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PAIRS OF POSITIVE SOLUTIONS FOR RESONANT SINGULAR EQUATIONS WITH THE p-LAPLACIAN

NIKOLAOS S. PAPAGEORGIOU, VICENŢIU D. RĂDULESCU, DUŠAN D. REPOVŠ

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Abstract. We consider a nonlinear elliptic equation driven by the Dirichlet p-Laplacian with a singular term and a $(p-1)$-linear perturbation which is resonant at $+\infty$ with respect to the principal eigenvalue. Using variational tools, together with suitable truncation and comparison techniques, we show the existence of at least two positive smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear elliptic problem with singular reaction

$$\begin{gathered} -\Delta_p u(z) = u(z)^{-\mu} + f(z, u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \quad 1 < p < \infty,\ 0 < \mu < 1. \end{gathered} \tag{1.1}$$

In this problem, Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < \infty.$$

In the reaction term, $u^{-\mu}$ (with $0 < \mu < 1$) is the singular part and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$ the map $x \mapsto f(z, x)$ is continuous) which exhibits $(p-1)$-linear growth near $+\infty$.

Using variational tools, together with suitable truncation and comparison techniques, we prove a multiplicity theorem establishing the existence of two positive smooth solutions. Such multiplicity theorems for singular problems were proved by Hirano, Saccon and Shioji [7], Papageorgiou and Rădulescu [12], Sun, Wu and Long [16] (semilinear problems driven by the Laplacian) and Giacomoni and Saudi [4], Giacomoni, Schindler and Takac [5], Kyritsi and Papageorgoiu [6], Papageorgiou and Smyrlis [13, 14], Perera and Zhang [15] (nonlinear problems). In all these papers the reaction term is parametric. The presence of the parameter permits a more precise control of the nonlinearity as the positive parameter λ becomes small.

A complete overview of the theory of singular elliptic equations can be found in the book by Ghergu and Rădulescu [3].

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2. Mathematical background and hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

> Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ. It leads to a deformation theorem from which we can deduce the minimax theory of the critical values of φ. One of the main results of this theory is the so-called "mountain pass theorem", which we recall here.

Theorem 2.1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $0 < \rho < \|u_0 - u_1\|$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \geq m_\rho$ and c is a critical value of φ (that is, there exists $u_0 \in X$ such that $\varphi(u_0) = c$ and $\varphi'(u_0) = 0$).

In the analysis of problem (1.1) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. In what follows, we denote by $\|\cdot\|$ the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+(\overline{\Omega}) = C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \Omega\}.$$

This cone has a nonempty interior

$$\operatorname{int} C_+ = \big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0\big\}.$$

Here, $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

This map has the following properties (see, for example, Motreanu, Motreanu and Papageorgiou [11, p. 40]).

Proposition 2.2. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega).$$

We will also need some facts about the spectrum of the Dirichlet p-Laplacian. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \hat{\lambda} m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Here, $m \in L^\infty(\Omega), m \ge 0, m \ne 0$. We say that $\hat{\lambda}$ is an "eigenvalue", if the above problem admits a nontrivial solution $\hat{u}$ known as an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. The nonlinear regularity theory (see, for example, Gasinski and Papageorgiou [2, pp. 737-738]), implies that $\hat{u} \in C_0^1(\overline{\Omega})$. There exists a smallest eigenvalue $\hat{\lambda}_1(m)$ such that:

- $\hat{\lambda}_1(m) > 0$ and is isolated in the spectrum $\hat{\sigma}(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega), m)$ (that is, there exists $\epsilon > 0$ such that $(\hat{\lambda}_1(m), \hat{\lambda}_1(m) + \epsilon) \cap \hat{\sigma}(p) = \emptyset$);
- $\hat{\lambda}_1(m) > 0$ is simple in the sense that if $\hat{u}, \hat{v}$ are two eigenfunctions corresponding to $\hat{\lambda}_1(m) > 0$, then $\hat{u} = \xi\hat{v}$ for some $\xi \in \mathbb{R}\backslash\{0\}$;
-
$$\hat{\lambda}_1(m) = \inf\Big[\frac{\|Du\|_p^p}{\int_\Omega m(z)|u|^p dz} : u \in W_0^{1,p}(\Omega), u \ne 0\Big]. \tag{2.1}$$

