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HIGH AND LOW PERTURBATIONS OF CHOQUARD EQUATIONS WITH CRITICAL REACTION AND VARIABLE GROWTH

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Abstract. We are concerned with the existence of ground state solutions to the nonhomogeneous perturbed Choquard equation

$$\begin{aligned}&-\Delta_{p(x)}u+V(x)|u|^{p(x)-2}u\\&=\left(\int_{\mathbb{R}^N} r(y)^{-1}|u(y)|^{r(y)}|x-y|^{-\lambda(x,y)}dy\right)|u|^{r(x)-2}u+g(x,u) \text{ in } \mathbb{R}^N,\end{aligned}$$

where the exponent $r(\cdot)$ is critical with respect to the Hardy-Littlewood-Sobolev inequality for variable exponents. We first consider the case where the perturbation $g(\cdot,\cdot)$ is subcritical and we distinguish between the superlinear and sublinear cases. In both situations we establish the existence of solutions and we prove the asymptotic behavior of low-energy solutions in the case of high perturbations. Next, we study the case where the nonlinearity $g(\cdot,\cdot)$ is critical. We prove the existence of solutions both for low and high perturbations and we establish asymptotic properties of low-energy solutions.

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1. **Introduction.** This paper is devoted to the qualitative and asymptotic analysis of solutions for the Choquard equation with variable exponents. The features of this paper are the following:

(i) the analysis is developed in the anisotropic case, corresponding to a differential operator with nonstandard growth;

(ii) the exponent associated to the nonlocal term is critical with respect to the anisotropic Hardy-Littlewood-Sobolev inequality;

(iii) the main results are concerned both with subcritical and critical perturbations of the nonlocal term;

(iv) the analysis in the subcritical setting corresponds to the non-autonomous case (for instance, coercive and bounded from below positive potentials V), while the critical case is analyzed in the autonomous framework;

(v) we establish sufficient conditions for the existence of solutions in the case of low or high perturbations.

To the best of our knowledge, this is the first paper dealing with Choquard equations with variable exponents and critical anisotropic reaction.

1.1. **Historical comments.** The Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3. \tag{1}$$

was first introduced in the pioneering work of Fröhlich [10] and Pekar [22] for the modeling of a quantum polaron at rest. This model corresponds to the study of free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (1) to describe an electron trapped in its own hole, see Lieb [17].

The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with non-relativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [9], Giulini and Großardt [13], Jones [14], and Schunck and Mielke [27]. Penrose [23, 24] proposed equation (1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon. Beyond physical motivations, ground state solutions of problem (1) are of particular interest because of connections with stochastic analysis, see Donsker and Varadhan [7].

As pointed out by Lieb [17], Choquard used equation (1) to study steady states of the one component plasma approximation in the Hartree-Fock theory. Classification of solutions of (1) was first studied by Ma and Zhao [19]. Pointwise bounds and blow-up for Choquard-Pekar inequalities at isolated singularities have been studied by Ghergu and Taliaferro [12]. For the Choquard-type equation and related problems, we refer to [5, 19, 25, 29] for the existence of solutions and multiplicity properties, to [6, 33] for existence of sign-changing solutions, and to [4, 30] for semiclassical solutions.

If the reaction of problem (1) is perturbed, then we obtain the Choquard equation

$$-\Delta u + Vu = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^r}{|x-y|^\lambda} dy\right) |u|^{r-2}u + g(u) \text{ in } \mathbb{R}^N \ (N \geq 3), \tag{2}$$

where $\lambda \in (0, N)$, V is a positive potential, and g is a suitable perturbation. The Hardy-Littlewood-Sobolev inequality implies that $\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^r|u(y)|^r}{|x-y|^\lambda}dxdy$ is well defined for $u \in H^1(\mathbb{R}^N)$ if $r \in \left[\frac{2N-\lambda}{N}, \frac{2N-\lambda}{N-2}\right]$. Usually, $\frac{2N-\lambda}{N}$ is called the *lower critical exponent* and $\frac{2N-\lambda}{N-2}$ is the *upper critical exponent* of the Choquard equation. The upper critical exponent plays a similar role as the Sobolev critical exponent in the local semilinear equations, while the lower critical exponent is related to the bubbling at infinity phenomenon. Several existence and nonexistence properties of solutions have been established for various values of r. For instance, in view of the Pohozaev identity, the autonomous Choquard equation (2) (with $V = 1$ and $g = 0$) has no nontrivial solutions is either $r \leq \frac{2N-\lambda}{N}$ or $r \geq \frac{2N-\lambda}{N-2}$. For more details we refer to Li and Ma [16] and the references therein.

1.2. **Related notions and properties.** In the sequel, we set

$$C^+(\mathbb{R}^N) := \{h \in C(\mathbb{R}^N) : \ 1 < h^- \leq h^+ < +\infty\},$$

where

$$h^- := \inf_{x\in\mathbb{R}^N} h(x) \text{ and } h^+ := \sup_{x\in\mathbb{R}^N} h(x).$$

For $p \in C^+(\mathbb{R}^N)$, we define the following anisotropic Lebesgue space

$$L^{p(x)}(\mathbb{R}^N) := \left\{u : \mathbb{R}^N \mapsto \mathbb{R};\ u \text{ is a measurable and } \int_{\mathbb{R}^N}|u(x)|^{p(x)}dx < +\infty\right\}.$$

We equip this function space with the following "Luxembourg norm"

$$\|u\|_{L^{p(x)}(\mathbb{R}^N)} = \inf\left\{\eta > 0 : \int_{\mathbb{R}^N}\left|\frac{u}{\eta}\right|^{p(x)} dx \leq 1\right\}.$$

We also consider the following Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ with variable exponent

$$W^{1,p(x)}(\mathbb{R}^N) := \left\{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}.$$

We refer to the monograph by Rădulescu and Repovš [26] for more details on Lebesgue and Sobolev spaces with variable exponent. We refer to Mingione and Rădulescu [20] for a survey on recent developments in problems with nonstandard growth and nonuniform ellipticity.

Throughout this paper, we are concerned with the anisotropic counterpart of problem (2), namely we study problems of the type

$$\begin{aligned} &-\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \\ &= \left(\int_{\mathbb{R}^N}\frac{r(y)^{-1}|u(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}}dy\right)|u|^{r(x)-2}u + g(x,u) \text{ in } \mathbb{R}^N\ (N \geq 3). \end{aligned} \tag{3}$$

The main results will be described in the next section of the present paper. At this stage, we point out that an important role in our analysis will be played by the following Hardy-Littlewood-Sobolev inequality for variable exponents, see Alves and Tavares [2, Proposition 2.4].

Theorem 1.1. *Let* ζ, $q \in C^+(\mathbb{R}^N)$, $\zeta \in L^{\xi^+}(\mathbb{R}^N) \cap L^{\xi^-}(\mathbb{R}^N)$ *and* $\upsilon \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$. *Assume that* $\lambda : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ *is a continuous function satisfying*

$$0 < \lambda^- := \inf_{x,y \in \mathbb{R}^N} \lambda(x,y) \leq \lambda^+ := \sup_{x,y \in \mathbb{R}^N} \lambda(x,y) < N$$

and

$$\frac{1}{\xi(x)} + \frac{\lambda(x,y)}{N} + \frac{1}{q(y)} = 2 \quad \text{for every } x,\ y \in \mathbb{R}^N.$$

Then the following inequality

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\zeta(x)\upsilon(y)}{|x-y|^{\lambda(x,y)}} dxdy \right| \leq C \left(\|\zeta\|_{L^{\xi^+}(\mathbb{R}^N)} \|\upsilon\|_{L^{q^+}(\mathbb{R}^N)} + \|\zeta\|_{L^{\xi^-}(\mathbb{R}^N)} \|\upsilon\|_{L^{q^-}(\mathbb{R}^N)} \right)$$

holds, where $C > 0$ *is a constant not depending on* ζ *and* υ.

2. Main results.

In the first part of this paper we are interested in the existence of solutions to the following Choquard problem with variable exponents and critical growth:

$$\begin{cases} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u &= \left(\displaystyle\int_{\mathbb{R}^N} \frac{r(y)^{-1}|u(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}} dy \right) |u|^{r(x)-2}u \\ & \qquad +\mu f(x,u), \\ u \in W_V^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{P_μ}$$

where μ is a positive parameter, $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a scalar potential, and $p : \mathbb{R}^N \mapsto \mathbb{R}$ is Lipschitz continuous function satisfying

$$1 < p^- := \inf_{x \in \mathbb{R}^N} p(x) \leq p(x) \leq p^+ := \sup_{x \in \mathbb{R}^N} p(x) < N.$$

We denote by $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ the $p(x)$-Laplace operator with variable exponent. We also assume that $\lambda : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function verifying

$$0 < \lambda^- \leq \lambda^+ < N.$$

Let $p^*(x) := Np(x)/(N - p(x))$ be the critical Sobolev exponent associated to $p(x)$. Throughout this paper we assume that $r \in C^+(\mathbb{R}^N)$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality for variable exponent (see Theorem 1.1), that is,

$$r(x) := \frac{2N - \lambda^+}{2N} p^*(x) \text{ for all } x \in \mathbb{R}^N. \tag{4}$$

In view of [2, Corollary 2.1], we additionally impose the restriction

$$r(x) \geq \frac{2N - \lambda^-}{2N} p(x) \text{ for all } x \in \mathbb{R}^N. \tag{5}$$

Relations (4) and (5) provide a relationship between the requested growth of the variable exponents $p(x)$ and $\lambda(x,y)$, namely

$$p(x) \geq N \frac{\lambda^+ - \lambda^-}{2N - \lambda^-} \text{ for all } x \in \mathbb{R}^N.$$

This relation is automatically fulfilled if λ is a constant function. We also point out that in the semilinear isotropic case corresponding to $p(x) = 2$ for all $x \in \mathbb{R}^N$ and $\lambda(x,y) \equiv \lambda$ for all $x, y \in \mathbb{R}^N$, relations (4) and (5) assert that our framework corresponds to the *critical* case (in relationship with the Hardy-Littlewood-Sobolev inequality), see [16].

For problem (P_μ), the appropriate Sobolev space is $W_V^{1,p(x)}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|u\|_{L_V^{p(x)}(\mathbb{R}^N)},$$

where

$$\|u\|_{L_V^{p(x)}(\mathbb{R}^N)} = \inf\left\{\eta > 0 : \int_{\mathbb{R}^N} V(x)\left|\frac{u}{\eta}\right|^{p(x)} dx \le 1\right\}.$$

We assume that the potential V satisfies the following hypotheses:

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $\inf\limits_{x\in\mathbb{R}^N} V(x) := V_0 > 0$.

(V_1) $V(x) \to +\infty$ as $|x| \to +\infty$.

By Alves [1, Lemma 4.2], condition (V_1) implies that the Sobolev embedding

$$W_V^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N)$$

is compact for all $s \in C^+(\mathbb{R}^N)$ and $p \ll s \ll p^*$. The notation $h_1 \ll h_2$ means that $\inf\{h_2(x) - h_1(x) : x \in \mathbb{R}^N\} > 0$.

Throughout this paper we assume that the nonlinear term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $f(x,t)t \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. The following hypotheses are required in the superlinear case.

(H_1) For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t|^{p(x)-1} + C_\varepsilon|t|^{\tau(x)-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $p \ll \tau \ll p^*$, where $\tau \in C^+(\mathbb{R}^N)$ and $\tau^- > p^+$.

(H_2) There exists $2r^- > \sigma > p^+$ with $p^- r^- > p^+$ such that

$$0 < F(x,t) := \int_0^t f(x,s)ds \le \frac{1}{\sigma} f(x,t)t \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \setminus \{0\}.$$

In the sublinear case we assume that the following hypotheses are fulfilled:

(H_3) $|f(x,t)| \le \beta(x)|t|^{\alpha(x)-1}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $\alpha \in C^+(\mathbb{R}^N)$, $\alpha \ll p$ and

$$0 \le \beta \in L^{p^*(x)/(p^*(x)-\alpha(x))}(\mathbb{R}^N).$$

(H_4) There exist $\kappa \in C^+(\mathbb{R}^N)$ with $\kappa^+ < p^-$, $a > 0$, $b > 0$ and open set $\emptyset \ne U \subset \mathbb{R}^N$ such that

$$F(x,t) \ge at^{\kappa(x)},\ \forall\,(x,t) \in U \times (0,b).$$

(H_5) $\min\{2r^-,\ p^- r^-\} > p^+$.

The main results of the first part of this paper provide existence properties both for high and low perturbations, as well as an asymptotic energy decay of solutions in the first case.

Theorem 2.1. *Assume that hypotheses* (H_1) – (H_2) *and* (V_0) – (V_1) *are fulfilled. Then there exists* $\mu^* > 0$ *such that for all* $\mu \in [\mu^*, +\infty)$ *problem* (P_μ) *has a nontrivial solution* $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$ *with* $\|u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \to 0$ *as* $\mu \to +\infty$.

Theorem 2.2. *Assume that hypotheses* (H_3) – (H_5) *and* (V_0) *are fulfilled. Then there exists* $\mu_* > 0$ *such that for all* $\mu \in (0, \mu_*]$ *problem* (P_μ) *has a nontrivial solution* $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$.

