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# CONCENTRATION PHENOMENA FOR MAGNETIC KIRCHHOFF EQUATIONS WITH CRITICAL GROWTH

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Abstract. In this paper, we study the following nonlinear magnetic Kirchhoff equation with critical growth

$$\begin{cases} -\left(a\epsilon^2 + b\epsilon\,[u]^2_{A/\epsilon}\right)\Delta_{A/\epsilon}u + V(x)u = f(|u|^2)u + |u|^4u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$ is a parameter, $a, b > 0$ are constants, $V : \mathbb{R}^3 \to \mathbb{R}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ are continuous potentials, and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear term with subcritical growth. Under a local assumption on the potential $V$, combining variational methods, penalization techniques and the Ljusternik-Schnirelmann theory, we establish multiplicity and concentration properties of solutions to the above problem for $\varepsilon$ small. A feature of this paper is that the function $f$ is assumed to be only continuous, which allows to consider larger classes of nonlinearities in the reaction.

1. **Introduction.** In this paper, we are concerned with multiplicity and concentration phenomena of the solutions for the following nonlinear magnetic Kirchhoff equation with critical growth

$$\begin{cases} -\left(a\epsilon^2 + b\epsilon\,[u]^2_{A/\epsilon}\right)\Delta_{A/\epsilon}u + V(x)u = f(|u|^2)u + |u|^4u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases} \tag{1.1}$$

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where $\epsilon > 0$ is a parameter, $a, b > 0$ are constants, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, the magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ is Hölder continuous with exponent $\alpha \in (0, 1]$, and $-\Delta_A u$ is the magnetic Laplace operator with the following form

$$-\Delta_A u := \Big(\frac{1}{i}\nabla - A(x)\Big)^2 u = -\Delta u - \frac{2}{i} A(x) \cdot \nabla u + |A(x)|^2 u - \frac{1}{i} u \operatorname{div}(A(x)),$$

while the definition of $[u]_A^2$ will be given in Section 2.

For problem (1.1), there is a vast literature concerning the existence and concentration of solutions for the case without magnetic potential, that is, if $A \equiv 0$. The first result in this direction was given by Floer and Weinstein in [8], where the case $N = 1$ and $f = i_{\mathbb{R}}$ is considered. Later on, by using different methods, several authors generalized this result to larger values of $N$, see [6, 21, 22, 24, 27]. In [6], del Pino and Felmer studied the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{cases}$$

where $\Omega$ is a possibly unbounded domain in $\mathbb{R}^N$, $N \geq 3$. By introducing a penalization method, they found solutions of the above problem that concentrate around the local minimum of $V$. More precisely, the authors assumed that there exists a bounded open set $\Lambda \subset \Omega$ such that

$$\inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x), \tag{1.2}$$

and the nonlinearity $f$ satisfies the subcritical growth condition. In [10], He and Zou considered the following fractional Schrödinger equation

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + u^{2_s^*-1}, \ x \in \mathbb{R}^N,$$

where $V$ satisfies the local assumption (1.2), and $f$ is subcritical. By using the Nehari manifold method and the Ljusternik-Schnirelmann category theory, the authors obtained the multiplicity of positive solutions. We notice that $f$ is only continuous in [10], hence the Nehari manifold is only a topological manifold. Thus, the critical point theory in $C^1$ manifold cannot be applied in this framework. To overcome this difficulty, He and Zou [10] used some variants of critical point theorems developed by Szulkin and Weth [25]. There are also many results about the existence, multiplicity and concentration of solutions for Kirchhoff equations, that is, provided that $A = 0$ and $a, b > 0$, see [9, 12, 18, 23, 29, 30] (see also [11] for the fractional case). In [23], Perera and Zhang studied the existence of solutions for Kirchhoff equation by using the Yang index and critical groups. Later on, He and Zou [9] studied the existence, multiplicity and concentration properties of positive solutions for the problem (1.1) without critical term and magnetic field by using the Nehari manifold method, the penalization technique and the Ljusternik-Schnirelmann category theory. We notice that the nonlinear term $f$ is a $C^1$ function in this paper, which allows to apply the critical point theory in $C^1$ manifolds. Next, He and Zou [10] applied the method introduced by Szulkin and Weth [25] and studied multiplicity and concentration behavior of positive solutions for a fracional Kirchhoff equation where the nonlinear term $f$ is only continuous.

On the other hand, when $a = b = 0$, the nonlinear magnetic Schrödinger equation (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [1, 2, 3, 4, 5, 7, 13, 14, 16, 20, 31] and references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [7]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we also refer to Ji and Rădulescu [15] who used Nehari manifold analysis, penalization techniques and the Ljusternik-Schnirelmann category theory to study multiplicity and concentration results for a magnetic Schrödinger equation in which the subcritical nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$. After that, Ji and Rădulescu [16] studied multiplicity and concentration of the solutions for the magnetic Schrödinger equation with critical growth. To the best of our knowledge, there are few results in the literature on the magnetic Kirchhoff equations. Recently, Ji and Rădulescu [17] considered multiplicity and concentrating phenomena of nontrivial solutions for magnetic Kirchhoff equations with subcritical growth and assuming that nonlinearity $f$ is only continuous.

It is quite natural to consider multiplicity and concentrating phenomena of nontrivial solutions for the problem (1.1) with *critical* growth. Inspired by [15, 17], the main purpose of this paper is to investigate qualitative properties of solutions to problem (1.1) by combining a local assumption on $V$ and adapting the penalization method and Ljusternik-Schnirelmann category theory.

Throughout the paper, we make the following assumptions on the potential $V$:

$(V_1)$ There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$;
$(V_2)$ There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let $f \in C(\mathbb{R}, \mathbb{R})$ be a nonlinearity satisfying:

$(f_1)$ $f(t) = 0$ if $t \leq 0$, and $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$;
$(f_2)$ There exist $\sigma, q \in (4, 6)$ and $\mu > 0$ such that

$$f(t) \geq \mu t^{\frac{\sigma-2}{2}} \quad \forall t > 0, \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0;$$

$(f_3)$ There exists a positive constant $4 < \theta < 6$ such that

$$0 < \frac{\theta}{2} F(t) \leq t f(t), \ \forall\, t > 0, \ \text{where } F(t) = \int_0^t f(s) ds;$$

$(f_4)$ The mapping $t \mapsto \frac{f(t)}{t}$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following multiplicity and concentration properties of solutions.

**Theorem 1.1.** *Assume that $V$ satisfies $(V_1)$, $(V_2)$ and $f$ satisfies $(f_1)$–$(f_4)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

*there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by $u_{\varepsilon_n}$ one of these solutions of (1.1) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^3$ is the global maximum point of $|u_{\varepsilon_n}|$, then*

$$\lim_{\varepsilon_n \to 0^+} V(\eta_{\varepsilon_n}) = V_0.$$

The proof of Theorem 1.1 uses some ideas introduced in [15, 17]. Notice that, due to the presence of the magnetic field $A(x)$, the problem (1.1) cannot be changed into a pure real-valued problem, hence we must deal directly with a complex-valued problem. This fact creates several new difficulties in employing the methods to deal with our problem. On the other hand, due to the presence of the nonlocal term, it is possible that the weak limit of a bounded Palais-Smale sequence of the Kirchhoff equation is not a solution of this equation. Moreover, since the problem we deal with has critical growth, we need more refined estimates to overcome the lack of compactness.

The present paper is organized as follows. In Section 2 we introduce the functional setting and give some preliminaries. In Section 3, we study an auxilliary problem. We prove the Palais-Smale condition for the auxilliary functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the associated autonomous problem. This allows us to show that the auxilliary problem has multiple solutions. Finally, in Section 5, we give the proof of Theorem 1.1.

**Notation.**

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- $B_R(y)$ denotes the open ball centered at $y \in \mathbb{R}^3$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R})$, $L^q(\mathbb{R}^3, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$. $\langle \cdot, \cdot \rangle_0$ denotes the inner product of the space $H^1(\mathbb{R}^3, \mathbb{R})$.

2. **Variational framework and the limit problem.** For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u.$$

Consider the function spaces

$$D_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R})\},$$

and

$$H_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in D_A^1(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C})\}.$$

Then $D_A^1(\mathbb{R}^3, \mathbb{C})$ and $H_A^1(\mathbb{R}^3, \mathbb{C})$ are Hilbert spaces endowed with the scalar products

$$\langle u, v\rangle_D := \mathrm{Re} \int_{\mathbb{R}^3} \nabla_A u \overline{\nabla_A v} dx, \quad \text{for any } u, v \in D_A^1(\mathbb{R}^3, \mathbb{C}),$$

$$\langle u, v\rangle_H := \mathrm{Re} \int_{\mathbb{R}^3} \Big(\nabla_A u \overline{\nabla_A v} + u\overline{v}\Big) dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Let $\|u\|_D$ and $\|u\|_A$ denote the norms induced by inner products $\langle u, v\rangle_D$ and $\langle u, v\rangle_A$, and $\langle u, u\rangle_D = [u]_A^2$.