The infimum in (2.1) is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(m)$. From the above properties it follows that the elements of this eigenspace have constant sign. We denote by $\hat{u}_1(m)$ the L^p-normalized (that is, $\|\hat{u}_1(m)\|_p = 1$) positive eigenfunction for the eigenvalue $\hat{\lambda}_1(m)$. As we have already mentioned, $\hat{u}_1(m) \in C_+$. In fact, the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [2, p. 738]) implies that $\hat{u}_1(m) \in \operatorname{int} C_+$. If $m \equiv 1$, then we write

$$\hat{\lambda}_1(1) = \hat{\lambda}_1 > 0 \quad \text{and} \quad \hat{u}_1(1) = \hat{u}_1 \in \operatorname{int} C_+.$$

The map $m \mapsto \hat{\lambda}_1(m)$ exhibits the following strict monotonicity property.

Proposition 2.3. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \le m_1(z) \le m_2(z)$ for almost all $z \in \Omega$ and $m_1 \ne 0, m_2 \ne m_1$, then $\hat{\lambda}_1(m_2) < \hat{\lambda}_1(m_1)$.*

We mention that every eigenfunction $\hat{u}$ corresponding to an eigenvalue $\hat{\lambda} \ne \hat{\lambda}_1(m)$, is necessarily nodal (that is, sign changing). For details on the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega), m)$ we refer to [2, 11].

For $x \in \mathbb{R}$ we define $x^\pm = \max\{\pm x, 0\}$. Then, given $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given a measurable function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we denote by $N_g(\cdot)$ the Nemitsky (superposition) operator corresponding to g, that is,

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that $z \mapsto N_g(u)(z) = g(z, u(z))$ is measurable.

The hypotheses on the perturbation term $f(z, x)$ are the following:

(H1): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \le a_\rho(z) \quad \text{for almost all } z \in \Omega, \text{ all } 0 \le x \le \rho$$

and there exists $w \in C^1(\overline{\Omega})$ such that

$$w(z) \ge \hat{c} > 0 \text{ for all } z \in \overline{\Omega} \text{ and } -\Delta_p w \ge 0 \text{ in } W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$$

and for every compact $K \subseteq \Omega$ we can find $c_K > 0$ such that

$$w(z)^{-\mu} + f(z, w(z)) \le -c_K < 0 \text{ for almost all } z \in K;$$

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exists $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1 \le \liminf_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \le \limsup_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \le \eta(z) \text{ uniformly for almost all } z \in \Omega,$$

$$f(z, x)x - pF(z, x) \to -\infty \text{ as } x \to +\infty \text{ uniformly for almost all } z \in \Omega;$$

(iii) there exists $\delta \in (0, \hat{c})$ such that for all compact $K \subseteq \Omega$ we have

$$f(z, x) \ge \hat{c}_K > 0 \quad \text{for almost all } z \in K, \text{ all } 0 < x \le \delta;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the mapping

$$x \mapsto f(z, x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 2.4. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that

$$f(z, x) = 0 \text{ for almost all } z \in \Omega, \text{ all } x \le 0. \tag{2.2}$$

Hypothesis (H1)(ii) permits resonance with respect to the principal eigenvalue $\hat{\lambda}_1 > 0$. The second convergence condition in (H1)(ii) implies that the resonance at $+\infty$ with respect to $\hat{\lambda}_1 > 0$, is from the right of the principle eigenvalue in the sense that

$$\hat{\lambda}_1 x^{p-1} - pF(z, x) \to -\infty \text{ as } x \to +\infty \text{ uniformly for almost all } z \in \Omega$$

(see the proof of Proposition 3.2). This makes the problem noncoercive and so the direct method of the calculus of variations is not applicable.

Hypothesis (H1)(iv) is satisfied if for example $f(z, \cdot)$ is differentiable and the derivative $f'_x(z, \cdot)$ satisfies for some $\rho > 0$

$$f'_x(z, x) \ge -\tilde{c}_\rho x^{p-2} \text{ for almost all } x \in \Omega, \text{ for all } 0 \le x \le \rho \text{ and some } \tilde{c}_\rho > 0.$$

Example 2.5. The following function satisfies hypotheses (H1). For the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} x^{p-1} - 2x^{r-1} & \text{if } 0 \le x \le 1 \\ \eta x^{p-1} + x^{\tau-1} - (2 + \eta)x^{q-1} & \text{if } 1 < x, \end{cases}$$

with $\eta \ge \hat{\lambda}_1$ and $1 < \tau,\ q < p < r < \infty$.

