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Research Article

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Coercive and noncoercive nonlinear Neumann problems with indefinite potential

Abstract: We consider nonlinear Neumann problems driven by a nonhomogeneous differential operator and an indefinite potential. In this paper we are concerned with two distinct cases. We first consider the case where the reaction is $(p-1)$-sublinear near $\pm\infty$ and $(p-1)$-superlinear near zero. In this setting the energy functional of the problem is coercive. In the second case, the reaction is $(p-1)$-superlinear near $\pm\infty$ (without satisfying the Ambrosetti–Rabinowitz condition) and has a $(p-1)$-sublinear growth near zero. Now, the energy functional is indefinite. For both cases we prove “three solutions theorems” and in the coercive setting we provide sign information for all of them. Our approach combines variational methods, truncation and perturbation techniques, and Morse theory (critical groups).

Keywords: Concave terms, superlinear reaction, nodal solution, critical groups, nonlinear regularity theory, indefinite potential

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann problem:

$$\begin{aligned} -\operatorname{div} a(Du(z)) + \beta(z)|u(z)|^{p-2}u(z) &= f(z,u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$. Also, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous strictly monotone map that satisfies certain other regularity conditions. The precise conditions on the map $a(\cdot)$ are listed in Hypothesis $\mathrm{H}(a)_1$. These assumptions are general enough to include some important classes of nonlinear differential operators. In particular, they incorporate the p-Laplace differential operator. However, we stress that in contrast to the p-Laplacian, the differential operator in (1.1) is not necessarily homogeneous and this is a source of difficulties, especially when we look for nodal (that is, sign changing) solutions. The potential (weight) function $\beta(\cdot)$ belongs to $L^\infty(\Omega)$ and may change sign (indefinite potential). Finally, the reaction $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous).

Our aim is to prove a “three solutions theorem” for problem (1.1) providing, if possible, sign information for all the solutions. We present two such multiplicity theorems under complementary conditions on the reaction $f(z,x)$. In the first multiplicity theorem, we assume that $f(z,\cdot)$ is $(p-1)$-linear near $\pm\infty$, while near zero it exhibits a “concave” term (that is, a $(p-1)$-superlinear term). In the second multiplicity theorem, $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$, while near zero it is $(p-1)$-linear. In the first case, the energy functional of the problem is coercive, while in the second case it is indefinite.

In the past, such multiplicity results were proved for equations driven by the p-Laplacian. We refer to the works of Liu [25], Liu and Liu [24] (Dirichlet problems) and Aizicovici, Papageorgiou and Staicu [3], Kyritsi and Papageorgiou [21] (Neumann problems) for the coercive case and by Bartsch and Liu [4], Bartsch, Liu and Weth [5], Filippakis, Kristaly and Papageorgiou [13], Liu [26], Sun [36] (Dirichlet problems) and Aizicovici, Papageorgiou and Staicu [1, 2] (Neumann equations) for the noncoercive case with superlinear reaction. In the aforementioned works, the hypotheses on the reaction $f(z, x)$ are in general more restrictive and they do not always provide sign information for all the solutions produced. We mention that another class of coercive Dirichlet equations with a nonlinear nonhomogeneous differential operator was studied recently by the authors in [32]. Finally, resonant semilinear equations with an indefinite and unbounded potential were investigated by Papageorgiou and Rădulescu [31].

Our approach combines variational methods based on the critical point theory, together with truncation and perturbation techniques, and Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools which will be used in the sequel.

2 Mathematical background. Auxiliary results

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that it satisfies the "Cerami condition" (the "C-condition" for short) if the following is true (see [7]):

Condition. Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that
(1) $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded,
(2) $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in X^* as $n \to \infty$
admits a strongly convergent subsequence.

This compactness-type condition is more general than the usual Palais–Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem that develops the minimax theory for certain critical values of φ. In particular, we can have the following version of the well-known "mountain pass theorem" (see, for example, Gasinski and Papageorgiou [15], Kristaly, Rădulescu and Varga [20], and Rădulescu [35]).

Theorem 2.1. *Let X be a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$ with $\|x_1 - x_0\| > r$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r,$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}$. Then $c \geq \eta_r$ and c is a critical value of φ.

The analysis of problem (1.1) will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. The latter function space is an ordered Banach space with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use in the following some facts about the spectrum of $(-\Delta_p + \widehat{\beta} I, W^{1,p}(\Omega))$ with $\widehat{\beta} \in L^q(\Omega)$ for $1 < q \leq \infty$. So, we consider the following nonlinear Neumann eigenvalue problem:

$$\begin{aligned} -\Delta_p u(z) + \widehat{\beta}(z)|u(z)|^{p-2}u(z) &= \lambda|u(z)|^{p-2}u(z) && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

This eigenvalue problem was studied recently by Mugnai and Papageorgiou [29]. Among other qualitative properties, they proved that if $\widehat{\beta} \in L^q(\Omega)$ with $q > Np'$ $(1/p + 1/p' = 1)$, then problem (2.1) has a smallest eigenvalue $\widehat{\lambda}_1(p, \widehat{\beta})$ which is simple, isolated and admits the following characterization:

$$\widehat{\lambda}_1(p, \widehat{\beta}) = \inf\left\{\frac{\mathcal{E}(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\right\}, \tag{2.2}$$

where

$$\mathcal{E}(u) = \|Du\|_p^p + \int_\Omega \widehat{\beta}(z)(z)|u(z)|^p\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The infimum in (2.2) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(p,\widehat{\beta})$. From (2.2) it is clear that the elements of this eigenspace have constant sign. By $\widehat{u}_1(p,\widehat{\beta}) \in W^{1,p}(\Omega)$ we denote the positive L^p-normalized (that is, $\|\widehat{u}_1(p,\widehat{\beta})\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p,\widehat{\beta})$. The interior regularity theory implies that $\widehat{u}_1(p,\widehat{\beta}) \in C^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$. If $\widehat{\beta} \in L^\infty(\Omega)$, then $\widehat{u}_1(p,\widehat{\beta}) \in \operatorname{int} C_+$ (see [29]).

Let $\eta \in C^1(0,\infty)$ and assume that

$$\begin{aligned} &0 < \widehat{c} \le \frac{t\eta'(t)}{\eta(t)} \le c_0 && \text{for all } t > 0,\\ &c_1 t^{p-1} \le \eta(t) \le c_2(1 + t^{p-1}) && \text{for all } t > 0, \text{ with } c_1, c_2 > 0. \end{aligned} \tag{2.3}$$

Hypothesis H$(a)_1$. The hypotheses on the map $a(\cdot)$ are the following:

$$a(y) = a_0(\|y\|)y \quad \text{for all } y \in \mathbb{R}^N$$

with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto ta_0(t)$, is strictly increasing, $ta_0(t) \to 0$ as $t \to 0^+$ and $\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} > -1$,

(ii) $\|\nabla a(y)\| \le c_3 \frac{\eta(\|y\|)}{\|y\|}$ for all $y \in \mathbb{R}^N\setminus\{0\}$ and some $c_3 > 0$,

(iii) $\frac{\eta(\|y\|)}{\|y\|}\|\xi\|^2 \le (\nabla a(y)\xi,\xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N\setminus\{0\}$ and all $\xi \in \mathbb{R}^N$,

(iv) if $G_0(t) = \int_0^t s a_0(s)\,ds$ $(t > 0)$, then there exists $\tau \in (1,p)$ such that $t \mapsto G_0(t^{1/\tau})$ is convex on $(0,+\infty)$,

$$\lim_{t\to 0^+} \frac{\tau G_0(t)}{t^\tau} < +\infty$$

and $t^2 a_0(t) - \tau G_0(t) \ge \tilde{c}\, t^p$ for all $t > 0$ and some $\tilde{c} > 0$.

Remark 2.2. Let $G(y) = G_0(\|y\|)$, $y \in \mathbb{R}^N$. Then for all $y \in \mathbb{R}^N\setminus\{0\}$, we have

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)y = a(y).$$

Hence, $G(\cdot)$ is primitive of $a(\cdot)$. Hypotheses H$(a)_1$ have some interesting consequences, which we present below. We first observe that $a(\cdot)$ is strictly monotone. Indeed, for all $y, y' \in \mathbb{R}^N$,

$$\begin{aligned} (a(y) - a(y'), y - y')_{\mathbb{R}^N} &= \int_0^1 \Big(\frac{d}{dt}a(y' + t(y - y')), y - y'\Big)_{\mathbb{R}^N} dt\\ &= \int_0^1 (\nabla a(y' + t(y - y'))(y - y'), y - y')_{\mathbb{R}^N}\, dt\\ &\ge c_1\|y' + t(y - y')\|^{p-2}\|y - y'\|^2 \quad \text{(see Hypothesis H}(a)_1\text{ (iii) and (2.3)).} \end{aligned}$$

It follows that the primitives $G(\cdot)$ and $G_0(\cdot)$ are strictly convex functions and $G_0(\cdot)$ is strictly increasing, too. In addition, we have for all $y \in \mathbb{R}^N$ and some $c_4 > 0$,

$$a(y) = \int_0^1 \frac{d}{dt}a(ty)\,dt = \int_0^1 \nabla a(ty)y\,dt \implies \|a(y)\| \le \int_0^1 \|\nabla a(ty)\|\|y\|\,dt \le c_4(1 + \|y\|^{p-1}). \tag{2.4}$$

Moreover, using Hypothesis H$(a)_1$ (iii) and (2.3), we obtain for all $y \in \mathbb{R}^N$,

$$(a(y), y) = \int_0^1 \Big(\frac{d}{dt}a(ty), y\Big)_{\mathbb{R}^N} dt = \int_0^1 (\nabla a(ty)y, y)_{\mathbb{R}^N}\,dt \ge \frac{c_1}{p-1}\|y\|^p. \tag{2.5}$$

Since $\nabla G(y) = a(y)$ for all $y \in \mathbb{R}^N$ (recall $a(0) = 0$, $G(0) = 0$), we have

$$G(y) = \int_0^1 \frac{d}{dt} G(ty)\,dt = \int_0^1 (a(ty), y)_{\mathbb{R}^N}\,dt.$$

Then using (2.4) and (2.5), we have for all $y \in \mathbb{R}^N$ and some $c_5 > 0$,

$$\frac{c_1}{p(p-1)}\|y\|^p \le G(y) \le c_5(1 + \|y\|^p). \tag{2.6}$$

Hypothesis $\mathrm{H}(a)_1$ are general enough to incorporate in our framework differential operators of interest.

Example 2.3. The following maps satisfy Hypothesis $\mathrm{H}(a)_1$:
(i) $a(y) = \|y\|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$,
(ii) $a(y) = \|y\|^{p-2}y + \mu\|y\|^{q-2}y$ with $1 < q < p$, $\mu \ge 0$,
(iii) $a(y) = (1 + \|y\|^2)^{(p-2)/2}y$ with $1 < p < \infty$.

In case (i) we take

$$G_0(t) = \frac{t^p}{p} \quad \text{for all } t \ge 0$$

and the corresponding differential operator is the p-Laplacian defined by

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

For this operator we take $\eta(t) = (p-1)t^{p-1}$ for all $t > 0$ if $1 < p \le 2$ and $\eta(t) = t^{p-1}$ for all $t > 0$ if $2 < p$ and $1 < \tau < p$ (see Hypothesis $\mathrm{H}(a)_1$ (iv)).

In case (ii) we take

$$G_0(t) = \frac{t^p}{p} + \frac{\mu t^q}{q} \quad \text{for all } t \ge 0$$

and the corresponding differential operator is the (p, q)-Laplacian defined by

$$\Delta_p u + \mu\Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

For this operator we have for all $t \ge 0$,

$$\begin{aligned} \eta(t) &= (p-1)t^{p-1} + \mu(q-1)t^{q-1} && \text{when } 1 < q < p < 2,\\ \eta(t) &= t^{p-1} + \mu(q-1)t^{q-1} && \text{when } 1 < q < 2 \le p,\\ \eta(t) &= t^{p-1} + \mu t^{q-1} && \text{when } 2 \le q < p. \end{aligned}$$

Indeed, for all $y \in \mathbb{R}^N\backslash\{0\}$,

$$\nabla a(y) = \|y\|^{p-2}\Big(I + (p-2)\frac{y \otimes y}{\|y\|^2}\Big) + \mu\|y\|^{q-2}\Big(I + (q-2)\frac{y \otimes y}{\|y\|^2}\Big).$$

First assume that $1 < q < p < 2$. Then

$$\|\nabla a(y)\| \le (p-1)\|y\|^{p-2} + \mu(q-1)\|y\|^{q-2}.$$

Also, for all $\xi \in \mathbb{R}^N$, we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \big((p-1)\|y\|^{p-2} + \mu(q-1)\|y\|^{q-2}\big)\,\|\xi\|^2.$$

Therefore with $\eta(t) = (p-1)t^{p-1} + \mu(q-1)t^{q-1}$ for $t \ge 0$, Hypotheses $\mathrm{H}(a)_1$ (ii)–(iii) are fulfilled.

Next, we assume that $1 < q < 2 \le p$. Then for all $y \in \mathbb{R}^N\backslash\{0\}$ and all $\xi \in \mathbb{R}^N$,

$$\|\nabla a(y)\| \le (p-1)\|y\|^{p-2} + \mu(q-1)\|y\|^{q-2} \le (p-1)[\|y\|^{p-2} + \mu(q-1)\|y\|^{q-2}]$$

and

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \big(\|y\|^{p-2} + \mu(q-1)\|y\|^{q-2}\big)\,\|\xi\|^2.$$

Therefore with $\eta(t) = t^{p-1} + \mu(q-1)t^{q-1}$ for $t \ge 0$, Hypotheses $\mathrm{H}(a)_1$ (ii)–(iii) are satisfied.