On $H_A^1(\mathbb{R}^3, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see e.g. [19, Theorem 7.21]):

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Moreover, making a simple change of variables, since $\Delta_{A_\varepsilon} = \epsilon^2 \Delta_{A/\epsilon}$ and $[u]^2_{A_\varepsilon} = \frac{1}{\epsilon}[u]^2_{A/\epsilon}$, we can see that the problem (1.1) is equivalent to

$$-\left(a + b\,[u]^2_{A_\varepsilon}\right)\Delta_{A_\varepsilon} u + V_\varepsilon(x)u = f(|u|^2)u + |u|^4 u \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Let $D_\varepsilon$ and $H_\varepsilon$ be the Hilbert spaces obtained as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the scalar products

$$\langle u, v\rangle_{D_\epsilon} := \operatorname{Re}\int_{\mathbb{R}^3} a\nabla_{A_\varepsilon} u\overline{\nabla_{A_\varepsilon} v}dx$$

and

$$\langle u, v\rangle_{\epsilon} := \operatorname{Re}\int_{\mathbb{R}^3} \left(a\nabla_{A_\varepsilon} u\overline{\nabla_{A_\varepsilon} v} + V_\varepsilon(x)u\overline{v}\right)dx$$

respectively, where $a > 0$. We denote by $\|\cdot\|_{D_\varepsilon}$ and $\|\cdot\|_\varepsilon$ the norms induced by inner products $\langle\cdot,\cdot\rangle_{D_\epsilon}$ and $\langle\cdot,\cdot\rangle_\epsilon$.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_{A_\varepsilon}(\mathbb{R}^3, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\|u\| \le C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^3, \mathbb{C})$ is continuous for $2 \le r \le 6$ and the embedding $H_\varepsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^3, \mathbb{C})$ is compact for $1 \le r < 6$.

For the compact supported functions in $H^1(\mathbb{R}^3, \mathbb{R})$, we have the following result, which will be very uesful for some estimates below.

**Lemma 2.1.** *If $u \in H^1(\mathbb{R}^3, \mathbb{R})$ and $u$ has compact support, then $\omega := e^{iA(0)\cdot x}u \in H_\varepsilon$.*

*Proof.* Assume that $\operatorname{supp}(u) \subset B_R(0)$. Since $V$ is continuous, it is clear that

$$\int_{\mathbb{R}^3} V_\varepsilon(x)|\omega|^2 dx = \int_{B_R(0)} V_\varepsilon(x)|\omega|^2 dx \le C\|u\|_2^2 < +\infty.$$

Moreover, since $V$ and $A$ are continuous, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla_{A_\epsilon}\omega|^2 dx &= \int_{\mathbb{R}^3} |\nabla\omega|^2 dx + \int_{\mathbb{R}^3} |A_\varepsilon(x)|^2|\omega|^2 dx + 2\operatorname{Re}\int_{\mathbb{R}^3} iA_\varepsilon(x)\overline{\omega}\nabla\omega dx \\ &\le 2\int_{\mathbb{R}^3} |\nabla\omega|^2 dx + 2\int_{\mathbb{R}^3} |A_\varepsilon(x)|^2|\omega|^2 dx \\ &\le C\left[\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |u|^2 dx\right] < +\infty\end{aligned}$$

and we conclude. □

## 3. **The auxilliary problem.**

To study problem (1.1), or equivalently, the problem (2.2) by variational methods, we shall modify suitably the nonlinearity $f$ so that, for $\varepsilon > 0$ small enough, the solutions of such auxilliary problem are also solutions of the original one. More precisely, choosing $K > 2$, then there exists a unique number $a_0 > 0$ verifying $f(a_0) + a_0^2 = V_0/K$ by $(f_4)$, where $V_0$ is given in $(V_1)$. Consider the function

$$\tilde{f}(t) := \begin{cases} f(t) + (t^+)^2, & t \le a_0, \\ V_0/K, & t > a_0, \end{cases}$$

and introduce the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x, t) := \chi_\Lambda(x)(f(t) + (t^+)^2) + (1 - \chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where $\chi_\Lambda$ is the characteristic function on $\Lambda$ and $G(x,t) := \displaystyle\int_0^t g(x,s)ds$.

In view of $(f_1)$–$(f_4)$, we have that $g$ is a Carathéodory function satisfying the following properties:

$(g_1)$ $g(x,t) = 0$ for each $t \leq 0$;
$(g_2)$ $\lim\limits_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;
$(g_3)$ $g(x,t) \leq f(t) + t^2$ for all $t \geq 0$ and any $x \in \mathbb{R}^3$;
$(g_4)$ $0 < \theta G(x,t) \leq 2g(x,t)t$ for each $x \in \Lambda$ and $t > 0$;
$(g_5)$ $0 < G(x,t) \leq g(x,t)t \leq V_0 t/K$, for each $x \in \Lambda^c$, $t > 0$;
$(g_6)$ for each $x \in \Lambda$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0,+\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0,a_0)$.

Next, we consider the *auxilliary* problem

$$-\Big(a + b\,[u]^2_{A_\varepsilon}\Big)\Delta_{A_\varepsilon} u + V_\varepsilon(x)u = g(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{3.2}$$

Note that, if $u$ is a nontrivial solution of the problem (3.2) with

$$|u(x)|^2 \leq a_0 \quad \text{for all } x \in \Lambda^c_\varepsilon, \quad \Lambda_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in \Lambda\},$$

then $u$ is a nontrivial solution of the problem (2.2).

The energy functional associated to the problem (3.2) is

$$J_\varepsilon(u) := \frac{a}{2}[u]^2_{A_\varepsilon} + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2dx + \frac{b}{4}[u]^4_{A_\varepsilon} - \frac{1}{2}\int_{\mathbb{R}^3} G(\varepsilon x, |u|^2)dx$$

for all $u \in H_\varepsilon$. By standard arguments, $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its critical points are the weak solutions of the auxilliary problem (3.2).

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of $J_\varepsilon$, that is,

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \backslash \{0\} : J'_\varepsilon(u)[u] = 0\},$$

and define the number $c_\varepsilon$ by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Let $H^+_\varepsilon$ be an open subset $H_\varepsilon$ given by

$$H^+_\varepsilon = \{u \in H_\varepsilon : |\mathrm{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S^+_\varepsilon = S_\varepsilon \cap H^+_\varepsilon$, where $S_\varepsilon$ is the unit sphere of $H_\varepsilon$. Note that $S^+_\varepsilon$ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on $H_\varepsilon$ and contained in $H^+_\varepsilon$. Therefore, $H_\varepsilon = T_u S^+_\varepsilon \bigoplus \mathbb{R}u$ for each $u \in T_u S^+_\varepsilon$, where $T_u S^+_\varepsilon = \{v \in H_\varepsilon : \langle u, v\rangle_\epsilon = 0\}$.

Now we show that the functional $J_\varepsilon$ satisfies the Mountain Pass Geometry (see [28]).

**Lemma 3.1.** *For any fixed $\varepsilon > 0$, the functional $J_\varepsilon$ satisfies the following properties:*

(i) *there exist $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$;*
(ii) *there exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > r$ such that $J_\varepsilon(e) < 0$.*

*Proof.* (i) By $(g_2)$, $(g_3)$ and $(f_2)$, for any $\zeta > 0$ small, there exists $C_\zeta > 0$ such that

$$G(\varepsilon x, |u|^2) \leq \zeta|u|^4 + C_\zeta|u|^6 \quad \text{for all } x \in \mathbb{R}^3.$$

By the Sobolev embedding it follows that

$$J_\varepsilon(u) \geq \frac{a}{2}[u]^2_{A_\varepsilon} + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2dx + \frac{b}{4}[u]^4_{A_\varepsilon} - \frac{\zeta}{2}\int_{\mathbb{R}^3}|u|^4dx - \frac{C_\zeta}{2}\int_{\mathbb{R}^3}|u|^6dx$$
$$\geq \frac{1}{2}\|u_n\|^2_\varepsilon - C_1\zeta\|u_n\|^4_\varepsilon - C_2C_\zeta\|u_n\|^6_\varepsilon.$$

Hence we can choose some $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$ small.
(ii) For each $u \in H_\varepsilon \setminus \{0\}$ with $\operatorname{supp}(u) \subset \Lambda_\varepsilon$. By the definition of $g$ and $(f_3)$, we have

$$J_\varepsilon(tu) = \frac{t^2}{2}\|u\|^2_\varepsilon + \frac{bt^4}{4}[u]^4_{A_\varepsilon} - \frac{1}{2}\int_{\Lambda_\varepsilon} F(t^2|u|^2)dx - \frac{t^6}{6}\int_{\Lambda_\varepsilon}|u|^6dx,$$
$$\leq \frac{t^2}{2}\|u\|^2_\varepsilon + \frac{bt^4}{4}[u]^4_{A_\varepsilon} - \frac{t^6}{6}\int_{\Lambda_\varepsilon}|u|^6dx,$$

hence $J_\varepsilon(tu) \to -\infty$ as $t \to +\infty$ and the conclusion follows. □

Since $f$ is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $S^+_\varepsilon$.