3. Pair of positive solutions

In this section we prove the existence of two positive smooth solutions for problem (1.1). We start by considering the auxiliary singular Dirichlet problem

$$-\Delta_p u(z) = u(z)^{-\mu} \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u > 0. \tag{3.1}$$

By Papageorgiou and Smyrlis [14, Proposition 5], we know that problem (3.1) has a unique positive solution $\tilde{u} \in \operatorname{int} C_+$.

Let $\delta > 0$ be as postulated by hypothesis (H1)(iii) and let

$$0 < t \le \min\big\{1, \frac{\delta}{\|u\|_\infty}\big\}.$$

We set $\underline{u} = t\tilde{u}$. Then $\underline{u} \in \operatorname{int} C_+$ and we have

$$\begin{aligned} -\Delta_p \underline{u}(z) = t^{p-1}[-\Delta_p \tilde{u}(z)] &= t^{p-1}\tilde{u}(z)^{-\mu} \\ &\le \underline{u}(z)^{-\mu} \quad (\text{since } 0 < t \le 1) \\ &\le \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) \text{ for almost all } z \in \Omega \end{aligned} \tag{3.2}$$

(see [14], note that $\underline{u}(z) \in (0, \delta]$ for all $z \in \overline{\Omega}$ and see hypothesis (H1)(iii)). Also note that $\underline{u} \le w$.

We introduce the following truncation of the reaction term in (1.1):

$$\hat{f}(z,x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) & \text{if } x < \underline{u}(z) \\ x^{-\mu} + f(z,x) & \text{if } \underline{u}(z) \le x \le w(z) \\ w(z)^{-\mu} + f(z, w(z)) & \text{if } w(z) < x. \end{cases} \tag{3.3}$$

This is a Carathéodory function. We set $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)ds$ and consider the functional $\hat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \hat{F}(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

By Papageorgiou and Smyrlis [14, Proposition 3] we have $\hat{\varphi} \in C^1(W_0^{1,p}(\mathbb{R}))$.

In what follows, we denote by $[\underline{u}, w]$ the order interval

$$[\underline{u}, w] = \{u \in W_0^{1,p}(\Omega) : \underline{u}(z) \le u(z) \le w(z) \text{ for almost all } z \in \Omega\}.$$

Also, we denote by $\operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w]$ the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[\underline{u}, w] \cap C_0^1(\overline{\Omega})$.

In the next proposition we produce a positive smooth solution located in the above order interval.

Proposition 3.1. *If hypotheses* (H1) *hold, then problem* (1.1) *has a positive solution* $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w]$.

Proof. We know that $\underline{u} \in \operatorname{int} C_+$. So, using Marano and Papageorgiou [10, Proposition 2.1] we can find $c_0 > 0$ such that

$$\hat{u}_1^{1/p'} \le c_0 \underline{u} \quad \Rightarrow \quad \underline{u}^{-\mu} \le c_0^\mu \hat{u}_1^{-\mu/p'}.$$

Hence using the lemma of Lazer and McKenna [9], we have that

$$\underline{u}^{-\mu} \in L^{p'}(\Omega).$$

Therefore by (3.2) we see that $\hat{\varphi}(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\hat{\varphi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\hat{\varphi}(u_0) = \inf[\hat{\varphi}(u) : u \in W_0^{1,p}(\Omega)],\\
&\Rightarrow \hat{\varphi}'(u_0) = 0,\\
&\Rightarrow \langle A(u_0), h\rangle = \int_\Omega \hat{f}(z,u_0)hdz \text{ for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{3.4}$$

In (3.4) we first choose $h = (\underline{u} - u_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
\langle A(u_0), (\underline{u} - u_0)^+\rangle &= \int_\Omega [\underline{u}^{-\mu} + f(z,\underline{u})](\underline{u} - u_0)^+ dz \quad \text{(see (3.3))}\\
&\geq \langle A(\underline{u}), (\underline{u} - u_0)^+\rangle \quad \text{(see (3.2))}
\end{aligned}$$

which implies

$$\langle A(\underline{u}) - A(u_0), (\underline{u} - u_0)^+\rangle \leq 0,$$

and this implies $\underline{u} \leq u_0$.

