$. The positivity of $\widetilde{u}_0$ is a consequence of a maximum principle that can be found in Proposition 2.17 [49]. $\square$

Acknowledgments

The authors are grateful to the anonymous reviewers for the careful reading of the manuscript and for valuable comments and suggestions. Xiaoming He is supported by the National Science Foundation of China (No. 11771468, No. 11971027). The work of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2020-0068, within PNCDI III.

Appendix

As we pointed out in Introduction, the nonlocal term is involved with critical growth, and we shall encounter the problem of the convergence of integral with nonlocal term. To this end, we present a technical lemma which is useful in the proof of Lemma 3.1 and Theorem 3.2.

Lemma A. *Let (ξ_n) be a bounded sequence in $D^{s,2}(\mathbb{R}^N)$ such that $\xi_n \to 0$ a.e. in $\mathbb{R}^N$. Denote by $A(u) = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u$. Then for each $w \in D^{s,2}(\mathbb{R}^N)$, we have the following estimates:*

$$\int_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2^*_s}{2^*_s-1}} dx = o_n(1).$$

Proof. By assumption, we have

$$\begin{aligned}
&|A(\xi_n + w) - A(\xi_n)| \\
&= \left| \int_0^1 \frac{d}{dt} \left\{ (I_\alpha * |\xi_n + tw|^{2^*_{\alpha,s}})|\xi_n + tw|^{2^*_{\alpha,s}-2}[\xi_n + tw] \right\} dt \right| \\
&= \Bigg| \int_0^1 (I_\alpha * (2^*_{\alpha,s}|\xi_n + tw|^{2^*_{\alpha,s}-2}[\xi_n + tw]w))|\xi_n + tw|^{2^*_{\alpha,s}-2}[\xi_n + tw] dt \\
&\quad + (2^*_{\alpha,s} - 1) \int_0^1 (I_\alpha * |\xi_n + tw|^{2^*_{\alpha,s}})|\xi_n + tw|^{2^*_{\alpha,s}-2} w dt \Bigg| \\
&\le 2^*_{\alpha,s} \int_0^1 (I_\alpha * (|\xi_n + tw|^{2^*_{\alpha,s}-1}|w|))|\xi_n + tw|^{2^*_{\alpha,s}-1} dt \\
&\quad + (2^*_{\alpha,s} - 1) \int_0^1 (I_\alpha * |\xi_n + tw|^{2^*_{\alpha,s}})|\xi_n + tw|^{2^*_{\alpha,s}-2} |w| dt \\
&\le C_1 (I_\alpha * ((|\xi_n|^{2^*_{\alpha,s}-1} + |w|^{2^*_{\alpha,s}-1})|w|))[|\xi_n|^{2^*_{\alpha,s}-1} + |w|^{2^*_{\alpha,s}-1}] \\
&\quad + C_2 (I_\alpha * (|\xi_n|^{2^*_{\alpha,s}} + |w|^{2^*_{\alpha,s}})(|\xi_n|^{2^*_{\alpha,s}-2} + |w|^{2^*_{\alpha,s}-2})|w| \\
&\le C_1 (I_\alpha * (\varepsilon|\xi_n|^{2^*_{\alpha,s}} + C_\varepsilon|w|^{2^*_{\alpha,s}}))[|\xi_n|^{2^*_{\alpha,s}-1} + |w|^{2^*_{\alpha,s}-1}]
\end{aligned}$$

$$
\begin{aligned}
&+ C_2(I_\alpha * (|\xi_n|^{2^*_{\alpha,s}} + |w|^{2^*_{\alpha,s}})(\varepsilon|\xi_n|^{2^*_{\alpha,s}-1} + C_\varepsilon|w|^{2^*_{\alpha,s}-1})\\
\le{}& \varepsilon C_3\Big[(I_\alpha * |\xi_n|^{2^*_{\alpha,s}})|\xi_n|^{2^*_{\alpha,s}-1} + (I_\alpha * |w|^{2^*_{\alpha,s}})|\xi_n|^{2^*_{\alpha,s}-1} + (I_\alpha * |\xi_n|^{2^*_{\alpha,s}})|w|^{2^*_{\alpha,s}-1}\Big]\\
&+ C_\varepsilon C_4\Big[(I_\alpha * |\xi_n|^{2^*_{\alpha,s}})|w|^{2^*_{\alpha,s}-1} + (I_\alpha * |w|^{2^*_{\alpha,s}})|\xi_n|^{2^*_{\alpha,s}-1}\Big]\\
&+ C_\varepsilon C_5(I_\alpha * |w|^{2^*_{\alpha,s}})|w|^{2^*_{\alpha,s}-1}\\
\triangleq{}& \mathcal{Q}_{\varepsilon,n}(x) + C_\varepsilon C_5(I_\alpha * |w|^{2^*_{\alpha,s}})|w|^{2^*_{\alpha,s}-1}. \qquad (A.1)
\end{aligned}
$$

