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NORMALIZED SOLUTIONS FOR NONLINEAR COUPLED FRACTIONAL SYSTEMS: LOW AND HIGH PERTURBATIONS IN THE ATTRACTIVE CASE

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Abstract. In this paper, we study the following coupled nonlocal system

$$\begin{cases}(-\Delta)^s u-\lambda_1 u=\mu_1|u|^\alpha u+\beta|u|^{\frac{\alpha-2}{2}}u|v|^{\frac{\alpha+2}{2}} & \text{in } \mathbb{R}^N,\\ (-\Delta)^s v-\lambda_2 v=\mu_2|v|^\alpha v+\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha-2}{2}}v & \text{in } \mathbb{R}^N,\end{cases}$$

satisfying the additional conditions

$$\int_{\mathbb{R}^N} u^2dx=b_1^2 \text{ and } \int_{\mathbb{R}^N} v^2dx=b_2^2,$$

where $(-\Delta)^s$ is the fractional Laplacian, $0<s<1$, μ_1, $\mu_2>0$, $N>2s$, and $\frac{4s}{N}<\alpha\le\frac{2s}{N-2s}$. We are concerned with the attractive case, which corresponds to $\beta>0$. In the case of low perturbations of the coupling parameter, by using two-dimensional linking arguments, we show that there exists $\beta_1>0$ such that when $0<\beta<\beta_1$, then the system has a positive radial solution. Next, in the case of high perturbations of the coupling parameter, we prove that there exists $\beta_2>0$ such that the system has a mountain-pass type solution for all $\beta>\beta_2$. These results correspond to low and high perturbations with respect to the values of the coupling parameter β. This paper extends and complements the main results established in [2] for the particular case $N=3$, $s=1$, $\alpha=2$.

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Key words and phrases. fractional Laplacian; normalized solution; ground state solution.

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1. **Introduction and main results.** In recent years, the normalized solutions for various classes of Schrödinger equations or systems have been widely investigated and there are many results, both for their particular interest from a physical point of view and for their relevance in models arising in nonlinear optics and Bose-Einstein condensation.

Consider the following system of coupled cubic Schrödinger equations:

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

satisfying the additional condition

$$\int_{\mathbb{R}^3} u^2 dx = a_1^2 \text{ and } \int_{\mathbb{R}^3} v^2 dx = a_2^2. \tag{2}$$

This problem was studied by Bartsch, Jeanjean and Soave in [2, 3, 4]. In [2], they considered the attractive case $\beta > 0$ and proved that for arbitrary masses a_i and parameter μ_i, there exists $\beta_2 > \beta_1 > 0$ such that for both $0 < \beta < \beta_1$ and $\beta > \beta_2$ system (1)–(2) has a positive radial solution. In the case $0 < \beta < \beta_1$ the solution is obtained based on a two-dimensional linking, while for the case $\beta > \beta_2$ this solution is of mountain pass type. For the repulsive case $\beta < 0$, by introducing a natural constraint, Bartsch and Soave [3] proved the existence of positive radial symmetric solutions of system (1)–(2). In [4], they considered the symmetric problem of system (1)–(2) with $\mu_1 = \mu_2$ and $a_1 = a_2$ and proved the existence of infinitely many solutions.

Since λ_1 and λ_2 are parts of the unknown, the Nehari manifold method is not available in the framework of normalized solutions. At the same time, the classical method used to prove the boundedness of any Palais-Smale sequence for the unconstrained problem does not work. Thus, the main difficulty in dealing with the normalized solutions is that the existence of bounded Palais-Smale sequence requires new arguments. However, if we find a bounded Palais-Smale sequence, according to the compactness of the embedding $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)$, we just get a strongly convergent subsequence in $L^{\alpha+2}(\mathbb{R}^N)$, but we cannot deduce the strong convergence in $L^2(\mathbb{R}^N)$. Hence we require new arguments to overcome the lack of compactness of the embedding $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$.

Compared to the semilinear case that corresponds to the Laplace operator, the fractional Laplacian problem is nonlocal and more challenging. This type of fractional Schrödinger equations or systems is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. Recently, a great attention has been focused on the study of equations or systems driven by the fractional Laplacian and with nonlinear reaction, both for their interesting theoretical structure and their concrete applications; see [1, 7, 15] and the references therein. This integro-differential operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others, see[18, 22] and references therein.

For fractional Laplacian equations or systems with fixed λ_i, the existence and non-degeneracy of solutions has been studied by many researchers; see, e.g., [1, 8, 11, 12, 26, 27, 28]. However, very few papers deal with the normalized solutions for fractional Laplacian systems or equations. To the best of our knowledge, this paper is the first to consider the existence of normalized solutions for fractional Laplacian systems.

The present paper is concerned with the existence of normalized solutions to the following class of critical systems driven by the fractional Laplace operator:

$$\begin{cases}(-\Delta)^s u - \lambda_1 u = \mu_1 |u|^\alpha u + \beta |u|^{\frac{\alpha-2}{2}} u |v|^{\frac{\alpha+2}{2}} & \text{in } \mathbb{R}^N,\\ (-\Delta)^s v - \lambda_2 v = \mu_2 |v|^\alpha v + \beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha-2}{2}} v & \text{in } \mathbb{R}^N,\end{cases} \tag{3}$$

satisfying the additional condition

$$\int_{\mathbb{R}^N} u^2 = b_1^2 \text{ and } \int_{\mathbb{R}^N} v^2 = b_2^2, \tag{4}$$

where $(-\Delta)^s$ is the fractional Laplacian, $0 < s < 1,\ \mu_1,\ \mu_2 > 0,\ 2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, $N > 2s,\ 2 < \alpha + 2 < 2_s^*$.

One refers to this type of solutions as to normalized solutions, since conditions (4) impose a normalization on the L^2-masses of u and v. This fact implies that λ_1 and λ_2 cannot be determined *a priori*, but are part of the unknown.

Our purpose is to establish the existence of normalized solutions of problem (3)–(4) under suitable conditions on the coupling parameter β.

The fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N$$

with a suitable positive normalizing constant $C(N,s)$.

From the mathematical point of view, problem (3)–(4) is nonlocal since the appearance of the operator $(-\Delta)^s$ indicates that equations describing (3) are not pointwise identities. This kind of problem has been paid much attention after the pioneering work of Lions [17], in which an abstract functional analysis framework was introduced. Nowadays, since physicists are interested in normalized solutions, mathematical researchers began to focus on solutions having a prescribed L^2-norm, that is, solutions which satisfy $\|u\|_2^2 = c$ for a priori given c. To the best of our knowledge, the study of solutions with prescribed norm was initiated by Jeanjean [16] in the framework of semilinear elliptic equations. We also refer to Bartsch, Zhong and Zou [5], Bellazzini, Jeanjean and Luo [6] and Cingolani and Jeanjean [10] (for normalized solutions of the Schrödinger-Poisson system), and Chen, Rădulescu and Tang [9] (for normalized solutions of nonautonomous Kirchhoff problems).

Let $D_s(\mathbb{R}^N)$ be Hilbert space obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ equipped with the norm

$$\|u\|^2_{D_s(\mathbb{R}^N)} = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|y-x|^{N+2s}} dxdy.$$

The energy functional associated with problem (3)–(4) is given by

$$J(u,v) = \frac{1}{2} ||(u,v)||^2_{\mathcal{D}} - \frac{1}{\alpha+2} \int_{\mathbb{R}^N} (\mu_1 |v|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}}) dx,$$

on the constraint $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, where for $b \in \mathbb{R}$ we define

$$\mathrm{T}_b := \left\{ u \in D_s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = b^2 \right\}.$$

We set $\mathcal{D} := D_s(\mathbb{R}^N) \times D_s(\mathbb{R}^N)$, which is endowed with the norm $||(u,v)||^2_{\mathcal{D}} = \|u\|^2_{D_s(\mathbb{R}^N)} + \|v\|^2_{D_s(\mathbb{R}^N)}$.

Our first result is concerned with the case of low perturbations of the coupling parameter.

Theorem 1.1. *Let $b_1, b_2, \mu_1, \mu_2 > 0$ be fixed and define $\beta_1 > 0$ by*

$$\max\left\{b_1^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_1^{-\frac{4s}{N\alpha-4s}}, b_2^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_2^{-\frac{4s}{N\alpha-4s}}\right\} \tag{5}$$

$$= b_1^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}(\mu_1+\beta_1)^{-\frac{4s}{N\alpha-4s}} + b_2^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}(\mu_2+\beta_1)^{-\frac{4s}{N\alpha-4s}}.$$

If $0 < \beta < \beta_1$, then system (3)–(4) has a solution $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{u}, \widetilde{v})$ such that $\widetilde{\lambda}_1, \widetilde{\lambda}_2 < 0$ and $\widetilde{u}$ and $\widetilde{v}$ are both positive and radial.

In order to state our next main result, we define

$$V := \{(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : G(u, , v) = 0\}, \tag{6}$$

where

$$G(u, v) = \|(u, v)\|^2_{\mathcal{D}} - \frac{N\alpha}{2s(\alpha+2)} \int_{\mathbb{R}^N} \left(\mu_1|u|^{\alpha+2} + \mu_2|v|^{\alpha+2} + 2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}\right) dx.$$

By Pohozaev's identity, we know that V contains all solutions of system (3)–(4). To obtain ground state solutions of system (3)–(4), we define

$$\mathcal{R}_b(u, v) \tag{7}$$

$$= \frac{\alpha N - 4s}{2N\alpha}\left(\frac{2s(\alpha+2)}{\alpha N}\right)^{\frac{4s}{\alpha N-4s}} \frac{\left(||u||^2_{D_s(\mathbb{R}^N)} + ||v||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{\alpha N}{\alpha N-4s}}}{\left(\int_{\mathbb{R}^N}\left(\mu_1 u^{\alpha+2} + \mu_2 v^{\alpha+2} + 2\beta u^{\frac{\alpha+2}{2}} v^{\frac{\alpha+2}{2}}\right) dx\right)^{\frac{4s}{\alpha N-4s}}}. \tag{8}$$

Our second main result of this paper deals with high perturbations of the coupling parameter.