Next, in (3.4) we choose $h = (u_0 - w)^+ \in W_0^{1,p}(\Omega)$ (see hypothesis (H1)(i)). Then

$$\begin{aligned}
\langle A(u_0), (u_0 - w)^+\rangle &= \int_\Omega [w^{-\mu} + f(z,w)](u_0 - w)^+ dz\\
&\leq \langle A(w), (u_0 - w)^+\rangle \quad \text{(see hypothesis (H1)(i))},
\end{aligned}$$

which implies

$$\langle A(u_0) - A(w), (u_0 - w)^+\rangle \leq 0,$$

and this implies $u_0 \leq w$. So, we have proved that

$$u_0 \in [\underline{u}, w] = \{u \in W_0^{1,p}(\Omega) : \underline{u}(z) \leq u_0(z) \leq w(z) \quad \text{for almost all } z \in \Omega\}. \tag{3.5}$$

Clearly, $u_0 \neq \underline{u}$ (see hypothesis (H1)(iii)) and $u_0 \neq w$ (see hypothesis (H1)(i)). From (3.3), (3.4), (3.5), we have

$$\langle A(u_0), h\rangle = \int_\Omega [u_0^{-\mu} + f(z,u_0)]hdz, \quad 0 \leq u_0^{-\mu} \leq \underline{u}^{-\mu} \in L^p(\Omega)$$

which implies

$$-\Delta_p u_0(z) = u_0(z)^{-\mu} + f(z,u_0(z)) \quad \text{for a.a. } z \in \Omega,\ u_0|_{\partial\Omega} = 0, \tag{3.6}$$

see [14].

Also, by Gilbarg and Trudinger [6, Lemma 14.16 p. 355] we know that there exists small $\delta_0 > 0$ such that, if $\Omega_{\delta_0} = \{z \in \Omega : d(z,\partial\Omega) < \delta_0\}$, then

$$d \in \operatorname{int} C_+(\overline{\Omega}_{\delta_0}),$$

where $d(\cdot) = d(\cdot, \partial\Omega)$. Let $D^* = \overline{\Omega}\backslash\Omega_{\delta_0}$. Setting $C(D^*)_+ = \{h \in C(D^*) : h(z) \geq 0 \text{ for all } z \in D^*\}$, we have $d \in \operatorname{int} C(D^*)_+ \subseteq \operatorname{int} C_+(D^*)$. Then as before, via Marano and Papageorgiou [10, Proposition 2.1] we find $0 < c_1 < c_2$ such that

$$c_1 d \leq \underline{u} \leq c_2 d. \tag{3.7}$$

Then by (3.6), (3.7), hypotheses (H1)(i), (H1)(iv) and Giacomoni and Saudi [4, Theorem B.1], we have

$$u_0 \in \operatorname{int} C_+.$$

Now let $\rho = \|w\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis (H1)(iv). We have

$$\begin{aligned}
&-\Delta_p u_0(z) - u_0(z)^{-\mu} + \hat{\xi}_\rho u_0(z)^{p-1}\\
&= f(z, u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1} \quad \text{(see (3.6))}\\
&\geq f(z, \underline{u}(z)) + \hat{\xi}_\rho \underline{u}(z)^{p-1} \quad \text{(see (3.5) and hypothesis } (H1)(iv))\\
&> \hat{\xi}_\rho \underline{u}(z)^{p-1} \quad \text{(see hypothesis(H1)(ii))}\\
&\geq -\Delta_p \underline{u}(z) - \underline{u}(z)^{-\mu} + \hat{\xi}_\rho \underline{u}(z)^{p-1} \quad \text{(see (3.2)) for almost all } z \in \Omega.
\end{aligned}$$

Hence, invoking Proposition 3.1 of Papageorgiou and Smyrlis [14], we have

$$u_0 - \underline{u} \in \operatorname{int} C_+.$$

From the hypothesis on the function $w(\cdot)$ (see (H1)(i)), we see that

$$D_0 = \{z \in \Omega : u_0(z) = w(z)\} \text{ is compact in } \Omega.$$

Then we can find an open set $\mathcal{U} \subseteq \Omega$ with Lipschitz boundary, such that

$$D_0 \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \Omega \text{ and } d(z, D_0) \leq \delta_1 \text{ for all } z \in \overline{\mathcal{U}}, \text{ with } \delta_1 > 0.$$

Let $\epsilon > 0$ be such that

$$u_0(z) + \epsilon \leq w(z) \text{ for all } z \in \partial\mathcal{U} \tag{3.8}$$

(such an $\epsilon > 0$ exists since $\partial\Omega$ is compact and $w - u_0 \in C(\overline{\Omega})$).