Finally, assume that $2 \le q < p$. Then for all $y \in \mathbb{R}^N \backslash \{0\}$ and all $\xi \in \mathbb{R}^N$,

$$\|\nabla a(y)\| \le (p-1)\|y\|^{p-2} + \mu(q-1)\|y\|^{q-1} \le (p-1)[\|y\|^{p-2} + \mu\|y\|^{q-2}]$$

and

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \left(\|y\|^{p-2} + \mu\|y\|^{q-2}\right)\|\xi\|^2.$$

Therefore with $\eta(t) = t^{p-1} + \mu t^{q-1}$ for $t \ge 0$, we see that Hypotheses $\mathrm{H}(a)_1$ (ii)–(iii) are fulfilled. Moreover, in Hypothesis $\mathrm{H}(a)_1$ (iv), we have $\tau = q$.

In case (iii) we take

$$G_0(t) = \frac{1}{p}\left[(1+t^2)^{\frac{p}{2}} - 1\right] \quad \text{for all } t \ge 0$$

and the corresponding differential operator is the generalized p-mean curvature differential operator

$$(1 + \|Du\|^2)^{\frac{p-2}{2}} Du \quad \text{for all } u \in W^{1,p}(\Omega).$$

Note that

$$\nabla a(y) = (1 + \|y\|^2)^{\frac{p-4}{2}}[(p-2)y \otimes y + (1 + \|y\|^2)I] \quad \text{for all } y \in \mathbb{R}^N \backslash \{0\}.$$

We first assume that $1 < p < 2$. Then for all $y \in \mathbb{R}^N \backslash \{0\}$ and for all $\xi \in \mathbb{R}^N$,

$$\|\nabla a(y)\| \le (1 + \|y\|^2)^{\frac{p-4}{2}}[(2-p)\|y\|^2 + 1 + \|y\|^2] \le c_*(1 + (p-1)\|y\|^2)^{\frac{p-2}{2}} \quad \text{for some } c_* > 0$$

and

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge (1 + \|y\|^2)^{\frac{p-4}{2}}[(1 + \|y\|^2)\|\xi\|^2 + (p-2)\|y\|^2\|\xi\|^2] \ge (1 + (p-1)\|y\|^2)^{\frac{p-2}{2}}\|\xi\|^2.$$

Therefore with $\eta(t) = (1 + (p-1)t^2)^{\frac{p-2}{2}} t$ for $t \ge 0$, Hypotheses $\mathrm{H}(a)_1$ (ii)–(iii) are satisfied.

Next, assume that $2 \le p$. Then for all $y \in \mathbb{R}^N \backslash \{0\}$ and for all $\xi \in \mathbb{R}^N$,

$$\|\nabla a(y)\| \le c_*(1 + \|y\|^2)^{\frac{p-2}{2}},$$
$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge (1 + \|y\|^2)^{\frac{p-2}{2}}\|\xi\|^2.$$

Therefore with $\eta(t) = (1 + t^2)^{\frac{p-2}{2}} t$, Hypotheses $\mathrm{H}(a)_1$ (ii)–(iii) are fulfilled. Moreover, in Hypothesis $\mathrm{H}(a)_1$ (iv), we have $1 < \tau < p$.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega (a(Du), Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } u, y \in W^{1,p}(\Omega). \tag{2.7}$$

From Gasinski and Papageorgiou [16], we have:

Proposition 2.4. *Assume that Hypotheses* $\mathrm{H}(a)_1$ (i)–(iii) *are fulfilled. Then the map* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.7) *is bounded (that is, it maps bounded sets into bounded sets), continuous, maximal monotone, and of type* $(S)_+$*, that is, if* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W^{1,p}(\Omega)$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, that is,

$$|f_0(z, x)| \le a(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}$$

with $a \in L^\infty(\Omega)_+$, $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p. \end{cases}$$

Let

$$F_0(z, x) = \int_0^x f_0(z, s)\, ds$$

and let $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))\, dz - \int_\Omega F_0(z, u(z))\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The following result relates Hölder and Sobolev local minimizers of a C^1-functional.

Proposition 2.5. *Assume that Hypotheses* $\mathrm{H}(a)_1$ (i)–(iii) *hold and* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1(\bar{\Omega})$*-minimizer of* φ_0*, that is, there exists some* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\bar{\Omega}),\ \|h\|_{C^1(\bar{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\beta}(\bar{\Omega})$ *with* $\beta \in (0,1)$ *and* u_0 *is also a local* $W^{1,p}(\Omega)$*-minimizer of* φ_0*, that is, there exists some* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega),\ \|h\| \leq \rho_1.$$

Remark 2.6. In the above result and in the sequel, $\|\cdot\|$ denotes the norm of the Sobolev space $W^{1,p}(\Omega)$, that is,

$$\|u\| = (\|u\|_p^p + \|Du\|_p^p)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 2.5 was first proved for the Dirichlet Sobolev space $H_0^1(\Omega)$ and for $G(y) = \frac{1}{2}\|y\|^2$ $(y \in \mathbb{R}^N)$ by Brezis and Nirenberg [6] and it was extended to the Dirichlet Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and for $G(y) = \frac{1}{p}\|y\|^p$ $(y \in \mathbb{R}^N)$ by Garcia Azorero, Manfredi and Peral Alonso [14]. Proposition 2.5 can be found in Motreanu and Papageorgiou [28].

Another mathematical tool that we will use in the sequel is the Morse theory and in particular critical groups. So, let us recall some basic definitions and facts from that theory.

Let X be a Banach space and $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ we denote by $H_k(Y_1, Y_2)$ the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. We recall that $H_k(Y_1, Y_2) = 0$ for all integers $k < 0$.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \qquad \dot{\varphi}^c = \{x \in X : \varphi(x) < c\},$$
$$K_\varphi = \{x \in X : \varphi'(x) = 0\}, \quad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The critical groups of φ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$ (that is, $x_0 \in K_\varphi^c$) are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{0\}) \quad \text{for all } k \geq 0.$$

Here U is a neighborhood of $x_0 \in X$ such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The second deformation theorem (see, e.g., Gasinski and Papageorgiou [15, p. 628]) implies that the above definition of critical groups of φ at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We introduce the following polynomials in $t \in \mathbb{R}$:

$$M(t, x) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \text{for all } x \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k.$$

The Morse relation says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t), \tag{2.8}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series with nonnegative integer coefficients β_k.

As we already mentioned, by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. The same notation will also be used to denote the norm of $\mathbb{R}^N$. However, no confusion is possible, since it will always be clear from the context which norm is used. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm(\cdot) \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ = u^-.$$

Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, we introduce the map

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}(\Omega)$$

(the Nemytskii map corresponding to h). Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3 Coercive problems

In this section, we examine problem (1.1) under hypotheses on $f(z,x)$ that make the energy functional coercive. We prove a "three solutions theorem" providing sign information for all the solutions. First we fix the hypotheses on the potential $\beta(\cdot)$:

Hypothesis H_0. We have $\beta \in L^\infty(\Omega)$.

Set $\widehat{\beta}(z) = \frac{p-1}{c_1}\beta(z)$.

Hypothesis H_1. We assume that the reaction term $f(z,x)$ is a Carathéodory function such that $f(z,0) = 0$ a.e. in Ω and

(i) $|f(z,x)| \le a(z)(1 + |x|^{p-1})$ a.e. in Ω, for all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+$,

(ii) there exists a function $\vartheta \in L^\infty(\Omega)$ such that $\vartheta(z) \le \frac{c_1}{p-1}\widehat{\lambda_1}(p,\widehat{\beta})$ a.e. in Ω, $\vartheta \ne \frac{c_1}{p-1}\widehat{\lambda_1}(p,\widehat{\beta})$ and
$$\limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \vartheta(z)$$
uniformly for a.a. $z \in \Omega$,

(iii) there exist $q \in (1,\tau)$ (see Hypothesis $\mathrm{H}(a)_1$ (iv)), $\tilde{c}_0 > 0$ and $\delta > 0$ such that $\tilde{c}_0|x|^q \le f(z,x)x \le qF(z,x)$ for a.a. $z \in \Omega$, all $0 < |x| \le \delta$,

(iv) for every $\rho < 0$, there exists $\epsilon_\rho > 0$ such that for a.a. $z \in \Omega$, the mapping $x \mapsto f(z,x) + \epsilon_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$.

Remark 3.1. Hypothesis H_1 (i) implies that asymptotically at $\pm\infty$ $f(z,\cdot)$ is $(p-1)$-sublinear. Hypothesis H_1 (ii) will make the energy functional coercive. Hypothesis H_1 (iii) implies the existence of a concave term near zero. Finally, Hypothesis H_1 (iv) is weaker than assuming the monotonicity of $f(z,\cdot)$ and is a one-sided Lipschitz condition on $f(z,\cdot)$.

Example 3.2. The following functions satisfy Hypothesis H_1. For the sake of simplicity, we drop the z-dependence:

$$f_1(x) = \vartheta|x|^{p-2}x + |x|^{q-2}x \qquad \text{with } \vartheta < \frac{c_1}{p-1}\widehat{\lambda_1}(p,\widehat{\beta}),\ 1 < q < p,$$

$$f_2(x) = \begin{cases} \vartheta(|x|^{q-2}x - |x|^{r-2}x) & \text{if } |x| \le 1, \\ \vartheta(|x|^{p-2}x - |x|^{\mu-2}x) & \text{if } |x| > 1, \end{cases} \qquad \text{with } \vartheta < \frac{c_1}{p-1}\widehat{\lambda_1}(p,\widehat{\beta}),\ 1 < q,\mu < p, r.$$

The next auxiliary result is useful to establish the coercivity of the energy functional of problem (1.1). This property can be found in Mugnai and Papageorgiou [29, Lemma 4.11].

Lemma 3.3. *Assume that $\tilde{\beta}, \tilde{\vartheta} \in L^\infty(\Omega)$ and $\tilde{\vartheta}(z) \le \widehat{\lambda_1}(p,\tilde{\beta})$ a.e. in Ω, $\tilde{\vartheta} \ne \widehat{\lambda_1}(p,\tilde{\beta})$. Then there exists some $c_6 > 0$ such that*
$$\tilde{\varepsilon}(u) - \int_\Omega \tilde{\vartheta}(z)|u(z)|^p\,dz \ge c_6\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega),$$
where $\tilde{\varepsilon}(u) = \|Du\|_p^p + \int_\Omega \tilde{\beta}(z)|u(z)|^p\,dz$ for all $u \in W^{1,p}(\Omega)$.

Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional associated to problem (1.1), namely
$$\varphi(u) = \int_\Omega G(Du(z))\,dz - \int_\Omega F(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently, $\varphi \in C^1(W^{1,p}(\Omega))$.

Let $\lambda > \|\beta^-\|_\infty$ and consider the following truncations-perturbations of $f(z,\cdot)$:

$$\widehat{f_+}(z,x) = f(z,x^+) + \lambda(x^+)^{p-1} \quad \text{and} \quad \widehat{f_-}(z,x) = f(z,-x^-) - \lambda(x^-)^{p-1}.$$

We set

$$\widehat{F_\pm}(z,x) = \int_0^x \widehat{f_\pm}(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi_\pm} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi_\pm}(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega (\beta(z)+\lambda)|u(z)|^p\,dz - \int_\Omega \widehat{F_\pm}(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 3.4. *Assume that Hypotheses* $\mathrm{H}(a)_1$, H_0 *and* H_1 *are fulfilled. Then problem* (1.1) *has at least two nontrivial constant sign solutions* $u_0 \in \operatorname{int} C_+$ *and* $v_0 \in -\operatorname{int} C_+$, *both local minimizers of the energy functional* φ.

Proof. By virtue of Hypotheses H_1 (i)–(ii), given $\epsilon > 0$, we can find $c_+ = c_+(\epsilon) > 0$ such that

$$F(z,x) \le \frac{1}{p}(\vartheta(z)+\epsilon)|x|^p + c_7 \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.1}$$

Thus, for all $u \in W^{1,p}(\Omega)$,

$$\begin{aligned}
\widehat{\varphi_+}(u) &= \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega (\beta(z)+\lambda)|u|^p\,dz + \int_\Omega \widehat{F_\pm}(z,u)\,dz \\
&\ge \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \frac{1}{p}\int_\Omega (\vartheta(z)+\epsilon)|u|^p\,dz + c_7|\Omega|_N \qquad \text{(see [10])} \\
&= \frac{c_1}{p(p-1)}\Big[\|Du\|_p^p + \int_\Omega \widehat{\beta}(z)|u|^p\,dz + \frac{p-1}{c_1}\int_\Omega \vartheta(z)|u|^p\,dz\Big] - \frac{\epsilon}{p}\|u\|^p + c_7|\Omega|_N \\
&\ge \frac{1}{p}\Big[\frac{c_1c_6}{p-1} - \epsilon\Big]\|u\|^p + c_7|\Omega|_N \qquad \text{(see Lemma 3.3).}
\end{aligned} \tag{3.2}$$

Choosing $\epsilon \in (0, \frac{c_1c_6}{p-1})$, we deduce from (3.2) that $\widehat{\varphi_+}$ is coercive. Also, via the Sobolev embedding theorem, we see that $\widehat{\varphi_+}$ is sequentially weakly lower semi-continuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi_+}(u_0) = \inf\{\widehat{\varphi_+}(u) : u \in W^{1,p}(\Omega)\} = \widehat{m_+}. \tag{3.3}$$

By virtue of Hypothesis $\mathrm{H}(a)_1$ (iv), we can find $c_8 > 0$ and $\delta_0 \in (0,\delta]$ such that

$$G(y) \le \frac{c_8}{\tau}\|y\|^\tau \quad \text{for all } \|y\| \le \delta_0. \tag{3.4}$$