Theorem 1.2. *Let $b_1, b_2, \mu_1, \mu_2 > 0$ be fixed and $\beta_2 > 0$ be fixed by*

$$\frac{(b_1^2 + b_2^2)^{\frac{N\alpha}{N\alpha-4s}}}{\left(\mu_1 b_1^{\alpha+2} + \mu_2 b_2^{\alpha+2} + 2\beta_2 b_1^{\frac{\alpha+2}{2}} b_2^{\frac{\alpha+2}{2}}\right)^{\frac{4s}{N\alpha-4s}}} \tag{9}$$

$$= \min\left\{b_1^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_1^{-\frac{4s}{N\alpha-4s}}, b_2^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_2^{-\frac{4s}{N\alpha-4s}}\right\}.$$

If $\beta > \beta_2$, then system (3)–(4) has a solution $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{u}, \widetilde{v})$ such that $\widetilde{\lambda}_1, \widetilde{\lambda}_2 < 0$ and $\widetilde{u}$ and $\widetilde{v}$ are both positive and radial. Moreover, $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{u}, \widetilde{v})$ is a ground state solution in the sense that

$$J(\widetilde{u}, \widetilde{v}) = \inf\{J(u, v) : (u, v) \in V\} = \inf_{(u,v)\in(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}} \mathcal{R}_b(u, v)$$

$$= \inf\{J(u, v) : (u, v) \textit{ is a solution of } (3) - (4) \textit{ for some } \lambda_1, \lambda_2\}.$$

Remark 1. For the special case $N = 3$, $s = 1$, $\alpha = 2$, the results in this paper are the same as those in [2]. In fact, we use in this paper some ideas introduced by Bartsch *et al.* in [2], where they considered the coupled cubic semilinear Schrödinger system on $\mathbb{R}^3$. Here, we would like to point out that the assumption $\frac{4s}{N} < \alpha \leq \frac{2s}{N-2s}$ is an essential condition in the present paper.

Finally, let us sketch the proof of Theorems 1.1–1.2. The solution in Theorem 1.1 is obtained by two-dimensional linking arguments. It is well known that the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is not compact for $2 \leq p \leq 2^*$. Hence, the associated functional of problem (3)–(4) does not satisfy the Palais-Smale condition. In order to overcome the lack of compactness, we first set our working space in $H^s_r(\mathbb{R}^N) \times H^s_r(\mathbb{R}^N)$, where

$$H^s_r(\mathbb{R}^N) = \{\varphi \in H^s(\mathbb{R}^N) : \varphi \text{ is radial}\}$$

and $H^s_r(\mathbb{R}^N)$ is endowed with the $H^s(\mathbb{R}^N)$ topology, that is, $\|\varphi\|_{H^s_r(\mathbb{R}^N)} = \|\varphi\|_{H^s(\mathbb{R}^N)}$. Then we search for solutions of (3)–(4) as critical points of J constrained on $\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$, where S_b is defined by

$$\mathrm{S}_b := \left\{ w \in H^s_r(\mathbb{R}^N) : \int_{\mathbb{R}^N} w^2 = b^2 \right\}.$$

Eventually, by Palais' principle of symmetric criticality [23, Theorem 1.28], we know that the solutions for of (3)–(4) for J on $\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$ are also the critical points of J on $T_{b_1} \times T_{b_2}$. To this end, we first define a minimax class and show that the energy functional has a minimax structure, then we apply the minimax principle (see Theorem 2.2) to J on Γ and hence we are able to obtain a Palais-Smale sequence for J on $\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$. However, the boundedness of the Palais-Smale sequence will be still unknown. Furthermore, in order to overcome the lack of compactness for $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, we need a Liouville-type result, which can be found in [13]. The solution in Theorem 1.2 is a mountain pass solution of J constrained to $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$ and the main novelty is to introduce a suitable minimax class so that we can use the mountain pass lemma.

The paper is organized as follows. In Section 2, we introduce some preliminaries that will be used to prove theorems. In Section 3, we prove Theorem 1.1. Finally, Theorem 1.2 will be proved in Section 4.

2. Preliminaries.

We first list some well-known results, which will be used to prove Theorem 1.1. To this end, we first give the following definition.

Definition 2.1. (see [14, Definition 3.1]) Let B be a closed subset of X. We shall say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with boundary B provided

- (a) every set in $\mathcal{F}$ contains B.
- (b) for any set A in $\mathcal{F}$ and any $\eta \in ([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (0 \times X) \cup ([0,1] \times B)$, we have $\eta(1 \times A) \in \mathcal{F}$.

Theorem 2.2. *(see [14, Theorem 3.2]) Let φ be a C^1 function on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy-stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B. Set $c = c(\varphi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x)$ and suppose that*

$$\sup \varphi(B) < c.$$

Then, for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ such that $\lim_n \sup_{A_n} \varphi = c$, there exists a sequence $(x_n)_n$ in X such that (i) $\lim_n \varphi(x_n) = c$ (ii)$\lim_n \|d\varphi(x_n)\| = 0$ (iii)$\lim_n dist(x_{n,A_n}) = 0$.

Moreover, if $d\varphi$ is uniformly continuous, then x_n can be chosen to be in A_n for each n.

In order to apply Theorem 2.2, we consider the following auxiliary problem

$$\begin{cases} (-\Delta)^s w + w = |w|^\alpha w & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(0) = \max w \text{ and } w \in D_s(\mathbb{R}^N). \end{cases} \tag{10}$$

This problem has a unique positive ground state solution, denoted by w_0, which is radial; see [11] for the one-dimensional case and [12] for $N \geq 2$.

Set

$$C_0 := \int_{\mathbb{R}^N} w_0^2 dx \text{ and } C_1 := \int_{\mathbb{R}^N} w_0^{\alpha+2} dx. \tag{11}$$

We observe that $w_{\lambda,\mu} = (-\lambda)^{\frac{1}{\alpha}} \mu^{-\frac{1}{\alpha}} w_0((-\lambda)^{\frac{1}{2s}} x)$, with $\lambda < 0$, is a solution of the problem

$$\begin{cases} (-\Delta)^s w - \lambda w = \mu |w|^\alpha w & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(0) = \max w \text{ and } \int_{\mathbb{R}^N} w^2 = b^2. \end{cases} \tag{12}$$

When λ appears as a Lagrange multiplier, the solution of (12) can be found as a critical point of the following energy functional associated with (12):

$$I_\mu(w) = \frac{1}{2} ||w||^2_{D_s(\mathbb{R}^N)} - \frac{\mu}{\alpha+2} \int_{\mathbb{R}^N} |w|^{\alpha+2} dx.$$

Define the set

$$\mathcal{P}(b,\mu) := \left\{ w \in \mathrm{T}_b : ||w||^2_{D_s(\mathbb{R}^N)} = \frac{N\alpha\mu}{2s(\alpha+2)} \int_{\mathbb{R}^N} |w|^{\alpha+2} dx \right\}. \tag{13}$$

The following auxiliary result shows the role of $\mathcal{P}(b,\mu)$.

Lemma 2.3. *If w is a solution of (12), then $w \in \mathcal{P}(b,\mu)$ and the positive solution w of problem (12) minimizes I_μ on $\mathcal{P}(b,\mu)$.*

Proof. From [12], we find that the Pohozaev identity for (12) is

$$\frac{N-2s}{2} ||w||^2_{D_s(\mathbb{R}^N)} - \lambda \frac{N}{2} \int_{\mathbb{R}^N} w^2 dx = \frac{N}{\alpha+2} \int_{\mathbb{R}^N} \mu |w|^{\alpha+2} dx. \tag{14}$$

Since w is a solution of (12), we have

$$||w||^2_{D_s(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} w^2 dx = \int_{\mathbb{R}^N} \mu |w|^{\alpha+2} dx. \tag{15}$$

Combining (14) with (15), we obtain

$$||w||^2_{D_s(\mathbb{R}^N)} = \frac{\alpha N \mu}{2s(\alpha+2)} \int_{\mathbb{R}^N} |w|^{\alpha+2} dx.$$

Thus $w \in \mathcal{P}(b,\mu)$. By similar arguments as in Lemma 2.10 in [16], we can deduce the last assertion, so we omit the details. □

Lemma 2.4. *The unique positive solution of problem (12) is $(\lambda_{b,\mu}, w_{b,\mu})$, where*

$$\lambda_{b,\mu} = -\left(\left(\frac{C_0}{b^2} \right)^{\frac{\alpha}{2}} \frac{1}{\mu} \right)^{\frac{4s}{N\alpha - 4s}}$$

and

$$w_{b,\mu}(x)=\left(\left(\frac{C_0}{b^2}\right)^{2s}\frac{1}{\mu^N}\right)^{\frac{1}{N\alpha-4s}} w_0\left(\left(\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}\frac{1}{\mu}\right)^{\frac{2}{N\alpha-4s}}x\right)$$

and the function $w_{b,\mu}(x)$ satisfies

$$||w_{b,\mu}||^2_{D_s(\mathbb{R}^N)}=\frac{\alpha N}{2s(\alpha+2)}\left(\frac{C_0}{b^2}\right)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\mu^{-\frac{4s}{N\alpha-4s}}C_1, \tag{16}$$

$$\int_{\mathbb{R}^N} w_{b,\mu}^{\alpha+2}dx=\left(\frac{C_0}{b^2}\right)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\mu^{-\frac{N\alpha}{N\alpha-4s}}C_1, \tag{17}$$

$$\ell(b,\mu):=I_\mu(w_{b,\mu})=\frac{\alpha N-4s}{4s(\alpha+2)}\left(\frac{C_0}{b^2}\right)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\mu^{-\frac{4s}{N\alpha-4s}}C_1. \tag{18}$$

Proof. It is easy to check that $(\lambda_{b,\mu}, w_{b,\mu})$ is the unique positive solution of (12), where w_0 is a solution of (10). By the explicit expression of $w_{b,\mu}$ and change of variables, we have

$$\int_{\mathbb{R}^N} w_{b,\mu}^{\alpha+2}dx=\left(\frac{C_0}{b^2}\right)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\mu^{-\frac{N\alpha}{N\alpha-4s}}C_1.$$

Note that $\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}w_{b,\mu}|^2dx = ||w_{b,\mu}||^2_{D_s(\mathbb{R}^N)}$. Similarly, we can get (16) and (18). □

When $\mu=\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}$, then $w_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}$ is the unique positive solution of the problem

$$\begin{cases}(-\Delta)^s w+w=\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}|w|^\alpha w & \text{in } \mathbb{R}^N,\\ w>0 & \text{in } \mathbb{R}^N,\\ w(0)=\max w \text{ and } \int_{\mathbb{R}^N} w^2=b^2. & \end{cases}$$

By Lemma 2.3, we know that $w_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}$ is a minimizer of $I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}$ on $\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})$.

Define

$$\mathcal{R}_b(w)=\frac{\alpha N-4s}{2\alpha N}\left(\frac{2s(\alpha+2)}{\alpha N}\right)^{\frac{4s}{\alpha N-4s}}\frac{\left(||w||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{\alpha N}{\alpha N-4s}}}{\left(\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}\int_{\mathbb{R}^N}w^{\alpha+2}dx\right)^{\frac{4s}{\alpha N-4s}}}.$$

Our next result establishes that this level can also be characterized as an infimum of $\mathcal{R}_b(w)$.