Recall the Hardy-Littlewood-Sobolev inequality [36, Theorem 4.3]: if $\theta \in \left(1, \frac{N}{N-\alpha}\right)$ then for every $v \in L^\theta(\mathbb{R}^N)$, $I_\alpha * v \in L^{\frac{N\theta}{N-(N-\alpha)\theta}}(\mathbb{R}^N)$ and

$$
\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{N\theta}{N-(N-\alpha)\theta}} \le C\left(\int_{\mathbb{R}^N} |v|^\theta dx\right)^{\frac{N}{N-(N-\alpha)\theta}}. \qquad (A.2)
$$

For each $\varepsilon > 0$, let us consider the function $G_{\varepsilon,n}$ given by

$$
G_{\varepsilon,n}(x) = \max\{|A(\xi_n + w) - A(\xi_n) - A(w)| - \mathcal{Q}_{\varepsilon,n}(x), 0\},
$$

which satisfies

$$
G_{\varepsilon,n}(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N
$$

and using Hölder's inequality and $(A.2)$ we see that

$$
0 \le G_{\varepsilon,n}(x) \le C_6(I_\alpha * |w|^{2^*_{\alpha,s}})|w|^{2^*_{\alpha,s}-1} \in L^{\frac{2^*_s}{2^*_s-1}}(\mathbb{R}^N).
$$

Therefore, by the Lebesgue dominated convergence theorem we have

$$
\int_{\mathbb{R}^N} |G_{\varepsilon,n}(x)|^{\frac{2^*_s}{2^*_s-1}} dx \to 0 \quad \text{as } n \to \infty. \qquad (A.3)
$$

From the definition of $G_{\varepsilon,n}(x)$, we get

$$
|A(\xi_n + w) - A(\xi_n) - A(w)| \le \mathcal{Q}_{\varepsilon,n}(x) + C_7 G_{\varepsilon,n}(x),
$$

which yields that

$$
|A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2^*_s}{2^*_s-1}} \le |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{2^*_s}{2^*_s-1}} + C_8|G_{\varepsilon,n}(x)|^{\frac{2^*_s}{2^*_s-1}}.
$$

By $(A.3)$, we obtain the following estimates

$$\begin{aligned}
&\int_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2_s^*}{2_s^*-1}} dx \\
&\le \int_{\mathbb{R}^N} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{2_s^*}{2_s^*-1}} dx + C_8 \int_{\mathbb{R}^N} |G_{\varepsilon,n}(x)|^{\frac{2_s^*}{2_s^*-1}} dx. \\
&= \int_{\mathbb{R}^N} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{2_s^*}{2_s^*-1}} dx + o_n(1).
\end{aligned} \tag{A.4}$$