Exploiting the uniform continuity of the map $x \mapsto x^{p-1}$ on $[0, \rho]$ we can find $\delta_2 > 0$ such that

$$\hat{\xi}_\rho |x^{p-1} - v^{p-1}| \leq \epsilon \quad \text{for all } x, v \in [\min_{\overline{\mathcal{U}}} u_0, \max_{\overline{\mathcal{U}}} w],\ |x - v| \leq \delta_2. \tag{3.9}$$

Similarly, the uniform continuity of $x \mapsto x^{-\mu}$ on any compact subset of $(0, +\infty)$, implies that we can find $\delta_3 \in (0, \delta_2]$ such that

$$|x^{-\mu} - v^{-\mu}| \leq \epsilon \quad \text{for all } x, v \in \big[\frac{\hat{c}}{2}, \|w\|_\infty\big],\ |x - v| \leq \delta_2. \tag{3.10}$$

Then choosing $\delta_1 \in (0, \delta_3)$ small enough and $\tilde{\delta} \in (0, \delta_1)$ we have

$$\begin{aligned}
&-\Delta_p (u_0 + \tilde{\delta})(z) + \hat{\xi}_\rho (u_0 + \tilde{\delta})(z)^{p-1}\\
&\leq -\Delta_p u_0(z) + \tilde{\xi}_\rho u_0(z)^{p-1} + \epsilon \quad \text{(see (3.9))}\\
&= u_0(z)^{-\mu} + f(z, u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1} + \epsilon \quad \text{(see (3.6))}\\
&\leq w(z)^{-\mu} + f(z, w(z)) + \hat{\xi}_\rho w(z)^{p-1} + 2\epsilon \quad \text{(see (3.10), (3.5), (H1)(iv))}\\
&\leq -c_{\overline{\mathcal{U}}} + 2\epsilon + \hat{\xi}_\rho w(z)^{p-1} \text{ for almost all } z \in \Omega \quad \text{(see (H1)(i))}.
\end{aligned} \tag{3.11}$$

Choosing $\epsilon \in (0, c_{\overline{\mathcal{U}}}/2)$ and using once more hypothesis (H1)(i), we deduce from (3.11) that

$$-\Delta_p (u_0 + \tilde{\delta}) + \hat{\xi}_\rho (u_0 + \tilde{\delta})^{p-1} \leq -\Delta_p w + \hat{\xi}_\rho w^{p-1} \text{ in } W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega). \tag{3.12}$$

From (3.12), (3.8) and the weak comparison principle of Tolksdorf [17, Lemma 3.1], we have

$$(u_0 + \tilde{\delta})(z) \leq w(z) \text{ for all } z \in \overline{\mathcal{U}}.$$

But $D_0 \subseteq \overline{\mathcal{U}}$. Therefore $D_0 = \emptyset$ and so

$$0 < (w - u_0)(z) \text{ for all } z \in \overline{\Omega}.$$

We conclude that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w].$$

The proof is now complete. □

Next we produce a second positive smooth solution for problem (1.1).

Proposition 3.2. *If hypotheses* (H1) *hold, then* (1.1) *has a second positive solution* $\hat{u} \in \operatorname{int} C_+$.