In the last part of this paper we deal with the following critical version of problem (P_μ):

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u &= \left(\displaystyle\int_{\mathbb{R}^N} \frac{g(y)r(y)^{-1}|u(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}}dy\right) g(x)|u|^{r(x)-2}u \\ & \qquad + K(x)|u|^{p^*(x)-2}u + \mu f(x,u), \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{Q_μ}$$

where $p : \mathbb{R}^N \mapsto \mathbb{R}$ is a Lipschitz continuous and radially symmetric function satisfying

$$1 < p^- := \inf_{x\in\mathbb{R}^N} p(x) \le p(x) \le p^+ := \sup_{x\in\mathbb{R}^N} p(x) < N.$$

To achieve our aim, we require that the following hypotheses are fulfilled.

(H_6) There exists $\min\left\{2r^-, p^{*-}\right\} > \sigma > p^+$ with $p^-r^- > p^+$ such that

$$0 < \sigma F(x,t) \le f(x,t)t \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \setminus \{0\}.$$

(H_7) $f(x,t) = f(|x|,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(H_8) $g,\ K \in L^\infty(\mathbb{R}^N)$, $K \in C(\mathbb{R}^N)$, $g(x) = g(|x|)$, $K(x) = K(|x|)$ for any $x \in \mathbb{R}^N$, $g(x),\ K(x) \ge 0$, $K(0) = 0$ and

$$\lim_{|x|\to 0} g(x) = \lim_{|x|\to+\infty} g(x) = \lim_{|x|\to+\infty} K(x) = 0.$$

(H_9) $\min\left\{2r^-,\ p^-r^-,\ p^{*-}\right\} > p^+$.

In the critical case our main results can be stated as follows.

Theorem 2.3. *Assume that hypotheses* (H_1) *and* (H_6) $-$ (H_8) *are fulfilled. Then there exists $\mu^{**} > 0$ such that for all $\mu \in [\mu^{**}, +\infty)$ problem (Q_μ) has a nontrivial solution $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ with $\|u_\mu\|_{W^{1,p(x)}(\mathbb{R}^N)} \to 0$ as $\mu \to +\infty$.*

Theorem 2.4. *Assume that hypotheses* (H_3) $-$ (H_4) *and* (H_7) $-$ (H_9) *are fulfilled. Then there exists $\mu_{**} > 0$ such that for all $\mu \in (0, \mu_{**}]$ problem (Q_μ) has a nontrivial solution $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$.*

3. **Auxiliary properties.** Let $C_c(\mathbb{R}^N)$ be the subspace of functions in $C(\mathbb{R}^N)$ with compact support and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\varphi|_\infty = \sup\left\{|\varphi(x)| : x \in \mathbb{R}^N\right\}$. A finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. For any finite measure ν we define $\|\nu\| := \sup\{|(\nu,\varphi)| : \varphi \in C_0(\mathbb{R}^N),\ |\varphi|_\infty = 1\}$, where $(\nu,\varphi) = \displaystyle\int_{\mathbb{R}^N} \varphi d\nu$.

Let $\mathcal{M}(\mathbb{R}^N)$ be the space of finite non-negative Borel measures on $\mathbb{R}^N$. We say that $\nu_n \xrightarrow{w^*} \nu$ in $\mathcal{M}(\mathbb{R}^N)$ as $n \to \infty$, provided that $(\nu_n,\varphi) \to (\nu,\varphi)$ for all $\varphi \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 3.1. *Let $\{u_n\}$ be a bounded sequence in $L^{p^*(x)}(\mathbb{R}^N) \cap L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Then, the following relation*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{r(x)}|u_n(y)|^{r(y)} - |u_n(x)-u(x)|^{r(x)}|u_n(y)-u(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}}dxdy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}}dxdy$$

holds true.

Proof. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{r(y)}|u_n(x)|^{r(x)} - |u_n(y)-u(y)|^{r(y)}|u_n(x)-u(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dydx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(|u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)}\right)\left(|u_n(x)|^{r(x)} - |u_n(x)-u(x)|^{r(x)}\right)}{|x-y|^{\lambda(x,y)}}dydx \\
&+ 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(|u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)}\right)|u_n(x)-u(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dydx =: I_1^n + I_2^n.
\end{aligned} \tag{6}$$

We claim that

$$\lim_{n\to\infty} I_1^n = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{r(y)}|u(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dydx \tag{7}$$

and

$$\lim_{n\to\infty} I_2^n = 0. \tag{8}$$

Since $\{u_n\}$ is a bounded sequence in $L^{p^*(x)}(\mathbb{R}^N)\cap L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)$, there exists a positive constant C_1 such that for all $n\in\mathbb{N}$

$$\|u\|_{L^{p^*(x)}},\ \|u_n\|_{L^{p^*(x)}(\mathbb{R}^N)},\ \|u\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)},\ \|u_n\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)} \le C_1.$$

The above relations imply that

$$\sup_{n\in\mathbb{N}}\left\{\||u_n-u|^{r(\cdot)}\|_{L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)},\ \||u_n-u|^{r(\cdot)}\|_{L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)}\right\}\le C_2$$

for some constant $C_2>0$. Moreover, $u_n\to u$ a.e. in $\mathbb{R}^N$ as $n\to\infty$. It follows from Proposition 5.4.7 of Willem [32, p. 106] that $|u_n-u|^{r(x)}\xrightarrow{w} 0$ in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)$ and $|u_n-u|^{r(x)}\xrightarrow{w} 0$ in $L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ as $n\to\infty$.

Next, we show that

$$\int_{\mathbb{R}^N}\left||u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)} - |u(y)|^{r(y)}\right|^{\frac{2N}{2N-\lambda^+}}dy = 0 \tag{9}$$

and

$$\int_{\mathbb{R}^N}\left||u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)} - |u(y)|^{r(y)}\right|^{\frac{2N}{2N-\lambda^-}}dy = 0. \tag{10}$$

In order to prove relations (9) and (10), we first show that the following inequality.

(i) For any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$\left||u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)}\right| \le \varepsilon|u_n(y)-u(y)|^{r(y)} + C_\varepsilon|u(y)|^{r(y)},\ \forall\, y\in\mathbb{R}^N.$$

It is obvious to get the above inequality when $0<r(y)\le 1$.

Now, it remains to examine the case $r(y)>1$. For any fixed $y\in\mathbb{R}^N$, by Taylor's formula, we have

$$|u_n(y)|^{r(y)} = |u_n(y)-u(y)|^{r(y)} + r(y)\xi^{r(y)-1}\left(|u_n(y)| - |u_n(y)-u(y)|\right),$$

where ξ is a measurable function with values between $|u_n(y)|$ and $|u_n(y)-u(y)|$. It follows that

$$\left||u_n(y)|^{r(y)} - |u_n(y)-u(y)|^{r(y)}\right|$$

$$
\begin{aligned}
&= r(y)\xi^{r(y)-1}\left||u_n(y)|-|u_n(y)-u(y)|\right| \\
&\leq r(y)2^{r(y)-1}\left(|u(y)|^{r(y)-1}+|2\left(u_n(y)-u(y)\right)|^{r(y)-1}\right)|u(y)|.
\end{aligned}
$$

By Young's inequality, for some fixed $\varepsilon_1 \in (0,1)$ we have

$$
\begin{aligned}
|u_n(y)-u(y)|^{r(y)-1}|u(y)| &\leq \frac{(r(y)-1)\,\varepsilon_1}{r(y)}|u_n(y)-u(y)|^{r(y)} \\
&\quad + \frac{1}{r(y)}(\varepsilon_1)^{1-r(y)}|u(y)|^{r(y)} \\
&\leq \varepsilon_1|u_n(y)-u(y)|^{r(y)}+(\varepsilon_1)^{1-r^+}|u(y)|^{r(y)}.
\end{aligned}
$$

Thus, we derive that

$$
\begin{aligned}
&\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}\right| \\
&\leq r^+2^{r^+-1}\left(\varepsilon_1 2^{r^+-1}|u_n(y)-u(y)|^{r(y)}+\left(1+2^{r^+-1}(\varepsilon_1)^{1-r^+}\right)|u(y)|^{r(y)}\right).
\end{aligned}
$$

Choosing $\varepsilon = r^+2^{2r^+-2}\varepsilon_1$, we obtain

$$
\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}\right| \leq \varepsilon|u_n(y)-u(y)|^{r(y)}+C_\varepsilon|u(y)|^{r(y)}.
$$

The proof of (i) is now complete.

Let us denote

$$
w_{\varepsilon,n}(y)=\left(\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}-|u(y)|^{r(y)}\right|-\varepsilon|u_n(y)-u(y)|^{r(y)}\right)^+,
$$

where $w^+(y)=\max\{w(y),0\}$. Clearly, $w_{\varepsilon,n}(y)\to 0$ a.e. in $\mathbb{R}^N$ as $n\to\infty$.

Additionally, we can deduce from the above information that

$$
|w_{\varepsilon,n}(\cdot)|^{\frac{2N}{2N-\lambda^+}} \leq (1+C_\varepsilon)^{\frac{2N}{2N-\lambda^+}}|u(\cdot)|^{p^*(\cdot)} \in L^1(\mathbb{R}^N)
$$

and

$$
|w_{\varepsilon,n}(\cdot)|^{\frac{2N}{2N-\lambda^-}} \leq (1+C_\varepsilon)^{\frac{2N}{2N-\lambda^-}}|u(\cdot)|^{\frac{2Nr(\cdot)}{2N-\lambda^-}} \in L^1(\mathbb{R}^N).
$$

By the Lebesgue dominated convergence theorem, we obtain

$$
\int_{\mathbb{R}^N}|w_{\varepsilon,n}|^{\frac{2N}{2N-\lambda^+}}dy,\ \int_{\mathbb{R}^N}|w_{\varepsilon,n}|^{\frac{2N}{2N-\lambda^-}}dy \to 0 \text{ as } n\to\infty.
$$

So, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}-|u(y)|^{r(y)}\right|^{\frac{2N}{2N-\lambda^+}}dy \\
&\leq C_3\int_{\mathbb{R}^N}|w_{\varepsilon,n}|^{\frac{2N}{2N-\lambda^+}}dy+C_3\varepsilon^{\frac{2N}{2N-\lambda^+}}
\end{aligned}
$$

for some constant $C_3>0$. We conclude that relation (9) holds. Similarly we obtain relation (10).

Denote

$$
A_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}-|u(y)|^{r(y)}\right|\left||u_n(x)|^{r(x)}-|u_n(x)-u(x)|^{r(x)}\right|}{|x-y|^{\lambda^+}}dydx,
$$

$$
B_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left||u_n(y)|^{r(y)}-|u_n(y)-u(y)|^{r(y)}-|u(y)|^{r(y)}\right|\left||u_n(x)|^{r(x)}-|u_n(x)-u(x)|^{r(x)}\right|}{|x-y|^{\lambda^-}}dydx,
$$

$$C_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{r(y)}\left||u_n(x)|^{r(x)} - |u_n(x)-u(x)|^{r(x)} - |u(x)|^{r(x)}\right|}{|x-y|^{\lambda^+}}dydx,$$
$$D_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{r(y)}\left||u_n(x)|^{r(x)} - |u_n(x)-u(x)|^{r(x)} - |u(x)|^{r(x)}\right|}{|x-y|^{\lambda^-}}dydx.$$

Using the Hardy-Littlewood-Sobolev inequality (see Lieb and Loss [18, Theorem 4.3]), the boundedness of $\{u_n\}$ in $L^{p^*(x)}(\mathbb{R}^N) \cap L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)$ and relations (9)–(10), we deduce that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n = 0. \tag{11}$$

Relation (11) yields that relation (7) holds.

Since $|u_n - u|^{r(x)} \xrightarrow{w} 0$ in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)$ and $|u_n - u|^{r(x)} \xrightarrow{w} 0$ in $L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ as $n \to \infty$, together with relation (11), we can conclude that relation (8) is fulfilled. The proof is now completed. $\square$

Corollary 1. *Let $\{u_n\}$ be a bounded sequence in $L^{p^*(x)}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n\to\infty$. Then, the following relation*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n|^{p^*(x)} - |u_n-u|^{p^*(x)}\right)dx = \int_{\mathbb{R}^N}|u|^{p^*(x)}dx$$

holds.

Proof. Similar to the proof of relation (9) or (10), we can get the result. So, we omit the details of the proof of Corollary 1. $\square$

4. **Proof of Theorem 2.1.** To establish the existence of nontrivial solutions to problem (P_μ), we define the functional $\Upsilon_\mu : W_V^{1,p(x)}(\mathbb{R}^N) \mapsto \mathbb{R}$ as follows

$$\begin{aligned}\Upsilon_\mu(u) = &\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)dx - \mu\int_{\mathbb{R}^N}F(x,u)dx\\ &-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{r(x)|x-y|^{\lambda(x,y)}r(y)}dxdy, \quad \forall\ u \in W_V^{1,p(x)}(\mathbb{R}^N).\end{aligned}$$

Similar to the proof of Lemma 3.2 in Alves and Tavares [2], using hypothesis (H$_1$) we deduce that $\Upsilon_\mu \in C^1\left(W_V^{1,p(x)}(\mathbb{R}^N), \mathbb{R}\right)$ with

$$\begin{aligned}\langle \Upsilon_\mu'(u), v\rangle = &\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u\nabla v + V(x)|u|^{p(x)-2}uv\right)dx - \mu\int_{\mathbb{R}^N}f(x,u)vdx\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{r(y)^{-1}|u(y)|^{r(y)}|u(x)|^{r(x)-2}u(x)v(x)}{|x-y|^{\lambda(x,y)}}dxdy\end{aligned}$$

for all u, $v \in W_V^{1,p(x)}(\mathbb{R}^N)$.