Lemma 2.5. *We have*

$$\inf_{\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})} I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w)=\inf_{\mathrm{T}_b}\mathcal{R}_b(w).$$

Proof. Since $w\in\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})$, it follows that

$$||w||^2_{D_s(\mathbb{R}^N)}=\frac{\alpha N}{2s(\alpha+2)}\int_{\mathbb{R}^N}\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}w^{\alpha+2}dx.$$

This shows that

$$\frac{||w||^2_{D_s(\mathbb{R}^N)}}{\frac{\alpha N}{2s(\alpha+2)}\int_{\mathbb{R}^N}\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}} w^{\alpha+2}dx} = 1 \text{ and } I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w) = \left(\frac{1}{2}-\frac{2s}{\alpha N}\right)||w||^2_{D_s(\mathbb{R}^N)}.$$

Therefore

$$I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w) = \left(\frac{1}{2}-\frac{2s}{\alpha N}\right)||w||^2_{D_s(\mathbb{R}^N)}\left(\frac{||w||^2_{D_s(\mathbb{R}^N)}}{\frac{\alpha N}{2s(\alpha+2)}\int_{\mathbb{R}^N}\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}} w^{\alpha+2}dx}\right)^{\frac{4s}{\alpha N-4s}}$$
$$= \mathcal{R}_b(w).$$

Thus, the above equality implies that

$$\inf_{\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})} I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w) \geq \inf_{\mathrm{T}_b}\mathcal{R}_b(w).$$

Next, we need to show that $\inf_{\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})} I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w) \leq \inf_{\mathrm{T}_b}\mathcal{R}_b(w)$. For this purpose, we define

$$(t\star w)(x) = e^{\frac{Nt}{2}}w(e^t x) \text{ and } \Psi_w(t) = I_\mu(t\star w), \tag{19}$$

hence

$$\Psi_w(t) = I_\mu(t\star w) = \frac{e^{2st}}{2}||w||^2_{D_s(\mathbb{R}^N)} - \frac{1}{\alpha+2}e^{\frac{N\alpha t}{2}}\int_{\mathbb{R}^N}\mu w^{\alpha+2}dx. \tag{20}$$

It is easy to check

$$\mathcal{R}_b(t\star w) = \mathcal{R}_b(w) \text{ for all } t\in\mathbb{R},\ w\in\mathrm{T}_b.$$

By (20), we know that $\Psi_w(t)$ has a unique $t^*_w\in\mathbb{R}$ such that $t^*_w\star w\in\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})$ and t^*_w satisfies

$$e^{\left(\frac{N\alpha-4s}{2}\right)t^*_w} = \frac{2s(\alpha+2)||w||^2_{D_s(\mathbb{R}^N)}}{N\alpha\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}\int_{\mathbb{R}^N} w^{\alpha+2}dx}.$$

Moreover, t^*_w is the unique critical point of $\Psi_w(t)$, which is a strict maximum.

Consequently,

$$\mathcal{R}_b(w) = \mathcal{R}_b(t^*_w\star w) = I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(t^*_w\star w) \geq \inf_{\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})} I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w),$$

hence

$$\inf_{\mathcal{P}(b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}})} I_{b,\left(\frac{C_0}{b^2}\right)^{\frac{\alpha}{2}}}(w) \leq \inf_{\mathrm{T}_b}\mathcal{R}_b(w).$$

The proof is now complete. □

Consider the following fractional Gagliardo-Nirenberg-Sobolev inequality (see[12])

$$\int_{\mathbb{R}^N}|w|^{\alpha+2}dx \leq C_{opt}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}w|^2dx\right)^{\frac{N\alpha}{4s}}\left(\int_{\mathbb{R}^N}|w|^2dx\right)^{\frac{\alpha+2}{2}-\frac{N\alpha}{4s}}, \tag{21}$$

for all $w\in H^s(\mathbb{R}^N)$, where α is a positive number and $C_{opt}>0$ denotes the optimal constant depending only on α, N and s. In particular, the optimal constant C_{opt}

is defined as follows

$$\begin{aligned}\frac{1}{(C_{opt})^{\frac{4s}{\alpha N-4s}}} &= \inf_{w\in D_s(\mathbb{R}^N)\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^N}|w|^2dx\right)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\left(||w||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{\alpha N}{\alpha N-4s}}}{\left(\int_{\mathbb{R}^N}|w|^{\alpha+2}dx\right)^{\frac{4s}{N\alpha-4s}}}\\ &= \inf_{\mathrm{T}_b}(b^2)^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}\frac{\left(||w||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{\alpha N}{\alpha N-4s}}}{\left(\int_{\mathbb{R}^N}|w|^{\alpha+2}dx\right)^{\frac{4s}{N\alpha-4s}}}\\ &= \frac{2\alpha N}{\alpha N-4s}\left(\frac{\alpha N}{2s(\alpha+2)}\right)^{\frac{4s}{\alpha N-4s}} C_0^{\frac{4s}{\alpha N-4s}} b^{\frac{4s(\alpha+1)-2N\alpha}{\alpha N-4s}}\inf_{\mathrm{T}_b}\mathcal{R}_b(w).\end{aligned}$$

3. **Proof of Theorem 1.1.**

In this section, we prove Theorem 1.1, which is based on a two-dimensional linking argument. As mentioned earlier, we will only work in the radial function space. We search for solutions of problem (3)–(4) as critical points of J constrained on $\mathrm{S}_{b_1}\times\mathrm{S}_{b_2}$, where S_b is defined by

$$\mathrm{S}_b := \left\{w\in H^s_r(\mathbb{R}^N) : \int_{\mathbb{R}^N} w^2 = b^2\right\}.$$

Let us recall that $(t\star w)(x) = e^{\frac{Nt}{2}}w(e^t x)$ in (19). We have the following auxiliary property.

Lemma 3.1. *For every $\mu>0$ and $w\in D_s(\mathbb{R}^N)$, there holds*

$$I_\mu(t\star w) = \frac{e^{2st}}{2}||w||^2_{D_s(\mathbb{R}^N)} - \frac{\mu}{\alpha+2}e^{\frac{N\alpha t}{2}}\int_{\mathbb{R}^N} w^{\alpha+2}dx,$$

$$\frac{\partial}{\partial t}I_\mu(t\star w) = se^{2st}||w||^2_{D_s(\mathbb{R}^N)} - \frac{N\alpha\mu}{2(\alpha+2)}e^{\frac{N\alpha t}{2}}\int_{\mathbb{R}^N} w^{\alpha+2}dx.$$

In particular, if $w = w_{a,\mu}$, then

$$\frac{\partial}{\partial t}I_\mu(t\star w_{a,\mu}) \begin{cases} >0 & \text{if} \quad t<0,\\ =0 & \text{if} \quad t=0,\\ <0 & \text{if} \quad t>0.\end{cases}$$

Proof. By the definition of $t\star w$ and a change of variables, it is easy to obtain the two identities. Since

$$\frac{\partial}{\partial t}I_\mu(t\star w) = e^{2st}\left[s||w||^2_{D_s(\mathbb{R}^N)} - \frac{N\alpha\mu}{2(\alpha+2)}e^{\frac{(N\alpha-4s)t}{2}}\int_{\mathbb{R}^N} w^{\alpha+2}dx\right],$$

we have

$$\frac{\partial}{\partial t}I_\mu(t\star w_{a,\mu}) \text{ is} \begin{cases} >0 & \text{if} \quad t<t^\star,\\ =0 & \text{if} \quad t=t^\star,\\ <0 & \text{if} \quad t>t^\star,\end{cases}$$

where $t^\star$ satisfies

$$e^{\frac{(N\alpha-4s)}{2}t^\star} = \frac{2s(\alpha+2)||w||^2_{D_s(\mathbb{R}^N)}}{N\alpha\mu\int_{\mathbb{R}^N} w^{\alpha+2}dx}.$$

By Lemma 2.3, when $w = w_{a,\mu}$, we have $e^{\frac{(N\alpha-4s)}{2}t^\star} = 1$, thus $t^\star = 0$. □

Lemma 3.2. *Let $\beta_1 = \beta_1(b_1, b_2, \mu_1, \mu_2)$ is defined by (5). Then for all $0 < \beta < \beta_1$ is defined by (5), we have*

$$\inf\{J(u,v) : (u,v) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)\} > \max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\},$$

where $\ell(b_i, \mu_i)$ is defined by (18).

Proof. By Young's inequality, for $(u,v) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)$, we have

$$\begin{aligned} J(u,v) &= \frac{1}{2}||(u,v)||^2_{\mathcal{D}} - \frac{1}{\alpha+2}\int_{\mathbb{R}^N}\left(\mu_1|u|^{\alpha+2} + \mu_2|v|^{\alpha+2} + 2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}\right)dx \\ &= I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} 2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}dx \\ &\geq I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{\alpha+2}\int_{\mathbb{R}^N}\left(|u|^{\alpha+2} + |v|^{\alpha+2}\right)dx \\ &= I_{\mu_1+\beta}(u) + I_{\mu_2+\beta}(v) \geq \inf_{u\in\mathcal{P}(b_1,\mu_1+\beta)} I_{\mu_1+\beta}(u) + \inf_{v\in\mathcal{P}(b_2,\mu_2+\beta)} I_{\mu_2+\beta}(v) \\ &= \ell(b_1, \mu_1+\beta) + \ell(b_2, \mu_2+\beta). \end{aligned}$$

Thus, we need to show that

$$\max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\} < \ell(b_1, \mu_1+\beta) + \ell(b_2, \mu_2+\beta).$$

By Lemma 2.4, we have

$$\begin{aligned} &\max\left\{b_1^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_1^{-\frac{4s}{N\alpha-4s}}, b_2^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}\mu_2^{-\frac{4s}{N\alpha-4s}}\right\} \\ &\qquad < b_1^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}(\mu_1+\beta)^{-\frac{4s}{N\alpha-4s}} + b_2^{\frac{2N\alpha-4s(\alpha+2)}{N\alpha-4s}}(\mu_2+\beta)^{-\frac{4s}{N\alpha-4s}}, \end{aligned}$$

since $0 < \beta < \beta_1$. It follows that

$$\inf\{J(u,v) : (u,v) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)\} > \max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\}.$$

The proof is now complete. □

Now we fix $0 < \beta < \beta_1 = \beta_1(b_1, b_2, \mu_1, \mu_2)$ and choose $\epsilon > 0$ such that

$$\inf\{J(u,v) : (u,v) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)\} > \max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\} + \epsilon. \tag{22}$$

Define

$$w_1 = w_{b_1,\mu_1+\beta}, \quad w_2 = w_{b_2,\mu_2+\beta} \tag{23}$$

and

$$\varphi_i(t) = I_{\mu_i}(t \star w_i), \quad \psi_i(t) = \frac{\partial}{\partial t} I_{\mu_i+\beta}(t \star w_i) \text{ for } i = 1, 2. \tag{24}$$

By Lemma 3.1, it is easy to get the following property.

