HOUSTON JOURNAL OF MATHEMATICS
Volume 20, No. 4, 1994

PERIODIC SOLUTIONS OF THE EQUATION $-\Delta v = v(1-|v|^2)$ IN $\mathbb{R}$ AND $\mathbb{R}^2$

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1. Introduction.

We study in this paper the existence of periodic functions $v : \mathbb{R} \to \mathbb{C}$ which satisfy the equation

$$-v'' = v(1-|v|^2)\,. \tag{1}$$

As observed in [2], the functions

$$Ae^{ikx}, \text{ where } k \in \mathbb{R},\ A \in \mathbb{C},\ |A|^2 + k^2 = 1, \tag{2}$$

are such solutions.

For fixed T, we also study the number of solutions of (1) with principal period T. The problem is that (1) has too many solutions, that is, if v is a solution, then $\alpha v(x_0 \pm x)$ is also a solution when $|\alpha| = 1$ and $x_0 \in \mathbb{R}$. In order to avoid redundancy, we shall first obtain a "canonical form" of solutions of (1). Namely, let V be a periodic solution of (1). We may suppose that $x = 0$ is a maximum point for $|V|^2$. Then one can find ϵ obeying $|\epsilon| < 1$ and $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $v(x) = \alpha V(\epsilon x)$ satisfies (1) and the conditions

$$v_1(0) = a > 0,\quad v_1'(0) = 0,\quad v_2(0) = 0,\quad v_2'(0) = b \geq 0, \tag{3}$$

where $v = v_1 + iv_2$ and $a = \max|v|$. It is clear that the system (1) together with (3) gives all the geometrically distinct solutions of (1); that is solutions that cannot be obtained from one another by the preceding procedure.

In what follows, we shall simply write "T-periodic solutions" instead of "solutions of principal period T". Our first result concerns the existence and the multiplicity of "T-periodic solutions".

2. The main result.

Our main result is the following

Theorem. *(i) If $T \leq 2\pi$, there are no T-periodic solutions of (1).*

(ii) If $T > 2\pi$, there is exactly one real solution v of (1) and (3) , that is a solution for which $v_2 \equiv 0$. Moreover, v depends analytically on T.

(iii) There is some $T_1 > 2\pi$ such that, for $2\pi < T \leq T_1$, (1) and (3) has no other T-periodic solutions apart those given by (ii) above and (2), where $k = 2\pi T^{-1}$, $A = \sqrt{1-k^2}$.

(iv) When $T > T_1$, (1) and (3) has other T-periodic solutions besides those two.

(v) For any $T > 2\pi$, the number of T-periodic solutions is finite.

(vi) For large T, (1) and (3) has at least $5T^2/8+O(T\ \log T)$ T-periodic solutions.

Remark. In fact, we shall find *all* the solutions of (1) and (3). More precisely, we shall exhibit a set $\Omega = \overline{\Omega} \subset \mathbb{R}^2$ such that, roughly speaking,

(i) if $(a,b) \notin \Omega$, then the solution of (1) and (3) has a finite lifetime for positive *or* negative x.

(ii) if $(a,b) \in \partial\Omega$, we obtain the solutions given by (2) or (ii) of the theorem.

(iii) if $(a,b) \in$ Int Ω, then $v \neq 0$, v has a global existence, $|v|$ and $\frac{\mathrm{d}}{\mathrm{d}x}\frac{v}{|v|}$ are periodic functions. For such (a,b), if T_0 is the principal period of $|v|$ and φ is (globally) defined so that $v = e^{i\varphi}|v|$, then v is periodic if and only if $\varphi(T_0) - \varphi(0) \in \pi\mathbb{Q}$. Given $q = \frac{m}{n} \in \mathbb{Q}$, $q > 0$, $(m,n) = 1$, the set

$$\{(a,b) \in \text{ Int } \Omega \ ; \ \ \varphi(T_0) - \varphi(0) = \pi q\} \tag{4}$$

is a smooth curve, which for example can be parametrized as $(a, b(a))$, where $a_0 < a < 1$ and a_0 depends on q. If $T_0(a)$ denotes the principal period of $|v|$ for the initial data $(a, b(a))$, then $\lim T_0(a) = \infty$ and this curve raises a smooth curve of periodic solutions of (1) and (3), with principal period $T(a) = nT_0(a)$ (if m is even) or $T(a) = 2nT_0(a)$ (if m is odd).

The bifurcation diagram of the distinguished solutions is depicted in Figure 1.

At present we do not know whether the curves $q =$const. are similar to (1) or (2) in Figure 1. In other words, we do not know whether T increases or not along these curves. If the first possibility holds, the minimum number of solutions given by (35) is the exact one. After the proof of the theorem, we shall give a sufficient condition for this happens (see the Remarks following the proof).