Proof. Consider the following truncation of the reaction term in (1.1):

$$g(z,x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z,\underline{u}(z)) & \text{if } u \le \underline{u}(z) \\ x^{-\mu} + f(z,x) & \text{if } \underline{u}(z) < x. \end{cases} \tag{3.13}$$

This is a Carathéodory function. We set $G(z,x) = \int_0^x g(z,s)ds$ and consider the functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As before, Papageorgiou and Smyrlis [14, Proposition 3] implies that

$$\varphi_0 \in C^1(W_0^{1,p}(\Omega)).$$

Claim. φ_0 satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi_0(u_n)| \le M_1 \quad \text{for some } M_1 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.14}$$

$$(1 + \|u_n\|)\varphi_0'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.15}$$

From (3.14) we have

$$\big|\langle A(u_n), h\rangle - \int_\Omega g(z,u_n)hdz\big| \le \frac{\epsilon_n\|h\|}{1+\|u_n\|} \tag{3.16}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\epsilon_n \to 0^+$.

In (3.16) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p - \int_\Omega [\underline{u}^{-\mu} + f(z,\underline{u})](-u_n^-)dz \le \epsilon_n \quad \text{for all } n \in \mathbb{N}, \text{ (see (3.13))}$$

which implies

$$\begin{aligned} &\|u_n^-\|^p \le c_3\|u_n^-\| \quad \text{for some } c_3 > 0 \text{ and for all } n \in \mathbb{N}, \\ &\Rightarrow \{u_n^-\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned} \tag{3.17}$$

Suppose that $\{u_n^+\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. By passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to \infty \tag{3.18}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega),\ y \ge 0. \tag{3.19}$$

From (3.16) and (3.17) we have

$$\big|\langle A(u_n^+), h\rangle - \int_\Omega g(z,u_n^+)hdz\big| \le c_4\|h\| \quad \text{for some } c_4 > 0 \text{ and all } n \in \mathbb{N}$$

which implies

$$\big|\langle A(y_n),h\rangle - \int_\Omega \frac{N_g(u_n^+)}{\|u_n^+\|^{p-1}}hdz\big| \le \frac{c_4\|h\|}{\|u_n^+\|^{p-1}} \quad \text{for all } n\in\mathbb{N}. \tag{3.20}$$

Hypotheses (H1)(i) and (H1)(i)(ii) imply that there exists $c_5>0$ such that

$$|f(z,x)| \le c_5(1+x^{p-1}) \quad \text{for almost all } z\in\Omega \text{ and all } x>0.$$

From this growth estimate and (3.13), it follows that

$$\Big\{\frac{N_g(u_n^+)}{\|u_n^+\|^{p-1}}\Big\}_{n\ge 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

So, by passing to a suitable sequence if necessary and using hypothesis (H1)(ii) we have

$$\frac{N_g(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \tilde{\eta}(z)y^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n\to\infty, \tag{3.21}$$

with $\hat{\lambda}_1 \le \tilde{\eta}(z) \le \eta(z)$ for almost all $z\in\Omega$, see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 16)].

Recall that $\underline{u}^{-\mu}\in L^{p'}(\Omega)$. Therefore

$$\big|\int_\Omega \underline{u}^{-\mu}hdz\big| \le c_6\|h\| \quad \text{for some } c_6>0 \text{ and all } h\in W_0^{1,p}(\Omega)$$

which implies

$$\frac{1}{\|u_n^+\|^{p-1}}\int_\Omega \underline{u}^{-\mu}hdz \to 0 \quad \text{as } n\to\infty, \text{ (see (3.18)).} \tag{3.22}$$

If in (3.20) we choose $h=y_n-y\in W_0^{1,p}(\Omega)$ and pass to the limit as $n\to\infty$, then using (3.19), (3.21), (3.22) we have $\lim_{n\to\infty}\langle A(y_n),y_n-y\rangle=0$ which implies

$$y_n\to y \text{ in } W_0^{1,p}(\Omega), \quad \|y\|=1,\ y\ge 0 \text{ (see Proposition 2.2).} \tag{3.23}$$

So, if in (3.20) we pass to the limit as $n\to\infty$ and use (3.21), (3.22), (3.23) to obtain

$$\langle A(y),h\rangle = \int_\Omega \tilde{\eta}(z)y^{p-1}hdz \quad \text{for all } h\in W_0^{1,p}(\Omega)$$

which implies

$$-\Delta_p y(z) = \tilde{\eta}(z)y(z)^{p-1} \quad \text{for almost all } z\in\Omega, \quad y|_{\partial\Omega}=0. \tag{3.24}$$

Recall that

$$\hat{\lambda}_1 \le \tilde{\eta}(z) \le \eta(z) \quad \text{for almost all } z\in\Omega \text{ (see (3.21)).}$$

We first assume that $\hat{\lambda}_1 \not\equiv \tilde{\eta}$. Then using Proposition 2.3 we have

$$\hat{\lambda}_1(\tilde{\eta}) < \hat{\lambda}_1(\hat{\lambda}_1) = 1.$$

Also, from (3.24) and since $\|y\|=1$ (hence $y\ne 0$, see (3.23)), we infer that $y(\cdot)$ must be nodal, a contradiction to (3.19).