We first establish the mountain pass geometry.

Lemma 4.1. *The functional Υ_μ satisfies the following properties.*

(i) There exists $\rho > 0$ small enough such that $\Upsilon_\mu(u) \geq \eta$ for all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ with $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \rho$ for some $\eta > 0$.

(ii) There exists $e \in W_V^{1,p(x)}(\mathbb{R}^N)$ such that $\|e\|_{W_V^{1,p(x)}(\mathbb{R}^N)} > \rho$ and $\Upsilon_\mu(e) < 0$.

Proof. (i) By Theorem 1.1, we obtain for all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{r(x)|x-y|^{\lambda(x,y)}r(y)}dxdy \\
&\leq c_1\left(\||u(\cdot)|^{r(\cdot)}\|^2_{L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)} + \||u(\cdot)|^{r(\cdot)}\|^2_{L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)}\right) \\
&\leq c_1 \max\left\{\|u\|^{2r^+}_{L^{p^*(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{p^*(x)}(\mathbb{R}^N)}\right\} \\
&\quad + c_1 \max\left\{\|u\|^{2r^+}_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)}\right\},
\end{aligned}$$

where c_1 is a positive constant which is independent of $u \in W_V^{1,p(x)}(\mathbb{R}^N)$. By hypothesis (H_1) we deduce that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} F(x,u)dx\right| &\leq \frac{\varepsilon}{p^-}\int_{\mathbb{R}^N}|u|^{p(x)}dx + \frac{C_\varepsilon}{p^-}\int_{\mathbb{R}^N}|u|^{\tau(x)}dx \\
&\leq \frac{\varepsilon}{p^-}\int_{\mathbb{R}^N}|u|^{p(x)}dx + \frac{C_\varepsilon}{p^-}\max\left\{\|u\|^{\tau^+}_{L^{\tau(x)}(\mathbb{R}^N)}, \|u\|^{\tau^-}_{L^{\tau(x)}(\mathbb{R}^N)}\right\}
\end{aligned}$$

for all $x \in \mathbb{R}^N$ and $u \in W_V^{1,p(x)}(\mathbb{R}^N)$.

Due to (V_0), $r(x) \geq (Np(x) - p(x)\lambda^-/2)/N$ and $p \ll \tau \ll p^*$, combining the continuous embeddings $W_V^{1,p(x)}(\mathbb{R}^N) \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}$ $(p(x) \leq s(x) \leq p^*(x))$, there exist positive constants c_2, c_3 (c_2 and c_3 are independent of $u \in W_V^{1,p(x)}(\mathbb{R}^N)$) such that

$$\begin{aligned}
&\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq c_2\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)}, \\
&\|u\|_{L^{p(x)}(\mathbb{R}^N)},\ \|u\|_{L^{p^*(x)}(\mathbb{R}^N)},\ \|u\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)},\ \|u\|_{L^{\tau(x)}(\mathbb{R}^N)} \leq c_3\|u\|_{W^{1,p(x)}(\mathbb{R}^N)}.
\end{aligned}$$

Also, we need the following elementary inequality

$$(a+b)^\theta \leq 2^{\theta-1}a^\theta + 2^{\theta-1}b^\theta \text{ for all } a,\ b > 0 \text{ and } \theta \geq 1.$$

Taking $\varepsilon = \dfrac{V_0 p^-}{2\mu p^+}$, for $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} < \dfrac{1}{c_2}$ we obtain

$$\begin{aligned}
\Upsilon_\mu(u) &\geq \int_{\mathbb{R}^N}\left(\frac{1}{p^+}|\nabla u|^{p(x)} + \frac{V_0}{2p^+}|u|^{p(x)}\right)dx \\
&\quad - c_4\max\left\{\|u\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\} \\
&\quad - \frac{\mu C_\varepsilon}{p^-}\max\left\{\|u\|^{\tau^+}_{L^{\tau(x)}(\mathbb{R}^N)}, \|u\|^{\tau^-}_{L^{\tau(x)}(\mathbb{R}^N)}\right\} \\
&\geq c_5\left(\|\nabla u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N)}\right) \\
&\quad - c_6\left(\|\nabla u\|^{2r^-}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{2r^-}_{L^{p(x)}(\mathbb{R}^N)}\right) \\
&\quad - c_6\left(\|\nabla u\|^{\tau^-}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{\tau^-}_{L^{p(x)}(\mathbb{R}^N)}\right),
\end{aligned}$$

where c_i $(i = 4,5,6)$ are some positive constants that do not depend on u. Since $2r^-$, $\tau^- > p^+$ and $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq c_2\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)}$, the result of (i) follows by fixing $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \rho$ with ρ sufficiently small.

(ii) For each $t > 1$ and $e' \in W_V^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$ with $\|e'\|_{W_V^{1,p(x)}(\mathbb{R}^N)} < 1$, by hypothesis (H_2), we have

$$\Upsilon_\mu(te') \leq \frac{2t^{p^+}}{p^-}\|e'\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{t^{2r^-}}{2(r^+)^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e'(x)|^{r(x)}|e'(y)|^{r(y)}}{|x-y|^{\lambda(x,y)}}dxdy.$$

Since $2r^- > p^+$, we can get the conclusion for $t > 1$ sufficiently large.

The proof is now complete. □

Now we discuss the compactness property for the functional Υ_μ, given by the (PS) condition at a suitable level. For this goal, we fix $\mu > 0$ and define

$$c_\mu := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Upsilon_\mu(\gamma(t)), \tag{12}$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], W_V^{1,p(x)}(\mathbb{R}^N)\right) : \gamma(0) = 0,\ \gamma(1) = e\right\}.$$

Clearly, using Lemma 4.1 we know that $c_\mu > 0$. Furthermore, we have the following result.

Lemma 4.2. *Assume that* (V_0), (H_1) *and* (H_2) *hold. Then we have*

$$\lim_{\mu\to+\infty} c_\mu = 0,$$

where c_μ *is given in* (12).

Proof. For e given in Lemma 4.1, there exists $t_\mu > 0$ satisfying

$$\Upsilon_\mu(t_\mu e) = \max_{t\geq 0}\Upsilon_\mu(te)$$

and

$$\int_{\mathbb{R}^N}(t_\mu)^{p(x)}\left(|\nabla e|^{p(x)} + V(x)|e|^{p(x)}\right)dx = \mu\int_{\mathbb{R}^N} f(x,t_\mu e)t_\mu e dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e(x)|^{r(x)}|e(y)|^{r(y)}(t_\mu)^{r(x)+r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy.$$

Using this equality, hypothesis (H_2) and $2r^- > p^+$ we conclude that $\{t_\mu\}$ is bounded.

Let $\{\mu_n\}$ be a sequence such that $\mu_n \to +\infty$ as $n \to \infty$. Since $\{t_{\mu_n}\}$ is bounded, passing to a subsequence, still denoted by $\{t_{\mu_n}\}$, we may assume that there exists $t_0 \geq 0$ such that $t_{\mu_n} \to t_0$ as $n \to \infty$. Thus, there exists a positive constant c_7 such that

$$\int_{\mathbb{R}^N}(t_{\mu_n})^{p(x)}\left(|\nabla e|^{p(x)} + V(x)|e|^{p(x)}\right)dx \leq c_7$$

for all $n \in \mathbb{N}$.

We assert that $t_0 = 0$. Indeed, if $t_0 > 0$, then by hypotheses $(\mathrm{H}_1) - (\mathrm{H}_2)$ and the boundedness of $\{t_{\mu_n}\}$ we obtain

$$0 < f(x,t_{\mu_n}e)t_{\mu_n}e \leq c_8\left(|e|^{p(x)} + |e|^{\tau(x)}\right) \in L^1(\mathbb{R}^N)$$

for some constant $c_8 > 0$. Clearly, $f(x,t_{\mu_n}e)t_{\mu_n}e \to f(x,t_0e)t_0e$ as $n \to \infty$ by the continuity of $f(x,\cdot)$. So, using Lebesgue's dominated convergence theorem and hypothesis (H_2) we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,t_{\mu_n}e)t_{\mu_n}e dx = \int_{\mathbb{R}^N} f(x,t_0e)t_0e dx > 0.$$

By this equality we deduce that

$$\mu_n \int_{\mathbb{R}^N} f(x, t_{\mu_n} e) t_{\mu_n} e dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|e(x)|^{r(x)} |e(y)|^{r(y)} (t_{\mu_n})^{r(x)+r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \to +\infty \text{ as } n \to \infty,$$

which is a contradiction. So, $t_0 = 0$.

Hence, we have $t_\mu \to 0$ as $\mu \to +\infty$. Since

$$0 \le c_\mu \le \Upsilon_\mu(t_\mu e) \le 2(t_\mu)^{p^-} \left(\|e\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)} + \|e\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \right)$$

for sufficiently large $\mu > 0$, it follows that $c_\mu \to 0$ as $\mu \to +\infty$.

The proof is now complete. □

Lemma 4.3. *There exists $\mu^* > 0$ such that Υ_μ satisfies the $(\mathrm{PS})_{c_\mu}$ condition on $W_V^{1,p(x)}(\mathbb{R}^N)$ for all $\mu \ge \mu^*$.*

Proof. Let $\{u_n\} \subset W_V^{1,p(x)}(\mathbb{R}^N)$ be a $(\mathrm{PS})_{c_\mu}$ sequence of the functional Υ_μ, that is, $\Upsilon_\mu(u_n) \to c_\mu$ and $\Upsilon'_\mu(u_n) \to 0$ as $n \to \infty$. We first prove that $\{u_n\}$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$. Using hypothesis (H_2), for large enough $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
& c_\mu + O(1)\|u_n\|_{W_V^{1,p(x)}(\mathbb{R}^N)} + o_n(1) \\
= & \Upsilon_\mu(u_n) - \frac{1}{\sigma}\langle \Upsilon'_\mu(u_n), u_n \rangle \\
= & \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{\sigma} \right) \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) dx \\
+ & \mu \int_{\mathbb{R}^N} \left(\frac{1}{\sigma} f(x, u_n) u_n - F(x, u_n) \right) dx \\
+ & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{1}{\sigma} - \frac{1}{2r(x)} \right) \frac{|u_n(x)|^{r(x)} |u_n(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \\
\ge & \int_{\mathbb{R}^N} \left(\frac{1}{p^+} - \frac{1}{\sigma} \right) \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) dx \\
+ & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{1}{\sigma} - \frac{1}{2r^-} \right) \frac{|u_n(x)|^{r(x)} |u_n(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \\
\ge & \int_{\mathbb{R}^N} \left(\frac{1}{p^+} - \frac{1}{\sigma} \right) \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) dx. \qquad (13)
\end{aligned}$$

The above inequality implies that $\{u_n\}$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$. Using the boundedness of $\{u_n\}$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and Sobolev embeddings we can find a positive constant c_9 such that

$$\int_{\mathbb{R}^N} ||u_n|^{r(x)-2} u_n|^{\frac{p^*(x)}{r(x)-1}} dx = \int_{\mathbb{R}^N} ||u_n|^{r(x)}|^{\frac{2N}{2N-\lambda^+}} dx = \int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx \le c_9.$$

As $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)$ and $L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ are uniformly convex, the Banach space

$$\left(L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N), \max\left\{ \|\cdot\|_{L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)}, \|\cdot\|_{L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)} \right\} \right)$$

is also uniformly convex, hence reflexive. The boundedness of $\{u_n\}$ in $W_V^{1,p(\cdot)}(\mathbb{R}^N)$ yields that the sequence $\left\{ |u_n|^{r(\cdot)} \right\}$ is bounded in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$.

Next, we claim that there exists $u_\mu \in W_V^{1,p(\cdot)}(\mathbb{R}^N)$ such that, up to a subsequence, still denoted by $\{u_n\}$, $u_n \to u_\mu$ a.e. in $\mathbb{R}^N$ and $|u_n|^{r(\cdot)} \xrightarrow{w} |u_\mu|^{r(\cdot)}$ in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ as $n \to \infty$. Indeed, since $\left\{|u_n|^{r(\cdot)}\right\}$ is bounded in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$, so there exists $T \in L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ such that $|u_n|^{r(\cdot)} \xrightarrow{w} T$ in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ as $n \to \infty$. Fix $v \in C_0^\infty(\mathbb{R}^N)$ and consider the continuous linear functional

$$I_v(w) = \int_{\mathbb{R}^N} wv dy, \; w \in L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N).$$

Then, we obtain

$$I_v(|u_n|^{r(\cdot)}) \to \int_{\mathbb{R}^N} T(y)v(y)dy \text{ as } n \to \infty. \tag{14}$$

Using Proposition 5.4.7 of Willem [32, p. 106], we obtain

$$|u_n|^{r(\cdot)} \xrightarrow{w} |u_\mu|^{r(\cdot)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \text{ as } n \to \infty,$$

hence

$$I_v(|u_n|^{r(\cdot)}) = \int_{\mathbb{R}^N} |u_n|^{r(y)}v(y)dy \to \int_{\mathbb{R}^N} |u_\mu|^{r(y)}v(y)dy \text{ as } n \to \infty. \tag{15}$$

It follows from relations (14) and (15) that $|u_\mu|^{r(\cdot)} = T(\cdot)$ a.e. in $\mathbb{R}^N$. Thus, we get the claim. Moreover, by Theorem 1.1, we know that the functional

$$G(w) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w(y)|u_\mu|^{r(x)}}{|x-y|^{\lambda(x,y)}}dxdy, \; w \in L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$$

is linear and continuous. Due to $|u_n|^{r(y)} \xrightarrow{w} |u_\mu|^{r(y)}$ in $L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N)$ as $n \to \infty$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{r(y)}|u_\mu(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dxdy \\ \to &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dxdy \text{ as } n \to \infty. \end{aligned}$$

Now, we can assume that there exist $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$ and δ_μ, $\varrho_\mu \geq 0$ such that, passing to a subsequence, still denoted by $\{u_n\}$,

$$\left\{ \begin{array}{l} u_n \xrightarrow{w} u_\mu \text{ in } W_V^{1,p(x)}(\mathbb{R}^N), \quad \|u_n\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \to \delta_\mu, \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy \to \varrho_\mu, \\ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N, \quad |u_n|^{r(x)} \xrightarrow{w} |u_\mu|^{r(x)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N), \\ |u_n|^{r(x)-2}u_n \xrightarrow{w} |u_\mu|^{r(x)-2}u_\mu \text{ in } L^{\frac{p^*(x)}{r(x)-1}}(\mathbb{R}^N), \end{array} \right. \tag{16}$$

as $n \to \infty$.