Finally, the last paragraph is devoted to the existence, in the whole $\mathbb{R}^2$, of 2-periodic solutions which are geometrically distinct to the real ones. Some existence and non-existence results are obtained.

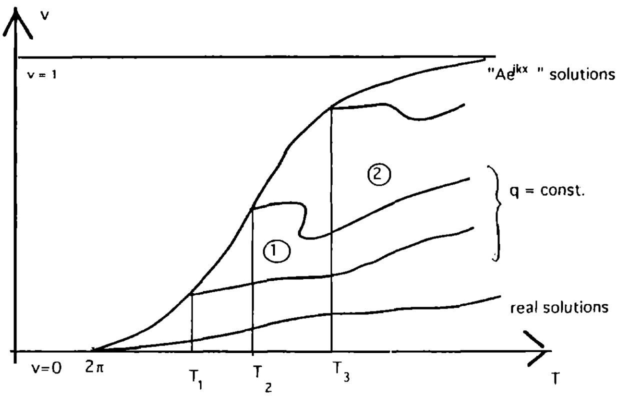

Figure 1

3. Proof of Theorem.

Let us note first that

$$a \leq 1 . \tag{5}$$

Suppose the contrary. Let $M > 1$ be such that $\min|v| < M < \max|v|$.

Let I be an interval such that $|v| > M$ in I and $|v| = M$ on ∂I. (Note that such an interval is necessarily finite). Since

$$(|v|^2)'' \geq 2|v|^2(|v|^2 - 1) > 0$$

in I, it follows that $|v| \leq M$ in I, which contradicts our choice of I.

Next we shall prove that

$$b^2 \leq a^2(1-a^2) . \tag{6}$$

Indeed, for small x we have

$$v_1(x) = a - \tfrac{a}{2}(1-a^2)x^2 + O(x^3),\ \ v_2(x) = bx + O(x^3),$$

so that (6) follows from the fact that $x = 0$ is a local maximum.

Now let $\Omega = \{(a,b) \in (0,1] \times [0,1];\ \ b^2 \le a^2(1-a^2)\}$.

We have obtained that if (1) and (3) raises a non-null periodic solution such that $x = 0$ is a local maximum, then necessarily $(a,b) \in \Omega$.

We shall first study the case $(a,b) \in \partial\Omega$.

Case 1 : If $b = a\sqrt{1-a^2}$, it follows that $v(x) = ae^{ikx}$, where $k = \sqrt{1-a^2}$. Indeed, (2) provides a solution for (1) and (3) in this case.

Case 2 : If $b = 0$, one gets easily that $v_2 = 0$. If $a = 1$, we get the trivial solution $v(x) \equiv 1$, so that in what follows we shall assume that $a \in (0,1)$.

Note first that v_1 cannot be positive (negative) on an infinite interval if v is periodic. For, otherwise, v_1 would be a periodic concave (convex) function, that is a constant function. This is impossible for our choice of a and b.

Let x_1, x_2 be two consecutive zeros of v_1. We may suppose that $v(x) > 0$ if $x_1 < x < x_2$, so that $v'(x_1) > 0,\ v'(x_2) < 0$. If x_3 is the smallest $x > x_2$ such that $v(x_3) = 0$, it follows that $v(x) < 0$ if $x_2 < x < x_3$.

If we prove that $x_2 - x_1 > \pi$, it will also follow that $x_3 - x_1 > 2\pi$ and that there is no $x \in (x_1, x_3)$ such that $v(x) = 0$ and $v'(x) > 0$. We will get that the principal period of v *must* be $> 2\pi$. This will be done in

Lemma 1. *Let $f : \mathbb{R} \to [0,1]$ be such that $\{x;\ f(x) = 0$ or $f(x) = 1\}$ contains only isolated points. Let v be a real function such that $v(x_1) = v(x_2) = 0$, and $v(x) > 0$ in $(x_1,\ x_2)$. If, for $x \in [x_1, x_2]$,*

$$-v'' = vf\,, \tag{7}$$

then $x_2 - x_1 > \pi$.

Proof. We may assume that $x_1 = 0$. Multiplying (7) by $\varphi(x) := \sin \frac{\pi x}{x_2}$ and integrating by parts, we obtain that

$$\int_0^{x_2} v\varphi > \int_0^{x_2} vf\varphi = (\frac{\pi}{x_2})^2 \int_0^{x_2} v\varphi\,,$$

that is $x_2 > \pi$. $\square$

Incidentally, this proves (i) of the Theorem.