**Lemma 3.2.** *Assume that $(V_1)$–$(V_2)$ and $(f_1)$–$(f_4)$ are satisfied, then the following properties hold:*

- (A1) *For any $u \in H^+_\varepsilon$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\varepsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g'_u(t) > 0$ in $(0, t_u)$ and $g'_u(t) < 0$ in $(t_u, \infty)$;*
- (A2) *There is a $\tau > 0$ independent on $u$ such that $t_u \geq \tau$ for all $u \in S^+_\varepsilon$. Moreover, for each compact $\mathcal{W} \subset S^+_\varepsilon$ there is $C_\mathcal{W}$ such that $t_u \leq C_\mathcal{W}$, for all $u \in \mathcal{W}$;*
- (A3) *The map $\widehat{m}_\varepsilon : H^+_\varepsilon \to \mathcal{N}_\varepsilon$ given by $\widehat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \widehat{m}_\varepsilon|_{S^+_\varepsilon}$ is a homeomorphism between $S^+_\varepsilon$ and $\mathcal{N}_\varepsilon$. Moreover, $m^{-1}_\varepsilon(u) = \frac{u}{\|u\|_\varepsilon}$;*
- (A4) *If $\{u_n\} \subset S^+_\varepsilon$ is a sequence such that $\operatorname{dist}(u_n, \partial S^+_\varepsilon) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

*Proof.* $(A1)$ Arguing as in the proof of Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ verifying $g'_u(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Now, we show that $t_u$ is unique. Arguing by contradiction, suppose that there exist $t_1 > t_2 > 0$ such that $g'_u(t_1) = g'_u(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i a[u]^2_{A_\varepsilon} + t_i\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2dx + t_i^3 b[u]^4_{A_\varepsilon} = \int_{\mathbb{R}^3} g(\varepsilon x, t_i^2|u|^2)t_i|u|^2dx.$$

Hence,

$$\frac{a[u]^2_{A_\varepsilon} + \int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2dx}{t_i^2} + b[u]^4_{A_\varepsilon} = \int_{\mathbb{R}^3}\frac{g(\varepsilon x, t_i^2|u|^2)|u|^2}{t_i^2}dx,$$

which implies that

$$\begin{aligned}&\Big(\frac{1}{t_1^2}-\frac{1}{t_2^2}\Big)\Big(a[u]_{A_\varepsilon}^2+\int_{\mathbb{R}^3}V_\varepsilon(x)|u|^2dx\Big)\\
=&\int_{\mathbb{R}^3}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
\geq&\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
&+\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
=&\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2}-\frac{f(t_2^2|u|^2)+t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4dx\\
&+\frac{1}{K}(\frac{1}{t_1^2}-\frac{1}{t_2^2})\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}V_0|u|^2dx.\end{aligned}$$

Since $t_1>t_2>0$, we have

$$\begin{aligned}&\Big(a[u]_{A_\varepsilon}^2+\int_{\mathbb{R}^3}V_\varepsilon(x)|u|^2dx\Big)\\
\leq&\frac{t_1^2t_2^2}{t_2^2-t_1^2}\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2}-\frac{f(t_2^2|u|^2)+t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4dx\\
&+\frac{1}{K}\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}V_0|u|^2dx\\
\leq&\frac{1}{K}\int_{\Lambda_\varepsilon^c}V_0|u|^2dx\leq\frac{1}{K}\|u\|_\varepsilon^2,\end{aligned}$$

which is a contradiction. Therefore, $\max_{t\geq 0}g_u(t)$ is achieved at a unique $t=t_u$ so that $g_u'(t)=0$ and $t_uu\in\mathcal{N}_\varepsilon$.

$(A2)$ For $\forall\, u\in S_\varepsilon^+$, by $(A1)$, there exists a unique $t_u>0$ such that

$$t_u+t_u^3b[u]_{A_\varepsilon}^4=\int_{\mathbb{R}^3}g(\varepsilon x,t_u^2|u|^2)t_u|u|^2dx.$$

From $(g_2)$, the Sobolev embeddings and $q>4$, we get

$$t_u\leq\zeta t_u^3\int_{\mathbb{R}^3}|u|^4dx+C_\zeta t_u^{q-1}\int_{\mathbb{R}^3}|u|^qdx+t_u^5\int_{\mathbb{R}^3}|u|^6dx\leq C_1\zeta t_u^3+C_2C_\zeta t_u^{q-1}+C_3t_u^5,$$

which implies that $t_u\geq\tau$ for some $\tau>0$. If $\mathcal{W}\subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\}\subset\mathcal{W}$ with $t_n:=t_{u_n}\to\infty$. Since $\mathcal{W}$ is compact, there exists a $u\in\mathcal{W}$ such that $u_n\to u$ in $H_\varepsilon$. Moreover, using the proof of Lemma 3.1(ii), we have that $J_\varepsilon(t_nu_n)\to-\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\varepsilon$, from the definition of $g$ and $(g_4)$, $(g_5)$ and $4 < \theta < 6$, it yields that

$$\begin{aligned}
J_\varepsilon(v_n) =& J_\varepsilon(v_n) - \frac{1}{\theta} J'_\varepsilon(v_n)[v_n] \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|v_n\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{\theta}) b[v_n]_{A_\varepsilon}^4 \\
&+ \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |v_n|^2)|v_n|^2 - \frac{1}{2} G(\varepsilon x, |v_n|^2)\Big) dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|v_n\|_\varepsilon^2 - \frac{1}{K}\int_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 dx\Big) \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K})\|v_n\|_\varepsilon^2.
\end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\varepsilon = t_n$, we obtain

$$0 < \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K}) \leq \frac{J_\varepsilon(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves $(A2)$.

$(A3)$ First of all, we note that $\widehat{m}_\varepsilon$, $m_\varepsilon$ and $m_\varepsilon^{-1}$ are well defined. Indeed, by $(A2)$, for each $u \in H_\varepsilon^+$, there is a unique $\widehat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in H_\varepsilon^+$. Otherwise, we have $|\mathrm{supp}(u) \cap \Lambda_\varepsilon| = 0$ and by $(g_5)$, it follows

$$\begin{aligned}
\|u\|_\varepsilon^2 + b[u]_{A_\varepsilon}^4 &= \int_{\mathbb{R}^3} g(\varepsilon x, |u|^2)|u|^2 dx \\
&= \int_{\Lambda_\varepsilon^c} g(\varepsilon x, |u|^2)|u|^2 dx \\
&\leq \frac{1}{K}\int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx \\
&\leq \frac{1}{K}\|u\|_\varepsilon^2
\end{aligned}$$

which is impossible since $K > 2$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$ is well defined and continuous. From

$$m_\varepsilon^{-1}\big(m_\varepsilon(u)\big) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\varepsilon} = u, \ \ \forall\, u \in S_\varepsilon^+,$$

we conclude that $m_\varepsilon$ is a bijection. We now prove that $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous. Let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ be such that $u_n \to u$ in $H_\varepsilon$. By $(A2)$, there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\varepsilon$, we obtain

$$t_n^2 a[u_n]_{A_\varepsilon}^2 + t_n^2\int_{\mathbb{R}^3} V_\varepsilon(x)|u_n|^2 dx + t_n^4 b[u_n]_{A_\varepsilon}^4 = \int_{\mathbb{R}^3} g(\varepsilon x, t_n^2|u_n|^2)t_n^2|u_n|^2 dx, \ \ \forall\, n \in N,$$

and passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$t_0^2 a[u]_{A_\varepsilon}^2 + t_0^2\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2 dx + t_0^4 b[u]_{A_\varepsilon}^4 = \int_{\mathbb{R}^3} g(\varepsilon x, t_0^2|u|^2)t_0^2|u|^2 dx.$$

We obtain that $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. This proves $\widehat{m}_\varepsilon(u_n) \to \widehat{m}_\varepsilon(u)$ in $H_\varepsilon^+$. Thus, $\widehat{m}_\varepsilon$ and $m_\varepsilon$ are continuous functions and $(A3)$ is proved.

$(A4)$ Let $\{u_n\} \subset S_\varepsilon^+$ be a subsequence such that $\mathrm{dist}\,(u_n, \partial S_\varepsilon^+) \to 0$, then for each $v \in \partial S_\varepsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in $\Lambda_\varepsilon$. Therefore, by $(V_1)$,

$(V_2)$ and the Sobolev embedding, for any $r \in [2,6]$, there exists a constant $C_r > 0$ such that

$$\begin{aligned}\|u_n\|_{L^r(\Lambda_\varepsilon)} &\leq \inf_{v\in\partial S_\varepsilon^+}\|u_n - v\|_{L^r(\Lambda_\varepsilon)} \\ &\leq C_r\Big(\inf_{v\in\partial S_\varepsilon^+}\int_{\Lambda_\varepsilon}(|\nabla_{A_\varepsilon}u_n - v|^2 + V_\varepsilon(x)|u_n - v|^2)dx\Big)^{\frac{1}{2}} \\ &\leq C_r \operatorname{dist}(u_n, \partial S_\varepsilon^+)\end{aligned}$$

for all $n \in N$. By $(g_2)$, $(g_3)$ and $(g_5)$, for each $t > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} G(\varepsilon x, t^2|u_n|^2)dx &\leq \int_{\Lambda_\varepsilon}\Big(F(t^2|u_n|^2) + \frac{t^6|u_n|^6}{6}\Big)dx + \frac{t^2}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2dx \\ &\leq C_1t^4\int_{\Lambda_\varepsilon}|u_n|^4dx + C_2t^q\int_{\Lambda_\varepsilon}|u_n|^qdx + \frac{t^6}{6}\int_{\Lambda_\varepsilon}|u_n|^6dx + \frac{t^2}{K}\|u_n\|_\varepsilon^2 \\ &\leq C_3t^4\operatorname{dist}(u_n, \partial S_\varepsilon^+)^4 + C_4t^q\operatorname{dist}(u_n, \partial S_\varepsilon^+)^q + C_5t^6\operatorname{dist}(u_n, \partial S_\varepsilon^+)^6 + \frac{t^2}{K}.\end{aligned}$$

Therefore,

$$\limsup_n \int_{\mathbb{R}^3} G(\varepsilon x, t^2|u_n|^2)dx \leq \frac{t^2}{K},\ \forall\, t > 0.$$

On the other hand, from the definition of $m_\varepsilon$ and the last inequality, for all $t > 0$, one has

$$\begin{aligned}\liminf_n J_\varepsilon(m_\varepsilon(u_n)) &\geq \liminf_n J_\varepsilon(tu_n) \\ &\geq \liminf_n \frac{t^2}{2}\|u_n\|_\varepsilon^2 - \frac{t^2}{K} \\ &= \frac{K-2}{2K}t^2,\end{aligned}$$

this implies that

$$\liminf_n \Big\{\frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 + \frac{b}{4}[m_\varepsilon(u_n)]_{A_\varepsilon}^4\Big\} \geq \liminf_n J_\varepsilon(m_\varepsilon(u_n)) \geq \frac{K-2}{2K}t^2,\ \forall\, t > 0.$$

From the arbitrary of $t > 0$, it is easy to see that $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. We conclude the proof of Lemma 3.2. $\square$

Now we define the function

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R},$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and we denote $\Psi_\varepsilon := (\widehat{\Psi}_\varepsilon)|_{S_\varepsilon^+}$.