Note that for any $\Omega \subset \mathbb{R}^N$, using Hölder's inequality (see Musielak [21]) and the Sobolev inequality we have

$$\begin{aligned} &\int_\Omega ||u_n|^{r(x)-2}u_n u_\mu|^{\frac{2N}{2N-\lambda^+}}dx \\ = &\int_\Omega |u_n|^{\frac{2N(r(x)-1)}{2N-\lambda^+}}|u_\mu|^{\frac{2N}{2N-\lambda^+}}dx \end{aligned}$$

$$\begin{aligned}
&\leq c_{10}\||u_n|^{\frac{2N(r(\cdot)-1)}{2N-\lambda^+}}\|_{L^{\frac{r(x)}{r(x)-1}}(\Omega)}\||u_\mu|^{\frac{2N}{2N-\lambda^+}}\|_{L^{r(x)}(\Omega)}\\
&\leq c_{11}\max\left\{\left(\int_{\mathbb{R}^N}|u_\mu|^{p^*(x)}dx\right)^{1/r^+},\left(\int_{\mathbb{R}^N}|u_\mu|^{p^*(x)}dx\right)^{1/r^-}\right\}\\
&\leq c_{11}\max\left\{\|u_\mu\|_{L^{p^*(x)}(\Omega)}^{\frac{2N}{2N-\lambda^+}},\|u_\mu\|_{L^{p^*(x)}(\Omega)}^{\frac{p^{*-}}{r^+}},\|u_\mu\|_{L^{p^*(x)}(\Omega)}^{\frac{p^{*+}}{r^-}}\right\} \qquad (17)
\end{aligned}$$

for some constants c_{10}, $c_{11} > 0$. Similarly, there exists a constants $c_{12} > 0$ such that

$$\begin{aligned}
&\int_\Omega ||u_n|^{r(x)-2}u_nu_\mu|^{\frac{2N}{2N-\lambda^-}}dx\\
=&\int_\Omega |u_n|^{\frac{2N(r(x)-1)}{2N-\lambda^-}}|u_\mu|^{\frac{2N}{2N-\lambda^-}}dx\\
\leq&\, c_{12}\max\left\{\|u_\mu\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\Omega)}^{\frac{2N}{2N-\lambda^-}},\|u_\mu\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\Omega)}^{\frac{2Nr^-}{(2N-\lambda^-)r^+}},\|u_\mu\|_{L^{\frac{2Nr(x)}{2N-\lambda^-}}(\Omega)}^{\frac{2Nr^+}{(2N-\lambda^-)r^-}}\right\}. \qquad (18)
\end{aligned}$$

Combining inequalities (17)–(18), $u_\mu \in L^{p^*(x)}(\mathbb{R}^N)$ and $u_\mu \in L^{\frac{2Nr(x)}{2N-\lambda^-}}(\mathbb{R}^N)$, we know that the two sequences $\left\{||u_n|^{r(x)-2}u_nu_\mu|^{\frac{2N}{2N-\lambda^+}}\right\}$ and $\left\{||u_n|^{r(x)-2}u_nu_\mu|^{\frac{2N}{2N-\lambda^-}}\right\}$ are equi-integrable in $L^1(\mathbb{R}^N)$. Additionally, $|u_n|^{r(x)-2}u_nu_\mu \to |u_\mu|^{r(x)}$ a.e. in $\mathbb{R}^N$ as $n\to\infty$. Therefore, by Vitali's convergence theorem (see, Bogachev [3, Corollary 4.5.5]),

$$|u_n|^{r(x)-2}u_nu_\mu \to |u_\mu|^{r(x)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N)\cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N) \text{ as } n\to\infty.$$

So, combining the above information, Theorem 1.1, the boundedness of $\{u_n\}$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and Sobolev embeddings, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{r(y)}|u_n(x)|^{r(x)-2}u_n(x)u_\mu(x)}{|x-y|^{\lambda(x,y)}}dxdy\\
=&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dxdy. \qquad (19)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)-2}u_\mu(x)u_n(x)}{|x-y|^{\lambda(x,y)}}dxdy\\
=&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)}}{|x-y|^{\lambda(x,y)}}dxdy. \qquad (20)
\end{aligned}$$

Fix $\varepsilon > 0$. Using hypothesis (H_1), Hölder's inequality (see Musielak [21]), the boundedness of $\{u_n\}$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and Sobolev inequalities we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}f(x,u_n)(u_n-u_\mu)dx\right|\\
&\leq \varepsilon\int_{\mathbb{R}^N}|u_n|^{p(x)-1}|u_n-u_\mu|dx + C_\varepsilon\int_{\mathbb{R}^N}|u_n|^{\tau(x)-1}|u_n-u_\mu|dx\\
&\leq \varepsilon c_{13}\||u_n|^{p(\cdot)-1}\|_{L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)}\|u_n-u_\mu\|_{L^{p(x)}(\mathbb{R}^N)}\\
&\quad + C_\varepsilon c_{13}\||u_n|^{\tau(\cdot)-1}\|_{L^{\frac{\tau(x)}{\tau(x)-1}}(\mathbb{R}^N)}\|u_n-u_\mu\|_{L^{\tau(x)}(\mathbb{R}^N)}
\end{aligned}$$

$$\leq \varepsilon c_{14} + C_\varepsilon c_{14}\|u_n - u_\mu\|_{L^{\tau(x)}(\mathbb{R}^N)}$$

for some constants c_{13}, $c_{14} > 0$.

Thanks to $p \ll \tau \ll p^*$, hypothesis (V_1) implies that $\|u_n - u_\mu\|_{L^{\tau(x)}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u_\mu)dx = 0. \tag{21}$$

Similar to the proofs of relations (17) and (21), we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_\mu)(u_n - u_\mu)dx = 0 \tag{22}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)u_\mu dx = \int_{\mathbb{R}^N} f(x, u_\mu)u_\mu dx. \tag{23}$$

Let us define the following linear continuous functional

$$\langle \mathcal{L}(u), v\rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right) dx$$

for u, $v \in W_V^{1,p(x)}(\mathbb{R}^N)$. Thus, by $u_n \xrightarrow{w} u_\mu$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \langle \mathcal{L}(u_\mu), u_n - u_\mu\rangle = 0. \tag{24}$$

On the other hand, since $\{u_n\}$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$, it follows that $\{\mathcal{L}(u_n)\}$ is bounded in $\left(W_V^{1,p(x)}(\mathbb{R}^N)\right)'$. Passing to a subsequence, still denoted by $\{\mathcal{L}(u_n)\}$, we may assume that there exists an element $\omega \in \left(W_V^{1,p(x)}(\mathbb{R}^N)\right)'$ such that

$$\lim_{n\to\infty} \langle \mathcal{L}(u_n), v\rangle = \langle \omega, v\rangle \tag{25}$$

for all $v \in W_V^{1,p(x)}(\mathbb{R}^N)$. Using $\langle \Upsilon'_\mu(u_n), u_\mu\rangle \to 0$ as $n \to \infty$ and relations (16), (19), (23), (25), we deduce that

$$\langle \omega, u_\mu\rangle = \mu \int_{\mathbb{R}^N} f(x, u_\mu)u_\mu dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy. \tag{26}$$

From relation (26) and hypothesis (H_2) we have that $\langle \omega, u_\mu\rangle \geq 0$.

Since $\{u_n\}$ is a $(PS)_{c_\mu}$ sequence, combining Lemma 3.1 and relations (16), (19)$-$(25), for large enough $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
o_n(1) &= \langle \Upsilon'_\mu(u_n) - \Upsilon'_\mu(u), u_n - u_\mu\rangle \\
&= \langle \mathcal{L}(u_n), u_n\rangle - \langle \mathcal{L}(u_n), u_\mu\rangle - \langle \mathcal{L}(u_\mu), u_n - u_\mu\rangle \\
&- \mu \int_{\mathbb{R}^N} \left(f(x, u_n) - f(x, u_\mu)\right)(u_n - u_\mu)dx \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{r(y)}|u_n(x)|^{r(x)-2}u_n(x)(u_n(x) - u_\mu(x))}{r(y)|x-y|^{\lambda(x,y)}} dxdy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\mu(y)|^{r(y)}|u_\mu(x)|^{r(x)-2}u_\mu(x)(u_n(x) - u_\mu(x))}{r(y)|x-y|^{\lambda(x,y)}} dxdy \\
&= \langle \mathcal{L}(u_n), u_n\rangle - \langle \omega, u_\mu\rangle \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)}|u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy + o_n(1)
\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathcal{L}(u_n) - \mathcal{L}(u_\mu), u_n - u_\mu \rangle \\
&\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)} |u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy + o_n(1). \qquad (27)
\end{aligned}
$$

Relation (27) implies that

$$
\begin{aligned}
&\lim_{n\to\infty} \langle \mathcal{L}(u_n) - \mathcal{L}(u_\mu), u_n - u_\mu \rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)} |u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy. \qquad (28)
\end{aligned}
$$

By Theorem 1.1 and Sobolev embedding inequalities, there exists $c_{15} > 0$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)} |u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \\
&\leq c_{15} \max \left\{ \|u_n - u_\mu\|^{2r^+}_{W_V^{1,p(x)}(\mathbb{R}^N)}, \|u_n - u_\mu\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \right\}. \qquad (29)
\end{aligned}
$$

Also, we can deduce that there exist two positive constants c_{16}, c_{17} such that

$$
\begin{aligned}
&c_{16} \min \left\{ \|u_n - u_\mu\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)}, \|u_n - u_\mu\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \right\} \\
&\leq \int_{\mathbb{R}^N} \left(|\nabla u_n - \nabla u_\mu|^{p(x)} + V(x)|u_n - u_\mu|^{p(x)} \right) dx \\
&\leq c_{17} \max \left\{ \|u_n - u_\mu\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)}, \|u_n - u_\mu\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \right\}. \qquad (30)
\end{aligned}
$$

Denoting

$$
\Omega_1 = \left\{ x \in \mathbb{R}^N : 1 < p(x) < 2 \right\} \quad \text{and} \quad \Omega_2 = \left\{ x \in \mathbb{R}^N : p(x) \geq 2 \right\},
$$

we allow the case that one of these sets is empty. Then it is clear that $\mathbb{R}^N = \Omega_1 \cup \Omega_2$.

From Kim-Kim [15, Proposition 3.3], we see that the following estimate

$$
(|\xi|^{p(x)-2}\xi - |\zeta|^{p(x)-2}\zeta, \xi - \zeta)_{\mathbb{R}^N} \geq \begin{cases} (|\xi| + |\zeta|)^{p(x)-2}|\xi - \zeta|^2 & \text{if } x \in \Omega_1, \\ 4^{1-p^+}|\xi - \zeta|^{p(x)} & \text{if } x \in \Omega_2 \end{cases} \qquad (31)
$$

holds for all ξ, $\zeta \in \mathbb{R}^N$.

We distinguish the following three cases.

Case 1. $\Omega_2 = \mathbb{R}^N$. By relations (29), (30) and (31), we have

$$
\begin{aligned}
&\langle \mathcal{L}(u_n) - \mathcal{L}(u_\mu), u_n - u_\mu \rangle \\
&\geq 4^{1-p^+} \int_{\mathbb{R}^N} \left(|\nabla u_n - \nabla u_\mu|^{p(x)} + V(x)|u_n - u_\mu|^{p(x)} \right) dx \\
&\geq c_{18} \min \left\{ \|u_n - u_\mu\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)}, \|u_n - u_\mu\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \right\} \\
&\geq c_{19} \min \left\{ \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)} |u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \right)^{\frac{p^-}{2r^+}}, \right. \\
&\quad \left. \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y) - u_\mu(y)|^{r(y)} |u_n(x) - u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \right)^{\frac{p^+}{2r^-}} \right\} \qquad (32)
\end{aligned}
$$

for some positive constants c_{18} and c_{19}. By relations (28) and (32), for $\varrho_\mu > 0$ the following estimate

$$\max\left\{(\varrho_\mu)^{1-\frac{p^+}{2r^-}},\,(\varrho_\mu)^{1-\frac{p^-}{2r^+}}\right\} \geq c_{19} \tag{33}$$

holds true. By a similar argument as in relation (13) and Lemma 4.2, we can deduce that

$$\lim_{\mu\to+\infty}\delta_\mu = 0. \tag{34}$$

Since $u_n \xrightarrow{w} u_\mu$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ as $n\to\infty$, combining relation (34) we have

$$\lim_{\mu\to+\infty}\|u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \leq \lim_{\mu\to+\infty}\lim_{n\to\infty}\|u_n\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \lim_{\mu\to+\infty}\delta_\mu = 0. \tag{35}$$

Denote

$$\mu^* = \sup\left\{\mu > 0 : \varrho_\mu > 0\right\}, \tag{36}$$

where ϱ_μ is given in (16).