Returning to the Case 2, we shall explicitely integrate (1) and (3) as one usually does for the Weierstrass Elliptic Functions. Multiplying (1) by v_1', we find

$$v_1'^2 = -v_1^2 + \tfrac{1}{2}v_1^4 + a^2 - \tfrac{1}{2}a^4. \tag{8}$$

It follows that, as far as the solution of (1) and (3) exists, we have $|v_1| \leq a$ and $|v_1'| \leq \sqrt{a^2 - \frac{1}{2}a^4}$. Hence the solution of (1) and (3) is globally defined.

Note that $v_1'(0) = 0$, $v_1''(0) < 0$, so that v_1 decreases for small $x > 0$. Moreover, $v_1'(x) < 0$ for $0 < x < \tau$, where

$$\tau = \sup\{x > 0;\;\; v_1(y) > 0 \;\text{ for all }\; 0 < y < x\}.$$

Indeed, suppose the contrary. Then, using (8), we obtain the existence of some $\tau_0 > 0$ such that $v_1(\tau_0) = a$, $\tau_0 < \tau$. If we consider the smallest $\tau_0 > 0$ such that the above equality occurs, we have $v_1(x) < a$ if $0 < x < \tau_0$. Since $v_1(0) = v_1(\tau_0) = a$, it follows that there exists some $0 < \tau_1 < \tau_0$ such that $v_1'(\tau_1) = 0$, which is the desired contradiction. Hence we have

$$v_1' = -\sqrt{a^2 - \tfrac{1}{2}a^4 - v_1^2 + \tfrac{1}{2}v_1^4} < 0 \;\text{ in }\; (0, \tau). \tag{9}$$

It follows that, if $0 < x < \tau$, then

$$\int_{v(x)}^{a} \frac{1}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}}\,dt = x, \quad \text{which gives}$$

$$\tau = \int_0^a \frac{dt}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} := \tau(a). \tag{10}$$

From (1), we obtain $v_1(\tau + x) = -v_1(x)$, $v_1(2\tau - x) = v_1(x)$, $v_1(4\tau + x) = v_1(x)$, so it is easy to see that v is periodic of principal period $T(a) = 4\tau(a)$.

Now (10) can be rewritten as

$$\tau(a) = \int_0^1 \frac{1}{\sqrt{(1-\xi^2)[1 - \frac{a^2}{2}(1+\xi^2)]}}\,d\xi, \tag{11}$$

so that τ increases with a and

$$\lim_{a\searrow 0}\tau(a)=\frac{\pi}{2},\quad \lim_{a\nearrow 1}\tau(a)=+\infty\,.$$

Since $\tau'(a)>0$, it follows that the mapping $T(a)\longmapsto a:=a(T)$ is analytic, so that (ii) is completely proved. Moreover,

$$\lim_{T\searrow 2\pi}a(T)=0 \quad\text{and}\quad \lim_{T\nearrow\infty}a(T)=1,$$

so that the diagram of "real" solutions is that depicted in Figure 1.

Next we return to the points (a,b) which are interior to Ω.

Case 3 : Let $(a,b)\in$ Int Ω. Write, for small x,

$$v(x)=e^{i\varphi(x)}w(x)\quad\text{with } \varphi(0)=0 \text{ and } w>0. \tag{12}$$

Then w satisfies

$$-w''=w(1-w^2)-\frac{a^2b^2}{w^3},\quad\text{and} \tag{13}$$

$$w(0)=a,\qquad w'(0)=0,\quad\text{while } \varphi \text{ is given by} \tag{14}$$

$$\varphi'=\frac{ab}{w^2},\quad \varphi(0)=0. \tag{15}$$

Hence, if the system (13) and (14) has a global *positive* solution, it follows that (12) is global. Moreover, if w is periodic of period T_0, then

$$v(nT_0+x)=e^{in\varphi(T_0)}e^{i\varphi(x)}w(x)\ \text{ for }\ 0\le x<T_0,\quad \text{for } n=0,1,... \tag{16}$$

so that (1) and (3) gives a periodic solution if and only if $\varphi(T_0)\in\pi\mathbb{Q}$.

We shall prove the global existence in

Lemma 2. *If $(a,b)\in$ Int Ω, then (13) and (14) has a global positive periodic solution.*

Proof. Note that the assumption made on (a,b) implies that $w''(0)<0$, so that, multiplying as above (13) by w', we obtain , for small $x>0$,

$$w'^2=-w^2+\tfrac{1}{2}w^4-\tfrac{a^2b^2}{w^2}+a^2-\tfrac{1}{2}a^4+b^2,\quad\text{and} \tag{17}$$

$$w' = -\sqrt{-w^2 + \tfrac{1}{2}w^4 - a^2b^2w^{-2} + a^2 - \tfrac{1}{2}a^4 + b^2}. \tag{18}$$

Now (17) implies that w and w' are bounded as far as the solution exists and, moreover, that $\inf\{w(x);\ \ w \ \text{ exists }\} > 0$.