From Lemma 3.2, arguing as in [26, Corollary 10] we may obtain the following result.

**Lemma 3.3.** *Assume that $(V_1)$–$(V_2)$ and $(f_1)$–$(f_4)$ are satisfied, then*

*(B1)* $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\varepsilon'(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\epsilon}{\|u\|_\epsilon}J_\varepsilon'(\widehat{m}_\varepsilon(u))[v],\ \ \forall\, u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon;$$

*(B2)* $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*

$$\Psi_\varepsilon'(u)v = \|m_\varepsilon(u)\|_\epsilon J_\varepsilon'(\widehat{m}_\varepsilon(u))[v],\ \ \forall\, v \in T_uS_\varepsilon^+;$$

*(B3) If $\{u_n\}$ is a $(PS)_c$ sequence of $\Psi_\varepsilon$, then $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence of $J_\varepsilon$. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_c$ sequence of $J_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}$ is a $(PS)_c$ sequence of $\Psi_\varepsilon$;*

$(B4)$ *$u$ is a critical point of $\Psi_\varepsilon$ if and only if $m_\varepsilon(u)$ is a critical point of $J_\varepsilon$. Moreover, the corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon. = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

As in [26], we have the following variational characterization of the infimum of $J_\varepsilon$ over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u\in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu). \tag{3.3}$$

The following result is important to prove the $(PS)_c$ condition for the functional $J_\varepsilon$.

**Lemma 3.4.** *Let $c > 0$ and $\{u_n\}$ is a $(PS)_c$ sequence for $J_\epsilon$, then $\{u_n\}$ is bounded in $H_\epsilon$.*

*Proof.* Assume that $\{u_n\} \subset H_\epsilon$ is a $(PS)_c$ sequence for $J_\epsilon$, that is, $J_\epsilon(u_n) \to c$ and $J'_\epsilon(u_n) \to 0$. By $(g_4)$, $(g_5)$ and $4 < \theta < 6$, we have

$$\begin{aligned}
c + o_n(1) + o_n(1)\|u_n\|_\varepsilon =& J_\varepsilon(u_n) - \frac{1}{\theta} J'_\varepsilon(u_n)[u_n] \\
=& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{\theta}) b[u_n]_{A_\varepsilon}^4 \\
&+ \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2}\int_{\Lambda_\varepsilon^c} G(\varepsilon x, |u_n|^2) dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2K}\int_{\mathbb{R}^3} V(\varepsilon x)|u_n|^2 dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2K}\Big)\|u_n\|_\varepsilon^2.
\end{aligned}$$

Since $K > 2$, from the last inequality we obtain that $\{u_n\}$ is bounded in $H_\varepsilon$. $\square$

Define

$$c_0 = \frac{abS^3}{4} + \frac{S^6}{24}\Big((b^2 + 4aS^{-3})^{3/2} + b^3\Big), \tag{3.4}$$

where $S$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^6(\mathbb{R}^3, \mathbb{R})$.

The following lemma provides a range of levels in which the functional $J_\epsilon$ verifies the Palais-Smale condition.

**Lemma 3.5.** *The functional $J_\epsilon$ satisfies the $(PS)_c$ condition for any $c \in (0, c_0)$.*

*Proof.* Let $(u_n) \subset H_\epsilon$ be a $(PS)_c$ for $J_\epsilon$. By Lemma 3.4, $(u_n)$ is bounded in $H_\epsilon$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in $H_\epsilon$ and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ for all $1 \leq r < 6$ as $n \to +\infty$.

*Step 1:* For the fixed $\epsilon > 0$, let $R > 0$ be such that $\Lambda_\epsilon \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$, for $R$ large enough,

$$\limsup_n \int_{B_R^c(0)} (a|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2) dx \leq \zeta. \tag{3.5}$$

Let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0 \leq \phi_R \leq 1, \quad \text{and} \quad |\nabla\phi_R| \leq C/R$$

where $C>0$ is a constant independent of $R$. Since the sequence $(\phi_R u_n)$ is bounded in $H_\epsilon$, we have

$$J'_\epsilon(u_n)[\phi_R u_n]=o_n(1).$$

Therefore

$$a\,\mathrm{Re}\int_{\mathbb{R}^3}\nabla_{A_\epsilon}u_n\overline{\nabla_{A_\epsilon}(\phi_R u_n)}dx+\int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\phi_R dx+b[u_n]^2_{A_\epsilon}\,\mathrm{Re}\int_{\mathbb{R}^3}\nabla_{A_\epsilon}u_n\overline{\nabla_{A_\epsilon}(\phi_R u_n)}dx$$
$$=\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)|u_n|^2\phi_R dx+o_n(1).$$

Since $\overline{\nabla_{A_\epsilon}(\phi_R u_n)}=i\overline{u_n}\nabla\phi_R+\phi_R\overline{\nabla_{A_\epsilon}u_n}$, using $(g_5)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)\phi_R dx\\&\le\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)|u_n|^2\phi_R dx-\Big(a+b[u_n]^2_{A_\epsilon}\Big)\,\mathrm{Re}\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R dx+o_n(1)\\&\le\frac{1}{K}\int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\phi_R dx+C\Big|\,\mathrm{Re}\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R dx\Big|+o_n(1).\end{aligned}$$

By the definition of $\phi_R$, the Hölder inequality and the boundedness of $(u_n)$ in $H_\varepsilon$, we obtain

$$\Big(1-\frac{1}{K}\Big)\int_{\mathbb{R}^3}(a|\nabla_{A_\varepsilon}u_n|^2+V_\varepsilon(x)|u_n|^2)\phi_R dx\le\frac{C}{R}\|u_n\|_2\|\nabla_{A_\varepsilon}u_n\|_2+o_n(1)\le\frac{C_1}{R}+o_n(1)$$

and so (3.5) holds.

*Step 2:* For any $R>0$, the following limit holds

$$\limsup_n\int_{B_R(0)}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)dx=\int_{B_R(0)}(a|\nabla_{A_\epsilon}u|^2+V_\epsilon(x)|u|^2)dx. \tag{3.6}$$

Let $\phi_\rho\in C^\infty(\mathbb{R}^3,\mathbb{R})$ be a cut-off function such that

$$\phi_\rho=1\quad x\in B_\rho(0),\qquad \phi_\rho=0\quad x\in B^c_{2\rho}(0),\qquad 0\le\phi_\rho\le1,\quad\text{and}\quad |\nabla\phi_\rho|\le C/\rho$$

where $C>0$ is a constant independent of $\rho$. Let

$$P_n(x)=M(u_n)|\nabla_{A_\varepsilon}u_n-\nabla_{A_\varepsilon}u|^2+V_\epsilon(x)|u_n-u|^2$$

where $M(u_n)=a+b\int_{\mathbb{R}^3}|\nabla_{A_\varepsilon}u_n|^2dx$. For the fixed $R>0$, choosing $\rho>R>0$, we have

$$\begin{aligned}\int_{B_R}P_n(x)dx\le&\int_{\mathbb{R}^3}P_n(x)\phi_\rho(x)dx\\&=M(u_n)\int_{\mathbb{R}^3}|\nabla_{A_\varepsilon}u_n-\nabla_{A_\varepsilon}u|^2\phi_\rho(x)dx+\int_{\mathbb{R}^3}V_\epsilon(x)|u_n-u|^2\phi_\rho(x)dx\\&=J^1_{n,\rho}-J^2_{n,\rho}+J^3_{n,\rho}+J^4_{n,\rho},\end{aligned}\tag{3.7}$$

where

$$J^1_{n,\rho}=M(u_n)\int_{\mathbb{R}^3}|\nabla_{A_\varepsilon}u_n|^2\phi_\rho(x)dx+\int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\phi_\rho(x)dx-\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)|u_n|^2\phi_\rho dx,$$

$$J^2_{n,\rho}=M(u_n)\,\mathrm{Re}\int_{\mathbb{R}^3}\nabla_{A_\epsilon}u_n\overline{\nabla_{A_\epsilon}u}\phi_\rho(x)dx+\mathrm{Re}\int_{\mathbb{R}^3}V_\epsilon(x)u_n\overline{u}\phi_\rho(x)dx-\mathrm{Re}\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)u_n\overline{u}\phi_\rho(x)dx,$$

$$J^3_{n,\rho}=-M(u_n)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n-\nabla_{A_\epsilon}u)\overline{\nabla_{A_\epsilon}u}\phi_\rho(x)dx-\mathrm{Re}\int_{\mathbb{R}^3}V_\epsilon(x)(u_n-u)\overline{u}\phi_\rho(x)dx,$$

and

$$J^4_{n,\rho}=\mathrm{Re}\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\phi_\rho(x)dx.$$