Lemma 3.3. *For $i = 1, 2$ there exists $\rho_i < 0$ and $R_i > 0$ such that*

(i) $0 < \varphi_i(\rho_i) < \epsilon$ *and* $\varphi_i(R_i) \leq 0$.
(ii) $\psi_i(t) > 0$ *for any* $t < 0$ *and* $\psi_i(t) < 0$ *for any* $t > 0$. *In particular,* $\psi_i(\rho_i) > 0$ *and* $\psi_i(R_i) > 0$.

Let $Q = [\rho_1, R_1] \times [\rho_2, R_2]$ and let

$$\gamma_0(k_1, k_2) = (k_1 \star w_1, k_2 \star w_2) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2} \quad \forall\ (k_1, k_2) \in \overline{Q}.$$

Define the minimax class

$$\Gamma := \{\gamma \in \mathcal{C}(\overline{Q}, \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}) : \gamma = \gamma_0 \text{ on } \partial Q\}.$$

From Lemma 3.2 it follows that if we want to apply Theorem 2.2, we need to prove the following technical lemmas.

Lemma 3.4. *There holds*

$$\sup_{\partial Q} J(\gamma_0) \leq \max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\} + \epsilon.$$

Proof. Since $\beta > 0$, for every $(u,v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$, we have

$$J(u,v) = I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} 2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}dx \leq I_{\mu_1}(u) + I_{\mu_2}(v). \quad (25)$$

Therefore, by (25) and Lemma 3.3, we infer that

$$J(k_1 \star w_1, \rho_2 \star w_2) \leq I_{\mu_1}(k_1 \star w_1) + I_{\mu_2}(\rho_2 \star w_2) \leq I_{\mu_1}(k_1 \star w_1) + \epsilon \quad (26)$$

$$\leq \sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_1) + \epsilon. \quad (27)$$

Next, we estimate $\sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_1)$, by Lemma 2.4, we obtain

$$w_{b_i,\mu_1} = t^\star \star w_i \text{ for } e^{\frac{(Ns\alpha-4s)}{2}t^\star} = \frac{2s(\alpha+2)||w_i||^2_{D_s(\mathbb{R}^N)}}{N\alpha\int_{\mathbb{R}^N}\mu_i w_i^{\alpha+2}dx} = \frac{\mu_i+\beta}{\mu_i}.$$

Since

$$t_1 \star (t_2 \star w) = (t_1+t_2) \star w,$$

we have

$$\sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_1) = \sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_{b_1,\mu_1}). \quad (28)$$

By Lemma 3.1, we know that the supremum of $\sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_{b_1,\mu_1})$ is achieved for $t = 0$. It follows that

$$J(k_1 \star w_1, \rho_2 \star w_2) \leq \sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_{b_1,\mu_1}) + \epsilon = \ell(b_1,\mu_1) + \epsilon, \quad \forall\ k_1 \in [\rho_1, R_1]. \quad (29)$$

Similarly,

$$J(\rho_1 \star w_1, k_2 \star w_2) \leq \sup_{t\in\mathbb{R}} I_{\mu_2}(t \star w_{b_2,\mu_2}) + \epsilon = \ell(b_2,\mu_2) + \epsilon, \quad \forall\ k_2 \in [\rho_2, R_2]. \quad (30)$$

By (28) and Lemma 3.3, we have

$$J(k_1 \star w_1, R_2 \star w_2) \leq I_{\mu_1}(k_1 \star w_1) + I_{\mu_2}(R_2 \star w_2) \quad (31)$$

$$\leq \sup_{t\in\mathbb{R}} I_{\mu_1}(t \star w_{b_1,\mu_1}) = \ell(b_1,\mu_1),\ \forall\ k_1 \in [\rho_1, R_1].$$

Similarly,

$$J(R_1 \star w_1, k_2 \star w_2) \leq I_{\mu_1}(R_1 \star w_1) + I_{\mu_2}(k_2 \star w_2) \quad (32)$$

$$\leq \sup_{t\in\mathbb{R}} I_{\mu_2}(t \star w_{b_2,\mu_2}) = \ell(b_2,\mu_2),\ \forall\ k_2 \in [\rho_2, R_2].$$

By (29)–(32), we complete the proof. □

Lemma 3.5. *For every $\gamma \in \Gamma$, there exists $(k_{1,\gamma}, k_{2,\gamma}) \in Q$ such that*

$$\gamma(k_{1,\gamma}, k_{2,\gamma}) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)\}.$$

Proof. For $\gamma \in \Gamma$, $\gamma(k_1,k_2) = (\gamma_1(k_1,k_2), \gamma_2(k_1,k_2)) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$, define

$$F_\gamma(k_1,k_2) := \left(\frac{\partial}{\partial t} I_{\mu_1+\beta}(t \star \gamma_1(k_1,k_2))|_{t=0}, \frac{\partial}{\partial t} I_{\mu_2+\beta}(t \star \gamma_2(k_1,k_2)) \,|_{t=0} \right).$$

We have

$$\begin{aligned}
&\frac{\partial}{\partial t} I_{\mu_i+\beta}(t \star \gamma_i(k_1,k_2))|_{t=0} \\
&= \left(se^{2st} ||\gamma_i(k_1,k_2)||^2_{D_s(\mathbb{R}^N)} - \frac{Ns\alpha(\mu_i+\beta)}{2(\alpha+2)} e^{\frac{Ns\alpha t}{2}} \int_{\mathbb{R}^N} \gamma_i^{\alpha+2}(k_1,k_2)dx \right) |_{t=0} \\
&= s||\gamma_i(k_1,k_2)||^2_{D_s(\mathbb{R}^N)} - \frac{Ns\alpha(\mu_i+\beta)}{2(\alpha+2)} \int_{\mathbb{R}^N} \gamma_i^{\alpha+2}(k_1,k_2)dx.
\end{aligned}$$

Thus, $F_\gamma(k_1,k_2) = 0$ if and only if $\gamma(k_1,k_2) \in \mathcal{P}(b_1, \mu_1+\beta) \times \mathcal{P}(b_2, \mu_2+\beta)$.

Since $\gamma = \gamma_0$ on ∂Q and

$$\begin{aligned}
F_{\gamma_0}(k_1,k_2) =& \Bigg(se^{2sk_1} ||w_1||^2_{D_s(\mathbb{R}^N)} - \frac{Ns\alpha(\mu_1+\beta)}{2(\alpha+2)} e^{\frac{Ns\alpha k_1}{2}} \int_{\mathbb{R}^N} w_1^{\alpha+2} dx, \\
& se^{2sk_2} ||w_2||^2_{D_s(\mathbb{R}^N)} - \frac{Ns\alpha(\mu_2+\beta)}{2(\alpha+2)} e^{\frac{Ns\alpha k_2}{2}} \int_{\mathbb{R}^N} w_2^{\alpha+2} dx \Bigg) \\
=& \left(\psi_1(k_1), \psi_2(k_2)\right).
\end{aligned}$$

By the definition of ψ_i in (24) and direct computation, we have

$$\deg\left(F_\gamma, Q, (0,0)\right) = \deg\left(F_{\gamma_0}, Q, (0,0\right) = 1.$$

Thus, according to the properties of the topological degree, there exists $(k_{1,\gamma}, k_{2,\gamma}) \in Q$ such that

$$F_\gamma(k_{1,\gamma}, k_{2,\gamma}) = (0,0).$$

The proof is now complete. □

Remark 2. By Lemmas 3.4 and 3.5, we can apply the minimax principle (see Theorem 2.2) to J on Γ and we obtain a Palais-Smale sequence for J on $\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$. However, the boundedness of the Palais-Smale sequence is unknown. For this purpose, we will use some ideas found in [16].

Lemma 3.6. *There exists a Palais-Smale sequence (u_n, v_n) for J on $\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$ at the level*

$$c := \inf_{\gamma \in \Gamma} \max_{(k_1,k_2)\in Q} J(\gamma(k_1,k_2)) > \max\{\ell(b_1,\mu_1), \ell(b_2,\mu_2)\}$$

and satisfying

$$\begin{aligned}
&\|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)} \\
&= \frac{N\alpha}{2s(\alpha+2)} \int_{\mathbb{R}^N} \left(\mu_1|u_n|^{\alpha+2} + \mu_2|v_n|^{\alpha+2} + 2\beta|u_n|^{\frac{\alpha+2}{2}}|v_n|^{\frac{\alpha+2}{2}} \right) dx + o(1),
\end{aligned} \tag{33}$$

where $o(1) \to 0$ as $n \to +\infty$.

Proof. Define

$$\widetilde{J}(s,u,v) := J(s \star u, s \star v), \ \forall\ (s,u,v)\ \in \mathbb{R} \times \mathrm{S}_{b_1} \times \mathrm{S}_{b_2},$$

$$\widetilde{\gamma}_0(k_1,k_2) := (0, \gamma_0(k_1,k_2)) = (0, k_1 \star w_1, k_2 \star w_2),$$

and

$$\widetilde{\Gamma} := \{\widetilde{\gamma} \in \mathcal{C}(Q, \mathbb{R} \times \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}) : \widetilde{\gamma} = \widetilde{\gamma}_0 \text{ on } \partial Q\}.$$

To use the minimax principle (Theorem 2.2) for the function $\widetilde{J}$ with the minimax class $\widetilde{\Gamma}$, we define

$$\widetilde{c} := \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \max_{(k_1,k_2)\in Q} \widetilde{J}(\widetilde{\gamma}(k_1,k_2)).$$

Since $\widetilde{J}(\widetilde{\gamma}_0) = J(\gamma_0)$ on ∂Q, we know by Lemmas 3.4 and 3.5 that if we show that $\widetilde{c} = c$, then the hypotheses of the minimax principle (Theorem 2.2) will be satisfied. On the one hand, since $\Gamma \subset \widetilde{\Gamma}$, we have $\widetilde{c} \le c$. But

$$\widetilde{\gamma}(k_1,k_2) = (s(k_1,k_2), \gamma_1(k_1,k_2), \gamma_2(k_1,k_2))$$

for any $\widetilde{\gamma} \in \widetilde{\Gamma}$ and $(k_1,k_2) \in Q$. So

$$\begin{aligned}\widetilde{J}(\widetilde{\gamma}(k_1,k_2)) &= J(s(k_1,k_2)\star\gamma_1(k_1,k_2), s(k_1,k_2)\star\gamma_2(k_1,k_2)),\\ &= \widetilde{J}(0, s(k_1,k_2)\star\gamma_1(k_1,k_2), s(k_1,k_2)\star\gamma_2(k_1,k_2))\end{aligned}$$

and $(s(k_1,k_2)\star\gamma_1(k_1,k_2), s(k_1,k_2)\star\gamma_2(k_1,k_2)) \in \Gamma$. Thus, $c = \widetilde{c}$. Since $\widetilde{J}(s,u,v) = \widetilde{J}(s,|u|,|v|)$, by the minimax principle (Theorem 2.2), we can choose the minimizing sequence $\widetilde{\gamma}_n = (s_n, \gamma_{1,n}, \gamma_{2,n})$ for $\widetilde{c}$ satisfying $s_n = 0, \gamma_{1,n}(k_1,k_2) \ge 0$, $\gamma_{2,n}(k_1,k_2) \ge 0$ a.e. in $\mathbb{R}^N$ for every $(k_1,k_2) \in Q$.