It follows that w is a global solution. Let

$$\tau = \sup\{x > 0;\ \ w'(y) < 0 \ \text{ for all } \ 0 < y < x\}.$$

Note that (18) is valid if $0 < x < \tau$. Let c be the only root of

$$f(x) := \tfrac{1}{2}x^4 - x^2 - a^2b^2x^{-2} + a^2 - \tfrac{1}{2}a^4 + b^2 = 0$$

which is positive and less than a.

Since $f(x) < 0$ if $0 < x < c$ or $x > a$, x close to a, (17) implies that

$$c \le w(x) \le a \ \text{ for all } \ x \in \mathbb{R}. \tag{19}$$

Claim 1. $\lim_{x \nearrow \tau} w(x) = c.$

Proof of Claim 1. If $\tau < \infty$, it follows that $w'(\tau) = 0$. Now (17) together with the definitions of τ and c show that $w(\tau) = c$. If $\tau = \infty$, then we have $\lim_{x\to\infty} w(x) \ge c$. If we would have $\lim_{x\to\infty} w(x) > c$, there would exist a constant $M > 0$ such that $w'(x) \le -M$ for each $x > 0$. The latest inequality contradicts (19) for large x.

As we did before, for $0 < x < \tau$, (18) gives

$$x = \int_{w(x)}^{a} \frac{dt}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}}, \quad \text{so that} \tag{20}$$

$$\tau = \int_{c}^{a} \frac{dt}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}} < \infty. \tag{21}$$

It follows by a reflection argument that $w(2\tau) = w(0) = a$, $w'(2\tau) = w'(0) = 0$, so that w is (2τ)-periodic. $\square$

Next, in order to simply the following computations, it is useful to replace the (a,b)-coordinates by other ones. When (a,c) as above, associate with (a,b) the point (A,C), where $A = a^2$, $C = c^2$. This change of coordinates maps $\operatorname{Int}\Omega$ analytically into

$$\omega := \{(A, C);\ \ 0 < C < A,\ 2A + C < 2\} \text{ (see Figure 2)} .$$

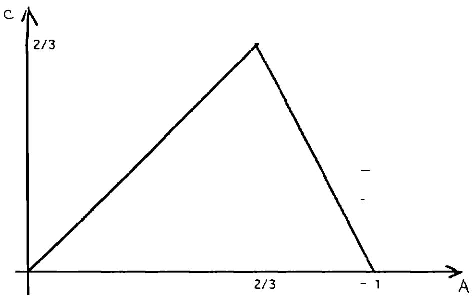

Figure 2

It follows from the above discussion that to each $(A, C) \in \omega$, there corresponds a solution (w, φ) of (13)-(15) such that w and φ' are periodic of period T_0 given by (after a suitable change of variables)

(22)
$$T_0 = T_0(A, C) = 2\sqrt{2} \int_0^\infty (y^2 + 1)^{-1} [(2 - 2A - C)y^2 + (2 - A - 2C)]^{-1} dy.$$

Moreover, $\varphi(0) = 0$ and

$$\varphi(T_0) = \sqrt{2AC(2 - A - C)} \int_0^{\tau(A,C)} w^{-2}(y)^{-1} dy \,, \tag{23}$$

where $\tau(A, C) = \frac{1}{2} T_0(A, C)$.

The change of variables $w(y) = t$ yields, with $\varphi(A\,, C) := \varphi(T_0\,(A\,, C))$,

(24) $\varphi(A, C) =$

$$\sqrt{2AC(2 - A - C)} \int_0^\infty \sqrt{\frac{y^2 + 1}{(2 - 2A - C)y^2 + (2 - A - 2C)}} \cdot \frac{dy}{Ay^2 + C},$$

and (22), (24) show that $(A, C) \longmapsto (T_0, \varphi)$ is an analytic map. Moreover, (22) gives that

$$T_0 > \pi, \qquad \lim_{(A,C) \to (0,0)} T_0(A, C) = \pi, \qquad \inf_{|(A,C)| \geq \varepsilon > 0} T_0(A, C) > \pi \,. \tag{25}$$

A lower estimate for φ is given by

Lemma 3. $\varphi > \frac{\pi}{\sqrt{2}}$ *and* $\lim_{(A,C)\to(0,0)} \varphi(A,C) = \frac{\pi}{\sqrt{2}}$.