It is easy to see that

$$J^1_{n,\rho} = J'_\epsilon(u_n)[\phi_\rho u_n] - M(u_n)\,\mathrm{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_\rho dx,$$

and

$$J^2_{n,\rho} = J'_\epsilon(u_n)[\phi_\rho u] - M(u_n)\,\mathrm{Re}\int_{\mathbb{R}^3} i\overline{u}\nabla_{A_\epsilon}u_n\nabla\phi_\rho dx.$$

Then

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}\left|J^1_{n,\rho}\right| = 0, \qquad \lim_{\rho\to\infty}\limsup_{n\to\infty}\left|J^2_{n,\rho}\right| = 0.$$

On the other hand, since the sequence $(u_n)$ is bounded in $H_\epsilon$, we assume that $\int_{\mathbb{R}^3}|\nabla_{A_\varepsilon}u_n|^2dx \to l^2$. Then

$$\begin{aligned}
J^3_{n,\rho} =& -(a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)\overline{\nabla_{A_\epsilon}(u\phi_\rho(x))}dx - \mathrm{Re}\int_{\mathbb{R}^3}V_\epsilon(x)(u_n-u)\overline{(u\phi_\rho(x))}dx\\
&+(a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)i\overline{u}\nabla\phi_\rho dx + o_n(1)\\
=& -(a+bl^2)\langle u_n-u, u\phi_\rho(x)\rangle_\epsilon + (a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)i\overline{u}\nabla\phi_\rho dx + o_n(1),
\end{aligned}$$

thus,

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}\left|J^3_{n,\rho}\right| = 0.$$

Now we prove that

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}\left|J^4_{n,\rho}\right| = 0. \tag{3.8}$$

First we show

$$\lim_n \int_{\Lambda_\varepsilon}|u_n|^6dx = \int_{\Lambda_\varepsilon}|u|^6dx. \tag{3.9}$$

Using the boundedness of $(u_n)$ in $D_\epsilon$ and the diamagnetic inequality (2.1), we may assume that

$$|\nabla|u_n||^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^6 \rightharpoonup \nu \tag{3.10}$$

in the sense of measures. By the concentration-compactness principle in [28], we can find an at most countable index $I$, sequences $(x_i)\subset\mathbb{R}^3, (\mu_i), (\nu_i)\subset(0,\infty)$ such that

$$\begin{aligned}
&\mu \geq |\nabla|u||^2dx + \sum_{i\in I}\mu_i\delta_{x_i},\\
&\nu = |u|^6 + \sum_{i\in I}\nu_i\delta_{x_i} \quad \text{and} \quad S\nu_i^{1/3} \leq \mu_i
\end{aligned} \tag{3.11}$$

for any $i\in I$, where $\delta_{x_i}$ is the Dirac mass at the point $x_i$. Let us show that $(x_i)_{i\in I}\cap\Lambda_\varepsilon = \emptyset$. Assume, by contradiction, that $x_i\in\Lambda_\varepsilon$ for some $i\in I$. For any $\rho>0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$ where $\psi\in C_0^\infty(\mathbb{R}^3,[0,1])$ is such that $\psi = 1$ in $B_1$, $\psi = 0$ in $\mathbb{R}^3\setminus B_2$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3,\mathbb{R})}\leq 2$. We suppose that $\rho>0$ is such that $\mathrm{supp}(\psi_\rho)\subset\Lambda_\varepsilon$. Since $(\psi_\rho u_n)$ is bounded in $H_\varepsilon$, we can see that $J'_\varepsilon(u_n)[\psi_\rho u_n] = o_n(1)$, that is

$$\begin{aligned}
&\left(a+b[u_n]^2_{A_\varepsilon}\right)\int_{\mathbb{R}^3}|\nabla_{A_\varepsilon}u_n|^2\psi_\rho dx + \left(a+b[u_n]^2_{A_\varepsilon}\right)\mathrm{Re}\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\varepsilon}u_n\nabla\psi_\rho dx + \int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\psi_\rho dx\\
&= \int_{\mathbb{R}^3}g(\varepsilon x,|u_n|^2)|u_n|^2\psi_\rho dx + o_n(1) = \int_{\mathbb{R}^N}f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^N}|u_n|^6\psi_\rho dx + o_n(1).
\end{aligned}$$

Using the diamagnetic inequality (2.1) again, it follows that

$$\begin{aligned}&\Big(a+b\int_{\mathbb{R}^3}|\nabla|u_n||^2\psi_\rho dx\Big)\int_{\mathbb{R}^3}|\nabla|u_n||^2\psi_\rho dx+\Big(a+b[u_n]^2_{A_\varepsilon}\Big)\operatorname{Re}\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\varepsilon}u_n\nabla\psi_\rho dx\\ \leq&\int_{\mathbb{R}^N}f(|u_n|^2)|u_n|^2\psi_\rho dx+\int_{\mathbb{R}^N}|u_n|^6\psi_\rho dx+o_n(1).\end{aligned}\tag{3.12}$$

Due to the fact that $f$ has the subcritical growth and $\psi_\rho$ has the compact support, we have that

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}f(|u_n|^2)|u_n|^2\psi_\rho dx=\lim_{\rho\to0}\int_{\mathbb{R}^3}f(|u|^2)|u|^2\psi_\rho dx=0.\tag{3.13}$$

Now, we show that

$$\lim_{\rho\to0}\limsup_{n\to\infty}\Big|\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\varepsilon}u_n\nabla\psi_\rho dx\Big|=0.\tag{3.14}$$

Because of the boundedness of $(u_n)$ in $H_\varepsilon$, using the Hölder inequality, the strong convergence of $(|u_n|)$ in $L^2_{\text{loc}}(\mathbb{R}^3,\mathbb{R})$, $|u|\in L^6(\mathbb{R}^3,\mathbb{R})$, $|\nabla\psi_\rho|\leq C\rho^{-1}$ and $|B_{2\rho}(x_i)|\sim\rho^3$, we have that

$$\begin{aligned}0\quad\leq&\quad\lim_{\rho\to0}\limsup_{n\to\infty}\Big|\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\varepsilon}u_n\nabla\psi_\rho dx\Big|\leq\lim_{\rho\to0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\overline{u_n}\nabla\psi_\rho||\nabla_{A_\varepsilon}u_n|dx\\ \leq&\quad\lim_{\rho\to0}\lim_{n\to\infty}\Big(\int_{B_{2\rho}(x_i)}|\overline{u_n}\nabla\psi_\rho|^2dx\Big)^{1/2}[u_n]_{A_\varepsilon}\\ \leq&\quad C\lim_{\rho\to0}\Big(\int_{B_{2\rho}(x_i)}|u|^2dx\Big)^{1/2}=0\end{aligned}$$

which shows that (3.14) holds.

Then, taking into account (3.10), (3.12), (3.13) and (3.14), we can conclude that $\nu_i\geq a\mu_i+b\mu_i^2$. Together with the inequality $S\nu_i^{1/3}\leq\mu_i$ in (3.11), we have

$$\mu_i\geq\frac{S^3}{2}(b+\sqrt{b^2+4aS^{-3}}).\tag{3.15}$$

Now, from $(f_3)$, $(g_4)$ and $(g_5)$, we have

$$\begin{aligned}c=&J_\varepsilon(u_n)-\frac{1}{4}J'_\varepsilon(u_n)[u_n]+o_n(1)\\ =&\frac{1}{4}\|u_n\|^2_\varepsilon+\int_{\mathbb{R}^3}\Big(\frac{1}{4}g(\varepsilon x,|u_n|^2)|u_n|^2-\frac{1}{2}G(\varepsilon x,|u_n|^2)\Big)dx+o_n(1)\\ \geq&\frac{1}{4}\|u_n\|^2_\varepsilon+\int_{\Lambda^c_\varepsilon}\Big(\frac{1}{4}g(\varepsilon x,|u_n|^2)|u_n|^2-\frac{1}{2}G(\varepsilon x,|u_n|^2)\Big)dx\\ &+\frac{1}{12}\int_{\Lambda_\varepsilon}|u_n|^6dx+o_n(1)\\ \geq&\frac{1}{4}\Big(\int_{\Lambda_\varepsilon}a\psi_\rho|\nabla|u_n||^2dx+\int_{\Lambda^c_\varepsilon}V_\varepsilon(x)|u_n|^2\Big)-\frac{1}{2}\int_{\Lambda^c_\varepsilon}G(\varepsilon x,|u_n|^2)dx\\ &+\frac{1}{12}\int_{\Lambda_\varepsilon}|u_n|^6dx+o_n(1)\\ \geq&\frac{a}{4}\int_{\Lambda_\varepsilon}\psi_\rho|\nabla|u_n||^2dx+(\frac{1}{4}-\frac{1}{2K})\int_{\Lambda^c_\varepsilon}V_\varepsilon(x)|u_n|^2dx+\frac{1}{12}\int_{\Lambda_\varepsilon}\psi_\rho|u_n|^6dx+o_n(1)\\ \geq&\frac{a}{4}\int_{\Lambda_\varepsilon}\psi_\rho|\nabla|u_n||^2dx+\frac{1}{12}\int_{\Lambda_\varepsilon}\psi_\rho|u_n|^6dx+o_n(1).\end{aligned}$$