Next, we show that $\mu^* < +\infty$. Indeed, if $\mu^* = +\infty$, we can assume that there exists a subsequence $\{\mu_k\}\subset\mathbb{R}$ with $\mu_k\to+\infty$ as $k\to\infty$, such that $\varrho_{\mu_k} > 0$ for all k. Without loss of generality, using relation (34) we can assume that $0 < \delta_{\mu_k} < 1$ for all k.

Using relations (27), (33) and $\langle\omega, u_\mu\rangle \geq 0$, we obtain

$$(\delta_{\mu_k})^{p^-\left(1-\frac{p^+}{2r^-}\right)} \geq \max\left\{(\varrho_{\mu_k})^{1-\frac{p^+}{2r^-}},\,(\varrho_{\mu_k})^{1-\frac{p^-}{2r^+}}\right\} \geq c_{20} > 0 \tag{37}$$

for some constant $c_{20} > 0$. This inequality and relation (34) imply that $0 > 0$, since $2r^- > p^+$. This is a contradiction. So, $\mu^* < +\infty$. Therefore, for all $\mu \geq \mu^*$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy = 0. \tag{38}$$

Relations (28) and (38) yield that

$$\lim_{n\to\infty}\langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu), u_n-u_\mu\rangle = 0. \tag{39}$$

Using relations (32) and (39) we conclude that

$$\lim_{n\to\infty}\|u_n-u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = 0.$$

Case 2. $\Omega_1 = \mathbb{R}^N$. Using relation (31), Hölder's inequality (see Musielak [21]), Sobolev's inequality and the boundedness of $\{u_n\}$ in $W_V^{1,p(x)}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(|\nabla u_n-\nabla u_\mu|^{p(x)}+V(x)|u_n-u_\mu|^{p(x)}\right)dx\\
&\leq c_{21}\int_{\mathbb{R}^N}\left((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_\mu|^{p(x)-2}\nabla u_\mu)(\nabla u_n-\nabla u_\mu)\right)^{p(x)/2}\\
&\quad\times\left(|\nabla u_n|^{p(x)}+|\nabla u_\mu|^{p(x)}\right)^{(2-p(x))/2}dx\\
&+c_{21}\int_{\mathbb{R}^N}V(x)\left((|u_n|^{p(x)-2}u_n-|u_\mu|^{p(x)-2}u_\mu)(u_n-u_\mu)\right)^{p(x)/2}\\
&\quad\times\left(|u_n|^{p(x)}+|u_\mu|^{p(x)}\right)^{(2-p(x))/2}dx
\end{aligned}$$

$$
\begin{aligned}
&\leq 2c_{21}\left\|\left((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_\mu|^{p(x)-2}\nabla u_\mu)(\nabla u_n-\nabla u_\mu)\right)^{p(x)/2}\right\|_{L^{2/p(x)}(\mathbb{R}^N)}\\
&\quad\times\left\|\left(|\nabla u_n|^{p(x)}+|\nabla u_\mu|^{p(x)}\right)^{(2-p(x))/2}\right\|_{L^{2/(2-p(x))}(\mathbb{R}^N)}\\
&\quad+2c_{21}\left\|\left(V(x)(|u_n|^{p(x)-2}u_n-|u_\mu|^{p(x)-2}u_\mu)(u_n-u_\mu)\right)^{p(x)/2}\right\|_{L^{2/p(x)}(\mathbb{R}^N)}\\
&\quad\times\left\|\left(V(x)|u_n|^{p(x)}+V(x)|u_\mu|^{p(x)}\right)^{(2-p(x))/2}\right\|_{L^{2/(2-p(x))}(\mathbb{R}^N)}\\
&\leq c_{22}\max\left\{\langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu),u_n-u_\mu\rangle^{p^+/2},\ \langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu),u_n-u_\mu\rangle^{p^-/2}\right\} \qquad (40)
\end{aligned}
$$

for some constants c_{21}, $c_{22}>0$.

By relations (29), (30) and (40), we obtain

$$
\begin{aligned}
&\max\left\{\langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu),u_n-u_\mu\rangle^{p^+/2},\ \langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu),u_n-u_\mu\rangle^{p^-/2}\right\}\\
&\geq\frac{1}{c_{22}}\int_{\mathbb{R}^N}\left(|\nabla u_n-\nabla u_\mu|^{p(x)}+V(x)|u_n-u_\mu|^{p(x)}\right)dx\\
&\geq c_{23}\min\left\{\|u_n-u_\mu\|^{p^+}_{W^{1,p(x)}_V(\mathbb{R}^N)},\ \|u_n-u_\mu\|^{p^-}_{W^{1,p(x)}_V(\mathbb{R}^N)}\right\}\\
&\geq c_{24}\min\Bigg\{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)-u_\mu(y)|^{r(y)}|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\right)^{\frac{p^-}{2r^+}},\\
&\qquad\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)-u_\mu(y)|^{r(y)}|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\right)^{\frac{p^+}{2r^-}}\Bigg\} \qquad (41)
\end{aligned}
$$

for some positive constants c_{23} and c_{24}.

Using relations (28) and (41), we get

$$
\max\left\{(\varrho_\mu)^{p^+/2},\ (\varrho_\mu)^{p^-/2}\right\}\geq c_{23}\min\left\{(\varrho_\mu)^{p^+/(2r^-)},\ (\varrho_\mu)^{p^-/(2r^+)}\right\}.
$$

This relation implies that

$$
\max\left\{(\varrho_\mu)^{p^+/2-p^-/(2r^+)},\ (\varrho_\mu)^{p^-/2-p^+/(2r^-)}\right\}\geq c_{24}>0 \qquad (42)
$$

for $\varrho_\mu>0$.

Similarly we prove that $\mu^*<+\infty$ (μ^* is given in (36)). Otherwise, if $\mu^*=+\infty$, we can assume that there exists a subsequence $\{\mu_k\}\subset\mathbb{R}$ with $\mu_k\to+\infty$ as $k\to\infty$, such that $\varrho_{\mu_k}>0$ for all k. Without loss of generality, by relation (34), we also can assume that $0<\delta_{\mu_k}<1$ for all k.

By relation (27) and $\langle\omega,u_\mu\rangle\geq 0$ again, together with relation (42), we deduce that

$$
(\delta_{\mu_k})^{p^-\left(\frac{p^-}{2}-\frac{p^+}{2r^-}\right)}\geq\max\left\{(\varrho_{\mu_k})^{p^+/2-p^-/(2r^+)},\ (\varrho_{\mu_k})^{p^-/2-p^+/(2r^-)}\right\}\geq c_{25}>0 \qquad (43)
$$

for some constant $c_{25}>0$. Since $p^-r^->p^+$, using relations (34), (43) we arrive at a contradiction. Hence, $\mu^*<+\infty$. Similar to the case $\Omega_2=\mathbb{R}^N$, for $\mu\geq\mu^*$ we can obtain

$$
\lim_{n\to\infty}\|u_n-u_\mu\|_{W^{1,p(x)}_V(\mathbb{R}^N)}=0.
$$

Case 3. $\Omega_1 \neq \emptyset$ and $\Omega_2 \neq \emptyset$.

Denote

$$p_1^+ = \sup_{x\in\Omega_1} p(x) \quad \text{and} \quad p_1^- = \inf_{x\in\Omega_1} p(x).$$

Arguing as in the discussions of relations (32) and (41), together with relations (29) and (30), we can conclude that there exist some constants c_{26}, c_{27}, $c_{28} > 0$ such that

$$\begin{aligned}
&\max\Bigg\{\langle \mathcal{L}(u_n)-\mathcal{L}(u_\mu), u_n-u_\mu\rangle,\, \langle \mathcal{L}(u_n)-\mathcal{L}(u_\mu), u_n-u_\mu\rangle^{p_1^+/2},\\
&\quad \langle \mathcal{L}(u_n)-\mathcal{L}(u_\mu), u_n-u_\mu\rangle^{p_1^-/2}\Bigg\}\\
&\geq c_{26}\int_{\mathbb{R}^N}\left(|\nabla u_n-\nabla u_\mu|^{p(x)}+V(x)|u_n-u_\mu|^{p(x)}\right)dx\\
&\geq c_{27}\min\left\{\|u_n-u_\mu\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)},\, \|u_n-u_\mu\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)}\right\}\\
&\geq c_{28}\min\Bigg\{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)-u_\mu(y)|^{r(y)}|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\right)^{\frac{p^-}{2r^+}},\\
&\quad \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)-u_\mu(y)|^{r(y)}|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\right)^{\frac{p^+}{2r^-}}\Bigg\}.
\end{aligned} \tag{44}$$

Using the main formula (28) and relation (44), we get

$$\max\left\{\varrho_\mu,\, (\varrho_\mu)^{p_1^+/2},\, (\varrho_\mu)^{p_1^-/2}\right\} \geq c_{28}\min\left\{(\varrho_\mu)^{p^+/(2r^-)},\, (\varrho_\mu)^{p^-/(2r^+)}\right\}.$$

This relation yields that

$$\max\left\{(\varrho_\mu)^{1-\frac{p^-}{2r^+}},\, (\varrho_\mu)^{\frac{p^-}{2}-\frac{p^+}{2r^-}}\right\} \geq c_{28} > 0 \tag{45}$$

for $\varrho_\mu > 0$. Arguing as in the above cases, we obtain that $\mu^* < +\infty$ (μ^* is given in (36)). So, for $\mu \geq \mu^*$ we deduce that

$$\lim_{n\to\infty}\|u_n-u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = 0.$$

In conclusion, we deduce that there exists some constant $\mu^* > 0$ such that Υ_μ satisfies the $(\mathrm{PS})_{c_\mu}$ condition on $W_V^{1,p(x)}(\mathbb{R}^N)$ for all $\mu \geq \mu^*$.

The proof is now complete. □

Proof of Theorem 2.1. Using Lemmas 4.1 and 4.3, there exists $\mu^* > 0$ such that for all $\mu \geq \mu^*$ the functional Υ_μ has a nontrivial critical point $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$. More precisely, the critical point u_μ is a mountain pass solution of problem (P_μ). Moreover, relation (34) implies that $\|u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \to 0$ as $\mu \to +\infty$.

The proof of Theorem 2.1 is now complete. □

5. **Proof of Theorem 2.2.** We first establish some auxiliary properties.

Lemma 5.1. *Assume that hypotheses* (H_3) *and* (H_5) *hold. Then there exist* $0 < \rho_0 < 1$ *and* $\mu_0 = \mu_0(\rho_0) > 0$, $\eta_0 > 0$, *such that* $\Upsilon_\mu(u) \geq \eta_0$ *for all* $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ *with* $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \rho_0$ *and for all* $\mu \leq \mu_0$.

Proof. For all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ with $\|u\|_{L^{p^*(x)}(\mathbb{R}^N)} \leq c_{29}\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \leq 1$ ($c_{29} > 1$ does not depend on u), we deduce from hypothesis (H_3), Theorem 1.1, Hölder's inequality (see Musielak [21]) and the Young inequality that there exists $c_{30} > 0$ such that for any $\varepsilon > 0$

$$\begin{aligned}
\Upsilon_\mu(u) &\geq \frac{1}{2^{p^+-1}p^+}\|u\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)} - c_{30}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \mu\int_{\mathbb{R}^N}\frac{\beta(x)}{\alpha(x)}|u|^{\alpha(x)}dx \\
&\geq \frac{1}{2^{p^+-1}p^+}\|u\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)} - c_{30}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \\
&\quad - 2\mu c_{29}^{\frac{\alpha^-p^{*-}}{p^{*+}}}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\|u\|^{\frac{\alpha^-p^{*-}}{p^{*+}}}_{W_V^{1,p(x)}(\mathbb{R}^N)} \\
&\geq \left(\frac{1}{2^{p^+-1}p^+} - \varepsilon\right)\|u\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)} - c_{30}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \\
&\quad - \varepsilon^{-\frac{\alpha^-p^{*-}}{p^+p^{*+}-\alpha^-p^{*-}}}\left(2\mu c_{29}^{\frac{\alpha^-p^{*-}}{p^{*+}}}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\right)^{\frac{p^+p^{*+}}{p^+p^{*+}-\alpha^-p^{*-}}}, \qquad (46)
\end{aligned}$$

since $\alpha^- < p^+$. Taking $\varepsilon = 2^{-p^+}p^{+^{-1}}$, relation (46) yields that

$$\begin{aligned}
\Upsilon_\mu(u) &\geq 2^{-p^+}p^{+^{-1}}\|u\|^{p^+}_{W_V^{1,p(x)}(\mathbb{R}^N)} - c_{30}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \\
&\quad - \left(2^{p^+}p^+\right)^{\frac{\alpha^-p^{*-}}{p^+p^{*+}-\alpha^-p^{*-}}}\left(2\mu c_{29}^{\frac{\alpha^-p^{*-}}{p^{*+}}}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\right)^{\frac{p^+p^{*+}}{p^+p^{*+}-\alpha^-p^{*-}}}.
\end{aligned}$$