Next, we assume that $\tilde{\eta}(z)=\hat{\lambda}_1$ for almost all $z\in\Omega$. It follows from (3.24) that

$$y=\vartheta\hat{u}_1 \quad \text{with } \vartheta>0, \text{ see (3.23).}$$

Then $y\in \operatorname{int}C_+$ and so $y(z)>0$ for all $z\in\Omega$. Therefore

$$u_n^+(z)\to+\infty \text{ for all } z\in\Omega \quad \text{as } n\to\infty, \tag{3.25}$$

which implies

$$f(z,u_n^+(z))u_n^+(z)-pF(z,u_n^+(z))\to-\infty$$

for almost all $z\in\Omega$ as $n\to\infty$, see hypothesis (H1)(ii). This in turn implies

$$\int_\Omega [f(z,u_n^+)u_n^+-pF(z,u_n^+)]dz\to-\infty \quad \text{(by Fatou's lemma).} \tag{3.26}$$

From (3.16) with $h=u_n^+\in W_0^{1,p}(\Omega)$, we have

$$-\|Du_n^+\|_p^p+\int_\Omega g(z,u_n^+)u_n^+dz\geq-\epsilon_n \quad \text{for all } n\in\mathbb{N}. \tag{3.27}$$

On the other hand, from (3.14) and (3.17), we have

$$\|Du_n^+\|_p^p-\int_\Omega pG(z,u_n^+)dz\geq-M_2 \quad \text{for some } M_2>0 \text{ and all } n\in\mathbb{N}. \tag{3.28}$$

Adding (3.27) and (3.28), we obtain

$$\int_\Omega [g(z,u_n^+)u_n^+-pG(z,u_n^+)]dz\geq-M_3 \quad \text{for some } M_3>0 \text{ and all } n\in\mathbb{N}$$

which implies

$$\int_\Omega [f(z,u_n^+)u_n^+-pF(z,u_n^+)]dz\geq-M_4 \tag{3.29}$$

for some $M_4>0$ and all $n\in\mathbb{N}$ (see (3.13) and (3.25)).

Comparing (3.26) and (3.29), we have a contradiction. This proves that

$$\begin{aligned}&\{u_n^+\}_{n\geq1}\subseteq W_0^{1,p}(\Omega) \text{ is bounded,}\\ &\Rightarrow \{u_n\}_{n\geq1}\subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (3.17)).}\end{aligned}$$

So, we assume that

$$u_n\xrightarrow{w}u \text{ in } W_0^{1,p}(\Omega) \quad\text{and}\quad u_n\to u \text{ in } L^p(\Omega).$$

Then we obtain

$$\int_\Omega g(z,u_n)(u_n-u)dz\to0 \quad \text{as } n\to\infty. \tag{3.30}$$

If in (3.16) we choose $h=u_n-u\in W_0^{1,p}(\Omega)$, then

$$\begin{aligned}&\lim_{n\to\infty}\langle A(u_n),u_n-u\rangle=0,\\ &\Rightarrow u_n\to u \quad \text{in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.2).}\end{aligned}$$

This proves the claim.

Note that

$$\hat\varphi\big|_{[\underline{u},w]}=\varphi_0\big|_{[\underline{u},w]} \quad \text{(see (3.3) and (3.13)).} \tag{3.31}$$

From the proof of Proposition 3.1 we know that $u_0\in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u},w]$ is a minimizer of $\hat\varphi$. Hence it follows from (3.31) that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0. Invoking Giacomoni and Saudi [4, Theorem 1.1], we can say that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ_0. Using (3.13) we can easily see that

$$\begin{aligned}K_{\varphi_0}&=\{u\in W_0^{1,p}(\Omega):\varphi_0'(u)=0\}\subseteq[\underline{u})\cap C_+\\ &=\{u\in C_0^1(\overline{\Omega}):\underline{u}(z)\leq u(z) \text{ for all } z\in\overline{\Omega}\}.\end{aligned}$$

So, we may assume that K_{φ_0} is finite or otherwise we already have an infinity of positive smooth solutions of (1.1). Since u_0 is a local minimizer of φ_0 we can find $\rho \in (0,1)$ small such that

$$\varphi_0(u_0) < \inf[\varphi_0(u) : \|u - u_0\| = \rho] = m_\rho \tag{3.32}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29]).