In conclusion, by the minimax principle (Theorem 2.2), there exists a Palais-Smale sequence $(\widetilde{s}_n, \widetilde{u}_n, \widetilde{v}_n)$ for $\widetilde{J}$ on $\mathbb{R} \times \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$ at level $\widetilde{c}$, and such that

$$\lim_{n\to+\infty} |\widetilde{s_n}| + dist((\widetilde{u}_n, \widetilde{v}_n), \widetilde{\gamma}_n(Q)) = 0. \tag{34}$$

In order to obtain a Palais-Smale sequence for J at level c satisfying (33), we can argue as in Lemma 2.4 of [16] with minor changes. The fact that $u_n^-, v_n^- \to 0$ a.e in $\mathbb{R}^N$ as $n \to \infty$ comes from (34). The proof is now complete. □

It is well known that the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact. To overcome the lack of such compactness, we need following Liouville-type property.

Lemma 3.7. *(see [13, Theorem 1.3]) Let w be a nonnegative solution of the fractional inequality*

$$(-\Delta)^s w \ge w^{\alpha+1} \text{ in } \mathbb{R}^N.$$

If $0 < \alpha \le \frac{2s}{N-2s}$ and $N > 2s$, then $w = 0$.

By Lemma 3.6, we obtain a Palais-Smale sequence (u_n, v_n) for J at level c satisfying (33). Next, we show the (u_n, v_n) is bounded.

Lemma 3.8. *The Palais-Smale sequence (u_n, v_n) for J at level c is bounded in $H^s(\mathbb{R}^N, \mathbb{R}^2)$. Furthermore, there exists $\widetilde{C} > 0$ such that*

$$\|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)} \ge \widetilde{C} \textit{ for all } n.$$

Proof. By (33), it is easy to obtain

$$J(u_n, v_n) = \frac{N\alpha - 4s}{2N\alpha}\left(\|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)}\right) \to c > 0,$$

where $o(1) \to 0$ as $n \to +\infty$. Therefore, the Palais-Smale sequence (u_n, v_n) for J at level c is bounded in $H^s(\mathbb{R}^N, \mathbb{R}^2)$ and there exists $\widetilde{C} > 0$ such that

$$\|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)} \ge \widetilde{C} \text{ for all } n.$$

The proof is now complete. □

By Lemma 3.8, (u_n, v_n), up to a subsequence, $(u_n, v_n) \to (\widetilde{u}, \widetilde{v})$ weakly in $H^s(\mathbb{R}^N)$. By the compact embedding of $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)$, we have $(u_n, v_n) \to (\widetilde{u}, \widetilde{v})$ strongly in $L^{\alpha+2}(\mathbb{R}^N)$. Note that $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not a compact embedding. So, we cannot conclude that $(\widetilde{u}, \widetilde{v}) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$.

Since $dJ|_{\mathrm{S}_{b_1} \times \mathrm{S}_{b_2}} \to 0$, then there exist two sequences of real numbers $(\lambda_{1,n})$ and $(\lambda_{2,n})$ such that

$$\begin{aligned} &\langle u_n, \varphi\rangle + \langle v_n, \psi\rangle - \int_{\mathbb{R}^N} (\lambda_{1,n} u_n \varphi + \lambda_{2,n} v_n \psi) dx \\ &- \int_{\mathbb{R}^N} (\mu_1 |u_n|^{\alpha} u_n \varphi + \mu_2 |v_n|^{\alpha} v_n \psi +) dx \\ &- \int_{\mathbb{R}^N} (\beta |u_n|^{\frac{\alpha-2}{2}} u_n \varphi |v_n|^{\frac{\alpha+2}{2}} + \beta |u_n|^{\frac{\alpha+2}{2}} |v_n|^{\frac{\alpha-2}{2}} v_n \psi) dx = o(1) \|(\varphi, \psi)\|, \end{aligned} \tag{35}$$

where

$$\langle u_n, \varphi\rangle = \frac{C(N,s)}{2} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy$$

for every $(\varphi, \psi) \in H^s(\mathbb{R}^N, \mathbb{R}^2)$, with $o(1) \to 0$ as $n \to +\infty$.

Lemma 3.9. *Both sequences $(\lambda_{1,n})$ and $(\lambda_{2,n})$ are bounded and at least one of them is convergent, up to a subsequence, to a strictly negative value.*

Proof. By (35), we deduce that

$$\lambda_{1,n} b_1^2 = \|u_n\|^2_{D_s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (\mu_1 |u_n|^{\alpha+2} + \beta |u_n|^{\frac{\alpha+2}{2}} |v_n|^{\frac{\alpha+2}{2}} dx - o(1),$$

$$\lambda_{2,n} b_2^2 = \|v_n\|^2_{D_s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (\mu_2 |v_n|^{\alpha+2} + \beta |u_n|^{\frac{\alpha+2}{2}} |v_n|^{\frac{\alpha+2}{2}} dx - o(1),$$

with $o(1) \to 0$ as $n \to +\infty$. Since (u_n, v_n) is bounded in $H^s(\mathbb{R}^N)$ and $L^{\alpha+2}(\mathbb{R}^N)$, then $(\lambda_{1,n})$ and $(\lambda_{2,n})$ are bounded sequences. By (33) and Lemma 3.8, we have

$$\begin{aligned} \lambda_{1,n} b_1^2 + \lambda_{2,n} b_2^2 &= \|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)} \\ &\quad - \int_{\mathbb{R}^N} (\mu_1 |u_n|^{\alpha+2} + \mu_2 |v_n|^{\alpha+2} + 2\beta |u_n|^{\frac{\alpha+2}{2}} |v_n|^{\frac{\alpha+2}{2}} dx - o(1) \\ &= \frac{N\alpha - 2s(\alpha+2)}{N\alpha} (\|u_n\|^2_{D_s(\mathbb{R}^N)} + \|v_n\|^2_{D_s(\mathbb{R}^N)}) \\ &\le \frac{N\alpha - 2s(\alpha+2)}{N\alpha} \widetilde{C} \end{aligned}$$

for every n large enough. Note that the last inequality holds because of the fact that $\frac{4s}{N} < \alpha \le \frac{2s}{N-2s}$. Thus, at least one of the two limits is strictly negative. $\square$

Let us consider the convergent subsequences $\lambda_{1,n} \to \lambda_1 \in \mathbb{R}$ and $\lambda_{2,n} \to \lambda_2 \in \mathbb{R}$. Then we obtain the following crucial lemma to deal with the lack of the compact embedding of $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$.

Lemma 3.10. *If $\lambda_1 < 0$ (or $\lambda_2 < 0$) then $u_n \to \widetilde{u}$ (or $v_n \to \widetilde{v}$) strongly in $H^s(\mathbb{R}^N)$.*

Proof. Since u_n is bounded in $H^s(\mathbb{R}^N)$, then $u_n \to \widetilde{u}$ weakly in $H^s(\mathbb{R}^N)$ and strongly in $L^{\alpha+2}(\mathbb{R}^N)$. By (35), we have

$$\begin{aligned} o(1) &= (dJ(u_n, v_n) - dJ(\widetilde{u}, \widetilde{v}))[(u_n - \widetilde{u}, 0)] - \lambda_1 \int_{\mathbb{R}^N} (u_n - \widetilde{u})^2 dx \\ &= \|u_n - \widetilde{u}\|^2_{D_s(\mathbb{R}^N)} - \lambda_1 \int_{\mathbb{R}^N} (u_n - \widetilde{u})^2 dx + o(1) \end{aligned}$$

with $o(1) \to 0$ as $n \to +\infty$. Since $\lambda_1 < 0$, we obtain that $u_n \to \widetilde{u}$ strongly in $H^s(\mathbb{R}^N)$. Similarly, we can prove that if $\lambda_2 < 0$, then $v_n \to \widetilde{v}$ strongly in $H^s(\mathbb{R}^N)$. □

3.1. **Proof of Theorem 1.1 completed.** Since (u_n, v_n) is bounded in $H^s(\mathbb{R}^N, \mathbb{R}^2)$, then $(u_n, v_n) \to (\widetilde{u}, \widetilde{v})$ weakly in $H^s(\mathbb{R}^N)$. So $(\widetilde{u}, \widetilde{v})$ is a solution of (3).

Next, we show that $(\widetilde{u}, \widetilde{v})$ satisfies (4). Without loss of generality, we assume that $\lambda_1 < 0$, then by Lemma 3.10, $u_n \to \widetilde{u}$ strongly in $H^s(\mathbb{R}^N)$. If $\lambda_2 < 0$, then $v_n \to \widetilde{v}$ strongly in $H^s(\mathbb{R}^N)$, which completes the proof of Theorem 1.1. Assume by contradiction that $\lambda_2 \geq 0$, and $v_n \to \widetilde{v}$ not strongly in $H^s(\mathbb{R}^N)$. Since $\widetilde{u}, \widetilde{v} \geq 0$ in $\mathbb{R}^N$, we have

$$(-\Delta)^2 \widetilde{v} = \lambda_2 \widetilde{v} + \mu_2 |\widetilde{v}|^\alpha \widetilde{v} + \beta |\widetilde{u}|^{\frac{\alpha+2}{2}} |\widetilde{v}|^{\frac{\alpha-2}{2}} \widetilde{v} \geq \mu_2 |\widetilde{v}|^{\alpha+1} \quad \text{in} \quad \mathbb{R}^N.$$

By Lemma 3.7, $\widetilde{v} = 0$, which implies that $\widetilde{u}$ solves

$$\begin{cases} (-\Delta)^s \widetilde{u} - \lambda_1 \widetilde{u} = \mu_1 |\widetilde{u}|^\alpha \widetilde{u} & \text{in} \quad \mathbb{R}^N, \\ \widetilde{u} > 0 & \text{in} \quad \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\widetilde{u}|^2 = b_1, \end{cases}$$

so $\widetilde{u} \in \mathcal{P}(b_1, \mu_1)$ and $I_{\mu_1}(\widetilde{u}) = \ell(b_1, \mu_1)$. By (33), we have

$$\begin{aligned} c &= \lim_{n\to\infty} J(u_n, v_n) \\ &= \lim_{n\to\infty} \frac{N\alpha - 4s}{4s(\alpha+2)} \int_{\mathbb{R}^N} \left(\mu_1 |u_n|^{\alpha+2} + \mu_2 |v_n|^{\alpha+2} + 2\beta |u_n|^{\frac{\alpha+2}{2}} |v_n|^{\frac{\alpha+2}{2}} \right) dx \\ &= \lim_{n\to\infty} \frac{N\alpha - 4s}{4s(\alpha+2)} \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}|^{\alpha+2} dx = I_{\mu_1}(\widetilde{u}) = \ell(b_1, \mu_1), \end{aligned}$$

which contradicts Lemma 3.6. The proof is now complete. □

4. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is based on a mountain pass argument. For any $(u, v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$, we consider the function

$$\begin{aligned} & J(t \star (u,v)) \\ &= \frac{e^{2st}}{2} ||(u,v)||^2_{\mathcal{D}_s(\mathbb{R}^N)} - \frac{e^{\frac{N\alpha t}{2}}}{\alpha+2} \int_{\mathbb{R}^N} (\mu_1 |u|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}}) dx, \end{aligned}$$

where $t \star (u, v) = (t \star u, t \star v)$ and $t \star u$ is defined in (19). We observe that if $(u, v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$, then $t \star (u, v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$. By the definition of $t \star u$, it is easy to prove the following lemma.