Set

$$\ell(t) = 2^{-p^+}p^{+^{-1}}t^{p^+} - c_{30}t^{2r^-},\ \ 0 \leq t \leq \frac{1}{c_{29}}.$$

Since $2r^- > p^+$, we have

$$\ell(\rho_0) = \max_{0\leq t\leq\frac{1}{c_{29}}}\ell(t) > 0,\ \text{ with } \rho_0 = \min\left\{\frac{1}{c_{29}}, \left(\frac{1}{2^{p^++1}c_{30}r^-}\right)^{1/(2r^--p^+)}\right\}.$$

Denote

$$\mu_0 = \left(\frac{\ell(\rho_0)}{2}\right)^{\frac{p^+p^{*+}-\alpha^-p^{*-}}{p^+p^{*+}}}\left(2^{-p^+}p^{+^{-1}}\right)^{\frac{\alpha^-p^{*-}}{p^+p^{*+}}}\left(2c_{29}^{\frac{\alpha^-p^{*-}}{p^{*+}}}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\right)^{-1}.$$

Thus, for all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ with $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \rho_0$ and for all $\mu \leq \mu_0$, we have

$$\Upsilon_\mu(u) \geq \ell(\rho_0) - \left(2^{p^+}p^+\right)^{\frac{\alpha^-p^{*-}}{p^+p^{*+}-\alpha^-p^{*-}}}\left(2\mu c_{29}^{\frac{\alpha^-p^{*-}}{p^{*+}}}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\right)^{\frac{p^+p^{*+}}{p^+p^{*+}-\alpha^-p^{*-}}}$$

$$\geq \ell(\rho_0) - \left(2^{p^+}p^+\right)^{\frac{\alpha^- p^{*-}}{p^+p^{*+}-\alpha^-p^{*-}}} \left(2\mu_0 c_{29}^{\frac{\alpha^- p^{*-}}{p^{*+}}} \|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)}\right)^{\frac{p^+p^{*+}}{p^+p^{*+}-\alpha^-p^{*-}}}$$

$$= \frac{\ell(\rho_0)}{2} =: \eta_0 > 0,$$

being $\alpha^- < p^+$.

The proof is now complete. □

Lemma 5.2. *Assume that hypotheses* (H$_3$) − (H$_5$) *are fulfilled, then* $c_\mu < 0$ *for any* $\mu \in (0, \mu_0]$, *where* $c_\mu = \inf\left\{\Upsilon_\mu(u) : u \in \overline{B}_{\rho_0}\right\}$, $B_{\rho_0} = \left\{u \in W_V^{1,p(x)}(\mathbb{R}^N) : \|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} < \rho_0\right\}$ *and the numbers* ρ_0 *and* μ_0 *are given in Lemma 5.1.*

Proof. For a fixed $x_0 \in U$, let R be so small such that $B_{2R}(x_0) \subset U$, where U is given in (H$_4$). Then, we choose a function $\psi \in C_0^\infty(B_{2R}(x_0))$ such that $0 \leq \psi \leq 1$,

$$0 < \|\psi\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \leq \rho_0$$

and

$$\int_{B_{2R}(x_0)} \psi^{\kappa(x)} dx > 0.$$

For each fixed $\mu \in (0, \mu_0]$, by hypothesis (H$_4$), we obtain for all $0 < t < \min\{b, 1\}$

$$\Upsilon_\mu(t\psi) \leq \frac{2\rho_0^{p^-} t^{p^-}}{p^-} - \mu a \left(\int_{B_{2R}(x_0)} \psi^{\kappa(x)} dx\right) t^{\kappa^+}.$$

Since $\kappa^+ < p^-$, we can find a fixed $t_0 > 0$ even small such that

$$t_0 < \min\left\{b, 1, \left(\frac{\mu a p^-}{2\rho_0^{p^-}} \int_{B_{2R}(x_0)} \psi^{\kappa(x)} dx\right)^{1/(p^- - k^+)}\right\},$$

consequently, $t_0\psi \in B_{\rho_0}$ and $\Upsilon_\mu(t_0\psi) < 0$. This implies that $c_\mu < 0$ for all $\mu \in (0, \mu_0]$. The proof is now complete. □

By Lemmas 5.1 and 5.2 and the Ekeland variational principle (see Ekeland [8, Theorem 1]), applied in $\overline{B}_{\rho_0}$, there exists a sequence $\{u_n\} \subset B_{\rho_0}$ such that $c_\mu \leq \Upsilon_\mu(u_n) \leq c_\mu + \dfrac{1}{n}$ and

$$\Upsilon_\mu(w) \geq \Upsilon_\mu(u_n) + \frac{\|u_n - w\|_{W_V^{1,p(x)}(\mathbb{R}^N)}}{n}$$

for all $w \in \overline{B}_{\rho_0}$.

Then, similarly with the proof of Corollary I.5.3 in Struwe [28] (see also Willem [31, Corollary 2.5]), we can deduce that $\{u_n\}$ is a $(\mathrm{PS})_{c_\mu}$ sequence of the functional Υ_μ.

Lemma 5.3. *There exists* $\mu_* > 0$ *such that* $\{u_n\}$ *admits a strongly convergent subsequence in* $W_V^{1,p(x)}(\mathbb{R}^N)$ *for all* $0 < \mu \leq \mu_*$.

Proof. Thanks to $\{u_n\} \subset B_{\rho_0}$, similarly to the proof of relation (16), we may assume that there exist $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$ and δ_μ, $\varrho_\mu \geq 0$ such that, passing to a subsequence, still denoted by $\{u_n\}$,

$$\begin{cases} u_n \xrightarrow{w} u_\mu \text{ in } W_V^{1,p(x)}(\mathbb{R}^N), \quad \|u_n\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \to \delta_\mu, \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy \to \varrho_\mu, \\ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N, \quad |u_n|^{r(x)} \xrightarrow{w} |u_\mu|^{r(x)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N), \\ |u_n|^{r(x)-2}u_n \xrightarrow{w} |u_\mu|^{r(x)-2}u_\mu \text{ in } L^{\frac{p^*(x)}{r(x)-1}}(\mathbb{R}^N) \end{cases} \tag{47}$$

as $n \to \infty$.

Note that $\{u_n\} \subset B_{\rho_0}$. For any $\Omega \subset \mathbb{R}^N$, it follows from hypothesis (H_3), Hölder's inequality (see Musielak [21]) and Sobolev inequality that

$$\begin{aligned} &\int_\Omega |f(x,u_n)(u_n-u_\mu)|dx \\ &\leq 2\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\Omega)} \||u_n|^{\alpha(\cdot)-1}(u_n-u_\mu)\|_{L^{\frac{p^*(x)}{\alpha(x)}}(\mathbb{R}^N)} \\ &\leq 2\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\Omega)} \max\left\{ \left(\int_{\mathbb{R}^N} |u_n|^{\frac{(\alpha(x)-1)p^*(x)}{\alpha(x)}}|u_n-u_\mu|^{\frac{p^*(x)}{\alpha(x)}}dx\right)^{\frac{\alpha^+}{p^{*-}}}, \right. \\ &\left. \left(\int_{\mathbb{R}^N} |u_n|^{\frac{(\alpha(x)-1)p^*(x)}{\alpha(x)}}|u_n-u_\mu|^{\frac{p^*(x)}{\alpha(x)}}dx\right)^{\frac{\alpha^-}{p^{*+}}} \right\} \\ &\leq 2\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\Omega)} \max\left\{ \left(2\||u_n|^{\frac{(\alpha(\cdot)-1)p^*(\cdot)}{\alpha(\cdot)}}\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}} \||u_n-u_\mu|^{\frac{p^*(x)}{\alpha(x)}}\|_{L^{\alpha(x)}}\right)^{\frac{\alpha^+}{p^{*-}}}, \right. \\ &\left. \left(2\||u_n|^{\frac{(\alpha(\cdot)-1)p^*(\cdot)}{\alpha(\cdot)}}\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}} \||u_n-u_\mu|^{\frac{p^*(x)}{\alpha(x)}}\|_{L^{\alpha(x)}}\right)^{\frac{\alpha^-}{p^{*+}}} \right\} \\ &\leq c_{31}\|\beta\|_{L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\Omega)} \end{aligned}$$

for some constant $c_{31} > 0$. By $\beta \in L^{\frac{p^*(x)}{p^*(x)-\alpha(x)}}(\mathbb{R}^N)$ we see that the sequence $\{f(x,u_n)(u_n-u_\mu)\}$ is equi-integrable in $L^1(\mathbb{R}^N)$.

Additionally, $f(x,u_n)(u_n-u_\mu) \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. So, it follows from Vitali's convergence theorem (see, Bogachev [3, Corollary 4.5.5]) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u_\mu)dx = 0. \tag{48}$$

Similarly we can conclude that relations (22)$-$(23) also hold true in this section. Furthermore, we also can use the argument produced in the proof of Lemma 4.3 to show that relations (19)$-$(20) also hold true for this setting.

Let $\mathcal{L}$ be defined as in the proof of Lemma 4.3, then relation (24) continues to remain unchanged, and we also can deduce that there is a functional $\omega \in \left(W_V^{1,p(x)}(\mathbb{R}^N)\right)'$ such that relation (25) holds for all $v \in W_V^{1,p(x)}(\mathbb{R}^N)$. Consequently, on account of the fact that $\{u_n\}$ is a $(\mathrm{PS})_{c_\mu}$ sequence, we can derive that relations (26), (27) and (28) are fulfilled.

Similar to relation (48), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n)dx = \int_{\mathbb{R}^N} F(x,u_\mu)dx. \tag{49}$$

Since $\{u_n\}$ is a minimizing $(\mathrm{PS})_{c_\mu}$ sequence, using relations (24), (25), (26), (28), (49) and Lemma 3.1, for sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned}
c_\mu &= \int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)dx - \mu\int_{\mathbb{R}^N} F(x,u_n)dx\\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{r(x)}|u_n(y)|^{r(y)}}{2r(x)|x-y|^{\lambda(x,y)}r(y)}dxdy + o_n(1)\\
&\geq \frac{1}{p^+}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)dx - \mu\int_{\mathbb{R}^N} F(x,u_n)dx\\
&- \frac{1}{2r^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{r(x)}|u_n(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy + o_n(1).
\end{aligned}$$

It follows that

$$\begin{aligned}
c_\mu &\geq \frac{1}{p^+}\langle \mathcal{L}(u_n) - \mathcal{L}(u_\mu), u_n - u_\mu\rangle + \frac{1}{p^+}\langle \mathcal{L}(u_n), u_\mu\rangle - \mu\int_{\mathbb{R}^N} F(x,u_n)dx\\
&- \frac{1}{2r^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{r(x)}|u_n(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy + o_n(1)\\
&= \frac{1}{p^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\\
&- \frac{1}{2r^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n|^{r(x)}|u_n(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy + \frac{\mu}{p^+}\int_{\mathbb{R}^N} f(x,u_\mu)u_\mu dx\\
&+ \frac{1}{p^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\mu(x)|^{r(x)}|u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy - \mu\int_{\mathbb{R}^N} F(x,u_n)dx + o_n(1)\\
&\geq \left(\frac{1}{p^+} - \frac{1}{2r^-}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\\
&+ \frac{\mu}{p^+}\int_{\mathbb{R}^N} f(x,u_\mu)u_\mu dx - \mu\int_{\mathbb{R}^N} F(x,u_\mu)dx + o_n(1),
\end{aligned} \tag{50}$$

since $2r^- > p^+$.

On account of the fact that $\|u_n\|_{W_V^{1,p(x)}(\mathbb{R}^N)} < \rho_0$ (where ρ_0 is independent of μ), hence, $\|u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \leq \rho_0$, and there exists some positive constant c_{32} (which does not depend on μ) such that

$$\int_{\mathbb{R}^N} F(x,u_\mu)dx \leq c_{32} \quad \text{and} \quad \int_{\mathbb{R}^N} f(x,u_\mu)u_\mu dx \leq c_{32}.$$

So, this relation together with relation (50) yields that

$$\begin{aligned}
\left(\frac{1}{p^+} - \frac{1}{2r^-}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\\
&\leq c_\mu + 2\mu c_{32} + o_n(1).
\end{aligned}$$

Combining this relation and Lemma 5.2, we deduce that

$$\lim_{\mu\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy = \lim_{\mu\to 0}\varrho_\mu = 0. \tag{51}$$

Denote

$$\mu_* = \begin{cases} \inf\{\mu \in (0,\mu_0] : \varrho_\mu > 0\} & \text{if } \varrho_\mu \not\equiv 0, \\ \mu_0 & \text{if } \varrho_\mu \equiv 0, \end{cases}$$

where ϱ_μ is given in (47).

According to the division of $\mathbb{R}^N$ and relation (31), we also can divide the discussion into three cases.