Proof. If we put $y = \sqrt{\frac{C}{A}}z$ in (24), we obtain

(26)
$$\varphi(A,C) = \sqrt{2(2-A-C)}\int_0^\infty \sqrt{\frac{Cz^2+A}{C(2-2A-C)z^2+A(2-A-2C)}}\frac{dz}{z^2+1},$$

so that the second assertion follows from the Lebesgue Dominated Convergence Theorem.

For the first one, it is enough to show that for given $0 < k < 1$, the function

$$(0, \frac{2}{k+2}) \ni A \overset{\psi}{\longmapsto} \varphi(A, kA)$$

is increasing.

A short computation, yields that

$$\psi'(A) = \frac{2k}{\sqrt{2-(k+1)A}} \int_0^\infty \sqrt{\frac{ky^2+1}{[k(2-(k+2)A)y^2+(2-(2k+1)A)]^3}}dy > 0. \quad \square \tag{27}$$

Incidentally, this shows that φ has no critical points and that the level curves φ=const. are analytic and can be parametrized as

$$(A(k), kA(k)). \tag{28}$$

Lemma 4. $\lim_{A\nearrow\frac{2}{k+2}} \psi(A) = \infty$.

Proof. It follows from (26) that

$$\psi(A) > \sqrt{2(2-\frac{2(k+1)}{k+2})}\int_0^\infty \sqrt{\frac{kx^2+1}{k(2-(k+2)A)z^2+2-(2k+1)A}}\frac{dz}{z^2+1},$$

and this integral tends monotonically to $+\infty$ by the Beppo Levi Theorem. $\square$

From the above Lemma, we see that the parametrization (28) is valid for $k \in (0,1)$. Moreover, (27) shows that the mapping

$$k \longmapsto A(k) \tag{29}$$

is analytic. Of course, the level line φ =const. is non-void if and only if const.$> \frac{\pi}{\sqrt{2}}$. This will be assumed in the sequel. We shall prove that (29) provides a decreasing mapping. Indeed, if we consider now ψ as $\psi(A,k)$, then it follows from (27) that $\frac{\partial\psi}{\partial A}$ increases with k. Hence, if $k_1 < k_2$, then

$$\psi(A,k_1) < \psi(A,k_2); \quad \text{that is } A(k) \text{ decreases with } k.$$

We obtain the existence of $\lim_{k\nearrow 1} A(k) := A_0$ and $\lim_{k\searrow 0} A(k) := A_1 > A_0$. From Lemma 3, $A_0 > 0$.

Claim 2. $A_1 = 1$.

Proof of Claim 2. It follows from (26) and the Lebesgue Dominated Convergence Theorem that

$$\lim_{(A,C)\to(A_2,0)} \varphi(A,C) = \frac{\pi}{\sqrt{2}} \quad \text{if } 0 < A_2 < 1\,,$$

so that, taking Lemma 3 into account, we obtain that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varphi(A,C) < \frac{\pi}{\sqrt{2}} + \varepsilon$$

if $0 < A < 1-\delta$, $0 < C < \delta$. This completes the proof of the claim. □

At this stage of the proof, we know that the level lines φ=const. are analytic, all of them "end" at (1,0) and "begin" at (A_0, A_0) for some suitable $0 < A_0 < 1$, A_0 depending on the constant. Moreover, if $q_1 < q_2$, the line $\varphi = q_1$ lies below the line $\varphi = q_2$ (see Figure 3).

Now A_0 can be found implicitly, as φ can be extended by continuity on the line segment MN. This shows that

$$q = \varphi(A_0, A_0) = \frac{\pi}{2}\sqrt{\frac{1-A_0}{2-3A_0}} \quad \text{or}\ , \tag{30}$$

$$A_0 = A_0(q) = \frac{8q^2 - \pi^2}{12q^2 - 3\pi^2}.$$

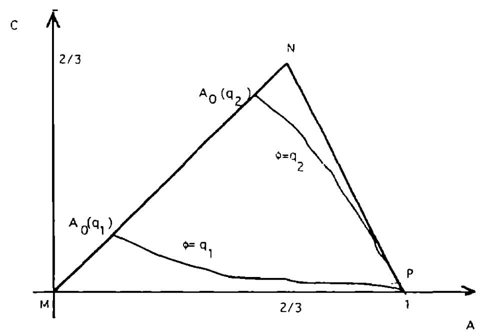

Figure 3

Returning to the proof of the theorem, note that (iii) and (iv) follow easily from the above calculation. Indeed, for small A and C, if $\varphi(A,C) = \pi\dfrac{m}{n}$ is a rational multiple of π, then $n \geq 4$, so that, taking into account the fact that $T_0(A,C) \geq \pi$, it follows that for small A the period of v is at least 4π. Now the existence of T_1 follows from (25).