Hypothesis H_1 (iii) yields

$$\frac{\tilde{c}_0}{q}|x|^q \le F(z,x) \quad \text{for all } z \in \Omega,\ \text{all } |x| \le \delta. \tag{3.5}$$

Recall that $\widehat{u_1}(p,\beta) \in \operatorname{int} C_+$. So, we can find $\eta \in (0,1)$ small such that

$$\eta|\widehat{u_1}(p,\beta)(z)|,\ \eta\|D\widehat{u_1}(p,\beta)(z)\| \in (0,\delta_0] \quad \text{for all } z \in \bar{\Omega}. \tag{3.6}$$

Therefore

$$\begin{aligned}
\widehat{\varphi_+}(\eta\,\hat{u}_1(p,\beta)) &= \int_\Omega G(\eta D\hat{u}_1(p,\beta))\,dz + \frac{\eta^p}{p}\int_\Omega \beta(z)|\hat{u}_1(p,\beta)|^p\,dz - \int_\Omega F(z,\eta\,\widehat{u_1}(p,\beta))\,dz \\
&\le \frac{c_8}{\tau}\eta^\tau\|D\hat{u}_1(p,\beta)\|_\tau^\tau + \frac{\eta^p}{p}\|\beta\|_\infty - \frac{\tilde{c}_0}{q}\eta^q\|\widehat{u_1}(p,\beta)\|_q^q,
\end{aligned} \tag{3.7}$$

see (3.4)–(3.6) and recall $\|\widehat{u_1}(p,\beta)\|_p = 1$. Since $1 < q < \tau < p$, by choosing $\eta \in (0,1)$ even smaller if necessary, relation (3.7) yields

$$\widehat{\varphi_+}(\eta\,\widehat{u_1}(p,\beta)) < 0 \implies \widehat{\varphi}_+(u_0) = \widehat{m}_+ < 0 = \widehat{\varphi}_+(0) \quad \text{(see (3.3))}, \quad \text{hence } u_0 \neq 0.$$

From (3.3) we have

$$\widehat{\varphi}_+'(u_0) = 0 \implies A(u_0) + (\beta + \lambda)|u_0|^{p-2}u_0 = N_{\widehat{f}_+}(u_0). \tag{3.8}$$

On (3.8) we act with $-u_0^- \in W^{1,p}(\Omega)$ and using (2.5), we obtain

$$\frac{c_1}{p-1}\|Du_0^-\|_p^p + \int_\Omega (\beta(z) + \lambda)u_0^-(z)^p\,dz \le 0.$$

Since $\lambda > \|\beta^-\|_\infty$, we infer that $u_0 \ge 0,\ u_0 \neq 0$. Therefore relation (3.8) becomes

$$A(u_0) + \beta u_0^{p-1} = N_f(u_0) \implies \begin{cases} -\operatorname{div} a(Du_0(z)) + \beta(z)u_0(z)^{p-1} = f(z,u_0(z)) & \text{a.e. in } \Omega, \\ \dfrac{\partial u_0}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{(see [2]).}$$

From Hu and Papageorgiou [18], we know that $u_0 \in L^\infty(\Omega)$ and so we can apply the regularity result of Lieberman [23, p. 320] and deduce that $u_0 \in C_+\setminus\{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\epsilon_\rho > 0$ be as postulated by Hypothesis H_1 (iv). Then

$$-\operatorname{div} a(Du_0(z)) + \epsilon_\rho(u_0)(z)^{p-1} = f(z,u_0(z)) + \epsilon_\rho u_0(z)^{p-1} \ge 0 \quad \text{a.e. in } \Omega. \tag{3.9}$$

Let $\gamma_0(t) = ta_0(t)$. Hypothesis $H(a)_1$ (iii) implies the one-dimensional estimate

$$t\gamma_0'(t) = t^2a_0'(t) + ta_0(t) \ge c_9t^{p-1} \quad \text{for all } t > 0,\ \text{some } c_9 > 0,$$

and so

$$\int_0^t s\gamma_0'(s)\,ds = t\gamma_0(t) - \int_0^t \gamma_0(s)\,ds = t^2a_0(t) - G_0(t) \ge \frac{c_9}{p}t^p \quad \text{for all } t > 0.$$

This estimate and (3.9) permit the use of the strong maximum principle of Pucci and Serrin [34, p. 111] and so we have $u_0(z) > 0$ for all $z \in \Omega$. Finally, we apply the boundary point theorem of Pucci and Serrin [34, p. 120] and conclude that $u_0 \in \operatorname{int} C_+$. Note that $\widehat{\varphi}_+|_{C_+} = \varphi|_{C_+}$. So, $u_0 \in \operatorname{int} C_+$ is a local $C^1(\bar{\Omega})$-minimizer of φ. Invoking Proposition 2.5, we conclude that $u_0 \in \operatorname{int} C_+$ is a local $W^{1,p}(\Omega)$ -minimizer of φ.

Similarly, working this time with $\widehat{\varphi}_-$, we produce a nontrivial negative solution $v_0 \in -\operatorname{int} C_+$ of problem (1.1), which is a local minimizer of φ. □

In fact, we can show the existence of extremal nontrivial constant sign solutions for problem (1.1). Namely, we show that there exists a smallest nontrivial positive solution and a biggest nontrivial negative solution. Our argument follows closely the reasoning of Papageorgiou and Rădulescu [32], where the authors deal with Dirichlet (p,q)-equations. For the convenience of the reader, we present the proofs in detail.

Note that Hypotheses H_1 (i), (iii) imply that we can find $c_{10} > \|\beta\|_\infty$ and λ such that

$$f(z,x)x \ge \tilde{c}|x|^q - c_{10}|x|^p \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.10}$$

This unilateral growth condition on $f(z,\cdot)$ leads to the following auxiliary Neumann problem:

$$\begin{aligned} -\operatorname{div} a(Du(z)) &= \tilde{c}|u(z)|^{q-2}u(z) - c_{10}|u(z)|^{p-2}u(z) && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.11}$$

Proposition 3.5. *Assume that Hypothesis* $H(a)_1$ *hold. Then problem* (3.11) *has a unique nontrivial positive solution* $\tilde{u} \in \operatorname{int} C_+$ *and since* (3.11) *is add* $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ *is unique nontrivial negative solution of* (3.11).

Proof. First we show that problem (3.11) admits a nontrivial positive solution. To this end, let

$$\Psi_+ : W^{1,p}(\Omega) \to \mathbb{R}$$

be the C^1-functional defined by

$$\Psi_+(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega (\beta(z)+\lambda)|u(z)|^p\,dz - \frac{\tilde{c}}{q}\|u^+\|_q^q + \frac{c_{10}-\lambda}{p}\|u^+\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Recall that $\lambda > \|\beta^-\|_\infty$. From this fact and since $q < p$, we infer that $\widehat{\Psi}_+$ is coercive. Also, it is sequentially weakly lower semi-continuous. Therefore, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\Psi_+(\tilde{u}) = \inf\{\Psi_+(u) : u \in W^{1,p}(\Omega)\}.$$

As before (see the proof of Proposition 3.4), using Hypothesis $\mathrm{H}(a)_1$ (iv), we show that $\Psi_+(\tilde{u}) < 0 = \Psi_+(0)$, hence $\tilde{u} \neq 0$. Also, we have

$$\Psi_+'(\tilde{u}) = 0 \implies A(\tilde{u}) + (\beta+\lambda)|\tilde{u}|^{p-2}\tilde{u} = \tilde{c}(\tilde{u}^+)^{q-1} - (c_{10}-\lambda)(\tilde{u}^+)^{p-1}. \tag{3.12}$$

On (3.12) we act with $-\tilde{u}^- \in W^{1,p}(\Omega)$. Since $\lambda > \|\beta^-\|_\infty$, we see that $\tilde{u} \geq 0$, $\tilde{u} \neq 0$. Hence (3.12) becomes

$$A(\tilde{u}) + \beta\tilde{u}^{p-1} = \tilde{c}\tilde{u}^{q-1} - c_{10}\tilde{u}^{p-1},$$

which shows that $\tilde{u}$ is a nontrivial positive solution of problem (3.11). Moreover, as before using the nonlinear regularity theory, we obtain $\tilde{u} \in C_+\backslash\{0\}$. Also, we have

$$\operatorname{div} a(D\tilde{u}(z)) \leq (\|\beta\|_\infty + c_{10})\tilde{u}(z)^{p-1} \quad \text{a.e. in } \Omega,$$

hence $\tilde{u} \in \operatorname{int} C_+$ (see Pucci and Serrin [34, p. 120]).

Next, we show the uniqueness of this positive solution. For this purpose, we consider the integral functional $\sigma_+ : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_+(u) = \begin{cases} \int_\Omega G(Du^{\frac{1}{\tau}})\,dz & \text{if } u \geq 0,\ u^{\frac{1}{\tau}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Let $u_1, u_2 \in \operatorname{dom}\sigma_+$ and let $y = (tu_1 + (1-t)u_2)^{\frac{1}{\tau}}$ with $t \in [0,1]$. From Diaz and Saa [9, Lemma 1], we have

$$\|Dy(z)\| \leq \Big(t\|Du_1(z)^{\frac{1}{\tau}}\|^\tau + (1-t)\|Du_2(z)^{\frac{1}{\tau}}\|^\tau\Big)^{\frac{1}{\tau}}.$$

Recall that $G_0(\cdot)$ is increasing. Hence

$$\begin{aligned} G_0(\|Dy(z)\|) &\leq G_0\Big(\Big(t\|Du_1(z)^{\frac{1}{\tau}}\|^\tau + (1-t)\|Du_2(z)^{\frac{1}{\tau}}\|^\tau\Big)^{\frac{1}{\tau}}\Big) \\ &\leq tG_0\Big(\|Du_1(z)^{\frac{1}{\tau}}\|\Big) + (1-t)G_0\Big(\|Du_2(z)^{\frac{1}{\tau}}\|\Big) \quad \text{a.e. in } \Omega \quad (\text{see Hypothesis } \mathrm{H}(a)_1 \text{ (iv)}), \end{aligned}$$

which implies

$$G(Dy(z)) \leq tG\Big(Du_1(z)^{\frac{1}{\tau}}\Big) + (1-t)G\Big(Du_2(z)^{\frac{1}{\tau}}\Big) \quad \text{a.e. in } \Omega$$

and hence σ_+ is convex. Moreover, via Fatou's lemma, we see that σ_+ is lower semi-continuous.

Suppose that $u, v \in W^{1,p}(\Omega)$ are two nontrivial positive solutions of (3.11). From the first part of the proof, we have $u, v \in \operatorname{int} C_+$. Therefore $u^\tau, v^\tau \in \operatorname{dom}\sigma_+$. Let $h \in C^1(\bar{\Omega})$. Then for $t \in [-1,1]$ with $|t|$ small, we have $u^\tau + th, v^\tau + th \in \operatorname{dom}\sigma_+$ and so the Gâteaux derivatives of σ_+ at u^τ and at v^τ in the direction h exist. Moreover, via the chain rule, we have

$$\sigma_+'(u^\tau)(h) = \frac{1}{\tau}\int_\Omega \frac{-\operatorname{div} a(Du)}{u^{\tau-1}}h\,dz, \quad \sigma_+'(v^\tau)(h) = \frac{1}{\tau}\int_\Omega \frac{-\operatorname{div} a(Dv)}{v^{\tau-1}}h\,dz.$$

The convexity of σ_+ implies the monotonicity of σ_+^1. Therefore

$$\begin{aligned}0 &\leq \int_\Omega \Big(\frac{-\operatorname{div} a(Du)}{u^{\tau-1}} + \frac{\operatorname{div} a(Dv)}{v^{\tau-1}} \Big)(u^\tau - v^\tau)\,dz \\ &\leq \int_\Omega \Big(\frac{\tilde{c}u^{q-1} - \hat{c}_{10}u^{p-1}}{u^{\tau-1}} - \frac{\tilde{c}v^{q-1} - \hat{c}_{10}v^{p-1}}{v^{\tau-1}} \Big)(u^\tau - v^\tau)\,dz, \quad \hat{c}_{10} = c_{10} - \|\beta\|_\infty > 0, \\ &= \int_\Omega \Big[\tilde{c}\Big(\frac{1}{u^{\tau-q}} - \frac{1}{v^{\tau-q}} \Big) - \hat{c}_{10}(u^{p-\tau} - v^{p-\tau}) \Big](u^\tau - v^\tau)\,dz. \end{aligned} \tag{3.14}$$

Since $q < \tau < p$, the function $x \mapsto \frac{\tilde{c}}{x^{\tau-q}} - \hat{c}_{10}x^{p-\tau}$ is strictly decreasing on $(0, +\infty)$. So, from (3.14) it follows that $u = v$ and this proves the uniqueness of the nontrivial positive solution $\tilde{u} \in \operatorname{int} C_+$. Evidently, $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ is the unique nontrivial negative solution of problem (3.11). □

Using these unique constant sign solutions of (3.11), we can generate extremal constant sign solutions of (1.1).

Proposition 3.6. *Assume that Hypotheses* $\mathrm{H}(a)_1$ *and* H_1 *hold. Then problem* (1.1) *has a smallest nontrivial positive solution* $u_* \in \operatorname{int} C_+$ *and a biggest nontrivial negative solution* $v_* \in -\operatorname{int} C_+$.

Proof. Let S_+ be the set of nontrivial positive solutions of (1.1). From Proposition 3.4 we know that $S_+ \neq \emptyset$ and $S_+ \subseteq \operatorname{int} C_+$. Moreover, as in Aizicovici, Papageorgiou and Staicu [1], we have that S_+ is downward directed, that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$. So, without any loss of generality, we may assume that there exists $M > 0$ such that

$$\|u\|_\infty \leq M \quad \text{for all } u \in S_+. \tag{3.15}$$

Claim. *We have* $\tilde{u} \leq u$ *for all* $u \in S_+$.