Now we estimate the last integral of $(A.4)$. Since (ξ_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, $\xi_n \to 0$ a.e. in $\mathbb{R}^N$, then $\xi_n \rightharpoonup 0$ in $L^{2_s^*}(\mathbb{R}^N)$. By the definition of $\mathcal{Q}_{\varepsilon,n}(x)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\mathcal{Q}_{\varepsilon,n}(x)|^{\frac{2_s^*}{2_s^*-1}} dx \\
&\le \varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |\xi_n|^{2_{\alpha,s}^*})|\xi_n|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&\quad + \varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |w|^{2_{\alpha,s}^*})|\xi_n|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx + \varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |\xi_n|^{2_{\alpha,s}^*})|w|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&\quad + C_\varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |w|^{2_{\alpha,s}^*})|\xi_n|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx + C_\varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |\xi_n|^{2_{\alpha,s}^*})|w|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&= \varepsilon C \int_{\mathbb{R}^N} \left[(I_\alpha * |\xi_n|^{2_{\alpha,s}^*})|\xi_n|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&\quad + D_\varepsilon \int_{\mathbb{R}^N} \left[(I_\alpha * |w|^{2_{\alpha,s}^*})|\xi_n|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&\quad + D_\varepsilon \int_{\mathbb{R}^N} \left[(I_\alpha * |\xi_n|^{2_{\alpha,s}^*})|w|^{2_{\alpha,s}^*-1}\right]^{\frac{2_s^*}{2_s^*-1}} dx \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{A.5}$$

where $D_\varepsilon = C(\varepsilon + C_\varepsilon)$. Next we estimate the three integrals in the right-side of $(A.5)$.

For I_1, by Hölder's inequality, $(A.2)$ and (ξ_n) is bounded in $D^{s,2}(\mathbb{R}^N)$, we have

$$I_1 = \varepsilon C \int_{\mathbb{R}^N} (I_\alpha * |\xi_n|^{2_{\alpha,s}^*})^{\frac{2_s^*}{2_s^*-1}} |\xi_n|^{(2_{\alpha,s}^*-1)\times\frac{2_s^*}{2_s^*-1}} dx$$

$$
\begin{aligned}
&= \varepsilon C \int\limits_{\mathbb{R}^N} (I_\alpha * |\xi_n|^{2^*_{\alpha,s}})^{\frac{2N}{N+2s}} |\xi_n|^{\frac{N+2s-\alpha}{N-2s}\times\frac{2N}{N+2s}} dx \\
&\le \varepsilon C \left[\int\limits_{\mathbb{R}^N} (I_\alpha * |\xi_n|^{2^*_{\alpha,s}})^{\frac{2N}{\alpha}} dx\right]^{\frac{\alpha}{N+2s}} \left[\int\limits_{\mathbb{R}^N} |\xi_n|^{\frac{2N}{N-2s}} dx\right]^{\frac{N+2s-\alpha}{N+2s}} \\
&\le \varepsilon C_1 \left[\int\limits_{\mathbb{R}^N} |\xi_n|^{2^*_s} dx\right]^{\frac{2N-\alpha}{N+2s}} \left[\int\limits_{\mathbb{R}^N} |\xi_n|^{2^*_s} dx\right]^{\frac{N+2s-\alpha}{N+2s}} \\
&\le \varepsilon C_2 \left(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\xi_n|^2 dx\right)^{\frac{2N(3N+2s-2\alpha)}{(N+2s)(N-2s)}} \\
&\le \varepsilon C_3.
\end{aligned} \tag{A.6}
$$

For I_2, we have

$$
\begin{aligned}
I_2 &= D_\varepsilon \int\limits_{\mathbb{R}^N} (I_\alpha * |w|^{2^*_{\alpha,s}})^{\frac{2^*_s}{2^*_s-1}} |\xi_n|^{(2^*_{\alpha,s}-1)\times\frac{2^*_s}{2^*_s-1}} dx \\
&= D_\varepsilon \int\limits_{\mathbb{R}^N} (I_\alpha * |w|^{2^*_{\alpha,s}})^{\frac{2N}{N+2s}} |\xi_n|^{\frac{N+2s-\alpha}{N-2s}\times\frac{2N}{N+2s}} dx \\
&= o_n(1),
\end{aligned} \tag{A.7}
$$

by virtue of $(I_\alpha * |w|^{2^*_{\alpha,s}})^{\frac{2N}{N+2s}} \in L^{\frac{N+2s}{\alpha}}(\mathbb{R}^N)$ and $|\xi_n|^{\frac{N+2s-\alpha}{N-2s}\times\frac{2N}{N+2s}} \rightharpoonup 0$ in $L^{\frac{N+2s}{N+2s-\alpha}}(\mathbb{R}^N)$.