In order to prove (v), note that the level line $\varphi = q$ contains a T-periodic solution if and only if

$$\text{(31)} \qquad \begin{cases} q = \pi\frac{m}{n},\ (m,n) = 1 \text{ and there exists } (A,C) \text{ on the level line} \\ \text{such that} \quad T_0(A,C) = \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd} \end{cases} \end{cases}.$$

We shall prove that

$$\text{(32)} \qquad \lim_{2A+C\nearrow 2} T_0(A,C) = \infty\,.$$

Suppose (32) proved for the moment. Obviously, if $\varphi(A_n,\, C_n) \to \infty$, then $2A_n + C_n \to 2$. It follows from (32) that, for q large enough, $T_0(A,\, C) > T$ if $(A,\, C)$ is on the level line $\varphi = q$, so that (31) cannot hold for such q. Hence, in order to prove v) it remains to show that, for given q, T_0 , the set $\mathcal{M} = \{(A,C);\ \ \varphi(A,C) = q, T_0(A,C) = T_0\}$ is finite.

Let $\mathcal{C}_1 = \{(A,C);\ \ \varphi(A,C) = q\}$. Since $\mathcal{C}_1$ is an analytic curve, $\mathcal{M}$ is finite provided that (1,0) and $(A_0(q), A_0(q))$ are not cluster points of

$\mathcal{M}$. For (1,0), this follows from the fact that, according to (32), $T_0(A,C)$ approaches $+\infty$ as A approaches 1 along $\mathcal{C}_1$. In particular, $T_0(A,C)$ is not constant along $\mathcal{C}_1$. In order to see what happens in $(A_0(q), A_0(q))$, we perform the following trick: let

$$\omega_1 = \omega \cup \{(C,A);\ \ (A,C) \in \omega\} \cup \{(A,A);\ \ 0 < A < 1\}$$

(see Figure 4).

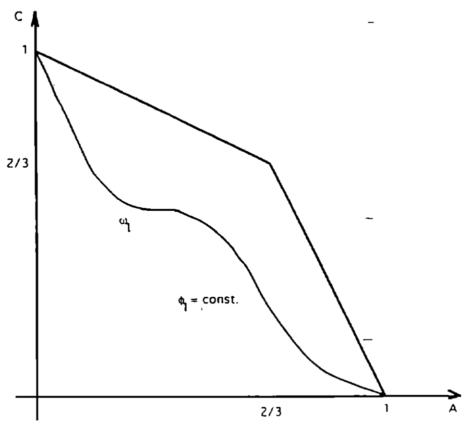

Figure 4

Obviously, (24) extends φ to an analytic function φ_1 in ω_1. The change of variables $z = \dfrac{1}{y}$ in (24) shows that $\varphi(A,C) = \varphi(C,A)$. Note also that (27) continues to hold for $k = 1$. This shows that φ_1 has no critical points and that $T_0(A,C)$ tends to $+\infty$ at the both ends of φ_1=const. Hence, φ can assume the same value only a finite number of times.

All that remains to do to complete the proof of (v) is the following

Proof of (32). Let $A_n < 1$, $0 < C_n < 1$ be such that $2A_n + C_n \nearrow 2$. Then

$$T_0(A_n, C_n) > 2\sqrt{2} \int_0^\infty \frac{dy}{\sqrt{(y^2+1)[(2-2A_n-C_n)y^2+2]}}, \tag{33}$$

and the right hand side of (33) tends to $+\infty$ from the Beppo Levi Theorem. $\square$

Next we return to the proof of (vi). Take $q = \pi\frac{m}{n}, (m,n) = 1,\ m > n/\sqrt{2}$. Then the level line $\varphi = q$ is nonempty and smooth. If we put

$$T_0(q) = 2\sqrt{2}\int_0^\infty \frac{dy}{\sqrt{(y^2+1)[(2-3A_0(q))y^2 + (2-3A_0(q))]}}$$

$$= \pi\sqrt{\frac{24q^2 - 6\pi^2}{16q^2 - 5\pi^2}}, \tag{34}$$

it follows from (32) that, along $\varphi = q$, T_0 assumes all the values between $T_0(q)$ and $+\infty$. Thus, for fixed T, (1) and (3) has at least one T-periodic solution corresponding to each q obeying

$$q = \pi\frac{m}{n},\ (m,n) = 1,\quad m > n/\sqrt{2},\ T_0(q) < \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd} \end{cases}. \tag{35}$$

Hence it suffices to count, for large T, the number of elements of $A \cup B$, where

$$A = \{(m,n);(m,n) = 1, m \text{ is even }, m > n/\sqrt{2}, \\ 24m^2n^2 - 6n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\} \text{ and} \tag{36}$$

$$B = \{(m,n);(m,n) = 1, m \text{ is odd }, m > n/\sqrt{2}, \\ 96m^2n^2 - 24n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\}. \tag{37}$$