From the above arguments, (3.11) and (3.15), we have

$$\begin{aligned} c &\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}} a\psi_\rho(x_i)\mu_i + \frac{1}{12}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\nu_i \\ &\geq \frac{a}{4}\mu_i + \frac{1}{12}\nu_i \\ &\geq \frac{aS}{4}\cdot\frac{bS^2+\sqrt{b^2S^4}+4aS}{2} + \frac{1}{12}\Big(\frac{bS^2+\sqrt{b^2S^4}+4aS}{2}\Big)^3 = c_0 \end{aligned}$$

which gives a contradiction. This means that (3.9) holds. Next, we observe that

$$\left|J^4_{n,\rho}\right| \leq \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap B_{2\rho}(0)}\left|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\right|dx + \int_{\Lambda_\varepsilon\cap B_{2\rho}(0)}\left|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\right|dx.$$

By the Sobolev compact embedding $H_\varepsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^3,\mathbb{C})$ for $1\leq r<6$, and $(g_5)$, we have

$$\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap B_{2\rho}(0)}\left|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\right|dx \to 0, \text{ as } n\to\infty.$$

Moreover, using the Sobolev compact embedding $H_\varepsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^3,\mathbb{C})$ for $1\leq r<6$, again, and (3.9), we have,

$$\int_{\Lambda_\varepsilon\cap B_{2\rho}(0)}\left|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\right|dx \to 0, \text{ as } n\to\infty.$$

Thus, (3.8) holds. Moreover, by (3.7), it follows that

$$0\leq \limsup_n \int_{B_R} P_n(x)dx \leq \limsup_n \Big(|J^1_{n,\rho}|+|J^2_{n,\rho}|+|J^3_{n,\rho}|+|J^4_{n,\rho}|\Big) = 0.$$

Then,

$$\limsup_n \int_{B_R} P_n(x)dx = 0.$$

Thus (3.6) holds.

*Step 3:* From (3.5) and (3.6), we have

$$\begin{aligned} \|u\|^2_\varepsilon &\leq \liminf_n \|u_n\|^2_\varepsilon \leq \limsup_n \|u_n\|^2_\varepsilon \\ &\leq \limsup_n \Big\{\int_{B_R(0)}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)dx + \int_{B^c_R(0)}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)dx\Big\} \\ &\leq \int_{B_R(0)}(a|\nabla_{A_\epsilon}u|^2+V_\epsilon(x)|u|^2)dx + \zeta. \end{aligned}$$

Passing to the limit as $\zeta\to 0$ we have $R\to\infty$, which implies that

$$\|u\|^2_\varepsilon \leq \liminf_n \|u_n\|^2_\varepsilon \leq \limsup_n \|u_n\|^2_\varepsilon \leq \|u\|^2_\varepsilon.$$

Then $u_n\to u$ in $H_\epsilon$ and we complete the proof. □

Since $f$ is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

**Corollary 3.1.** *The functional $\Psi_\varepsilon$ satisfies the $(PS)_c$ condition on $S^+_\varepsilon$ at any level $c\in(0,c_0)$.*

*Proof.* Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for $\Psi_\varepsilon$. Then $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi_\varepsilon'(u_n)\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n}S_\varepsilon^+)^*$. By Lemma 3.3($B3$), we know that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for $J_\varepsilon$ in $H_\varepsilon$. From Lemma 3.5, we know that there exists a $u \in S_\varepsilon^+$ such that, up to a subsequence, $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in $H_\varepsilon$. By Lemma 3.2($A3$), we obtain

$$u_n \to u \text{ in } S_\varepsilon^+,$$

and the proof is complete. □

## 4. **Multiple solutions of the auxilliary problem.**

4.1. **The autonomous problem.** For our scope, we need also to study the following *limit* problem

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V_0 u = f(|u|^2)u + |u|^4 u, \quad u : \mathbb{R}^3 \to \mathbb{R}, \tag{4.1}$$

whose associated $C^1$-functional, defined in $H^1(\mathbb{R}^3, \mathbb{R})$, is

$$I_{V_0}(u) := \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_0 u^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)dx - \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6 dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_{V_0}'(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_{V_0}(u).$$

By $(f_1)$ and $(f_4)$, for each $u \in H^1(\mathbb{R}^3, \mathbb{R})\setminus\{0\}$, there is a unique $t(u) > 0$ such that

$$I_{V_0}(t(u)u) = \max_{t \geq 0} I_{V_0}(tu) \quad \text{and} \quad t(u)u \in \mathcal{N}_{V_0}.$$

Then, using the assumptions on $f$, arguing as in [28, Lemma 4.1 and Theorem 4.2] we have that

$$0 < c_{V_0} = \inf_{u \in H^1(\mathbb{R}^3,\mathbb{R})\setminus\{0\}} \max_{t \geq 0} I_{V_0}(tu).$$

Now, we estimate the ground state energy $c_{V_0}$ and show that $c_{V_0} \in (0, c_0)$.

**Lemma 4.1.** *Assume that $(f_1)$–$(f_4)$ hold, then $0 < c_{V_0} < c_0$.*

*Proof.* For $\delta > 0$, the function $\omega_\delta : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\omega_\delta(x) = 3^{1/4}\frac{\delta^{1/4}}{(\delta + |x|^2)^{1/2}}$$

is a family of functions on which $S$ is attained. Let $\phi \in C_0^\infty(\mathbb{R}^3, [0,1])$ be a cut-off function with

$$\phi = 1 \quad x \in B_{\rho/2}(0), \quad \phi = 0 \quad x \in B_\rho^c(0),$$

then we define the test function by $v_\delta = \omega_\delta/\|\omega_\delta\|_6$, where $u_\delta = \phi v_\delta$. Since

$$I_{V_0}(tv_\delta) := \frac{t^2}{2}\int_{\mathbb{R}^3} (a|\nabla v_\delta|^2 + V_0 v_\delta^2)dx + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla v_\delta|^2 dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(t^2 v_\delta^2)dx - \frac{t^6}{6}\int_{\mathbb{R}^3} v_\delta^6 dx,$$

under the conditions $(f_1)$–$(f_4)$, arguing as in [29, Lemma 3.11], we can show that for small $\delta > 0$, $\max_{t>0} I_{V_0}(tv_\delta) < c_0$. Thus, since $0 < c_{V_0} < c_0$, we conclude the proof. □

Let $H_0 := H^1(\mathbb{R}^3, \mathbb{R})$ and define by $H_0^+$ the open set of $H_0$ given by

$$H_0^+ = \{u \in H_0 : |\text{supp}(u^+)| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where $S_0$ be the unit sphere of $H_0$.

As in Section 3, $S_0^+$ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on $H_0$ and contained in $H_0^+$. Therefore, $H_0 = T_u S_0^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_0^+$, where $T_u S_0^+ = \{v \in H_0 : \langle u, v\rangle_0 = 0\}$.

Arguing as in Lemma 3.2, we have the following property.

**Lemma 4.2.** *Let $V_0$ be given in $(V_1)$ and suppose that $(f_1)$–$(f_4)$ are satisfied, then the following properties hold:*

$(a1)$ *For any $u \in H_0^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_{V_0}(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in $(t_u, \infty)$;*

$(a2)$ *There is a $\tau > 0$ independent on $u$ such that $t_u > \tau$ for all $u \in S_0^+$. Moreover, for each compact $\mathcal{W} \subset S_0^+$ there is $C_{\mathcal{W}}$ such that $t_u \le C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*

$(a3)$ *The map $\widehat{m} : H_0^+ \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m_0 = \widehat{m}_0|_{S_0^+}$ is a homeomorphism between $S_0^+$ and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$;*

$(a4)$ *If there is a sequence $\{u_n\} \subset S_0^+$ such that $dist(u_n, \partial S_0^+) \to 0$, then $\|m(u_n)\|_0 \to \infty$ and $I_{V_0}(m(u_n)) \to \infty$.*

We shall consider the functional defined by

$$\widehat{\Psi}_0(u) = I_{V_0}(\widehat{m}(u)) \quad \text{and} \quad \Psi_0 := \widehat{\Psi}_0|_{S_0}.$$

Arguing as in [26, Proposition 9 and Corollary 10], we have that

**Lemma 4.3.** *Let $V_0$ be given in $(V_1)$ and suppose that $(f_1)$–$(f_4)$ are satisfied, then*

$(b1)$ *$\widehat{\Psi}_0 \in C^1(H_0^+, \mathbb{R})$ and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_{V_0}'(\widehat{m}(u))[v], \ \ \forall\, u \in H_0^+ \text{ and } \forall v \in H_0;$$

$(b2)$ *$\Psi_0 \in C^1(S_0^+, \mathbb{R})$ and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_{V_0}'(\widehat{m}(u))[v], \ \ \forall v \in T_u S_0^+;$$

$(b3)$ *If $\{u_n\}$ is a $(PS)_c$ sequence of $\Psi_0$, then $\{m(u_n)\}$ is a $(PS)_c$ sequence of $I_{V_0}$. If $\{u_n\} \subset \mathcal{N}_0$ is a bounded $(PS)_c$ sequence of $I_{V_0}$, then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence of $\Psi_0$;*

$(b4)$ *$u$ is a critical point of $\Psi_0$ if and only if $m(u)$ is a critical point of $I_{V_0}$. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_{V_0}.$$

Similar to the previous argument, we have the following variational characterization of the infimum of $I_{V_0}$ over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_{V_0}(u) = \inf_{u \in H_0^+} \sup_{t>0} I_{V_0}(tu) = \inf_{u \in S_0^+} \sup_{t>0} I_{V_0}(tu). \tag{4.2}$$

The next result is useful in later arguments.