Case $\Omega_2 = \mathbb{R}^N$. As in the proof of Lemma 4.3, i.e., similar to the proof of relation (33), we can get

$$\max\left\{ (\varrho_\mu)^{1-\frac{p^+}{2r^-}}, (\varrho_\mu)^{1-\frac{p^-}{2r^+}} \right\} \geq c_{19} \tag{52}$$

for $\varrho_\mu > 0$. If $\varrho_\mu \not\equiv 0$, relations (51) and (52) imply that $\mu_* = \inf\{\mu \in (0,\mu_0] : \varrho_\mu > 0\} > 0$. Otherwise, we can deduce that there exists a sequence $\{\mu_k\} \subset \mathbb{R}$ with $\varrho_{\mu_k} > 0$ such that $\mu_k \to 0$ as $k \to \infty$. By relations (51) and (52) we have

$$0 = \lim_{k\to\infty} \max\left\{ (\varrho_{\mu_k})^{1-\frac{p^+}{2r^-}}, (\varrho_{\mu_k})^{1-\frac{p^-}{2r^+}} \right\} \geq c_{19}.$$

This is a contradiction. Hence, $\varrho_\mu = 0$ for all $\mu \in (0,\mu_*]$, i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_\mu(x)|^{r(x)}|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy = 0 \tag{53}$$

for all $\mu \in (0,\mu_*]$. Therefore, by relations (28), (31) and (53) we get

$$\lim_{n\to\infty} \|u_n - u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = 0.$$

As for the cases $\Omega_1 = \mathbb{R}^N$ and $\Omega_1 \neq \emptyset \neq \Omega_2$, it follows as above and as in the proof of Lemma 4.3 that there exists $\mu_* > 0$ such that $\|u_n - u_\mu\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for all $\mu \in (0,\mu_*]$.

This proof is now complete. □

Proof of Theorem 2.2. Using Lemmas 5.1 and 5.2, we can deduce that there is a $(\mathrm{PS})_{c_\mu}$ sequence $\{u_n\}$ of the functional Υ_μ at the level $c_\mu < 0$ given in Lemma 5.2. Additionally, by Lemma 5.3, there exists $\mu_* > 0$ such that $u_n \to u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$ (up to a subsequence) as $n \to \infty$ for all $\mu \in (0,\mu_*]$. Furthermore, $\Upsilon_\mu(u_\mu) = c_\mu < 0$ and $\Upsilon'_\mu(u_\mu) = 0$, that is, problem (P_μ) has a nontrivial solution $u_\mu \in W_V^{1,p(x)}(\mathbb{R}^N)$.

This proof of Theorem 2.2 is now complete. □

6. **Proof of Theorem 2.3.**

In this section, we use $J_\mu : W_{rad}^{1,p(x)}(\mathbb{R}^N) \mapsto \mathbb{R}$ to denote the energy functional related to problem (Q_μ) defined by

$$\begin{aligned} J_\mu(u) = &\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \mu\int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} \frac{K(x)}{p^*(x)}|u|^{p^*(x)}dx \\ &- \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)|u(x)|^{r(x)}g(y)|u(y)|^{r(y)}}{r(x)|x-y|^{\lambda(x,y)}r(y)} dxdy. \end{aligned}$$

By hypothesis (H_1), we can demonstrate as Lemma 3.2 in Alves and Tavares [2] to infer that

$$J_\mu \in C^1\left(W_{rad}^{1,p(x)}(\mathbb{R}^N), \mathbb{R}\right),$$

with

$$\langle J'_\mu(u), v\rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right) dx - \mu \int_{\mathbb{R}^N} f(x,u)v dx$$

$$- \int_{\mathbb{R}^N} K(x)|u|^{p^*(x)-2}uv dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(y)|u(y)|^{r(y)}g(x)|u(x)|^{r(x)-2}u(x)v(x)}{r(y)|x-y|^{\lambda(x,y)}} dxdy$$

for all u, $v \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$.

Our first result establishes the mountain pass geometry.

Lemma 6.1. *The functional J_μ satisfies the following properties.*

(i) There exists $\rho_1 > 0$ small enough such that $J_\mu(u) \geq \eta_1$ for all $u \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \rho_1$ for some $\eta_1 > 0$.

(ii) There exists $e \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ such that $\|e\|_{W^{1,p(x)}(\mathbb{R}^N)} > \rho_1$ and $J_\mu(e) < 0$.

Proof. The proof of this lemma is similar to that of Lemma 4.1. So, we omit it here. $\square$

For some fixed $\mu > 0$, let us define

$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\mu(\gamma(t)), \tag{54}$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], W^{1,p(x)}_{rad}(\mathbb{R}^N)\right) : \gamma(0) = 0,\ \gamma(1) = e\right\}.$$

By Lemma 6.1, we obtain that $c_\mu > 0$. Furthermore, we have the following asymptotic behavior of these levels.

Lemma 6.2. *Assume that* (H$_1$) *and* (H$_6$) *are fulfilled. Then we have*

$$\lim_{\mu\to+\infty} c_\mu = 0,$$

where c_μ is given in (54).

Proof. The proof of this property is similar to that of Lemma 4.2. Therefore, we omit it here. $\square$

Lemma 6.3. *There exists $\mu^{**} > 0$ such that J_μ satisfies the* $(\mathrm{PS})_{c_\mu}$ *condition on $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ for all $\mu \geq \mu^{**}$.*

Proof. Let $\{u_n\} \subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$ be a $(\mathrm{PS})_{c_\mu}$ sequence of the functional J_μ, that is, $J_\mu(u_n) \to c_\mu$ and $J'_\mu(u_n) \to 0$ as $n \to \infty$. In a similar fashion to relation (13), by (H$_6$) we are able to conclude that the sequence $\{u_n\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. So, we may assume that there is an element $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ and δ_μ, $\varrho_\mu \geq 0$ such that, up to a subsequence, still denoted by $\{u_n\}$,

$$\begin{cases} u_n \xrightarrow{w} u_\mu \text{ in } W^{1,p(x)}_{rad}(\mathbb{R}^N), \quad \|u_n\|_{W^{1,p(x)}(\mathbb{R}^N)} \to \delta_\mu, \\ |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \xrightarrow{w^*} \upsilon \text{ in } \mathcal{M}(\mathbb{R}^N), \quad |u_n|^{p^*(x)} \xrightarrow{w^*} \nu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)|u_n(x)-u_\mu(x)|^{r(x)}g(y)|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \to \varrho_\mu, \\ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N, \quad |u_n|^{r(x)} \xrightarrow{w} |u_\mu|^{r(x)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N), \\ |u_n|^{r(x)-2}u_n \xrightarrow{w} |u_\mu|^{r(x)-2}u_\mu \text{ in } L^{\frac{p^*(x)}{r(x)-1}}(\mathbb{R}^N), \end{cases} \tag{55}$$

as $n \to \infty$.

It follows from the concentration-compactness principle for variable exponents (see Fu and Zhang [11, Theorem 2.2] that

$$\upsilon = |\nabla u_\mu|^{p(x)} + |u_\mu|^{p(x)} + \sum_{j\in J} \upsilon_j \delta_{x_j} + \widetilde{\upsilon} \quad \text{and} \quad \nu = |u_\mu|^{p^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j},$$

where J is a countable set, $\{\upsilon_j\}$, $\{\nu_j\} \subset [0, +\infty)$, $\{x_j\} \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\widetilde{\upsilon} \in \mathcal{M}(\mathbb{R}^N)$ is a non-atomic non-negative measure. Applying the concentration-compactness principle for variable exponents, we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty = \int_{\mathbb{R}^N} |u_\mu|^{p^*(x)} dx + \sum_{j\in J} \nu_j + \nu_\infty.$$

Next, we show that

$$\int_{\mathbb{R}^N} |u_n - u_\mu|^{p^*(x)} dx = 0. \tag{56}$$

According to the above discussion, we divide the proof into two parts, that is, $\nu_j = 0$ and $\nu_\infty = 0$.

(i) We first show that $\nu_j = 0$. For any $\varepsilon > 0$, there exists a radially symmetric function $\varphi \in C_0^\infty(B_{2\varepsilon}(0))$ such that $0 \le \varphi \le 1$, $|\nabla\varphi| \le 2/\varepsilon$; $\varphi = 1$ on $B_\varepsilon(0)$. Since $\{u_n\varphi\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, we obtain $\langle J'_\mu(u_n), u_n\varphi\rangle \to 0$ as $n \to \infty$. By straightforward computation we obtain

$$\begin{aligned}
\langle J'_\mu(u_n), u_n\varphi\rangle = &\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) \varphi dx - \mu \int_{\mathbb{R}^N} f(x, u_n) u_n \varphi \, dx \\
&- \int_{\mathbb{R}^N} K(x) |u_n|^{p^*(x)} \varphi dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\varphi u_n dx \\
&- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(y)|u_n(y)|^{r(y)} g(x) |u_n(x)|^{r(x)} \varphi(x)}{r(y)|x-y|^{\lambda(x,y)}} dx dy.
\end{aligned}$$

Similar to the proof of relation (17), we can get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n) u_n \varphi dx = \int_{\mathbb{R}^N} f(x, u_\mu) u_\mu \varphi dx.$$

So, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \Bigg(\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla\varphi u_n dx \\
&\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(y)|u_n(y)|^{r(y)} g(x) |u_n(x)|^{r(x)} \varphi(x)}{r(y)|x-y|^{\lambda(x,y)}} dx dy \Bigg) \\
= &\int_{\mathbb{R}^N} -\varphi d\upsilon + \mu \int_{\mathbb{R}^N} f(x, u_\mu) u_\mu \varphi dx + \int_{\mathbb{R}^N} K(x) \varphi d\nu.
\end{aligned} \tag{57}$$

Clearly, there exists some constant $c_{33} > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\varphi u_\mu|^{p(x)} dx \\
= &\int_{B_{2\varepsilon}(0)} |\nabla\varphi u_\mu|^{p(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq c_{33} \max\left\{ \left(\frac{4^N w_N}{N}\right)^{\frac{p^+}{N}}, \left(\frac{4^N w_N}{N}\right)^{\frac{p^-}{N}} \right\} \| |u_\mu|^{p(x)} \|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(0))} \\
&= o_\varepsilon(1), \text{ as } \varepsilon \to 0,
\end{aligned} \tag{58}
$$

where w_N is the surface area of the unit sphere in $\mathbb{R}^N$. On account of the fact that $u_n \to u_\mu$ in $L^{p(x)}(B_{2\varepsilon}(0))$ as $n \to \infty$, we can derive that $\|\nabla\varphi u_n\|_{L^{p(x)}(\mathbb{R}^N)} \to \|\nabla\varphi u_\mu\|_{L^{p(x)}(\mathbb{R}^N)}$ in $\mathbb{R}$ as $n \to \infty$. Hence, relation (58), Hölder's inequality (see Musielak [21]), the Sobolev inequality and the boundedness of $\{u_n\}$ yield that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi u_n dx\right| \\
\leq\ & \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-1}|\nabla\varphi u_n| dx \\
\leq\ & 2\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \| |\nabla u_n|^{p(x)-1} \|_{L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)} \|\nabla\varphi u_n\|_{L^{p(x)}(\mathbb{R}^N)} \\
\leq\ & c_{34}\lim_{\varepsilon\to 0} \|\nabla\varphi u_\mu\|_{L^{p(x)}(\mathbb{R}^N)} \\
=\ & 0
\end{aligned} \tag{59}
$$

for some constant $c_{34} > 0$.

Note that $\lim_{|x|\to 0} g(x) = 0$, for any $\eta > 0$, there exists $\delta = \delta(\eta) > 0$ such that $2\varepsilon < \delta$, we get

$$
\int_{B_{2\varepsilon(0)}} g(x)^{\frac{2N}{2N-\lambda^+}} |u_n|^{p^*(x)} dx \leq \eta^{\frac{2N}{2N-\lambda^+}} \int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx \leq c_{35}\eta^{\frac{2N}{2N-\lambda^+}}
$$

for some constant $c_{35} > 0$. Thus we have

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{B_{2\varepsilon(0)}} g(x)^{\frac{2N}{2N-\lambda^+}} |u_n|^{p^*(x)} dx = 0. \tag{60}
$$

Similarly we have

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{B_{2\varepsilon(0)}} g(x)^{\frac{2N}{2N-\lambda^-}} |u_n|^{\frac{2Nr(x)}{2N-\lambda^-}} dx = 0. \tag{61}
$$

Relations (60)−(61) together with Theorem 1.1 imply that

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(y)|u_n(y)|^{r(y)} g(x)|u_n(x)|^{r(x)}\varphi(x)}{r(y)|x-y|^{\lambda(x,y)}} dxdy = 0. \tag{62}
$$

Using (H_1) we can easily see that

$$
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} f(x,u_\mu)u_\mu\varphi dx = 0. \tag{63}
$$

Note that $K(0) = 0$, then it follows from relations (57), (59), (62) and (63) that $\upsilon(\{0\}) = 0$. This implies that 0 is not an atom of υ.

We now prove that $\nu_j = 0$ for any $j \in J$. We deduce from the above discussion that there exists $x_{j_0} \neq 0$ ($j_0 \in J$) such that $\nu_{j_0} = \nu_{j_0}(\{x_{j_0}\}) > 0$. Due to $\{u_n\} \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$, the measure ν is $O(N)$-invariant, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$. For any $g \in O(N)$, $\nu_{j_0}(\{gx_{j_0}\}) = \nu_{j_0}(\{x_{j_0}\}) > 0$. Additionally, we see that

$$
|O(N)| = \inf_{x\in\mathbb{R}^N,\, x\neq 0} |O(N)_x| = +\infty,
$$

where $|O(N)_x|$ denotes the cardinality of $\{gx : g \in O(N)\}$. Therefore, $\nu_{j_0}(\{gx_{j_0} : g \in O(N)\}) = +\infty$. But the measure ν is finite, hence we get a contradiction. So, for any $j \in J$ we deduce that $\nu_j = 0$.