Lemma 4.1. *Let* $(u, v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$. *Then*

$$\lim_{t\to-\infty} \left[||t \star u||^2_{D_s(\mathbb{R}^N)} + ||t \star v||^2_{D_s(\mathbb{R}^N)} \right] = 0,$$

$$\lim_{t\to\infty} \left[||t \star u||^2_{D_s(\mathbb{R}^N)} + ||t \star v||^2_{D_s(\mathbb{R}^N)} \right] = +\infty,$$

and

$$\lim_{t\to-\infty} J(t\star(u,v)) = 0^+, \quad \lim_{t\to\infty} J(t\star(u,v)) = -\infty.$$

Next, we construct the mountain pass structure of the problem.

Lemma 4.2. *There exists $K>0$ sufficiently small such that for all the sets*

$$A := \left\{(u,v)\in \mathrm{S}_{b_1}\times \mathrm{S}_{b_2} : \left(||u||^2_{D_s(\mathbb{R}^N)} + ||v||^2_{D_s(\mathbb{R}^N)}\right) \le K\right\}$$

and

$$B := \left\{(u,v)\in \mathrm{S}_{b_1}\times \mathrm{S}_{b_2} : \left(||u||^2_{D_s(\mathbb{R}^N)} + ||v||^2_{D_s(\mathbb{R}^N)}\right) = 2K\right\}$$

there hold

$$J(u,v)>0 \text{ on } A \text{ and } \sup_A J(u,v) < \inf_B J(u,v).$$

Proof. By the fractional Gagliardo-Nirenberg-Sobolev inequality (21), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\mu_1|u|^{\alpha+2}+\mu_2|v|^{\alpha+2}+2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}\right)dx\\ &\le C\int_{\mathbb{R}^N}\left(|u|^{\alpha+2}+|v|^{\alpha+2}\right)dx \le C\left(||u||^2_{D_s(\mathbb{R}^N)} + ||v||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{N\alpha}{4s}}\end{aligned}$$

for every $(u,v)\in \mathrm{S}_{b_1}\times \mathrm{S}_{b_2}$, where $C>0$ depends on $\mu_1,\mu_2,\beta,b_1,b_2$. If $(u_1,v_1)\in B$ and $(u_2,v_2)\in A$, we have

$$\begin{aligned}&J(u_1,v_1)-J(u_2,v_2)\\ &\ge \frac{1}{2}\left[\left(||u_1||^2_{D_s(\mathbb{R}^N)} + ||v_1||^2_{D_s(\mathbb{R}^N)}\right) - \left(||u_2||^2_{D_s(\mathbb{R}^N)} + ||v_2||^2_{D_s(\mathbb{R}^N)}\right)\right]\\ &- \frac{1}{\alpha+2}\int_{\mathbb{R}^N}\left(\mu_1|u_1|^{\alpha+2}+\mu_2|v_1|^{\alpha+2}+2\beta|u_1|^{\frac{\alpha+2}{2}}|v_1|^{\frac{\alpha+2}{2}}\right)dx\\ &\ge \frac{K}{2} - \frac{C}{\alpha+2}(2K)^{\frac{N\alpha}{4s}} \ge \frac{K}{4}\end{aligned}$$

provided that $K>0$ is small enough. Furthermore, we also have

$$\begin{aligned}&J(u_2,v_2) &(36)\\ &\ge \frac{1}{2}\left(||u_2||^2_{D_s(\mathbb{R}^N)} + ||v_2||^2_{D_s(\mathbb{R}^N)}\right) - \frac{C}{\alpha+2}\left(||u_2||^2_{D_s(\mathbb{R}^N)} + ||v_2||^2_{D_s(\mathbb{R}^N)}\right)^{\frac{N\alpha}{4s}} > 0 &(37)\end{aligned}$$

for every $(u_2,v_2)\in A$. The proof is now complete. □

In order to use the mountain pass lemma, we need to introduce a suitable minimax class. We recall $w_{b,\mu}$ defined in Lemma 2.4. Set

$$C := \left\{(u,v)\in \mathrm{S}_{b_1}\times \mathrm{S}_{b_2} : \left(||u||^2_{D_s(\mathbb{R}^N)} + ||v||^2_{D_s(\mathbb{R}^N)}\right) \ge 3K \text{ and } J(u,v)\le 0\right\}. \tag{38}$$

By Lemma 4.1, there exist $t_1<0$ and $t_2>0$ such that

$$t_1\star\left(w_{b_1,\left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2,\left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}}\right) := (\overline{u}_1,\overline{v}_1)\in A,$$

$$t_2 \star \left(w_{b_1, \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2, \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}} \right) := (\overline{u}_2, \overline{v}_2) \in C.$$

Finally, we define

$$\Gamma := \left\{ \gamma \in \mathcal{C}([0,1], \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}) : \gamma(0) = (\overline{u}_1, \overline{v}_1) \text{ and } \gamma(1) = (\overline{u}_2, \overline{v}_2) \right\}. \tag{39}$$

By Lemma 4.2, we can use the mountain pass lemma to J on the minimax class Γ to get a Palais-Smale sequence.

To conquer the difficulty from the lack of compactness of the embedding $H_s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, we establish the following property.

Lemma 4.3. *If $\beta > \beta_2$, then*

$$\sup_{t\in\mathbb{R}} J \left(t \star \left(w_{b_1, \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2, \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}} \right) \right) < \min \left\{ \ell(b_1, \mu_1), \ell(b_2, \mu_2) \right\}$$

where β_2 is defined by (9).

Proof. By the expression of $w_{b_i, \left(\frac{C_0}{b_i^2}\right)^{\frac{\alpha}{2}}}$ in Lemma 2.4 and change of variables, we obtain

$$\int_{\mathbb{R}^N} \left(t \star w_{b_1, \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}} \right)^{\frac{\alpha+2}{2}} \left(t \star w_{b_2, \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}} \right)^{\frac{\alpha+2}{2}} dx \tag{40}$$

$$= e^{\frac{N(\alpha+2)t}{2}} \left(\frac{C_0^2}{b_1^2 b_2^2} \right)^{\frac{s(\alpha+2)}{N\alpha-4s}} \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{4} \frac{-N(\alpha+2)}{N\alpha-4s}} \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{4} \frac{-N(\alpha+2)}{N\alpha-4s}} \int_{\mathbb{R}^N} w_0^{\alpha+2}(e^t x) dx$$

$$= e^{\frac{N\alpha t}{2}} \left(\frac{b_1^2 b_2^2}{C_0^2} \right)^{\frac{\alpha+2}{4}} C_1. \tag{41}$$

By Lemma 2.4 and direct calculation, we have

$$||t \star w_{b_1, \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}||^2_{D_s(\mathbb{R}^N)} = \frac{\alpha N}{2s(\alpha+2)} e^{2st} \frac{b_1^2}{C_0} C_1,$$

$$||t \star w_{b_2, \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}}||^2_{D_s(\mathbb{R}^N)} = \frac{\alpha N}{2s(\alpha+2)} e^{2st} \frac{b_2^2}{C_0} C_1,$$

$$\int_{\mathbb{R}^N} \left(t \star w_{b_1, \left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}} \right)^{\alpha+2} dx = e^{\frac{N\alpha t}{2}} \left(\frac{b_1^2}{C_0} \right)^{\frac{\alpha+2}{2}} C_1,$$

$$\int_{\mathbb{R}^N} \left(t \star w_{b_2, \left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}} \right)^{\alpha+2} dx = e^{\frac{N\alpha t}{2}} \left(\frac{b_2^2}{C_0} \right)^{\frac{\alpha+2}{2}} C_1.$$

Combining the above equalities, we get

$$J \left(t \star \left(w_{b_1, \frac{C_0}{b_1}}, w_{b_2, \frac{C_0}{b_2}} \right) \right) = \frac{\alpha N}{4s(\alpha+2)} e^{2st} \left[\frac{b_1^2}{C_0} C_1 + \frac{b_2^2}{C_0} C_1 \right]$$

$$- \frac{e^{\frac{N\alpha t}{2}}}{\alpha+2} \left[\mu_1 \left(\frac{b_1^2}{C_0} \right)^{\frac{\alpha+2}{2}} C_1 + \mu_2 \left(\frac{b_2^2}{C_0} \right)^{\frac{\alpha+2}{2}} C_1 + 2\beta \left(\frac{b_1^2 b_2^2}{C_0^2} \right)^{\frac{\alpha+2}{4}} C_1 \right].$$

By straightforward calculation, we have

$$\begin{aligned}&\max_{t\in\mathbb{R}} J\left(t\star\left(w_{b_1,\left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2,\left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}}\right)\right)\\&=\frac{\alpha N-4s}{4s(\alpha+2)}\frac{(b_1^2+b_2^2)^{\frac{N\alpha}{N\alpha-4s}}}{\left(\mu_1 b_1^{\alpha+2}+\mu_2 b_2^{\alpha+2}+2\beta b_1^{\frac{\alpha+2}{2}}b_2^{\frac{\alpha+2}{2}}\right)^{\frac{4s}{N\alpha-4s}}}C_0^{\frac{2s(\alpha+2)-N\alpha}{N\alpha-4s}}C_1.\end{aligned}$$

Consequently, by (18), if $\beta>\beta_2$, then

$$\sup_{t\in\mathbb{R}} J\left(t\star\left(w_{b_1,\left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2,\left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}}\right)\right)<\min\left\{\ell(b_1,\mu_1),\ell(b_2,\mu_2)\right\}$$

where β_2 is defined by (9). The proof is now complete. □

In view of Lemma 4.2 and the mountain pass lemma applied to J on the minimax class Γ, it is easy to deduce the following lemma with similar arguments as in the proof of Lemma 3.6.