**Lemma 4.4.** *Let $\{u_n\} \subset H_0$ be a $(PS)_{c_{V_0}}$ sequence for $I_{V_0}$, then the problem (4.1) has a positive ground state solution.*

*Proof.* By $(f_3)$, it follows that

$$c_{V_0} + o_n(1) + o_n(1)\|u_n\| = I_{V_0}(u_n) - \frac{1}{4}\langle I'_{V_0}(u_n), u_n\rangle \geq \frac{1}{4}\|u_n\|^2.$$

Thus, $(u_n)$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R})$. We can assume that there exists $u \in H^1(\mathbb{R}^3,\mathbb{R})$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3,\mathbb{R})$, $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^3,\mathbb{R})$ and $u_n \to u$ a.e. in $x \in \mathbb{R}^3$.

Now we show that $(u_n)$ is non-vanishing. Otherwise, assume that $(u_n)$ is vanishing, then Lions lemma (see [28]) implies that $u_n \to 0$ in $L^r(\mathbb{R}^3,\mathbb{R})$ for any $2 < r < 6$. By the compact embedding and $(f_1)$–$(f_2)$, we have $\int_{\mathbb{R}^3} F(u_n^2)dx \to 0$ and $\int_{\mathbb{R}^3} f(u_n^2)u_n^2 dx \to 0$ as $n \to \infty$. Moreover, using the former limits, it follows that

$$c_{V_0} = \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + V_0 u_n^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 - \frac{1}{6}\int_{\mathbb{R}^3}(u_n^+)^6 dx + o_n(1), \quad (4.3)$$

and

$$\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + V_0 u_n^2)dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 = \int_{\mathbb{R}^3}(u_n^+)^6 dx + o_n(1). \quad (4.4)$$

Suppose that

$$\lim_n \int_{\mathbb{R}^3}(u_n^+)^6 dx = l \geq 0.$$

If $l = 0$, then (4.3) and (4.4) imply that $c_{V_0} = 0$, which contradicts with $c_{V_0} > 0$. Then $l > 0$, using the definition of $S$, we have

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 \leq \int_{\mathbb{R}^3}(u_n^+)^6 dx + o_n(1) \leq S^{-3}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^3 + o_n(1). \quad (4.5)$$

If $\int_{\mathbb{R}^3}|\nabla u_n|^2 dx \to 0$, then

$$\int_{\mathbb{R}^3}(u_n^+)^6 dx \leq S^{-3}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^3 \to 0,$$

which contradicts with $l > 0$. Thus $\int_{\mathbb{R}^3}|\nabla u_n|^2 dx \not\to 0$. By (4.5), we have

$$\int_{\mathbb{R}^3}|\nabla u_n|^2 dx \geq \frac{S^3}{2}(b + \sqrt{b^2 + 4aS^{-3}}) + o_n(1). \quad (4.6)$$

Using (4.5) again, it follows that

$$\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 \geq aS^3 + \frac{bS^6}{2}(b + \sqrt{b^2 + 4aS^{-3}}) + o_n(1). \quad (4.7)$$

From (4.6) and (4.7), we can obtain

$$\begin{aligned} c_{V_0} &= \frac{1}{3}\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + V_0 u_n^2)dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 + o_n(1) \\ &\geq \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 + o_n(1) \geq c_0 + o_n(1) \end{aligned}$$

which contradicts with $c_{V_0} < c_0$. Thus, $(u_n)$ is non-vanishing, that is, there exist $R, \eta > 0$, and $(\tilde{y}_n) \subset \mathbb{R}^3$ such that

$$\lim_n \int_{B_R(\tilde{y}_n)} u_n^2 dx \geq \eta. \quad (4.8)$$

By the invariance of the translation of $I_{V_0}$, we may assume that $(\tilde{y}_n)$ is bounded, so $u \neq 0$.

Now we claim that $I'_{V_0}(u) = 0$. Assume that there exists $\iota \geq 0$ such that $\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \to \iota^2$. Since $I'_{V_0}(u_n) \to 0$, then $u$ is a solution of the following equation

$$-(a + b\iota^2)\Delta u + V_0 u = f(|u|^2)u + (u^+)^5, \quad u \in H^1(\mathbb{R}^3, \mathbb{R}). \tag{4.9}$$

It suffices to show that $\iota^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. From the weakly lower semi-continuous of the norm, it is easy to obtain that $\iota^2 \geq \int_{\mathbb{R}^3} |\nabla u|^2 dx$. Then, by (4.9), we have $\langle I'_{V_0}(u), u\rangle \leq 0$. By $(f_4)$, there exists $t \leq 1$ such that $tu \in \mathcal{N}_0$. By (4.9) again, it follows that

$$\begin{aligned}
c_{V_0} \leq & I_{V_0}(tu) - \frac{1}{4} I'_{V_0}(tu)[tu] \\
= & \frac{1}{4}\int_{\mathbb{R}^3} (a|\nabla tu|^2 + V_0(tu)^2)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{4} f(|tu|^2)|tu|^2 - \frac{1}{2}F(|tu|^2)\Big)dx + \frac{1}{12}\int_{\mathbb{R}^3} (tu^+)^6 dx \\
\leq & \frac{1}{4}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_0 u^2)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{4} f(|u|^2)|u|^2 - \frac{1}{2}F(|u|^2)\Big)dx + \frac{1}{12}\int_{\mathbb{R}^3} (u^+)^6 dx \\
\leq & \frac{1}{4}\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V_0 u_n^2)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{4} f(|u_n|^2)|u_n|^2 - \frac{1}{2}F(|u_n|^2)\Big)dx + \frac{1}{12}\int_{\mathbb{R}^3} (u_n^+)^6 dx + o_n(1) \\
= & I_{V_0}(u_n) - \frac{1}{4} I'_{V_0}(u_n)[u_n] + o_n(1) = c_{V_0} + o_n(1).
\end{aligned}$$

Thus $t = 1$ and

$$\begin{aligned}
&\lim_n \frac{1}{4}\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V_0 u_n^2)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{4} f(|u_n|^2)|u_n|^2 - \frac{1}{2}F(|u_n|^2)\Big)dx + \frac{1}{12}\int_{\mathbb{R}^3} (u_n^+)^6 dx \\
&= \frac{1}{4}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_0 u^2)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{4} f(|u|^2)|u|^2 - \frac{1}{2}F(|u|^2)\Big)dx + \frac{1}{12}\int_{\mathbb{R}^3} (u^+)^6 dx.
\end{aligned}$$

Then $\iota^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$ and $u_n \to \omega$ in $H^1(\mathbb{R}^3, \mathbb{R})$. Therefore $I'_{V_0}(u) = 0$ and $I_{V_0}(u) = c_{V_0}$. Then $u$ is a ground state of problem. From the assumption of $f$, $u \geq 0$. Moreover, using the standard argument, we may show that $u(x) > 0$ for $x \in \mathbb{R}^N$. The proof is complete. □

Arguing as in [30, Lemma 2.3], there exists a positive radial ground state solution of the problem (4.1), which implies that this solution decays exponentially at infinity with its gradient; moreover, this ground state solution is of class $C^2(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$.

**Lemma 4.5.** *Let $(u_n) \subset \mathcal{N}_0$ be such that $I_0(u_n) \to c_{V_0}$. Then $(u_n)$ has a convergent subsequence in $H_0$.*

*Proof.* Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.2($a3$), Lemma 4.3($b4$) and the definition of $c_{V_0}$, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \ \ \forall n \in N,$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{u \in S_0^+} \Psi_0(u).$$

Although $S_0^+$ is not a complete $C^1$ manifold, we still can use the Ekeland's variational principle to the functional $\mathcal{E}_0 : H \to \mathbb{R} \cup \{\infty\}$ defined by $\mathcal{E}_0(u) := \widehat{\Psi}_0(u)$ if $u \in S_0^+$ and $\mathcal{E}_0(u) := \infty$ if $u \in \partial S_0^+$, where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric $d(u, v) := \|u - v\|_0$. In fact, by Lemma 4.2(a4), $\mathcal{E}_0 \in C(H, \mathbb{R} \cup \{\infty\})$, and from Lemma 4.3(b4), $\mathcal{E}_0$ is bounded below. Therefore, there exists a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for $\Psi_0$ on $S_0^+$ and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Arguing as in Lemma 4.4 and Lemma 4.3, we conclude. □

Now, we show the relationship between $c_\epsilon$ and $c_{V_0}$.