(ii) Finally we prove that $\nu_\infty = 0$. For any $R > 0$, we choose a radially symmetric function $\xi_R \in C^\infty(\mathbb{R}^N)$ such that $0 \le \xi_R \le 1$, $|\nabla \xi_R| < 2/R$; $\xi_R = 1$ in $\mathbb{R}^N \setminus B_{2R}(0)$, $\xi_R = 0$ in $B_R(0)$. Clearly, $\{u_n \xi_R\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. Consequently, it follows that $\langle J'_\mu(u_n), u_n\xi_R\rangle \to 0$, as $n \to \infty$. Thus, we have

$$\begin{aligned}\langle J'_\mu(u_n), u_n\xi_R\rangle = & \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)\xi_R dx - \mu \int_{\mathbb{R}^N} f(x, u_n)u_n\xi_R \, dx \\ & - \int_{\mathbb{R}^N} K(x)|u_n|^{p^*(x)}\xi_R dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \xi_R u_n dx \\ & - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(y)|u_n(y)|^{r(y)}g(x)|u_n(x)|^{r(x)}\xi_R(x)}{r(y)|x-y|^{\lambda(x,y)}} dxdy. \end{aligned} \tag{64}$$

Similar to the proofs of relations (17) and (63), we deduce that

$$\lim_{R\to+\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n\xi_R dx = \lim_{R\to+\infty}\int_{\mathbb{R}^N} f(x,u_\mu)u_\mu\xi_R dx = 0. \tag{65}$$

By the definition of ξ_R, we can deduce that

$$\lim_{R\to+\infty}\int_{\mathbb{R}^N} |\nabla \xi_R u_\mu|^{p(x)} dx = 0,$$

since $1 < p^- \le p(x) \le p^+ < N$. Due to $u_n \to u_\mu$ strongly in $L^{p(x)}(B_{2R}(0)\setminus B_R(0))$, we observe that

$$\lim_{n\to\infty}\|\nabla \xi_R u_n\|_{L^{p(x)}(\mathbb{R}^N)} = \|\nabla \xi_R u_\mu\|_{L^{p(x)}(\mathbb{R}^N)}.$$

In a similar fashion to relation (58) we have

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\xi_R u_n dx\right| = 0. \tag{66}$$

Note that $\lim_{|x|\to+\infty} g(x) = 0$. Similar to the proof of relation (62), we also have

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)|u_n(y)|^{r(y)}g(x)|u_n(x)|^{r(x)}\xi_R(x)}{r(y)|x-y|^{\lambda(x,y)}}dxdy = 0. \tag{67}$$

Also, we observe that

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} K(x)|u_n|^{p^*(x)}\xi_R dx = 0, \tag{68}$$

being $\lim_{|x|\to+\infty} K(x) = 0$.

So, by relations (64)$-$(68) we derive that

$$\upsilon_\infty = \lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)\xi_R dx \le 0,$$

that is, $\upsilon_\infty = 0$. Then, we can easily deduce that

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla(u_n\xi_R)|^{p(x)} + |u_n\xi_R|^{p(x)}\right)dx = 0.$$

It follows that

$$\nu_\infty = \lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\xi_R u_n|^{p^*(x)}dx = 0.$$

Therefore, we deduce from (i) and (ii) that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx=\int_{\mathbb{R}^N}|u_\mu|^{p^*(x)}dx.$$

So, combining this relation and Corollary 1, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u_\mu|^{p^*(x)}dx=0. \tag{69}$$

Additionally, we also can use the argument produced in the proof of Lemma 4.3 to show that relations (17)–(23) also hold true for this setting.

Let $\mathcal{L}$ be defined in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ as in the proof of Lemma 4.3, then relation (24) continues to remain unchanged in this section, and we also can deduce that there is a functional $\omega\in\left(W^{1,p(x)}_{rad}(\mathbb{R}^N)\right)'$ such that relation (25) holds for all $v\in W^{1,p(x)}_{rad}(\mathbb{R}^N)$. Note that $\langle J'_\mu(u_n),u_\mu\rangle\to 0$ as $n\to\infty$, g, $K\geq 0$ and g, $K\in L^\infty(\mathbb{R}^N)$, together with relations (19), (23), (25), (55) and (69), we have

$$\begin{aligned}\langle\omega,u_\mu\rangle&=\mu\int_{\mathbb{R}^N}f(x,u_\mu)u_\mu dx+\int_{\mathbb{R}^N}K(x)|u_\mu|^{p^*(x)}dx\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)|u_\mu(y)|^{r(y)}g(x)|u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy.\end{aligned} \tag{70}$$

Using relation (70) and recalling that hypothesis (H_6), we know that $\langle\omega,u_\mu\rangle\geq 0$.

Consequently, on account of the fact that $\{u_n\}$ is a $(PS)_{c_\mu}$ sequence and g, $K\geq 0$ and g, $K\in L^\infty(\mathbb{R}^N)$, together with relations (19)–(25) and (69), we can conclude that

$$\begin{aligned}&\lim_{n\to\infty}\langle\mathcal{L}(u_n)-\mathcal{L}(u_\mu),u_n-u_\mu\rangle\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)|u_n(y)-u_\mu(y)|^{r(y)}g(x)|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\\&+\lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)|u_n-u_\mu|^{p^*(x)}dx\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)|u_n(y)-u_\mu(y)|^{r(y)}g(x)|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\end{aligned} \tag{71}$$

Similar to the proofs of relations (29)–(30), we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)|u_n(y)-u_\mu(y)|^{r(y)}g(x)|u_n(x)-u_\mu(x)|^{r(x)}}{r(y)|x-y|^{\lambda(x,y)}}dxdy\\&\leq c_{36}\max\left\{\|u_n-u_\mu\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)},\ \|u_n-u_\mu\|^{2r^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\}\end{aligned} \tag{72}$$

and

$$\begin{aligned}&c_{37}\min\left\{\|u_n-u_\mu\|^{p^+}_{W^{1,p(x)}(\mathbb{R}^N)},\ \|u_n-u_\mu\|^{p^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\}\\&\leq\int_{\mathbb{R}^N}\left(|\nabla u_n-\nabla u_\mu|^{p(x)}+|u_n-u_\mu|^{p(x)}\right)dx\\&\leq c_{38}\max\left\{\|u_n-u_\mu\|^{p^+}_{W^{1,p(x)}(\mathbb{R}^N)},\ \|u_n-u_\mu\|^{p^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\}\end{aligned} \tag{73}$$

for some constants c_{36}, c_{37}, $c_{38}>0$.

Denote

$$\mu^{**}=\sup\{\mu>0:\varrho_\mu>0\}, \tag{74}$$

where ϱ_μ is given in (55).

According to the division of $\mathbb{R}^N$ and relation (31), we also can divide the discussion into three cases. The rest of the proof is similar to the proof of Lemma 4.3, that is, using relations (70)−(73), we can use the argument used in the proof of Lemma 4.3 to show that $\mu^{**} < +\infty$. Thus, for all $\mu \in [\mu^{**}, +\infty)$ we can easily get $\|u_n - u_\mu\|_{W^{1,p(x)}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Moreover, we also have

$$\lim_{\mu\to+\infty} \|u_\mu\|_{W^{1,p(x)}(\mathbb{R}^N)} = 0. \tag{75}$$

In conclusion, there exists some constant $\mu^{**} > 0$ such that J_μ satisfies the $(\mathrm{PS})_{c_\mu}$ condition on $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ for all $\mu \geq \mu^{**}$. The proof is now complete. □

Proof of Theorem 2.3. Using Lemmas 6.1 and 6.3, we know that there exists a constant $\mu^{**} > 0$ such that for all $\mu \geq \mu^*$ the functional J_μ has a nontrivial critical point $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$. That is, the critical point u_μ is a mountain pass solution of problem (Q_μ). Moreover, relation (75) implies that $\|u_\mu\|_{W^{1,p(x)}_V(\mathbb{R}^N)} \to 0$ as $\mu \to +\infty$. The proof of Theorem 2.3 is now complete. □

7. **Proof of Theorem 2.4.** We first establish some auxiliary results.

Lemma 7.1. *Assume that hypotheses* (H_3), (H_8) *and* (H_9) *hold. Then there exist* $0 < \rho_2 < 1$ *and* $\mu_1 = \mu_1(\rho_2) > 0$, $\eta_2 > 0$, *such that* $J_\mu(u) \geq \eta_2$ *for all* $u \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ *with* $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \rho_2$ *and for all* $\mu \leq \mu_1$.

Proof. This follows with similar arguments as in the proof of Lemma 5.1. □

Lemma 7.2. *Assume that hypotheses* (H_3) − (H_4) *and* (H_8) − (H_9) *are fulfilled, then* $c_\mu < 0$ *for any* $\mu \in (0, \mu_1]$, *where* $c_\mu = \inf\left\{J_\mu(u) : u \in \overline{B}_{\rho_2}\right\}$,

$$B_{\rho_2} = \left\{u \in W^{1,p(x)}_{rad}(\mathbb{R}^N) : \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} < \rho_2\right\}$$

and the numbers μ_1 *and* ρ_2 *are given in Lemma 7.1.*

Proof. This follows with similar arguments as in the proof of Lemma 5.2. □

By Lemmas 7.1 and 7.2 and the Ekeland variational principle (see Ekeland [8, Theorem 1]), applied in $\overline{B}_{\rho_2}$, there exists a sequence $\{u_n\} \subset B_{\rho_2}$ such that $c_\mu \leq J_\mu(u_n) \leq c_\mu + \dfrac{1}{n}$ and

$$J_\mu(w) \geq J_\mu(u_n) + \frac{\|u_n - w\|_{W^{1,p(x)}(\mathbb{R}^N)}}{n}$$

for all $w \in \overline{B}_{\rho_2}$. Then, as in the proof of Corollary I.5.3 of Struwe [28] (see also [31, Corollary 2.5]), we deduce that $\{u_n\}$ is a $(\mathrm{PS})_{c_\mu}$ sequence of the functional J_μ.

Lemma 7.3. *There exists* $\mu_{**} > 0$ *such that* $\{u_n\}$ *possesses a strongly convergent subsequence in* $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ *for all* $0 < \mu \leq \mu_{**}$.

Proof. Since the argument is similar to the proofs Lemmas 5.3 and 6.3, we only give outline of the proof of Lemma 7.3. Thanks to $\{u_n\} \subset B_{\rho_2}$, similar to the proof of relation (55), we may assume that there exists an element $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ and $\delta_\mu,\ \varrho_\mu \geq 0$ such that, passing to a subsequence, still denoted by $\{u_n\}$,

$$\begin{cases} u_n \xrightarrow{w} u_\mu \text{ in } W^{1,p(x)}_{rad}(\mathbb{R}^N), \quad \|u_n\|_{W^{1,p(x)}(\mathbb{R}^N)} \to \delta_\mu, \\ |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \xrightarrow{w^*} \upsilon \text{ in } \mathcal{M}(\mathbb{R}^N), \quad |u_n|^{p^*(x)} \xrightarrow{w^*} \nu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)|u_n(x)-u_\mu(x)|^{r(x)} g(y)|u_n(y)-u_\mu(y)|^{r(y)}}{r(y)|x-y|^{\lambda(x,y)}} dxdy \to \varrho_\mu, \\ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N, \quad |u_n|^{r(x)} \xrightarrow{w} |u_\mu|^{r(x)} \text{ in } L^{\frac{2N}{2N-\lambda^+}}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-\lambda^-}}(\mathbb{R}^N), \\ |u_n|^{r(x)-2}u_n \xrightarrow{w} |u_\mu|^{r(x)-2}u_\mu \text{ in } L^{\frac{p^*(x)}{r(x)-1}}(\mathbb{R}^N), \end{cases} \tag{76}$$

as $n \to \infty$.

$$\mu_{**} = \begin{cases} \inf\{\mu \in (0,\mu_1] : \varrho_\mu > 0\} & \text{if } \varrho_\mu \not\equiv 0, \\ \mu_1 & \text{if } \varrho_\mu \equiv 0, \end{cases}$$

where ϱ_μ is given in (76).

As in the proof of relation (69), we can conclude that relation (69) also holds true for this setting. Then, using the argument produced in the proof of Lemma 5.3, we can show that $\mu_{**} > 0$. Eventually, for all $\mu \in (0, \mu_{**}]$ we get $\|u_n - u_\mu\|_{W^{1,p(x)}(\mathbb{R}^N)} \to 0$ as $n \to \infty$.

In conclusion, there exists some constant $\mu_{**} > 0$ such that J_μ satisfies the $(\mathrm{PS})_{c_\mu}$ condition on $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ for all $0 < \mu \leq \mu_{**}$. The proof is now complete. □

Proof of Theorem 2.4. By Lemmas 7.1 and 7.2, we deduce that there is a $(\mathrm{PS})_{c_\mu}$ sequence $\{u_n\}$ of the functional J_μ at the level $c_\mu < 0$ given in Lemma 7.1. Additionally, by Lemma 7.3, there exists $\mu_{**} > 0$ such that $u_n \to u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ (passing to a subsequence) as $n \to \infty$ for all $\mu \in (0, \mu_{**}]$. Furthermore, $J_\mu(u_\mu) = c_\mu < 0$ and $J'_\mu(u_\mu) = 0$, that is, problem (Q_μ) has a nontrivial solution $u_\mu \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$. The proof of Theorem 2.4 is now complete. □

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