Lemma 4.4. *There exists a Palais-Smale sequence (u_n,v_n) for J on $\mathrm{S}_{b_1}\times\mathrm{S}_{b_2}$ at the level*

$$d=\inf_{\gamma\in\Gamma}\max_{k\in[0,1]} J(\gamma(k))$$

satisfying

$$\begin{aligned}&\|u_n\|^2_{D_s(\mathbb{R}^N)}+\|v_n\|^2_{D_s(\mathbb{R}^N)}\\&=\frac{N\alpha}{2s(\alpha+2)}\int_{\mathbb{R}^N}\left(\mu_1|u_n|^{\alpha+2}+\mu_2|v_n|^{\alpha+2}+2\beta|u_n|^{\frac{\alpha+2}{2}}|v_n|^{\frac{\alpha+2}{2}}\right)dx+o(1),\end{aligned}\tag{42}$$

where $o(1)\to 0$ as $n\to+\infty$. Furthermore, $u_n^-,v_n^-\to 0$ a.e. in $\mathbb{R}^N$ as $n\to+\infty$.

By Lemmas 3.7–3.10 and the same arguments as in the proof of Theorem 1.1, if $v_n\to\widetilde{v}$ not strongly in $H^s(\mathbb{R}^N)$ and $\lambda_2\geq 0$, then we can deduce that $\widetilde{v}=0$ and $d=\ell(b_1,\mu_1)$.

To get a contradiction, let us consider the path

$$\gamma(k):=(((1-k)t_1+kt_2)\star(w_{b_1,\left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2,\left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}})).$$

We observe that $\gamma\in\Gamma$. Thus, by Lemma 4.3

$$d\leq\max_{k\in[0,1]} J(\gamma(k))\leq\sup_{t\in\mathbb{R}} J\left(t\star\left(w_{b_1,\left(\frac{C_0}{b_1^2}\right)^{\frac{\alpha}{2}}}, w_{b_2,\left(\frac{C_0}{b_2^2}\right)^{\frac{\alpha}{2}}}\right)\right)<\ell(b_1,\mu_1),$$

which is a contradiction. Thus, we obtain a nontrivial positive solution $(\widetilde{u},\widetilde{v})$. Next, we show that this solution is a ground state solution. For this purpose we prove that

$$\begin{aligned}J(\widetilde{u},\widetilde{v})&=\inf\left\{J(u,v):(u,v)\in V\right\}=\inf_{(u,v)\in\mathrm{S}_{b_1}\times\mathrm{S}_{b_2}}\mathcal{R}_b(u,v)\\&=\inf\left\{J(u,v):(u,v)\text{ is a solution of }(3)-(4)\text{ for some }\lambda_1,\lambda_2\right\},\end{aligned}$$

where V and $\mathcal{R}_b(u,v)$ have been defined in (6) and (7).

Recalling the definition of the sets A (see Lemma 4.2) and C (see relation (38)), we set

$$A^+ := \{(u,v) \in A : u,v \geq 0 \text{ a.e in } \mathbb{R}^N\}$$

and

$$C^+ := \{(u,v) \in C : u,v \geq 0 \text{ a.e in } \mathbb{R}^N\}.$$

For any $(u_1,v_1) \in A^+$ and $(u_2,v_2) \in C^+$, let

$$\Gamma(u_1,v_1,u_2,v_2) := \left\{\gamma \in \mathcal{C}([0,1], \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}) : \gamma(0) = (u_1,v_1) \text{ and } \gamma(1) = (u_2,v_2)\right\}.$$

Lemma 4.5. *The sets A^+ and C^+ are connected by arcs, so that*

$$d = \inf_{\gamma \in \Gamma(u_1,v_1,u_2,v_2)} \max_{k \in [0,1]} J(\gamma(k)) \tag{43}$$

for every $(u_1,v_1) \in A^+$ and $(u_2,v_2) \in C^+$.

Proof. The proof uses some ideas found in the proof of Lemma 2.8 in [16]. Once we show that A^+ and C^+ are connected by arcs, then we can get (43). Let $(u_1,v_1),(u_2,v_2) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$ be nonnegative functions such that

$$\|u_1\|^2_{D_s(\mathbb{R}^N)} + \|v_1\|^2_{D_s(\mathbb{R}^N)} = \|u_2\|^2_{D_s(\mathbb{R}^N)} + \|v_2\|^2_{D_s(\mathbb{R}^N)} = \alpha^2 \tag{44}$$

for some $\alpha > 0$. We define

$$h(t,k)(x) := (\cos k(t \star u_1)(x) + \sin k(t \star u_2)(x), \cos k(t \star v_1)(x) + \sin k(t \star v_2)(x))$$

for $t \in \mathbb{R}$ and $k \in [0, \frac{\pi}{2}]$.

Setting $h = (h_1,h_2)$, we observe that $h_1(t,k), h_2(t,k) \geq 0$ a.e. in $\mathbb{R}^N$. By straightforward computation we find

$$\int_{\mathbb{R}^N} h_1^2(t,k)dx = b_1^2 + \sin(2k)\int_{\mathbb{R}^N} u_1u_2dx,$$

$$\int_{\mathbb{R}^N} h_2^2(t,k)dx = b_2^2 + \sin(2k)\int_{\mathbb{R}^N} v_1v_2dx,$$

$$\|h_1(t,k)\|^2_{D_s(\mathbb{R}^N)} + \|h_2(t,k)\|^2_{D_s(\mathbb{R}^N)}$$
$$= e^{2st}\left[\alpha^2 + \sin(2k)\left[\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_1(-\Delta)^{\frac{s}{2}}u_2dx + \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}v_1(-\Delta)^{\frac{s}{2}}v_2dx\right]\right],$$

for all $t \in \mathbb{R}$ and $k \in [0, \frac{\pi}{2}]$. Since $(u_1,v_1),(u_2,v_2) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$ are nonnegative functions, by Hölder's inequality, there exists $C > 0$ such that

$$b_1^2 \leq \int_{\mathbb{R}^N} h_1^2(t,k)dx = b_1^2 + \sin(2k)\int_{\mathbb{R}^N} u_1u_2dx \leq 2b_1^2,$$

$$b_2^2 \leq \int_{\mathbb{R}^N} h_2^2(t,k)dx = b_2^2 + \sin(2k)\int_{\mathbb{R}^N} v_1v_2dx \leq 2b_2^2,$$

$$Ce^{2st} \leq \|h_1(t,k)\|^2_{D_s(\mathbb{R}^N)} + \|h_2(t,k)\|^2_{D_s(\mathbb{R}^N)} \leq 2\alpha^2e^{2st}.$$

Thus, for all $(t,k) \in \mathbb{R} \times [0, \frac{\pi}{2}]$, we define the function

$$\widehat{h}(t,k)(x) := \left(b_1\frac{h_1(t,k)}{\|h_1(t,k)\|_{L^2(\mathbb{R}^N)}}, b_2\frac{h_2(t,k)}{\|h_2(t,k)\|_{L^2(\mathbb{R}^N)}}\right). \tag{45}$$

We observe that $\widehat{h}(t,k)(x) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2}$. Thus, we obtain the following estimate

$$\frac{Ce^{2st}\min\{b_1^2, b_2^2\}}{2\max\{b_1^2, b_2^2\}} \le \|\widehat{h}_1(t,k)\|^2_{D_s(\mathbb{R}^N)} + \|\widehat{h}_2(t,k)\|^2_{D_s(\mathbb{R}^N)} \le \frac{2\alpha^2 e^{2st}\max\{b_1^2, b_2^2\}}{\min\{b_1^2, b_2^2\}}. \tag{46}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^N} \widehat{h}_1^{\alpha+2}(t,k)dx &= \frac{b_1^{\alpha+2}}{\|h_1(t,k)\|^{\alpha+2}_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} h_1^{\alpha+2}(t,k)dx \\ &= \frac{b_1^{\alpha+2}}{\|h_1(t,k)\|^{\alpha+2}_{L^2(\mathbb{R}^N)}} e^{\frac{N\alpha t}{2}} \int_{\mathbb{R}^N} (u_1\cos k + u_2 \sin k)^{\alpha+2}dx,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^N} \widehat{h}_2^{\alpha+2}(t,k)dx &= \frac{b_2^{\alpha+2}}{\|h_2(t,k)\|^{\alpha+2}_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} h_2^{\alpha+2}(t,k)dx \\ &= \frac{b_2^{\alpha+2}}{\|h_2(t,k)\|^{\alpha+2}_{L^2(\mathbb{R}^N)}} e^{\frac{N\alpha t}{2}} \int_{\mathbb{R}^N} (v_1\cos k + v_2 \sin k)^{\alpha+2}dx\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} \widehat{h}_1^{\frac{\alpha+2}{2}}(t,k)\widehat{h}_2^{\frac{\alpha+2}{2}}(t,k)dx \\ &= \frac{b_1^{\frac{\alpha+2}{2}} b_2^{\frac{\alpha+2}{2}}}{\|h_1(t,k)\|^{\frac{\alpha+2}{2}}_{L^2(\mathbb{R}^N)}\|h_2(t,k)\|^{\frac{\alpha+2}{2}}_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} h_1^{\frac{\alpha+2}{2}}(t,k)h_2^{\frac{\alpha+2}{2}}(t,k)dx \\ &= \frac{b_1^{\frac{\alpha+2}{2}} b_2^{\frac{\alpha+2}{2}}}{\|h_1(t,k)\|^{\frac{\alpha+2}{2}}_{L^2(\mathbb{R}^N)}\|h_2(t,k)\|^{\frac{\alpha+2}{2}}_{L^2(\mathbb{R}^N)}} \\ &\times e^{\frac{N\alpha t}{2}} \int_{\mathbb{R}^N} (u_1\cos k + u_2\sin k)^{\frac{\alpha+2}{2}}(v_1\cos k + v_2 \sin k)^{\frac{\alpha+2}{2}}dx.\end{aligned}$$

By these relations, there exists $C > 0$ small enough such that

$$\int_{\mathbb{R}^N} \widehat{h}_1^{\alpha+2}(t,k)dx \ge Ce^{\frac{N\alpha t}{2}} \text{ and } \int_{\mathbb{R}^N} \widehat{h}_2^{\alpha+2}(t,k)dx \ge Ce^{\frac{N\alpha t}{2}}, \tag{47}$$

$$\int_{\mathbb{R}^N} \widehat{h}_1^{\frac{\alpha+2}{2}}(t,k)\widehat{h}_2^{\frac{\alpha+2}{2}}(t,k)dx \ge Ce^{\frac{N\alpha t}{2}}, \tag{48}$$

for all $(t,k) \in \mathbb{R} \times [0, \frac{\pi}{2}]$.