Let $u \in S_+$ and consider the Carathéodory function

$$h_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \tilde{c}x^{q-1} - (c_{10} - \lambda)x^{p-1} & \text{if } 0 \leq x \leq u(z), \\ \tilde{c}u(z)^{q-1} - (c_{10} - \lambda)u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.16}$$

As before, $\lambda > \|\beta^-\|_\infty$. We set

$$H_+(z, x) = \int_0^x h_+(z, s)\,ds$$

and consider the C^1-functional $\gamma_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega [\beta(z) + \lambda]|u(z)|^p\,dz - \int_\Omega H_+(z_1 u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Relation (3.16) implies that γ_+ is coercive. Also, it is sequentially weakly lower semi-continuous. So, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\gamma_+(\tilde{u}_0) = \inf\{\gamma_+(u) : u \in W^{1,p}(\Omega)\}. \tag{3.17}$$

As before (see Proposition 3.5) and since $u \in \operatorname{int} C_+$, we have

$$\gamma_+(\tilde{u}_0) < 0 = \gamma_+(0),$$

hence $\tilde{u}_0 \neq 0$. From (3.11) we have

$$\gamma_+'(\tilde{u}_0) = 0 \implies A(\tilde{u}_0) + (\beta + \lambda)|\tilde{u}_0|^{p-2}\tilde{u}_0 = N_{h_+}(\tilde{u}_0). \tag{3.18}$$

On (3.18) we first act with $-\tilde{u}_0^- \in W^{1,p}(\Omega)$. Since $\lambda > \|\beta^-\|_\infty$, we obtain

$$\tilde{u}_0 \geq 0, \quad \tilde{u}_0 \neq 0.$$

Then on (3.18) we act with $(\tilde{u}_0 - u)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}\langle A(\tilde{u}_0), (\tilde{u}_0 - u)^+\rangle + \int_\Omega (\beta(z) + \lambda)\tilde{u}_0^{p-1}(\tilde{u}_0 - u)^+ \, dz &= \int_\Omega h_+(z, \tilde{u}_0)(\tilde{u}_0 - u)^+ \, dz \\ &= \int_\Omega [\tilde{c}u^{q-1} - c_{10}u^{p-1}](\tilde{u}_0 - u)^+ \, dz \qquad \text{(see (3.16))} \\ &\le \int_\Omega f(z,u)(\tilde{u}_0 - u)^+ \, dz \qquad \text{(see (3.10))} \\ &= \langle A(u), (\tilde{u}_0 - u)^+\rangle + \int_\Omega (\beta(z) + \lambda)u^{p-1}(\tilde{u}_0 - u)^+ \, dz,\end{aligned}$$

which implies

$$\langle A(\tilde{u}_0) - A(u), (\tilde{u}_0 - u)^+\rangle + \int_\Omega (\beta(z) + \lambda)(\tilde{u}_0^{p-1} - u^{p-1})(\tilde{u}_0 - u)^+ \, dz \le 0,$$

and so

$$|\{\tilde{u}_0 > u\}|_N = 0 \quad (\text{since } \lambda > \|\beta^-\|_\infty),$$

hence $\tilde{u}_0 \le u$. So, we have proved that

$$\tilde{u}_0 \in [0, u] = \{v \in W^{1,p}(\Omega) : 0 \le v(z) \le u(z) \text{ a.e. in } \Omega\}, \quad \tilde{u}_0 \ne 0.$$

Then relation (3.18) becomes

$$\begin{aligned}A(\tilde{u}_0) + \beta\tilde{u}_0^{p-1} = \tilde{c}\tilde{u}_0^{q-1} - c_{10}\tilde{u}_0^{p-1} \quad \text{(see (3.16))} &\implies \tilde{u}_0 \text{ is a nontrivial positive solution of (3.11)}, \\ &\implies \tilde{u}_0 = \tilde{u} \in \operatorname{int} C_+ \quad \text{(see Proposition 3.5)}.\end{aligned}$$

Therefore $\tilde{u} \le u$ for all $u \in S_+$ and this proves the claim.

Now, let $C \subseteq S_+$ be a chain (that is, a totally ordered subset of S_+). We know that we can find $\{u_n\}_{n\ge1} \subseteq C$ such that $\operatorname{int} C = \inf_{n\ge1} u_n$ (see Dunford and Schwartz [11, p. 336]). We have

$$A(u_n) + \beta u_n^{p-1} = N_f(u_n), \quad u_* \le u_n \le M \quad \text{for all } n \ge 1 \quad \text{(see (3.15))}. \tag{3.19}$$

So, $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded and we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega) \quad \text{as } n \to \infty. \tag{3.20}$$

On (3.19) we act with $u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.20). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \implies u_n \to u \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 2.4)}. \tag{3.21}$$

Passing to the limit as $n \to \infty$ in (3.19) and using (3.21), we obtain

$$A(u) + \beta u^{p-1} = N_f(u), \ \tilde{u} \le u \le M \implies u \in S_+, \ u = \inf C.$$

Since C is an arbitrary chain of S_+, from the Kuratowski–Zorn lemma we infer that we can find $u_* \in S_+$ a minimal element. Since S_+ is downward directed, we conclude that $u_* \in \operatorname{int} C_+$ is the smallest nontrivial positive solution of (1.1).

Similarly, let S_- be the set of nontrivial negative solutions of problem (1.1). From Proposition 3.4 we know that $S_- \ne \emptyset$ and $S_- \subseteq -\operatorname{int} C_+$. Also, S_- is upward directed, that is, if $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \le v_1$, $v_2 \le v$ (see [1]). Reasoning as above, via the Kuratowski–Zorn lemma, we produce $v_* \in -\operatorname{int} C_+$ the biggest nontrivial negative solution of (1.1). □

Using these extremal constant sign solutions of (1.1), we can produce a nodal solution. Via suitable truncation and perturbation techniques, we focus on the order interval $[v_*, u_*] = \{u \in W^{1,p}(\Omega) : v_* \le u \le u_* \text{ a.e. in } \Omega\}$. Then using variational methods coupled with Morse theory, we show that problem (1.1) admits a solution in $[v_*, u_*]$ distinct from $0, u_*, v_*$. Evidently, this a nodal solution.

To execute this solution plan, we need to compute the critical groups of φ at the origin.

Proposition 3.7. *Assume that Hypotheses* $\mathrm{H}(a)_1$, H_0 *and* H_1 *hold. Then* $C_k(\varphi, 0) = 0$ *for all* $k \geq 0$.

Proof. Recall that Hypotheses H_1 (i), (iii) imply that

$$F(z,x) \geq \tilde{c}_0|x|^q - c_{11}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_{11} > 0,\ r > p. \tag{3.22}$$

Also, from Hypothesis $\mathrm{H}(a)_1$ (iv) and (2.6), we have

$$G(y) \leq c_{12}(\|y\|^\tau + \|y\|^p) \quad \text{for all } y \in \mathbb{R}^N \text{ and some } c_{12} > 0. \tag{3.23}$$

Then for all $u \in W^{1,p}(\Omega)$ and all $t > 0$, we have

$$\begin{aligned} \varphi(tu) &= \int_\Omega G(tDu)\,dz + \frac{t^p}{p}\int_\Omega \beta(z)|u|^p\,dz - \int_\Omega F(z,tu)\,dz \\ &\leq c_{12}(t^\tau\|Du\|_\tau^\tau + t^p\|Du\|_p^p) + \frac{t^p}{p}\|\beta\|_\infty\|u\|_p^p + c_{11}t^r\|u\|_r^r - \tilde{c}_0 t^q\|u\|_q^q \quad \text{(see (3.22)–(3.23))}. \end{aligned} \tag{3.24}$$

Since $q < \tau < p < r$, from (3.24) it is clear that we can find $t^* = t^*(u) \in (0,1)$ such that

$$\varphi(tu) < 0 \quad \text{for all } t \in (0, t^*). \tag{3.25}$$

Let $u \in W^{1,p}(\Omega)$, $0 < \|u\| \leq 1$ and $\varphi(u) = 0$. We have

$$\begin{aligned} \frac{d}{dt}\varphi(tu)\Big|_{t=1} &= \langle \varphi'(u), u\rangle \\ &= \langle A(u), u\rangle + \int_\Omega \beta|u|^p\,dz - \int_\Omega f(z,u)u\,dz \\ &= \int_\Omega \big((a(Du), Du)_{\mathbb{R}^N} - \tau G(Du)\big)\,dz + \Big(1 - \frac{\tau}{p}\Big)\int_\Omega \beta(z)|u|^p\,dz \\ &\qquad + (\tau - q)\int_\Omega F(z,u)\,dz + \int_\Omega [qF(z,u) - f(z,u)u]\,dz \quad (\text{since } \varphi(u) = 0). \end{aligned} \tag{3.26}$$

By virtue of Hypothesis $\mathrm{H}(a)_1$ (iv), we have

$$(a(Du(z)), Du(z))_{\mathbb{R}^N} - \tau G(Du(z)) \geq \tilde{c}\|Du(z)\|^p \quad \text{for a.a. } z \in \Omega. \tag{3.27}$$

Hypothesis H_1 (iii) implies that for a.a. $z \in \Omega$ and all $|x| \leq \delta_1$ with $\delta_1 \in (0, \delta]$,

$$F(z,x) \geq \frac{\tilde{c}_0}{q}|x|^q \geq \frac{\tilde{c}_0}{\delta_1^{p-q}}|x|^p. \tag{3.28}$$

On the other hand, Hypothesis H_1 (i) implies that we can find $c_{13} = c_{13}(\delta_1, r) > 0$ such that

$$F(z,x) \geq -c_{13}|x|^r \quad \text{for a.a. } z \in \Omega \text{ and all } |x| > \delta_1. \tag{3.29}$$

Combining (3.28) and (3.29), we find $c_{14} = c_{14}(\delta_1, r) > 0$ such that

$$F(z,x) \geq \frac{\tilde{c}_0}{\delta_1^{p-q}}|x|^p - c_{14}|x|^r \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.30}$$

Moreover, from Hypotheses H_1 (i), (iii) we have

$$qF(z,x) - f(z,x)x \geq -c_{15}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_{15} > 0. \tag{3.31}$$

Returning to (3.26) and using (3.27), (3.30), and (3.31), we obtain

$$\frac{d}{dt}\varphi(tu)\Big|_{t=1} \geq \tilde{c}\|Du\|_p^p + \Big[\frac{\tilde{c}_0}{\delta_1^{p-q}} - \Big(1 - \frac{\tau}{p}\Big)\|\beta\|_\infty\Big]\|u\|_p^p - c_{16}\|u\|^r \quad \text{for some } c_{16} > 0. \tag{3.32}$$

We choose $\delta_1 \in (0, \delta]$ small such that

$$\frac{\tilde{c}_0}{\delta_1^{p-q}} > \Big(1 - \frac{\tau}{p}\Big)\|\beta\|_\infty.$$

Then from (3.32) we see that

$$\frac{d}{dt}\varphi(tu)\Big|_{t=1} \geq c_{17}\|u\|^p - c_{16}\|u\|^r \quad \text{for some } c_{17} > 0.$$

Since $p < r$, there exists some $\rho \in (0,1)$ small such that

$$\frac{d}{dt}\varphi(tu)\Big|_{t=1} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } 0 < \|u\| \leq \rho,\ \varphi(u) = 0. \tag{3.33}$$

Now, let $u \in W^{1,p}(\Omega)$, with $0 < \|u\| \leq \rho$ and $\varphi(u) = 0$. We show in what follows that

$$\varphi(tu) \leq 0 \quad \text{for all } t \in [0,1]. \tag{3.34}$$

We argue by contradiction. So, suppose that there is some $t_0 \in (0,1)$ such that $\varphi(t_0 u) > 0$. Since φ is continuous and $\varphi(u) = 0$, by Bolzano's theorem, we can find $t_1 \in (t_0, 1]$ such that $\varphi(t_1 u) = 0$. Let

$$t_* = \min\{t \in [t_0, 1] : \varphi(tu) = 0\} > t_0 > 0.$$

Then

$$\varphi(tu) > 0 \quad \text{for all } t \in [t_0, t_*). \tag{3.35}$$

Let $y = t_* u$. We have $0 < \|y\| \leq \|u\| \leq \rho$ and $\varphi(y) = 0$. Therefore, from (3.33) it follows that

$$\frac{d}{dt}\varphi(ty)\Big|_{t=1} > 0 \tag{3.36}$$

Also, from (3.35) we have

$$\varphi(y) = \varphi(t_* u) = 0 < \varphi(tu) \quad \text{for all } t \in [t_0, t_*) \implies \frac{d}{dt}\varphi(ty)\Big|_{t=1} = t_*\frac{d}{dt}\varphi(tu)\Big|_{t=t_*} = t_* \lim_{t\to t_*^-}\frac{\varphi(tu)}{t - t_*} \leq 0. \tag{3.37}$$

Comparing (3.36) and (3.37), we reach a contradiction. This proves (3.34).

By taking $\rho \in (0,1)$ even smaller if necessary, we may assume that $K_\varphi \cap \bar{B}_\rho = \{0\}$, where

$$\bar{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\| \leq \rho\}.$$

Let $h : [0,1] \times (\varphi^0 \cap \bar{B}_\rho) \to \varphi^0 \cap \bar{B}_\rho$ be the continuous function defined by

$$h(t,u) = (1-t)u \quad \text{for all } (t,u) \in [0,1] \times (\varphi^0 \cap \bar{B}_\rho).$$

From (3.34) we see that $h(\cdot,\cdot)$ is well-defined. This deformation shows that $\varphi^0 \cap \bar{B}_\rho$ is contractible in itself.