For I_3, we have

$$
\begin{aligned}
I_3 &= D_\varepsilon \int\limits_{\mathbb{R}^N} (I_\alpha * |\xi_n|^{2^*_{\alpha,s}})^{\frac{2^*_s}{2^*_s-1}} |w|^{(2^*_{\alpha,s}-1)\times\frac{2^*_s}{2^*_s-1}} dx \\
&= D_\varepsilon \int\limits_{\mathbb{R}^N} (I_\alpha * |\xi_n|^{2^*_{\alpha,s}})^{\frac{2N}{N+2s}} |w|^{\frac{N+2s-\alpha}{N-2s}\times\frac{2N}{N+2s}} dx \\
&= o_n(1),
\end{aligned} \tag{A.8}
$$

by virtue of $(I_\alpha * |\xi_n|^{2^*_{\alpha,s}})^{\frac{2N}{N+2s}} \rightharpoonup 0$ in $L^{\frac{N+2s}{\alpha}}(\mathbb{R}^N)$ and $|w|^{\frac{N+2s-\alpha}{N-2s}\times\frac{2N}{N+2s}} \in L^{\frac{N+2s}{N+2s-\alpha}}(\mathbb{R}^N)$. From $(A.4)-(A.8)$, we obtain

$$
\int\limits_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2^*_s}{2^*_s-1}} dx \to 0 \quad \text{as } n \to \infty,
$$

which completes the proof. □

References

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 248 (2004) 423–443.

[2] C.O. Alves, Existence of positive solutions for a problem with lack of compactness involving the p-Laplacian, Nonlinear Anal. 51 (2002) 1187–1206.

[3] C.O. Alves, G.M. Figueiredo, Existence of positive solution of the Choquard equation $-\Delta u + a(x)u = (I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu-2}u$ in $\mathbb{R}^N$, arXiv:1812.04875v1 [math.AP], 12 Dec 2018.

[4] D. Applebaum, Lévy processes-from probability to finance and quantum groups, Not. Am. Math. Soc. 51 (2004) 1336–1347.

[5] B. Barris, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (2015) 875–900.

[6] J. Bellazzini, M. Ghimenti, C. Mercuri, V. Moroz, J. van Schaftingen, Sharp Gagliardo-Nirenberg inequalities in fractional Coulomb-Sobolev spaces, Trans. Am. Math. Soc. 370 (2018) 8285–8310.

[7] V. Benci, G. Cerami, Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$, J. Funct. Anal. 88 (1990) 90–117.

[8] B. Buffoni, L. Jeanjean, C.A. Stuart, Existence of a nontrivial solution to a strongly indefinite semilinear equation, Proc. Am. Math. Soc. 119 (1993) 179–186.

[9] D. Cassani, J. Zhang, Choquard-type equations with Hardy-Littlewood-Sobolev upper-critical growth, Adv. Nonlinear Anal. 8 (1) (2019) 1184–1212.

[10] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, Calc. Var. Partial Differ. Equ. 3 (1992) 493–512.

[11] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys. 63 (2012) 233–248.

[12] M. Clapp, D. Salazar, Positive and sign changing solutions to a nonlinear Choquard equation, J. Math. Anal. Appl. 407 (2017) 1–15.

[13] J.N. Correa, G.M. Figueiredo, Existence of positive solution of the equation $(-\Delta)^s u + a(x)u = |u|^{2^*_s-2}u$, Calc. Var. Partial Differ. Equ. 58 (2019) 63.

[14] A. Cotsiolis, N.K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. 295 (2004) 225–236.

[15] P. d'Avenia, G. Siciliano, M. Squassina, On fractional Choquard equations, Math. Models Methods Appl. Sci. 25 (2015) 1447–1476.

[16] P. d'Avenia, G. Siciliano, M. Squassina, Existence results for a doubly nonlocal equation, São Paulo J. Math. Sci. 9 (2015) 311–324.

[17] A. Elgart, B. Schlein, Mean field dynamics of boson stars, Commun. Pure Appl. Math. 60 (2007) 500–545.

[18] H. Fröhlich, Theory of electrical breakdown in ionic crystal, Proc. R. Soc. Edinb., Sect. A 160 (901) (1937) 230–241.

[19] F. Gao, M. Yang, On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation, Sci. China Math. 61 (2018) 1219–1242.