Note that $A \cup B \supset \{(m,n);\ \ (m,n) = 1,\ m \geq n,\ m \leq \sqrt{\frac{5}{24}}\pi T\}$. It follows that there are at least

$$\sum_{1 \leq m \leq \sqrt{\frac{5}{24}}\pi T} \Phi(m) \tag{38}$$

solutions, where Φ is Euler's Function. Now a Theorem of Mertens (see [4]) asserts that the sum in (38) is

$$\frac{5}{8}T^2 + O(T\log T). \quad \square \tag{39}$$

Remarks. 1. It is obvious that (38) does not provide an accurate estimate. Nevertheless, one sees that the number of elements of $A \cup B$ is $O(T^2)$.

2. (35) counts *all* the T-periodic solutions if and only if T_0 increases along φ=q=const. as far as A increases from $A_0(q)$ to 1. A sufficient condition is that $(A, C) \longmapsto (T_0(A, C), \varphi(A, C))$ is a local diffeomorphism. This relies on the following fact: let ω be an open connected set of $\mathbb{R}^2$ and $f : \omega \rightarrow \mathbb{R}^2$ a *local* diffeomorfism. If the level lines f_2=const. are connected, then $f : \omega \rightarrow f(\omega)$ is a *global* diffeomorfism.

3. It follows from the proof that the diagram of bifurcation is, indeed, as in Figure 1. For example, the level line $\varphi = q$, $q = \pi \frac{m}{n}$, raises a branch of periodic solutions which starts from a solution of the form (2). Note that, on a level line, the solutions oscillate more and more as $A \nearrow 1$, in the sense that $\max |v|$ and $\min |v|$ approach 1 and 0 as A approaches 1. It is also easy to see that, in Figure 1, the points $T_1, T_2, T_3, \ldots$ are isolated.

4. One may prove that, if $a = \max |v|$ for a T-periodic solution, then

(i) $a^2 + (\frac{2\pi}{T})^2 = 1$ if v is given by (2);

(ii) $a^2 + (\frac{2\pi}{T})^2 > 1$ if v is a real solution;

(iii) $a^2 + (\frac{2\pi}{T})^2 < 1$ if v is a "complex" solution.

5. We have seen that the solution of (1) and (3) is globally existent if $(A, C) \in \overline{\omega}$. The same happens if $(A, C) \in \overline{\omega_1}$. There is nothing surprising in this, because starting with some $(A, C) \in \omega_1 \setminus \omega$ means considering the "canonical form" of (1) with $x = 0$ a local *minimum*, this time.

Let Ω_1 be the inverse image of ω_1 with respect to the mapping $(a, b) \longmapsto (A, C)$. Considering some point (a, b), $a \geq 0$, $b \geq 0$ such that $(a, b) \notin \overline{\Omega_1}$, it is easy to carry out once again (13)-(21) in order to prove that this time v has a finite left *or* right life time.

4. Existence of non-trivial periodic solutions in $\mathbb{R}^2$.

We are concerned with the existence of double periodic solutions, that is of functions $u : \mathbb{R}^2 \rightarrow \mathbb{C}$ solutions of

$$-\Delta u = u(1 - |u|^2), \quad u \in L^2_{loc}(\mathbb{R}^2)\,, \tag{40}$$

such that there exist $\omega_1, \omega_2 \in \mathbb{R}^2$ linearly independent with

$$u(x + \omega_j) = u(x), \qquad j = 1, 2\,. \tag{41}$$

Of course, we have already obtained such solutions: take $\omega_1 = (2T, 0)$ with $T > \pi$, ω_2 arbitrary and u a $2T$-periodic real solution. Even simpler, one may take u=const., $|u| = 0$ or 1.

Therefore, we shall look for non-trivial solutions, that is solutions enjoying the property

$$\text{(42)} \qquad \begin{cases} \text{there is no } v : \mathbb{R} \to \mathbb{C} \text{ solution of (1) such that} \\ u(x) = v(\alpha_1 x_1 + \alpha_2 x_2) \text{ for some } \alpha \in \mathbb{C},\ |\alpha| = 1. \end{cases}$$

We start with a non-existence result.

Proposition 1. *If $|\omega_1|, |\omega_2|$ are small enough, all the solutions of (40)-(41) are constant.*

We shall use in the proof

Lemma 5. *Let u be a solution of (40)-(41). Then $|u| \leq 1$ (so that u is smooth).*

Proof of Lemma 5. We follow an idea from [2]. It follows easily from (40) that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. Let

$$P = \{\lambda\omega_1 + \mu\omega_2;\ \ 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}.$$

Let φ be a $C_0^\infty(\mathbb{R}^2)$-function such that $\varphi \geq 0$, $\varphi = 1$ in a neighborhood of 0, and $\varphi_n(x) = \frac{1}{n^2}\varphi(\frac{x}{n})$ for $n \geq 1$.