Fix $u \in \bar{B}_\rho$ with $\varphi(u) > 0$. We show that there exists a unique $t(u) \in (0,1)$ such that

$$\varphi(t(u)u) = 0. \tag{3.38}$$

Note that $\varphi(u) > 0$ and $t \mapsto \varphi(tu)$ is continuous. So, the existence of some $t(u) \in (0,1)$ follows from Bolzano's theorem. We need to show the uniqueness of $t(u)$. Suppose there are $0 < \hat{t}_1 = t(u)_1 < \hat{t}_2 = t(u)_2 < 1$ such that $\varphi(\hat{t}_1, u) = \varphi(\hat{t}_2 u) = 0$. Then from (3.34), we have

$$k(t) = \varphi(t\hat{t}_2 u) \leq 0 \quad \text{for all } t \in [0,1].$$

Hence $\frac{\hat{t}_1}{\hat{t}_2} \in (0,1)$ is a maximizer of $k(\cdot)$ and so

$$\frac{d}{dt}k(t)\Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} = 0 \implies \frac{\hat{t}_1}{\hat{t}_2}\frac{d}{dt}\varphi(t\hat{t}_2 u)\Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} = \frac{d}{dt}\varphi(t\hat{t}_1 u)\Big|_{t=1} = 0,$$

which contradicts (3.33). This proves the uniqueness of $t(u)$.

From the uniqueness of $t(u) \in (0,1)$ and (3.34), we have

$$\begin{aligned} \varphi(tu) < 0 &\quad \text{if } t \in (0, t(u)), \\ \varphi(tu) > 0 &\quad \text{if } t \in (t(u), 1]. \end{aligned} \tag{3.39}$$

Now, let $\widehat{\epsilon}_1 : \bar{B}_\rho\backslash\{0\} \to (0,1]$ be defined by

$$\widehat{\epsilon}_1(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\rho\backslash\{0\},\ \varphi(u) \le 0, \\ t(u) & \text{if } u \in \bar{B}_\rho\backslash\{0\},\ \varphi(u) > 0. \end{cases} \tag{3.40}$$

We claim that $\widehat{\epsilon}_1$ is continuous. Evidently, we need to check the continuity at $u \in \bar{B}_\rho\backslash\{0\}$ with $\varphi(u) = 0$. Let $u_n \to u$ with $\varphi(u_n) > 0$ for all $n \ge 1$. Arguing by contradiction, suppose that by passing to a subsequence if necessary, we have $t(u_n) \le \widehat{t} < 1$ for all $n \ge 1$. From (3.39) we have

$$\begin{aligned} \varphi(tu_n) > 0 \quad \text{for all } t \in (\tilde{t}, 1] \text{ and all } n \ge 1 &\implies \varphi(tu) \ge 0 \quad \text{for all } t \in (\tilde{t}, 1] \\ &\implies \varphi(tu) = 0 \quad \text{for all } t \in (\tilde{t}, 1] \quad (\text{see } (3.34)) \\ &\implies \frac{d}{dt}\varphi(tu)\Big|_{t=1} = 0, \end{aligned}$$

which contradicts (3.33). This proves the continuity of $\widehat{\epsilon}_1$.

Next, consider the map $\widehat{\epsilon}_1 : \bar{B}_\rho\backslash\{0\} \to (\bar{B}_\rho \cap \varphi^0)\backslash\{0\}$ defined by

$$\widehat{\epsilon}_2(u) = \begin{cases} u & \text{if } u \in \bar{B}_\rho\backslash\{0\},\ \varphi(u) \le 0, \\ \widehat{\epsilon}_1(u)u & \text{if } u \in \bar{B}_\rho\backslash\{0\},\ \varphi(u) > 0. \end{cases}$$

Evidently, $\widehat{\epsilon}_2$ is continuous and

$$\widehat{\epsilon}_2|_{(\bar{B}_\rho \cap \varphi^0)\backslash\{0\}} = \mathrm{id}|_{(\bar{B}_\rho \cap \varphi^0)\backslash\{0\}}.$$

Therefore $\widehat{\epsilon}_2$ is a retraction of $\bar{B}_\rho\backslash\{0\}$ onto $(\bar{B}_\rho \cap \varphi^0)\backslash\{0\}$. But $\bar{B}_\rho\backslash\{0\}$ is contractible in itself. Hence the same holds for $(\bar{B}_\rho \cap \varphi^0)\backslash\{0\}$. Recall that we have seen that $\bar{B}_\rho \cap \varphi^0$ is contractible in itself. So, from Granas and Dugundji [17, p. 389], we deduce that

$$H_1(\bar{B}_\rho \cap \varphi^0, (\bar{B}_\rho \cap \varphi^0)\backslash\{0\}) = 0 \quad \text{for all } k \ge 0,$$

hence $C_k(\varphi, 0) = 0$ for all $k \ge 0$ (see Section 2). □

Remark 3.8. The first such computation of the critical groups of φ for equations with concave nonlinearities near the origin was conducted by Moroz [27] for Dirichlet problems driven by the Laplace operator (semilinear equations) with $\beta \equiv 0$. The conditions on $f(z,x)$ in Moroz [27] were more restrictive. The result of Moroz [27] was extended to Dirichlet problems driven by the p-Laplacian with $\beta \equiv 0$, by Jiu and Su [19]. Our proof here was inspired by these two works.

Now, we are ready to produce a nodal solution for problem (1.1).

Proposition 3.9. *Assume that Hypotheses* $\mathrm{H}(a)_1$, H_0 *and* H_1 *hold. Then problem* (1.1) *admits a nodal solution* $y_0 \in [v_*, u_*] \cap C^1(\bar{\Omega})$ *(here* $v_* \in -\operatorname{int} C_+$ *and* $u_* \in \operatorname{int} C_+$ *are the two extremal nontrivial constant sign solutions of* (1.1) *produced in Proposition 3.6).*

Proof. As before, let $\lambda > \|\beta^-\|_\infty$ and consider the following truncation-perturbation of the reaction $f(z,\cdot)$:

$$\widehat{f}(z,x) = \begin{cases} f(z, v_*(z)) + \lambda|v_*(z)|^{p-2}v_*(z) & \text{if } x < v_*(z), \\ f(z,x) + \lambda|x|^{p-2}x & \text{if } v_*(z) \le x \le u_*(z), \\ f(z, u_*(z)) + \lambda u_*(z)^{p-1} & \text{if } u_*(z) < x. \end{cases} \tag{3.41}$$

This is a Carathéodory function. We set

$$\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds$$

and consider the C^1-functional $\widehat{\Psi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\Psi}(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega [\beta(z) + \lambda]|u(z)|^p\,dz - \int_\Omega \widehat{F}(z, u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, we introduce the Carathéodory functions $\widehat{f}_{\pm}(z,x) = \widehat{f}(z, \pm x^{\pm})$, we set

$$\widehat{F}_{\pm}(z,x) = \int_0^x \widehat{f}_{\pm}(z,s)\,ds$$

and consider the C^1-functionals $\widehat{\Psi}_{\pm} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\Psi}_{\pm}(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega [\beta(z)+\lambda]|u(z)|^p\,dz - \int_\Omega \widehat{F}_{\pm}(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We can easily check that

$$K_{\widehat{\Psi}} \subseteq [v_*, u_*], \quad K_{\widehat{\Psi}_+} \subseteq [0, u_*], \quad K_{\widehat{\Psi}_-} \subseteq [v_*, 0].$$

The extremality of $v_* \in -\operatorname{int} C_+$ and of $u_* \in \operatorname{int} C_+$ implies that

$$K_{\widehat{\Psi}} \subseteq [v_*, u_*], \quad K_{\widehat{\Psi}_+} = \{0, u_*\}, \quad K_{\widehat{\Psi}_-} = \{v_*, 0\}. \tag{3.42}$$

Claim. *Both $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of $\widehat{\Psi}$.*

Evidently, $\widehat{\Psi}_+$ is coercive (see (3.41) and recall that $\lambda > \|\beta^-\|_\infty$). Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $\widehat{u}_* \in W^{1,p}(\Omega)$ such that

$$\widehat{\Psi}_+(\widehat{u}_*) = \inf\{\widehat{\Psi}_+(u) : u \in W^{1,p}(\Omega)\}. \tag{3.43}$$

As before (see the proof of Proposition 3.4), using Hypotheses $\mathrm{H}(a)_1$ (iv) and H_1 (iii), we show that

$$\widehat{\Psi}_+(\widehat{u}_*) < 0 = \widehat{\Psi}_+(0),$$

hence $\widehat{u}_* \neq 0$. From (3.43) we have

$$\widehat{\Psi}_+'(\widehat{u}_*) = 0 \implies A(\widehat{u}_*) + (\beta+\lambda)|\widehat{u}_*|^{p-2}\widehat{u}_* = N_{\widehat{f}_+}(\widehat{u}_*). \tag{3.44}$$

On (3.44) we act with $-u^- \in W^{1,p}(\Omega)$ and since $\lambda > \|\beta^-\|_\infty$, we obtain that $\widehat{u}_* \geq 0$, $\widehat{u}_* \neq 0$ (see (3.41)). Next on (3.44) we act with $(\widehat{u}_* - u_*)^+ \in W^{1,p}(\Omega)$ and we have

$$\begin{aligned}\langle A(\widehat{u}_*), (\widehat{u}_* - u_*)^+\rangle + \int_\Omega (\beta(z)+\lambda)\widehat{u}_*^{p-1}(\widehat{u}_* - u_*)^+\,dz &= \int_\Omega \widehat{f}_+(z,\widehat{u}_*)(\widehat{u}_* - u_*)^+\,dz \\ &= \int_\Omega [f(z,u_*) + \lambda u_*^{p-1}](\widehat{u}_* - u_*)^+\,dz \quad \text{(see (3.41))} \\ &= \langle A(u_*), (\widehat{u}_* - u_*)^+\rangle + \int_\Omega [\beta(z)+\lambda]u_*^{p-1}(\widehat{u}_* - u_*)^+\,dz,\end{aligned}$$

which implies

$$\int_{\{\widehat{u}_* > u_*\}} (a(D\widehat{u}_*) - a(Du_*),\ D\widehat{u}_* - Du_*)_{\mathbb{R}^N}\,dz + \int_{\{\widehat{u}_* > u_*\}} (\beta(z)+\lambda)(\widehat{u}_*^{p-1} - u_*^{p-1})(\widehat{u}_* - u_*)\,dz = 0$$

and so

$$|\{\widehat{u}_* > u_*\}|_N = 0,$$

hence $\widehat{u}_* \leq u_*$. Hence we have proved that $\widehat{u}_* \in K_{\widehat{\Psi}_+}$ and $\widehat{u}_* \in [0, u_*]$, $\widehat{u}_* \neq 0$, hence $\widehat{u}_* = u_*$ (see (3.42)). But

$$\widehat{\Psi}|_{C_+} = \widehat{\Psi}_+|_{C_+}.$$

Thus u_* is a local $C^1(\bar{\Omega})$-minimizer of $\widehat{\Psi}$, hence it is also a local $W^{1,p}(\Omega)$-minimizer of $\widehat{\Psi}$ (see Proposition 2.5).

Similarly for $v_* \in -\operatorname{int} C_+$ using this time the functional $\widehat{\Psi}_-$. This proves the claim.

Without any loss of generality, we may assume that $\widehat{\Psi}(v_*) \le \widehat{\Psi}(u_*)$ (the analysis is similar if the opposite inequality holds). From the claim we know that $u_* \in \operatorname{int} C_+$ is a local minimizer of $\widehat{\Psi}$. Hence we can find $\rho \in (0, 1)$ small such that

$$\widehat{\Psi}(v_*) \le \widehat{\Psi}(u_*) < \inf\{\widehat{\Psi}(u) : \|u - u_*\| = \rho\} = \widehat{\eta}_\rho, \quad \|v_* - u_*\| > \rho. \tag{3.45}$$

Recall that $\widehat{\Psi}$ is coercive, hence it satisfies the C-condition. Combining this fact and (3.45), we see that we can apply Theorem 2.1 (the mountain pass theorem) and find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\Psi}} \quad \text{and} \quad \widehat{\eta}_\rho \le \widehat{\Psi}(y_0). \tag{3.46}$$

From (3.42) and (3.46) it follows that $y_0 \in [v_*, u_*]$, hence y_0 is a solution of problem (1.1) (see (3.41)).

Since y_0 is a critical point of $\widehat{\Psi}$ of mountain pass type, we have

$$C_1(\widehat{\Psi}, y_0) \neq 0. \tag{3.47}$$

Also because $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$ and $\widehat{\Psi}|_{[v_*,u_*]} = \varphi|_{[v_*,u_*]}$, from the homotopy invariance of critical groups, we have

$$C_k(\widehat{\Psi}, 0) = C_k(\varphi, 0) = 0 \quad \text{for all } k \ge 0 \quad \text{(see Proposition 3.7).} \tag{3.48}$$

Comparing (3.47) and (3.48), we see that $y_0 \neq 0$. Hence y_0 is a nodal solution of (1.1) and the nonlinear regularity theory implies $y_0 \in C^1(\bar{\Omega})$. □

Now, we can state the following multiplicity theorem for problem (1.1).

Theorem 3.10. *Assume that Hypotheses* $\mathrm{H}(a)_1$, H_0 *and* H_1 *hold. Then problem* (1.1) *admits at least three nontrivial solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega})$ *nodal.*

Remark 3.11. Three solutions theorems for coercive nonlinear equations driven by the p-Laplacian (that is, we have $a(y) = \|y\|^{p-2}y$ for all $y \in \mathbb{R}^N$, $1 < p < \infty$) were proved by Liu [25], Liu and Liu [24] and Kyritsi and Papageorgiou [21]. In [25] and [24], the authors deal with Dirichlet problems with $\beta \equiv 0$ and the reaction $f(z, \cdot)$ satisfies a global sign condition. They prove a three solutions theorem, but they do not produce a nodal solution. In Kyritsi and Papageorgiou [21] the problem is Neumann, with $\beta(z) \equiv \beta \in (0, +\infty)$ for all $z \in \Omega$ and no nodal solution is obtained.