[20] A. Garroni, S. Müller, Γ-limit of a phase-field model of dislocations, SIAM J. Math. Anal. 36 (2005) 1943–1964.

[21] M. Ghimenti, J. van Schaftingen, Nodal solutions for the Choquard equation, J. Funct. Anal. 271 (2016) 107–135.

[22] D. Giulini, A. Großardt, The Schrödinger-Newton equation as a non-relativistic limit of self-gravitating Klein-Gordon and Dirac fields, Class. Quantum Gravity 29 (21) (2012) 215010.

[23] X. He, W. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, Calc. Var. Partial Differ. Equ. 55 (2016) 91.

[24] K.R.W. Jones, Gravitational self-energy as the litmus of reality, Mod. Phys. Lett. A 10 (8) (1995) 657–668.

[25] F. Lan, X. He, The Nehari manifold for a fractional critical Choquard equation involving sign-changing weight functions, Nonlinear Anal. 180 (2019) 236–263.

[26] N. Laskin, Fractional Schrödinger equations, Phys. Rev. E 66 (2002) 056108.

[27] S. Liang, P. Pucci, B. Zhang, Multiple solutions for critical Choquard-Kirchhoff type equations, Adv. Nonlinear Anal. 10 (1) (2021) 400–419.

[28] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976–1977) 93–105.

[29] E. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, 2001.

[30] P.-L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980) 1063–1072.

[31] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, Rev. Math. Iberoam. 1 (1) (1985) 145–201.

[32] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455–467.
[33] P. Ma, J. Zhang, Existence and multiplicity of solutions for fractional Choquard equations, Nonlinear Anal. 164 (2017) 100–117.
[34] X. Mingqi, V.D. Rădulescu, B. Zhang, A critical fractional Choquard-Kirchhoff problem with magnetic field, Commun. Contemp. Math. 21 (4) (2019) 1850.
[35] G. Molica Bisci, V.D. Rădulescu, R. Servadei, Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and Its Applications, vol. 162, Cambridge University Press, Cambridge, 2016.
[36] V. Moroz, J. van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.
[37] V. Moroz, J. van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (1) (2017) 773–813.
[38] T. Mukherjee, K. Sreenadh, Fractional Choquard equation with critical nonlinearities, Nonlinear Differ. Equ. Appl. 24 (2017) 63.
[39] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[40] S. Pekar, Untersuchung uber die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.
[41] R. Penrose, Quantum computation, entanglement and state reduction, Philos. Trans. R. Soc. Lond. A, Math. Phys. Eng. Sci. 356 (1743) (1998) 1927–1939.
[42] R. Penrose, The Road to Reality. A Complete Guide to the Laws of the Universe, Alfred A. Knopf Inc., New York, 2005.
[43] P. Pucci, M. Xiang, B. Zhang, Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian, Adv. Calc. Var. 12 (3) (2019) 253–275.
[44] D. Qin, V.D. Rădulescu, X. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, J. Differ. Equ. 275 (2021) 652–683.
[45] X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional Laplacian, Arch. Ration. Mech. Anal. 213 (2014) 587–628.
[46] F.E. Schunck, E.W. Mielke, General relativistic boson stars, Class. Quantum Gravity 20 (20) (2003) R301–R356.
[47] J. Seok, Limit profiles and uniqueness of ground states to the nonlinear Choquard equations, Adv. Nonlinear Anal. 8 (1) (2019) 1083–1098.
[48] Z. Shen, F. Gao, M. Yang, Ground states for nonlinear fractional Choquard equations with general nonlinearities, Math. Methods Appl. Sci. 39 (2016) 4082–4098.
[49] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Commun. Pure Appl. Math. 60 (2007) 67–112.
[50] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187 (1984) 511–517.
[51] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.
[52] M. Xiang, B. Zhang, Q. Hong, Existence of solutions for a critical fractional Kirchhoff type problem in $\mathbb{R}^N$, Sci. China Math. 60 (2017) 1647–1660.
[53] X. Zhang, B. Zhang, M. Xiang, Ground states for fractional Schrödinger equations involving a critical nonlinearity, Adv. Nonlinear Anal. 5 (2016) 293–314.