Multiplying (40) with $u(|u|^2-1)^+\varphi_n$ and integrating by parts, we get, as $n \to \infty$,

$$\int_{P\cap[|u|\geq 1]} |\nabla u|^2(|u|^2-1) + \int_{P\cap[|u|\geq 1]} |\nabla |u|^2|^2 \leq -\int_{P\cap[|u|\geq 1]} |u|^2(|u|^2-1)^2 ,$$

that is $|u| \leq 1$ a.e. It follows that $u \in L^\infty$, so that u may be supposed smooth. $\square$

Proof of Proposition 1. Let $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of eigenfunctions of $-\Delta$ in $H^1_p(P)$ (here "p" means periodic conditions on ∂P) with corresponding eigenvalues $(\lambda_n)_{n\geq 0}$. We may suppose $\varphi_0 = 1$, so that $\lambda_n > 0$ for all $n \geq 1$. If $|\omega_1|, |\omega_2|$ are small enough, then $\lambda_n > 2$ if $n \geq 1$.

Let u be a solution of (40)-(41) and write

$$u = \sum c_n \varphi_n \,, \quad u|u|^2 = \sum d_n \varphi_n \,.$$

Integrating (40) over P, we find that $c_0 = d_0$. Multiplying (40) by φ_n, $n \geq 1$ and integrating we obtain, if $d_n \neq 0$, $|d_n| = (\lambda_n - 1)|c_n| > |c_n|$. Since $|u| \leq 1$, we have

$$\int_P |u|^2 \geq \int_P |u|^6, \text{ that is } \sum |c_n|^2 \geq \sum |d_n|^2.$$

Examining these formulae, we see that $c_n = d_n = 0$ if $n \geq 1$ or u is constant. □

Concerning the existence of solutions of (40)-(42), we have been able to prove it if P is a rectangle large enough.

Proposition 2. *Let P be large enough such that the first eigenvalue of $-\Delta$ in $H_0^1(R)$ is inferior to 1, where $R = \frac{1}{2}P$, then (40)-(42) has solutions.*

Proof. Let $J : H_0^1(R) \to \mathbb{R}$, be defined by

$$J(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{4} \int (1 - |u|^2)^2.$$

Then J is a C^1-function (see [3]), even and bounded from below. It is not difficult to see that it satisfies the (PS)-condition:

(PS): if $(u_n) \subset H_0^1(R)$ is such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$ in $H^{-1}(R)$, then (u_n) is relatively compact in $H_0^1(R)$.

Now $J(0) = \frac{1}{4}|R|$ and, if φ_1 is the first eigenfunction of $-\Delta$ in $H_0^1(R)$, then $J(\varepsilon\varphi_1) < J(0)$ for small ε.

More generally, if the k-th eigenvalue is inferior to 1, one can easily see that there is some $r > 0$ such that $J(u) < J(0)$ if $u \in$ Sp $\{\varphi_1, ..., \varphi_k\}$ and $\|u\| = r$. Here φ_j denotes the eigenfunction corresponding to the k-th eigenvalue.

It follows from Theorem 8.10 in [6] that J has at least k pairs $(u_j, -u_j)$ of critical points which are different from 0. Let u_0 be a critical point of J in R. Suppose $R = (0\,, a) \times (0\,, b)$.Define $u : P \to \mathbb{C}$ by

$$u(x') = u(x'') = -u_0(x), \quad u(x''') = u_0(x)\,,$$

where $x = (x_1, x_2), x' = (2a - x_1, x_2), x'' = (x_1, 2b - x_2), x''' = (2a - x_1, 2b - x_2)$.

It is obvious that u satisfies (41). It is not hard to see that u_0 is regular (see [5]). It follows then by a simple calculation that u satisfies (40).

Finally, suppose (42) does not hold. Let $\vec{\beta} = (\alpha_2, -\alpha_1)$ where $\alpha = \alpha_1 + i\alpha_2$ is as in (42). Then u must be constant along each parallel to $\vec{\beta}$. Since any such line intersects the grid generated by P, it follows that $u \equiv 0$, which is not the case. $\square$

Acknowledgements. This work was done while the authors were at the Laboratoire d'Analyse Numérique, Université Paris 6, with a Tempus fellowship. We would like to thank Professor H. Brezis, who gave us this problem, for his very valuable advice and hearty encouragement.

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Received April 25, 1994

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