4 Noncoercive problems

In the previous section it was assumed that the reaction $f(z, \cdot)$ is $(p-1)$-sublinear near $\pm\infty$ and $(p-1)$-superlinear near zero (see Hypotheses H_1 (i), (iii)). In this section, we investigate the complementary situation. Namely, we consider nonlinearities which are $(p-1)$-superlinear near $\pm\infty$ and $(p-1)$-sublinear near 0. In this case the energy functional of the problem is indefinite. To express the $(p-1)$-superlinearity of $f(z, \cdot)$ near $\pm\infty$, we do not employ the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). Instead we use a more general condition which incorporates in our setting “superlinear” reactions with “slower” growth. These nonlinearities fail to satisfy the AR-condition.

We need to modify a little the conditions on the map $a(y)$:

Hypothesis H$(a)_2$. We have

$$a(y) = a_0(\|y\|)y \quad \text{for all } y \in \mathbb{R}^N$$

with $a_0(t) > 0$ for all $t > 0$, hypotheses (i)–(iii) are the same as the corresponding Hypotheses $\mathrm{H}(a)_1$ (i)–(iii) and

(iv) there exists some $q \in (1, p)$ such that the map $t \mapsto G_0(t^{\frac{1}{q}})$ is convex in $(0, +\infty)$ and there exists some $\gamma \in \mathbb{R}$ such that

$$\gamma \le pG_0(t) - t^2 a_0(t) \quad \text{for all } t > 0.$$

Remark 4.1. The examples presented in Section 2 satisfy Hypotheses $\mathrm{H}(a)_2$.

Hypothesis H_2. We assume that the reaction term $f(z,x)$ is a Carathéodory function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ such that $f(z,0)=0$ a.e. in Ω satisfying the following conditions:

(i) $|f(z,x)|\le a(z)(1+|x|^{r-1})$ a.e. in Ω, for all $x\in\mathbb{R}$, with $a\in L^\infty(\Omega)_+$ and $p<r<p^*$,

(ii) if $F(z,x)=\int_0^x f(z,s)\,ds$, then
$$\lim_{x\to\pm\infty}\frac{F(z,x)}{|x|^p}=\pm\infty$$
uniformly for a.a. $z\in\Omega$,

(iii) there exists some $\tau\in((r-p)\max\{\frac{N}{p},1\},p^*)$ such that
$$0<\beta_0\le\liminf_{x\to\pm\infty}\frac{f(z,x)x-pF(z,x)}{|x^\tau|}$$
uniformly for a.a. $z\in\Omega$,

(iv) there exists some $\vartheta\in L^\infty(\Omega)$, $\vartheta(z)\le\frac{c_1}{p-1}\widehat{\lambda}_1(p,\widehat{\beta})$ a.e. in Ω, $\vartheta\ne\frac{c_1}{p-1}\widehat{\lambda}_1(p,\widehat{\beta})$ (recall that $\widehat{\beta}=\frac{p-1}{c_1}\beta\in L^\infty(\Omega)$) such that
$$\limsup_{x\to 0}\frac{pF(z,x)}{|x|^p}\le\vartheta(z)$$
uniformly for a.a. $z\in\Omega$,

(v) for every $\rho>0$, there exists $\epsilon_\rho>0$ such that for a.a. $z\in\Omega$, the mapping $x\mapsto f(z,x)+\epsilon_\rho|x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$.

Remark 4.2. Hypotheses H_2 (ii)–(iii) imply that the reaction $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$. In the literature, "superlinear" problems are usually treated with the help of the so-called AR-condition. According to that condition, there exist $\eta>p$ and $M>0$ such that
$$0<\eta F(z,x)\le f(z,x)x\quad\text{for a.a. } z\in\Omega,\ \text{all } |x|\ge M,\quad\text{and}\quad\inf_\Omega F(\cdot,\pm M)>0.\tag{4.1}$$
A straightforward integration of (4.1) leads to
$$c_{18}|x|^\eta\le F(z,x)\quad\text{for a.a. } z\in\Omega,\ \text{all } |x|\ge M \text{ and some } c_{18}>0.\tag{4.2}$$
Clearly, (4.2) implies the much weaker condition H_2 (ii). Here, we replace (4.1) (the AR-condition) by Hypothesis H_2 (iii) which is weaker. Indeed, suppose that (4.1). We may assume that $\eta>(r-p)\max\{\frac{N}{p},1\}$. Then
$$\frac{f(z,x)x-pF(z,x)}{|x|^\eta}=\frac{f(z,x)x-\eta F(z,x)}{|x|^\eta}+(\eta-p)\frac{F(z,x)}{|x|^\eta}\ge(\eta-p)c_{18}\quad\text{for a.a. } z\in\Omega \text{ and all } |x|\ge M,$$
hence Hypothesis H_2 (iii) holds. Similar superlinearity conditions were used by Costa and Magalhaes [8], Fei [12] and Li, Wu and Zhou [22].

Example 4.3. The following functions satisfy Hypothesis H_2. For the sake of simplicity, we drop the z-dependence:
$$f_1(x)=\vartheta|x|^{p-2}x+|x|^{r-2}x\qquad\text{with } \vartheta<\widehat{\lambda}_1(p,\widehat{\beta}),\ 1<p<r<p^*,$$
$$f_2(x)=\begin{cases}\vartheta|x|^{p-2}x & \text{if } |x|<1,\\ |x|^{p-2}x\ln|x|+\vartheta|x|^{q-2}x & \text{if } |x|\ge 1,\end{cases}\qquad\text{with } \vartheta<\widehat{\lambda}_1(p,\widehat{\beta}),\ 1<q<p.$$
Note that f_1 satisfies the AR-condition (see (4.1)), but f_2 does not.

As before (see Section 3) with $\lambda>\|\beta^-\|_\infty$, we consider the truncations-perturbations of $f(z,\cdot)$, $\widehat{f}_\pm(z,x)$ and the corresponding C^1-functionals $\widehat{\varphi}_\pm:W^{1,p}(\Omega)\to\mathbb{R}$. Also $\varphi:W^{1,p}(\Omega)\to\mathbb{R}$ is the C^1-energy functional of problem (1.1) (see Section 3).

Proposition 4.4. *Assume that Hypotheses* $\mathrm{H}(a)_2$, H_0, H_2 *hold. Then the functionals* $\widehat{\varphi}_\pm$ *satisfy the C-condition.*

Proof. We do the proof for the functional $\widehat{\varphi}_+$, the arguments for $\widehat{\varphi}_-$ being similar. So, let $\{u_n\}_{n\ge1}\subseteq W^{1,p}(\Omega)$ be such that
$$|\widehat{\varphi}_+(u_n)|\le M_1\quad\text{for some } M_1>0 \text{ and all } n\ge 1\tag{4.3}$$
and
$$(1+\|u_n\|)\widehat{\varphi}_+'(u_n)\to 0\quad\text{in } W^{1,p}(\Omega)^*\quad\text{as } n\to\infty.\tag{4.4}$$

From (4.4), we have for all $h \in W^{1,p}(\Omega)$ and some $\epsilon_n \to 0^+$,

$$\left| \langle A(u_n), h\rangle + \int_\Omega (\beta(z) + \lambda)|u_n|^{p-2} u_n h\,dz - \int_\Omega \hat{f}_+(z, u_n) h\,dz \right| \le \frac{\epsilon_n \|h\|}{1 + \|u_n\|}. \tag{4.5}$$

In (4.5) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then by (2.5) we obtain for all $n \ge 1$,

$$\frac{c_1}{p-1}\|Du_n^-\|_p^p + \int_\Omega (\beta(z) + \lambda)(u_n^-)^p\,dz \le \epsilon_n \implies u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega) \quad (\text{recall } \lambda > \|\beta^-\|_\infty). \tag{4.6}$$

Next in (4.5) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. It follows that

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\,dz - \int_\Omega \beta(z)(u_n^+)^p\,dz + \int_\Omega f(z, u_n^+)u_n^+\,dz \le \epsilon_n \quad \text{for all } n \ge 1. \tag{4.7}$$

On the other hand, from (4.3) and (4.6) we have for all $n \ge 1$,

$$\int_\Omega pG(Du_n^+)\,dz + \int_\Omega \beta(z)(u_n^+)^p\,dz - \int_\Omega pF(z, u_n^+)\,dz \le M_2 \quad \text{for some } M_2 > 0. \tag{4.8}$$

Adding (4.7) and (4.8), we obtain for all $n \ge 1$,

$$\begin{aligned} \int_\Omega [pG(Du_n^+) - (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}]\,dz + \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]\,dz &\le M_3 \quad \text{for some } M_3 > 0, \\ \Longrightarrow \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]\,dz &\le M_4 \quad \text{for some } M_4 > 0 \end{aligned} \tag{4.9}$$

(see Hypothesis $H(a)_2$ (iv)). By virtue of Hypotheses H_2 (i), (iii), we can find $\beta_1 \in (0, \beta_0)$ and $c_{19} > 0$ such that

$$\beta_1 x^\tau - c_{19} \le f(z, x)x - pF(z, x) \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 0. \tag{4.10}$$

Using (4.10) in (4.9), we deduce that there is some $M_5 > 0$ such that for all $n \ge 1$,

$$\beta_1 \|u_n^+\|_\tau^\tau \le M_5, \tag{4.11}$$

hence $\{u_n^+\}_{n\ge1} \subseteq L^\tau(\Omega)$ is bounded.

First suppose that $N \ne p$. From Hypothesis H_2 (iii) it is clear that without any loss of generality we may assume that $\tau < r < p^*$. Hence, we can find $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}.$$

Invoking the interpolation inequality (see, for example, Gasinski and Papageorgiou [15, p. 905]), we have

$$\|u_n^+\|_r \le \|u_n^+\|_\tau^{1-t}\,\|u_n^+\|_{p^*}^t \implies \|u_n^+\|_r^r \le M_6 \|u_n^+\|^{tr} \quad \text{for some } M_6 > 0 \text{ and all } n \ge 1 \tag{4.12}$$

(see (3.46) and use the Sobolev embedding theorem). Hypothesis H_2 (i) implies that

$$f(z, x)x \le c_{20}(1 + x^r) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0 \text{ with } c_{20} > 0. \tag{4.13}$$

From (4.5) with $h = u_n^+ \in W^{1,p}(\Omega)$, we have for all $n \ge 1$,

$$\begin{aligned} &\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N} dz + \int_\Omega \beta(u_n^+)^p\,dz - \int_\Omega f(z, u_n^+)u_n^+\,dz \le \epsilon_n \\ &\implies \frac{c_1}{p-1}\|Du_n^+\|_p^p \le c_{21}(1 + \|u_n^+\|_r^r) \quad \text{for some } c_{21} > 0 \end{aligned} \tag{4.14}$$

(see (2.5), (4.13), Hypothesis H_0 and recall that $r > p$). We know that $u \to \|u\|_\tau + \|Du\|_p$ is on equivalent norm on $W^{1,p}(\Omega)$ (see, for example, Gasinski and Papageorgiou [15, p. 227]). So, from (4.11) and (4.14) and since $\tau < r$, we have for all $n \ge 1$,

$$\|u_n^+\|^p \le c_{22}(1 + \|u_n^+\|_r^r) \le c_{23}(1 + \|u_n^+\|^{tr}). \tag{4.15}$$

The hypothesis on τ (see Hypothesis H_2 (iii)) implies that $tr < p$. So, from (4.15) it follows that

$$\{u_n^+\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded } \implies \{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded} \quad \text{(see (4.6))}. \tag{4.16}$$

If $N = p$, then $p^* = +\infty$ and from the Sobolev embedding theorem, we have that $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \le s < \infty$. Then for the previous argument to work we replace p^* by $\eta > r > \tau$ and choose $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{\eta} \implies tr = \frac{\eta(r-\tau)}{\eta-\tau}.$$

Note that $\frac{\eta(r-\tau)}{\eta-\tau} \to r - \tau$ as $\eta \to p^* = +\infty$. But $r - \tau < p$ (see Hypothesis H_2 (iii)). Therefore for $\eta > r$ large, we can have $tr < p$ and so (4.16) holds.

By virtue of (4.16), we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega) \quad \text{as } n \to \infty. \tag{4.17}$$

In (4.5) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.17). Thus, by Proposition 2.4,

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \implies u_n \to u \text{ in } W^{1,p}(\Omega) \implies \widehat{\varphi}_+ \text{ satisfies the C-condition.}$$

Similarly for the functional $\widehat{\varphi}_-$. □

With some minor straightforward changes in the above proof, we can also have the following result.

Proposition 4.5. *Assume that Hypotheses* $H(a)_2$, H_0, H_2 *hold. Then the functional* φ *satisfies the C-condition.*

First we will produce two nontrivial constant sign solutions. This will be done by using Theorem 2.1 (the mountain pass theorem). To this end, we check the mountain pass geometry for the functionals $\widehat{\varphi}_\pm$.

Proposition 4.6. *Assume that Hypotheses* $H(a)_2$, H_0, H_1 *hold. Then* $u = 0$ *is a local minimizer for the three functionals* $\widehat{\varphi}_\pm$ *and* φ.