Let $(u_1,v_1), (u_2,v_2) \in A^+$ and $\widehat{h}(t,k)(x)$ defined as (45). By (46), there exists $t_0 > 0$ such that

$$\|\widehat{h}_1(-t_0,k)\|^2_{D_s(\mathbb{R}^N)} + \|\widehat{h}_2(-t_0,k)\|^2_{D_s(\mathbb{R}^N)} \le K \text{ for all } k \in [0, \frac{\pi}{2}],$$

where K has been defined in Lemma 4.2. By the choice of t_0, we let

$$\sigma_1(r) = \begin{cases} -r \star (u_1,v_1) = \widehat{h}(-r,0), & 0 \le r \le t_0, \\ \widehat{h}(-t_0, r-t_0), & t_0 < r \le t_0 + \frac{\pi}{2}, \\ (r - 2t_0 - \frac{\pi}{2}) \star (u_2,v_2) = \widehat{h}(r-2t_0-\frac{\pi}{2}, \frac{\pi}{2}), & t_0 + \frac{\pi}{2} < r \le 2t_0 + \frac{\pi}{2}. \end{cases}$$

It follows that σ_1 is a continuous path connecting (u_1, v_1) and (u_2, v_2) and lying in A^+. To conclude that A^+ is connected by arcs, it remains to analyze the cases when condition (44) is not satisfied. Suppose that

$$\|u_1\|^2_{D_s(\mathbb{R}^N)} + \|v_1\|^2_{D_s(\mathbb{R}^N)} > \|u_2\|^2_{D_s(\mathbb{R}^N)} + \|v_2\|^2_{D_s(\mathbb{R}^N)}.$$

Then, by Lemma 4.1, there exists $t_1 < 0$ such that

$$\|t_1 \star u_1\|^2_{D_s(\mathbb{R}^N)} + \|t_1 \star v_1\|^2_{D_s(\mathbb{R}^N)} = \|u_2\|^2_{D_s(\mathbb{R}^N)} + \|v_2\|^2_{D_s(\mathbb{R}^N)}.$$

Therefore, to connect (u_1, v_1) and u_2, v_2 by a path in A^+ we can first connect (u_1, v_1) with $t_1 \star (u_1, v_1)$, and then connect this point with (u_2, v_2).

To prove that C^+ is connected by arcs, let us fix $(u_1, v_1), (u_2, v_2) \in C^+$ and suppose that (44) holds. By relations (46)–(48), there exists $t_0 > 0$ such that

$$\|\widehat{h}_1(t_0, k)\|^2_{D_s(\mathbb{R}^N)} + \|\widehat{h}_2(t_0, k)\|^2_{D_s(\mathbb{R}^N)} \geq 3K \text{ and } J(\widehat{h}(t_0, k)) \leq 0,$$

for all $k \in [0, \frac{\pi}{2}]$, By the choice of t_0, we let

$$\sigma_2(r) = \begin{cases} r \star (u_1, v_1) = \widehat{h}(r, 0), & 0 \leq r \leq t_0, \\ \widehat{h}(t_0, r - t_0), & t_0 < r \leq t_0 + \frac{\pi}{2}, \\ (2t_0 + \frac{\pi}{2} - r) \star (u_2, v_2) = \widehat{h}(2t_0 + \frac{\pi}{2} - r, \frac{\pi}{2}), & t_0 + \frac{\pi}{2} < r \leq 2t_0 + \frac{\pi}{2}. \end{cases}$$

which is the desired continuous path connecting (u_1, v_1) and (u_2, v_2) in C^+. □

Let us recall the set

$$V := \{(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : G(u, , v) = 0\}$$

and its radial subset

$$V_{rad} := \{(u, v) \in \mathrm{S}_{b_1} \times \mathrm{S}_{b_2} : G(u, , v) = 0\},$$

where

$$G(u, v) = \|(u, v)\|^2_{\mathcal{D}} - \frac{N\alpha}{2s(\alpha + 2)} \int_{\mathbb{R}^N} \left(\mu_1 |u|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}} \right) dx.$$

Lemma 4.6. *If (u, v) is a solution of problem (3)–(4) for some $\lambda_1, \lambda_2 \in \mathbb{R}$, then $(u, v) \in V$.*

Proof. The Pohozaev identity for (3) is

$$\begin{aligned} &\frac{N - 2s}{2} ||(u, v)||^2_{\mathcal{D}} - \frac{N}{2} \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2) dx \\ &= \frac{N}{\alpha + 2} \int_{\mathbb{R}^N} \left(\mu_1 |u|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}} \right) dx. \end{aligned}$$

On the other hand,

$$||(u, v)||^2_{\mathcal{D}} - \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2) dx = \int_{\mathbb{R}^N} \left(\mu_1 |u|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}} \right) dx.$$

Combining the above two equalities, we get the desired result. □

For $(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, we set $\Psi_{u,v}(t) := J(t \star (u, v))$. The proof of the following lemma follows by straightforward computation and we shall omit it.

Lemma 4.7. *For every $(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, there exists a unique $t_{u,v} \in \mathbb{R}$ such that $t_{u,v} \star (u, v) \in V$ and $t_{u,v}$ is the unique critical point of $\Psi_{u,v}(t)$, which is a strict maximum.*

Lemma 4.8. *We have* $\inf_V J = \inf_{V_{rad}} J$.

Proof. Assume by contradiction that there exists $(u, v) \in V$ such that

$$0 < J(u, v) < \inf_{V_{rad}} J. \tag{49}$$

For $u \in H^s(\mathbb{R}^N)$, let u^* be its Schwartz spherical rearrangement. By the properties of the Schwartz symmetrization, $J(u^*, v^*) \leq J(u, v)$ and $G(u^*, v^*) \leq G(u, v)$. Thus, there exists $t_0 \leq 0$ such that $G(t_0 \star (u^*, v^*)) = 0$. We show that

$$J(t_0 \star (u^*, v^*)) \leq e^{2st_0} J(u, v).$$

Since $G(t_0 \star (u^*, v^*)) = G(u, v) = 0$, we have

$$\begin{aligned} J(t_0 \star (u^*, v^*)) &= \frac{N\alpha - 4s}{2N\alpha} e^{2st_0} ||(u^*, v^*)||^2_{\mathcal{D}} \\ &\leq \frac{N\alpha - 4s}{2N\alpha} e^{2st_0} ||(u, v)||^2_{\mathcal{D}} = e^{2st_0} J(u, v). \end{aligned}$$

Therefore

$$0 < J(u, v) < \inf_{V_{rad}} J \leq J(t_0 \star (u^*, v^*)) \leq e^{2st_0} J(u, v),$$

which contradicts the fact that $t_0 \leq 0$. □

4.1. **Proof of Theorem 1.2 completed.** Notice that any solution of problem (3)–(4) is in V. If we assume that

$$J(\overline{u}, \overline{v}) = d \leq \inf\{J(u, v) : (u, v) \in V_{rad}\}, \tag{50}$$

then, by Lemma 4.8, we obtain $J(\overline{u}, \overline{v}) = \inf_V J(u, v)$.

Next, we prove relation (50). We choose arbitrarily $(u, v) \in V_{rad}$ and show that $J(u, v) \geq d$. Firstly, since $(u, v) \in V_{rad}$, we have $(|u|, |v|) \in V_{rad}$ and $J(u, v) = J(|u|, |v|)$, hence we can suppose that $u, v \geq 0$ a.e in $\mathbb{R}^N$. Let us consider the function $\Psi_{u,v}$. By Lemma 4.1, there exists $t_0 \gg 1$ such that $-t_0 \star (u, v) \in A^+$ and $t_0 \star (u, v) \in C^+$. Therefore, the continuous path

$$\gamma(k)((2k - 1)t_0) \star (u, v), \quad t \in [0, 1]$$

connects A^+ and C^+. By Lemma 4.5 and Lemma 4.7, we have

$$d \leq \max_{k \in [0,1]} J(\gamma(k)) = J(u, v).$$

Thus, we have proved (50).

Finally, we claim that

$$\inf_V J = \inf_{\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}} \mathcal{R}_b(u, v).$$

In fact, since the proof is similar to that of Lemma 2.5, we give a concise treatment for the reader's convenience. Since $(u, v) \in V$, then

$$\frac{\|(u, v)\|^2_{\mathcal{D}}}{\frac{N\alpha}{2s(\alpha+2)} \int_{\mathbb{R}^N} \left(\mu_1 |u|^{\alpha+2} + \mu_2 |v|^{\alpha+2} + 2\beta |u|^{\frac{\alpha+2}{2}} |v|^{\frac{\alpha+2}{2}} \right) dx} = 1$$

and

$$J(u, v) = \frac{N\alpha - 4s}{2N\alpha} ||(u, v)||^2_{\mathcal{D}}.$$

Therefore

$$
\begin{aligned}
&J(u,v)\\
&=\frac{N\alpha-4s}{2N\alpha}||(u,v)||^2_{\mathcal{D}}\\
&\times\left(\frac{\|(u,v)\|^2_{\mathcal{D}}}{\frac{N\alpha}{2s(\alpha+2)}\int_{\mathbb{R}^N}\left(\mu_1|u|^{\alpha+2}+\mu_2|v|^{\alpha+2}+2\beta|u|^{\frac{\alpha+2}{2}}|v|^{\frac{\alpha+2}{2}}\right)dx}\right)^{\frac{4s}{\alpha N-4s}}\\
&=\frac{N\alpha-4s}{2N\alpha}\left(\frac{2s(\alpha+2)}{N\alpha}\right)^{\frac{4s}{N\alpha-4s}}\frac{\left(||(u,v)||^2_{\mathcal{D}}\right)^{\frac{\alpha N}{\alpha N-4s}}}{\left(\int_{\mathbb{R}^N}\left(\mu_1u^{\alpha+2}+\mu_2v^{\alpha+2}+2\beta u^{\frac{\alpha+2}{2}}v^{\frac{\alpha+2}{2}}\right)dx\right)^{\frac{4s}{\alpha N-4s}}}\\
&=\mathcal{R}_b(u,v),
\end{aligned}
$$

which implies that

$$\inf_V J\geq\inf_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}\mathcal{R}_b(u,v).$$

On the other hand, it is easy to check that

$$\mathcal{R}_b(t\star(u,v))=\mathcal{R}_b(u,v)\text{ for all }t\in\mathbb{R},\ (u,v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}.$$

By Lemma 4.7, we have

$$\mathcal{R}_b(u,v)=\mathcal{R}_b(t_{u,v}\star(u,v))=J(t_{u,v}\star(u,v))\geq\inf_V J$$

for every $(u,v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. The proof is now complete. □

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