Hypothesis (H1)(ii) implies that given any $\xi > 0$, we can find $M_5 = M_5(\xi) > 0$ such that

$$f(z,x)x - pF(z,x) \le -\xi \text{ for almost all } z \in \Omega \text{ and all } x \ge M_5. \tag{3.33}$$

We have

$$\begin{aligned} \frac{d}{dx}\Big(\frac{F(z,x)}{x^p}\Big) &= \frac{f(z,x)x^{2p} - px^{p-1}F(z,x)}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\ &\le -\frac{\xi}{x^{p+1}} \end{aligned}$$

for almost all $z \in \Omega$ and all $x \ge M_5$, see (3.33). This implies

$$\frac{F(z,x)}{x^p} - \frac{F(z,y)}{y^p} \le \frac{\xi}{p}\Big[\frac{1}{x^p} - \frac{1}{y^p}\Big], \tag{3.34}$$

for almost all $z \in \Omega$, for all $x \ge y \ge M_5$.

Hypothesis (H1)(iii) implies

$$\hat{\lambda}_1 \le \liminf_{x \to +\infty} \frac{pF(z,x)}{x^p} \le \limsup_{x \to +\infty} \frac{pF(z,x)}{x^p} \le \eta(z) \tag{3.35}$$

uniformly for almost all $z \in \Omega$.

In (3.34) we pass to the limit as $x \to +\infty$ and use (3.35). We obtain that $\hat{\lambda}_1 y^p - pF(z,y) \le -\xi$ for almost all $z \in \Omega$ and all $y \ge M_5$. This implies

$$\hat{\lambda}_1 y^p - pF(z,y) \to -\infty \quad \text{as } y \to +\infty \text{ uniformly for a.a .} z \in \Omega. \tag{3.36}$$

For $t > 0$ big (so that $t\hat{u}_1 \ge \underline{u}$, recall that $\hat{u}_1 \in \operatorname{int} C_+$), we have

$$\varphi_0(t\hat{u}_1) \le \frac{t^p}{p}\hat{\lambda}_1 - \int_\Omega F(z, t\hat{u}_1)dz + c_7 \quad \text{for some } c_7 > 0, \text{ see (3.13)}$$

which implies

$$p\varphi_0(t\hat{u}_1) \le \int_\Omega [\hat{\lambda}_1(t\hat{u}_1)^p - pF(z, t\hat{u}_1)]dz + pc_7,$$

which in turn implies

$$p\varphi_0(t\hat{u}_1) \to -\infty \quad \text{(see (3.36) and use Fatou's lemma).} \tag{3.37}$$

Then (3.32), (3.37) and the claim permit the use of Theorem 2.1 (the mountain pass theorem) and so we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_0} \quad \text{and} \quad m_\rho \le \varphi_0(\hat{u}). \tag{3.38}$$

It follows from (3.32) and (3.38) that $\hat{u} \neq u_0$, $\hat{u} \in [\underline{u}) \cap C_+$ and so $\hat{u} \in \operatorname{int} C_+$ is the second positive smooth solution of problem (1.1). $\square$

So, we can state the following multiplicity theorem for problem (1.1)

Theorem 3.3. *If hypotheses* (H1) *hold, then problem* (1.1) *has at least two positive smooth solutions* u_0 *and* $\hat{u}$ *in* $\operatorname{int} C_+$.

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Nikolaos S. Papageorgiou
National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr

Vicenţiu D. Rădulescu (corresponding author)
Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Department of Mathematics, University of Craiova, Craiova 200585, Romania
E-mail address: radulescu@inf.ucv.ro

Dušan D. Repovš
Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia
E-mail address: dusan.repovs@guest.arnes.si