Proof. We do the proof for the functional $\widehat{\varphi}_+$, the arguments for the functionals $\widehat{\varphi}_-$ and φ being similar. By virtue of Hypothesis H_2 (iv), given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \le \frac{1}{p}[\vartheta(z) + \epsilon]|x|^p \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \le \delta. \tag{4.18}$$

Let $u \in C^1(\bar{\Omega})$ such that $\|u\|_{C^1(\bar{\Omega})} \le \delta$. Then

$$\begin{aligned}
\widehat{\varphi}_+(u) &= \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega (\beta(z)+\lambda)|u|^p\,dz + \int_\Omega \widehat{F}_+(z,u)\,dz \\
&\ge \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \int_\Omega F(z,u)\,dz \qquad \text{(see (2.6))} \\
&\ge \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \frac{1}{p}\int_\Omega \vartheta(z)|u|^p\,dz - \frac{\epsilon}{p}\|u\|^p \\
&= \frac{c_1}{p(p-1)}\Big[\|Du\|_p^p + \int_\Omega \widehat{\beta}(z)|u|^p\,dz - \int_\Omega \frac{p-1}{c_1}\vartheta(z)|u|^p\,dz\Big] - \frac{\epsilon}{p}\|u\|^p \\
&\ge \frac{1}{p}\Big[\frac{c_1c_{25}}{p-1} - \epsilon\Big]\|u\|^p \quad \text{for some } c_{25} > 0 \qquad \text{(see Lemma 3.3).}
\end{aligned} \tag{4.19}$$

Choosing $\epsilon \in (0, \frac{c_1c_{25}}{p-1})$ from (4.19), we infer that $u = 0$ is a local $C^1(\bar{\Omega})$-minimizer of $\widehat{\varphi}_+$, hence $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\widehat{\varphi}_+$ (see Proposition 2.5).

Similarly for the functionals $\widehat{\varphi}_-$ and φ. □

The superlinearity of $F(z,\cdot)$ (see Hypothesis H_2 (ii)) leads to the following result.

Proposition 4.7. *Assume that Hypotheses* $H(a)_2$, H_0, H_2 *hold and* $u \in W^{1,p}(\Omega)$, $u \ge 0$, $u \ne 0$. *Then* $\widehat{\varphi}_\pm(tu) \to -\infty$ *as* $t \to \pm\infty$.

Proof. By virtue of Hypotheses H_2 (i)–(ii), given any $\eta > 0$, we can find $c_{26} = c_{26}(\eta) > 0$ such that

$$F(z,x) \geq \eta|x|^p - c_{26} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{4.20}$$

Then for all $t \geq 1$, we have

$$\hat{\varphi}_+(tu) = \int_\Omega G(tDu)\,dz + \frac{t^p}{p}\int_\Omega \beta(z)|u|^p\,dz - \int_\Omega F(z,tu)\,dz$$

(see (3.23), (4.20) and recall $t \geq 1$ and $q < p$)

$$\begin{aligned} &\leq c_{12}t^p(\|Du\|_q^q + \|Du\|_p^p) + c_{27}t^p\|u\|_p^p - \eta t^p\|u\|_p^p + c_{26}|\Omega|_N && \text{for some } c_{27} > 0 \\ &\leq t^p[c_{28}\|u\|^p - \eta\|u\|_p^p] + c_{26}|\Omega|_N && \text{for some } c_{28} > 0. \end{aligned} \tag{4.21}$$

Choosing

$$\eta > \frac{c_{28}\|u\|^p}{\|u\|_p^p},$$

from (4.21) we infer that $\hat{\varphi}_+(tu) \to -\infty$ as $t \to +\infty$. In a similar fashion we also show that $\hat{\varphi}_-(tu) \to -\infty$ as $t \to -\infty$. □

Now, we have the mountain pass geometry for the functionals $\hat{\varphi}_\pm$ and we can produce two nontrivial constant sign solutions of (1.1).

Proposition 4.8. *Assume that Hypotheses* $H(a)_2$, H_0, H_2 *hold. Then problem* (1.1) *has at least two nontrivial constant sign solutions* $u_0 \in \operatorname{int} C_+$ *and* $v_0 \in -\operatorname{int} C_+$.

Proof. Proposition 4.6 implies that we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}_+(0) = 0 < \inf\{\hat{\varphi}_+(u) : \|u\| = \rho\} = \hat{\eta}_+.$$

This fact together with Propositions 4.4 and 4.7 permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_+} \text{ and } \hat{\eta}_+ \leq \hat{\varphi}_+(u_0) \implies u_0 \neq 0 \text{ and } A(u_0) + (\beta + \lambda)|u_0|^{p-2}u_0 = N_{\hat{f}_+}(u_0). \tag{4.22}$$

On (4.22) we act with $-u_0^- \in W^{1,p}(\Omega)$ and since $\lambda > \|\beta^-\|_\infty$, we obtain $u_0 \geq 0$, $u_0 \neq 0$. Therefore (4.22) yields

$$A(u_0) + \beta u_0^{p-1} = N_f(u_0),$$

hence u_0 is a nontrivial positive solution of (1.1) and $u_0 \in C_+\backslash\{0\}$ (by the nonlinear regularity theory).

Let $\rho = \|u_0\|_\infty$ and let $\epsilon_\rho > 0$ be as postulated by Hypothesis H_2 (iv). Then

$$-\operatorname{div} a(Du_0(z)) + (\beta(z) + \epsilon_\rho)u_0(z)^{p-1} \geq f(z,u_0(z)) + \epsilon_\rho u_0(z)^{p-1} \geq 0 \quad \text{a.e. in } \Omega \implies u_0 \in \operatorname{int} C_+ \tag{4.23}$$

(see the proof of Proposition 3.4 and Pucci–Serrin [34, pp. 111, 120]). Similarly, working with the functional $\hat{\varphi}_-$, we produce $v_0 \in -\operatorname{int} C_+$ a nontrivial negative solution of (1.1). □

To produce a third nontrivial solution, we will use Morse theory (critical groups). In the next two propositions, we compute the critical groups at infinity for the functionals φ and $\hat{\varphi}_+$.

Proposition 4.9. *Assume that Hypotheses* $H(a)_2$, H_0, H_2 *hold. Then* $C_k(\varphi, \infty) = 0$ *for all* $k \geq 0$.

Proof. As in the proof of Proposition 4.7, we show that for every $u \in W^{1,p}(\Omega)$, $u \neq 0$, we have

$$\varphi(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.24}$$

By virtue of Hypotheses H_2 (i), (iii), we have for some $c_{29} > 0$ and $\beta_1 \in (0, \beta_0)$,

$$pF(z,x) - f(z,x)x \leq c_{29} - \beta_1|x|^\tau \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}.$$

For $u \in W^{1,p}(\Omega)$ and $t > 0$, we have

$$\begin{aligned}
\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u\rangle \\
&= \frac{1}{t}\langle \varphi'(tu), tu\rangle \\
&= \frac{1}{t}\Big[\int_\Omega (a(tDu), tDu)_{\mathbb{R}^N}\,dz + \int_\Omega \beta|tu|^p\,dz - \int_\Omega f(z,tu)(tu)\,dz\Big] \\
&\le \frac{1}{t}\Big[\int_\Omega pG(tDu)\,dz + |\gamma||\Omega|_N + \int_\Omega \beta|tu|^p\,dz\Big] + c_{29}|\Omega|_N \\
&\qquad - \int_\Omega pF(z,tu)\,dz - \beta_1\|tu\|_\tau^\tau \quad \text{(see Hypothesis H}(a)_2\text{ (iv) and (4.24))} \\
&\le \frac{1}{t}[p\varphi(tu) + (|\gamma| + c_{29})|\Omega|_N].
\end{aligned} \tag{4.25}$$

From (4.24) and (4.25), we deduce that

$$\frac{d}{dt}\varphi(tu) < 0 \quad \text{for all } t > 0 \text{ big enough.} \tag{4.26}$$

Then by virtue of the implicit function theorem, we can find a unique function $\epsilon \in C(\partial B_1)$ such that

$$\epsilon > 0 \quad \text{and} \quad \varphi(\epsilon(u)u) = \rho_* < -\frac{|\gamma| + c_{29}}{p} \quad \text{(see (4.25)).} \tag{4.27}$$

We extend ϵ on $W^{1,p}(\Omega)\setminus\{0\}$ as

$$\epsilon_0(u) = \frac{1}{\|u\|}\epsilon\Big(\frac{u}{\|u\|}\Big) \quad \text{for all } u \in W^{1,p}(\Omega)\setminus\{0\}.$$

Clearly, $\epsilon_0 \in C(W^{1,p}(\Omega)\setminus\{0\})$ and $\varphi(\epsilon_0(u)u) = \rho_*$ (see (4.27)). Moreover, if $\varphi(u) = \rho_*$, then $\epsilon_0(u) = 1$. Therefore, if we set

$$\epsilon^*(u) = \begin{cases} 1 & \text{if } \varphi(u) < \rho_*, \\ \epsilon_0(u) & \text{if } \varphi(u) \ge \rho_*, \end{cases} \tag{4.28}$$

then $\epsilon^* \in C(W^{1,p}(\Omega)\setminus\{0\})$.

We consider the homotopy $h : [0,1] \times (W^{1,p}(\Omega)\setminus\{0\}) \to W^{1,p}(\Omega)\setminus\{0\}$ defined by

$$h(t,u) = (1-t)u + t\epsilon^*(u)u \quad \text{for all } (t,u) \in [0,1] \times (W^{1,p}(\Omega)\setminus\{0\}).$$

Note that

$$\begin{aligned}
h(0,u) = u, \quad h(1,u) = \epsilon^*(u)u \in \varphi^{\rho_*} &\quad \text{for all } u \in W^{1,p}(\Omega)\setminus\{0\}, \\
h(t,\cdot)|_{\varphi^{\rho_*}} = \mathrm{id}|_{\varphi^{\rho_*}} &\quad \text{for all } t \in [0,1] \qquad \text{(see (4.28)).}
\end{aligned}$$

This shows that

$$\varphi^{\rho_*} \text{ is a strong deformation retract of } W^{1,p}(\Omega)\setminus\{0\}. \tag{4.29}$$

If we use the radial retraction

$$r_0(u) = \frac{u}{\|u\|} \quad \text{for all } u \in W^{1,p}(\Omega)\setminus\{0\},$$

we see that ∂B_1 is a retract of $W^{1,p}(\Omega)\setminus\{0\}$ and $W^{1,p}(\Omega)\setminus\{0\}$ is deformable onto ∂B_1. Therefore, [10, Theorem 6.5, p. 325] implies that

$$\partial B_1 \text{ is a deformation retract of } W^{1,p}(\Omega)\setminus\{0\}. \tag{4.30}$$

From (4.29) and (4.30) it follows that for all $k \ge 0$,

$$\varphi^{\rho_*} \text{ and } \partial B_1 \text{ are homotopically equivalent} \implies H_k(W^{1,p}(\Omega), \varphi^{\rho_*}) = H_k(W^{1,p}(\Omega), \partial B_1). \tag{4.31}$$

Since $W^{1,p}(\Omega)$ is infinite dimensional, we know that ∂B_1 is contractible in itself. Thus, by Granas and Dugundji [17, p. 389],

$$H_k(W^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \ge 0. \tag{4.32}$$

From (4.31) and (4.32) it follows that

$$H_k(W^{1,p}(\Omega), \varphi^{\rho_*}) = 0 \quad \text{for all } k \geq 0.$$

Choosing $\rho_* < -\frac{|\gamma|+c_{29}}{p}$ with $|\rho_*|$ large enough, we have for all $k \geq 0$,

$$C_k(\varphi, \infty) = H_k(W^{1,p}(\Omega), \varphi^{\rho_*}),$$

hence $C_k(\varphi, \infty) = 0$. This completes the proof. □

Remark 4.10. The first computation of the critical groups of the energy functional for problems with superlinear reaction was developed by Wang [37]. In that case the problem is Dirichlet, driven by the Laplacian, $\beta \equiv 0$, and the superlinear reaction f is autonomous (that is, $f(z, \cdot) = f(\cdot)$), $f \in C^1(\mathbb{R})$ and satisfies the AR-condition (see (4.1)). Our proof uses ideas from the proof of Wang [37].

We can obtain an analogous result for the functionals $\widehat{\varphi}_\pm$.

Proposition 4.11. *Assume that Hypotheses* $\mathrm{H}(a)_2$, H_0, H_2 *hold. Then* $C_k(\widehat{\varphi}_\pm, \infty) = 0$ *for all* $k \geq 0$.

Proof. Let $\widehat{\sigma}_+ = \widehat{\varphi}_+|_{C^1(\bar{\Omega})}$. From the nonlinear regularity theory (see Lieberman [23]), we have that $K_{\widehat{\varphi}_+} \subseteq C^1(\bar{\Omega})$ and in fact $K_{\widehat{\varphi}_+} \subseteq C_+$. Hence

$$K_{\widehat{\varphi}_+} = K_{\widehat{\sigma}_+} = K \subseteq C_+.$$

Since $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, from Palais [30] we have for $a < \inf_K \widehat{\varphi}_+ = \inf_K \widehat{\sigma}_+$,

$$H_k(W^{1,p}(\Omega), \dot{\widehat{\varphi}}{}_+^a) = H_k(C^1(\bar{\Omega}), \dot{\widehat{\sigma}}{}_+^a) \implies C_k(\widehat{\varphi}_+, \infty) = C_k(\widehat{\sigma}_+, \infty) \quad \text{for all } k \geq 0. \tag{4.33}$$

So, by virtue of (4.33), in order to prove the proposition, we need to show that

$$H_k(C^1(\bar{\Omega}), \widehat{\sigma_+^a}) = 0 \quad \text{for all } k \geq 0. \tag{4.34}$$

To this end, let

$$\partial B_1^C = \{u \in C^1(\bar{\Omega}) : \|u\|_{C^1(\bar{\Omega})} = 1\} \quad \text{and} \quad \partial B_{1,+}^C = \{u \in \partial B_1^C : u^+ \neq 0\}.$$

We consider the homotopy $h_+ : [0,1] \times \partial B_{1,+}^C \to \partial B_{1,+}^C$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_1(p,\beta)}{\|(1-t)u + t\widehat{u}_1(p,\beta)\|_{C^1(\bar{\Omega})}} \quad \text{for all } (t,u) \in [0,1] \times W^{1,p}(\Omega).$$

We have

$$h_+(1,u) = \frac{\widehat{u}_1(p,\beta)}{\|\widehat{u}_1(p,\beta)\|_{C^1(\bar{\Omega})}} \in \partial B_{1,+}^C,$$

hence $\partial B_{1,+}^C$ is contractible in itself. As a consequence of Hypothesis H_2 (ii) for every $u \in \partial B_{1,+}^C$, we have

$$\widehat{\sigma}_+(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.35}$$

For all $u \in \partial B_{1,+}^C$, we have

$$\begin{aligned}
\frac{d}{dt}\widehat{\sigma}_+(tu) &= \frac{1}{t}\Big[\int_\Omega (a(D(tu)), D(tu))_{\mathbb{R}^N}\,dz + \int_\Omega (\beta(z)+\lambda)|tu|^p\,dz - \int_\Omega \widehat{f}_+(z,tu)tu\,dz\Big] \\
&\leq \frac{1}{t}\Big[\int_\Omega pG(D(tu))\,dz + \int_\Omega (\beta(z)+\lambda)|tu|^p\,dz - \int_\Omega pF(z,tu)\,dz + c_{30}\Big] \quad \text{for some } c_{30} > 0
\end{aligned}$$

(see Hypothesis $\mathrm{H}(a)_2$ (iv) and (4.25))

$$\begin{aligned}
&= \frac{1}{t}[\widehat{\varphi}_+(tu) + c_{30}] \\
&= \frac{1}{t}[\widehat{\sigma}_+(tu) + c_{30}].
\end{aligned}$$

From (4.35) we see that for $t > 0$ big enough we have $\widehat{\sigma}_+(tu) < -\frac{c_{30}}{p}$. Hence

$$\frac{d}{dt}\widehat{\sigma}_+(tu) < 0 \quad \text{for all } t > 0 \text{ large enough.} \tag{4.36}$$

Let $\bar{B}_1^C = \{u \in C^1(\bar{\Omega}) : \|u\|_{C^1(\bar{\Omega})} \le 1\}$ and choose $a \in \mathbb{R}$ such that

$$a < \min\Big\{-\frac{c_{30}}{p}, \inf_{\bar{B}_1^C} \widehat{\sigma}_+\Big\}. \tag{4.37}$$

As before (see the proof of Proposition 4.9), from (4.37) and the implicit function theorem, we can find a unique $\mu \in C(\partial B_1^C)$, $\mu \ge 1$, such that

$$\widehat{\sigma}_+(tu) \begin{cases} > a & \text{if } t \in [0, \mu(u)), \\ = a & \text{if } t = \mu(u), \\ < a & \text{if } t > \mu(u). \end{cases} \tag{4.38}$$

From (4.37) and (4.38), we have

$$\widehat{\sigma}_+^a = \{tu : u \in \partial B_{1,+}^C, t \ge \mu(u)\}. \tag{4.39}$$

Let $E_+ = \{tu : u \in \partial B_{1,+}^C,\ t \ge 1\}$. From (4.39) we have $\widehat{\sigma}_+^a \subseteq E_+$. We consider the deformation $\widehat{h}_+ : [0,1] \times E_+ \to E_+$ defined by

$$\widehat{h}_+(s, tu) = \begin{cases} (1-s)tu + s\mu(u)u & \text{if } t \in [1, \mu(u)], \\ tu & \text{if } t > \mu(u). \end{cases}$$

Then we have

$$\widehat{h}_+(0, tu) = tu, \quad \widehat{h}_+(1, tu) \in \widehat{\sigma}_+^a \quad \text{(see (4.39))} \quad \text{and} \quad \widehat{h}_+(s, \cdot)|_{\widehat{\sigma}_+^a} = \mathrm{id}|_{\widehat{\sigma}_+^a} \quad \text{for all } s \in [0,1].$$

This means that $\widehat{\sigma}_+^a$ is a strong deformation retract of E_+. Hence

$$H_k(C^1(\bar{\Omega}), E_+) = H_k(C^1(\bar{\Omega}), \widehat{\sigma}_+^a) \quad \text{for all } k \ge 0. \tag{4.40}$$

Let $h_+^* : [0,1] \times E_+ \to E_+$ be the homotopy defined by

$$h_+^*(s, tu) = (1-s)tu + s\frac{tu}{\|tu\|_{C^1(\bar{\Omega})}}.$$

From Dugundji [10, p. 325], we obtain that $\partial B_{1,+}^C$ is a deformation retract of E_+. Therefore

$$\begin{aligned} & H_k(C^1(\bar{\Omega}), E_+) = H_k(C^1(\bar{\Omega}), \partial B_{1,+}^C) \quad \text{for all } k \ge 0 \\ \Longrightarrow\ & H_k(C^1(\bar{\Omega}), \widehat{\sigma}_+^a) = H_k(C^1(\bar{\Omega}), \partial B_{1,+}^C) \quad \text{for all } k \ge 0 \quad \text{(see (4.40))}. \end{aligned} \tag{4.41}$$

We have seen earlier in the proof that $\partial B_{1,+}^C$ is contractible in itself. Thus, by Granas and Dugundji [17, p. 389], $H_k(C^1(\bar{\Omega}), \partial B_{1,+}^C) = 0$, hence $H_k(C^1(\bar{\Omega}), \widehat{\sigma}_+^a) = 0$ for all $k \ge 0$ (see (4.41)).

So, we have proved relation (4.34) and from this it follows that for all $k \ge 0$, $C_k(\widehat{\varphi}_+, \infty) = C_k(\widehat{\sigma}_+, \infty) = 0$. Similarly, we show that $C_k(\widehat{\varphi}_-, \infty) = 0$ for all $k \ge 0$. □

Using this result, we can compute the critical groups of φ at $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$.

Proposition 4.12. *Assume that Hypotheses* $\mathrm{H}(a)_2$, H_0, H_2 *hold and* $K_\varphi = \{0, u_0, v_0\}$. *Then*

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \textit{for all } k \ge 0.$$

Proof. Note that $\widehat{\varphi}_+'|_{C_+} = \varphi'|_{C_+}$ and so $K_{\widehat{\varphi}_+} = \{0, u_0\}$.

Let $\eta < 0 < \lambda < \widehat{\varphi}_+(u_0) = \varphi(u_0)$ (since $u_0 \in \operatorname{int} C_+$). We consider the following triple of sets:

$$\widehat{\varphi}_+^\eta \subseteq \widehat{\varphi}_+^\lambda \subseteq W^{1,p}(\Omega) = W.$$

For this triple of sets, we consider the corresponding long exact sequence of homology groups

$$\cdots \longrightarrow H_k(W, \widehat{\varphi}_+^\eta) \xrightarrow{i_*} H_k(W, \widehat{\varphi}_+^\lambda) \xrightarrow{\partial_*} H_{k-1}(\widehat{\varphi}_+^\lambda, \widehat{\varphi}_+^\eta) \longrightarrow \cdots, \tag{4.42}$$

where i_* is the homomorphism induced by the inclusion $(W, \widehat{\varphi}_+^\eta) \hookrightarrow (W, \widehat{\varphi}_+^\lambda)$ and ∂_* is the boundary homomorphism. From the rank theorem and using the exactness of (4.42), we have

$$\operatorname{rank} H_k(W, \widehat{\varphi}_+^\lambda) = \operatorname{rank} \ker \partial_* + \operatorname{rank} \operatorname{im} \partial_* = \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} \partial_*. \tag{4.43}$$

From the choice of λ, we have

$$H_k(W, \widehat{\varphi}_+^{\lambda}) = C_k(\widehat{\varphi}_+, u_0) \quad \text{for all } k \geq 0. \tag{4.44}$$

Also, since $\eta < 0 = \widehat{\varphi}_+^{\eta}(0) < \widehat{\varphi}_+^{\eta}(u_0)$ and $K_{\widehat{\varphi}_+} = \{0, u_0\}$, we have for all $k \geq 0$,

$$H_k(W, \widehat{\varphi}_+^{\eta}) = C_k(\widehat{\varphi}_+, \infty) \implies H_k(W, \widehat{\varphi}_+^{\eta}) = 0 \text{ (see Proposition 4.11)} \implies \operatorname{im} i_* = \{0\}. \tag{4.45}$$

Similarly, we have

$$H_{k-1}(\widehat{\varphi}_+^{\lambda}, \widehat{\varphi}_+^{\eta}) = C_{k-1}(\widehat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} \quad \text{for all } k \geq 0 \quad \text{(see Proposition 4.6).} \tag{4.46}$$

We return to (4.43) and use (4.44), (4.45), (4.46). We obtain

$$\operatorname{rank} C_k(\widehat{\varphi}_+, u_0) \leq 1. \tag{4.47}$$

But recall that u_0 is a critical point of $\widehat{\varphi}_+$ of mountain pass type (see the proof of Proposition 4.8). Therefore

$$C_1(\widehat{\varphi}_+, u_0) \neq 0. \tag{4.48}$$

From (4.47) and (4.48) and since in (4.42) only the tail (that is, $k = 1$) is nontrivial, we have

$$C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.49}$$

Claim. *We have $C_k(\widehat{\varphi}_+, u_0) = C_k(\varphi, u_0)$ for all $k \geq 0$.*

We consider the homotopy

$$h(t, u) = (1 - t)\varphi(u) + t\widehat{\varphi}_+(u) \quad \text{for all } (t, u) \in [0, 1] \times W^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geq1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t, \quad u_n \to u_0 \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \quad \text{for all } n \geq 1. \tag{4.50}$$

From (4.50), we have

$$\begin{aligned} &A(u_n) + \beta|u_n|^{p-2}u_n + t_n\lambda|u_n|^{p-2}u_n = (1 - t_n)N_f(u_n) + t_nN_{\widehat{f}_+}(u_n) \\ &\implies \begin{cases} -\operatorname{div} a(Du_n(z)) + (\beta(z) + t_n\lambda)|u_n(z)|^{p-2}u_n(z) = (1 - t_n)f(z, u_n(z)) + t_n\widehat{f}_+(z, u_n(z)) & \text{a.e. in } \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \end{aligned}$$

From Hu and Papageorgiou [18], we know that we can find $M_7 > 0$ such that

$$\|u_n\|_\infty \leq M_7 \quad \text{for all } n \geq 1.$$

Then from Lieberman [23, p. 320], there are $\gamma \in (0, 1)$ and $M_8 > 0$ such that

$$u_n \in C^{1,\gamma}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\gamma}(\bar{\Omega})} \leq M_8 \quad \text{for all } n \geq 1. \tag{4.51}$$

Recall that $C^{1,\gamma}(\bar{\Omega})$ is embedded compactly in $C^1(\bar{\Omega})$. So, from (4.50) and (4.51) it follows that

$$u_n \to u_0 \text{ in } C^1(\bar{\Omega}) \implies u_n \in \operatorname{int} C_+ \text{ for all } n \geq n_0 \text{ (since } u_0 \in \operatorname{int} C_+). \tag{4.52}$$

We deduce that $\{u_n\}_{n\geq n_0} \subseteq K_\varphi$, a contradiction.

Invoking the homotopy invariance property of critical groups, we have

$$C_k(\widehat{\varphi}_+, u_0) = C_k(\varphi, u_0) \quad \text{for all } k \geq 0.$$

This proves the claim.

From the claim and (4.49), we have

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

In a similar fashion, using this time $\widehat{\varphi}_-$, we show that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

This completes the proof. □

Proposition 4.13. *Assume that Hypotheses* $\mathrm{H}(a)_2$, H_0, H_2 *are fulfilled. Then problem* (1.1) *admits a third nontrivial solution* $y_0 \in C^1(\bar{\Omega})$.

Proof. Arguing by contradiction, suppose that $K_\varphi = \{0, u_0, v_0\}$. From Proposition 4.12, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.53}$$

Next, Proposition 4.6 yields

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.54}$$

Finally, from Proposition 4.9

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0. \tag{4.55}$$

From (4.53), (4.54), (4.55) and the Morse relation with $t = -1$ (see (2.8)), we have $2(-1)^1 + (-1)^0 = 0$, a contradiction. So, we can find $y_0 \in K_\varphi$, $y_0 \notin \{0, u_0, v_0\}$. This is the third nontrivial solution of (1.1) and the nonlinear regularity theory implies that $y_0 \in C^1(\bar{\Omega})$. □

Therefore, we can state the following multiplicity theorem (three solutions theorem) for the noncoercive version of problem (1.1).

Theorem 4.14. *Assume that hypotheses* $\mathrm{H}(a)_2$, H_0, H_2 *hold. Then problem* (1.1) *has at least three nontrivial solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, *and* $y_0 \in C^1(\bar{\Omega})$.

Remark 4.15. It is an interesting open question, whether we can have the third nontrivial solution $y_0 \in C^1(\bar{\Omega})$ to be nodal. Nodal solutions for superlinear Neumann problems driven by the p-Laplacian with $\beta(\cdot) \equiv \beta$, where $\beta \in (0, +\infty)$, and a reaction satisfying the AR-condition, were obtained by Aizicovici, Papageorgiou and Staicu [1], under stronger conditions. Theorem 4.14 extends the multiplicity theorem of Wang [37], where the problem is semilinear (driven by the Laplacian), with Dirichlet boundary condition, $\beta \equiv 0$ and a superlinear reaction satisfying the AR-condition.

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