**Lemma 4.6.** *The numbers $c_\epsilon$ and $c_{V_0}$ satisfy the following inequality*

$$\lim_{\epsilon\to 0} c_\epsilon = c_{V_0} < c_0.$$

*Proof.* Let $\eta \in C_c^\infty(\mathbb{R}^3, [0,1])$ be a cut-off function such that $\eta = 1$ in $B_{\rho/2}$ and $\operatorname{supp}(\eta) = B_\rho \subset \Lambda$ for some $\rho > 0$. Let us define $\omega_\epsilon(x) := \eta_\epsilon(x)\omega(x)e^{iA(0)\cdot x}$, where $\eta_\epsilon(x) = \eta(\epsilon x)$ for $\epsilon > 0$, $\omega$ is a positive and radial ground state solution of the problem (4.1). We observe that $|\omega_\epsilon| = \eta_\epsilon \omega$ and $\omega_\epsilon \in H_\varepsilon$ in view of Lemma 2.1. Arguing as in [5, Lemma 4.1] or [15, Lemma 4.6], we obtain

$$\lim_{\epsilon\to 0} \|\omega_\epsilon\|_\epsilon^2 = \|\omega\|_{V_0}^2 \tag{4.10}$$

and

$$\lim_{\epsilon\to 0} [\omega_\epsilon]_{A_\varepsilon}^2 = \int_{\mathbb{R}^3} |\nabla \omega|^2 dx. \tag{4.11}$$

It is also easy to check that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^3} |\nabla \omega_\epsilon|^6 dx = \int_{\mathbb{R}^3} |\nabla \omega|^6 dx. \tag{4.12}$$

Let $t_\epsilon > 0$ be the unique number such that

$$J_\epsilon(t_\epsilon \omega_\epsilon) = \max_{t\geq 0} J_\epsilon(t\omega_\epsilon).$$

Then $t_\epsilon$ satisfies

$$\begin{aligned} t_\epsilon^2\Big(a[\omega_\epsilon]_{A_\varepsilon}^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|\omega_\epsilon|^2 dx\Big) + bt_\epsilon^4[\omega_\epsilon]_{A_\varepsilon}^4 &= \int_{\mathbb{R}^3} g(\epsilon x, t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2 dx \\ &= \int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^3} t_\epsilon^6|\omega_\epsilon|^6 dx, \end{aligned}$$

where we use $\operatorname{supp}(\eta) \subset \Lambda$ and the definition of $g(x,t)$. Moreover, combining the facts that $\eta = 1$ in $B_{\rho/2}$, $u$ is a positive continuous function and hypothesis $(f_4)$, we have

$$\begin{aligned} \frac{1}{t_\epsilon^2}\Big(a[\omega_\epsilon]_{A_\varepsilon}^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|\omega_\epsilon|^2 dx\Big) + b[\omega_\epsilon]_{A_\varepsilon}^4 &= \frac{1}{t_\epsilon^2}\int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^3} t_\epsilon^2|\omega_\epsilon|^6 dx \\ &\geq \frac{1}{t_\epsilon^2}\int_{\mathbb{R}^3} f(t_\epsilon^2\eta^2(|\epsilon x|)\omega^2(x))\eta^2(|\epsilon x|)\omega^2(x) dz \\ &\geq \frac{1}{t_\epsilon^2}\int_{B_{\rho/(2\epsilon)}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z) dz \\ &\geq \frac{1}{t_\epsilon^2}\int_{B_{\rho/2}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z) dz \\ &\geq \frac{f(t_\epsilon^2\gamma^2)}{t_\epsilon^2}\int_{B_{\rho/2}(0)} \omega^2(z) dz \end{aligned}$$

for all $0 < \epsilon < 1$ and where $\gamma = \min\{\omega(z) : |z| \leq \rho/2\}$.

If $t_\epsilon \to +\infty$ as $\epsilon \to 0$, by $(f_4)$, we deduce that $[\omega_\epsilon]_{A_\varepsilon}^4 \to +\infty$ which contradicts (4.11).

Therefore, up to a subsequence, we may assume that $t_\epsilon \to t_0 \geq 0$ as $\epsilon \to 0$.

If $t_\epsilon \to 0$, using the fact that $f$ is increasing, the Lebesgue dominated convergence theorem and relation (4.12), we obtain

$$a[\omega_\epsilon]_{A_\varepsilon}^2 + \int_{\mathbb{R}^3} V_\varepsilon(x)|\omega_\epsilon|^2 dx + bt_\epsilon^2[\omega_\epsilon]_{A_\varepsilon}^4 = \int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^3} t_\epsilon^4|\omega_\epsilon|^6 dx \to 0, \text{ as } \epsilon \to 0$$

which contradicts (4.10). Thus, we have $t_0 > 0$ and

$$t_0^2 \int_{\mathbb{R}^3} (a|\nabla \omega|^2 + V_0 \omega^2) dx + b t_0^4 \Big( \int_{\mathbb{R}^3} |\nabla \omega|^2 dx \Big)^2 = \int_{\mathbb{R}^2} f(t_0^2 \omega^2) t_0^2 \omega^2 dx + \int_{\mathbb{R}^3} t_0^6 |\omega|^6 dx,$$

so that $t_0 \omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} F(|t_\epsilon \omega_\epsilon|^2) dx = \int_{\mathbb{R}^2} F(\omega^2) dx.$$

Hence

$$\lim_{\epsilon \to 0} J_\epsilon(t_\epsilon \omega_\epsilon) = I_{V_0}(u) = c_{V_0}.$$

Since $c_\epsilon \leq \max_{t \geq 0} J_\epsilon(t\omega_\epsilon) = J_\epsilon(t_\epsilon \omega_\epsilon)$, we can conclude that $\limsup_{\epsilon \to 0} c_\epsilon \leq c_{V_0}$. Moreover, by (3.3), (4.2) and $I_{V_0}(|u|) \leq J_\epsilon(u)$ for any $u \in H_\varepsilon$, we have $c_{V_0} \leq c_\epsilon$. Then $c_{V_0} \leq \liminf_{\epsilon \to 0} c_\epsilon$. Combining with the previous arguments, we conclude that $\lim_{\epsilon \to 0} c_\epsilon = c_{V_0} < c_0$. □

**Remark 4.1.** *From Lemma 4.2 and Lemma 3.5, we see that for $\epsilon > 0$ small, the problem (3.2) has a ground state solution $u_\epsilon$ such that $J_\varepsilon(u_\epsilon) = c_\epsilon$ and $J'_\varepsilon(u_\epsilon) = 0$.*

4.2. **The technical results.** In this subsection, we prove a multiplicity result for the auxilliary problem (3.2) using the Ljusternik-Schnirelmann category theory. In order to get this property, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$.

For any $y \in M$, let us introduce the function

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|) \omega\Big(\frac{\varepsilon x - y}{\varepsilon}\Big) \exp\Big(i\tau_y\Big(\frac{\varepsilon x - y}{\varepsilon}\Big)\Big),$$

where

$$\tau_y(x) := \sum_i^3 A_i(y) x_i.$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t \geq 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

Note that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.

Let us define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Moreover, arguing as in Lemma 4.2, the energy of the above functions has the following behavior as $\varepsilon \to 0^+$.

**Lemma 4.7.** *The limit*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}$$

*holds uniformly in $y \in M$.*

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$\Upsilon(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_4^4} \int_{\mathbb{R}^3} \Upsilon(\varepsilon x)|u(x)|^4 dx.$$

We have the following asymptotic property.

**Lemma 4.8.** *The limit*

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$$

*holds uniformly in $y \in M$.*

*Proof.* Assume by contradiction that there exist $\kappa > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \kappa. \tag{4.13}$$

Using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\displaystyle\int_{\mathbb{R}^3} (\Upsilon(\varepsilon_n z + y_n) - y_n)\eta^4(|\varepsilon_n z|)\omega^4(z)dz}{\displaystyle\int_{\mathbb{R}^3} \eta^4(|\varepsilon_n z|)\omega^4(z)dz}.$$

Taking into account that $(y_n) \subset M \subset M_\delta \subset B_\rho$ and applying the Lebesgue dominated convergence theorem, we obtain

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.13). $\square$

Now, we prove the following useful compactness result.

**Proposition 4.1.** *Let $\varepsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that the sequence $(|v_n|) \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.*

*Proof.* Since $J'_{\varepsilon_n}(u_n)[u_n] = 0$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\varepsilon_n} \leq C$ for all $n \in \mathbb{N}$.

Arguing as in the proof of Lemma 4.4 and recalling that $c_{V_0} > 0$, we have that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \geq \beta. \tag{4.14}$$

Now, let us consider the sequence $\{|v_n|\} \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x+\tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$, and using (4.14), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ for some $v \neq 0$.

Let now $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \varepsilon_n \tilde{y}_n$.

Using the diamagnetic inequality (2.1) again, we have

$$c_{V_0} \leq I_0(\tilde{v}_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.

Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^3,\mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^3,\mathbb{R})$, then $(t_n)$ is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^3,\mathbb{R})$, that is, $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$.

Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^3,\mathbb{R})$, and, by Lemma 4.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3,\mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^3,\mathbb{R})$.

Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for $n$ large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - \varepsilon_n |x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using $(V_1)$ and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}\int_{\mathbb{R}^3} (a|\nabla |v_n||^2 + V_0|v_n|^2)dx &\leq a[v_n]^2_{A_\varepsilon} + \int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)|v_n|^2 dx + b[v_n]^4_{A_\varepsilon} \\ &\leq \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\ &\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} |v_n|^6 dx.\end{aligned} \tag{4.15}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^3,\mathbb{R})$ and $\tilde{f}(t) \leq V_0/K$, we can see that (4.15) yields

$$\min\Big\{1, V_0\Big(1 - \frac{1}{K}\Big)\Big\} \int_{\mathbb{R}^3} (|\nabla |v_n||^2 + |v_n|^2)dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^3,\mathbb{R})$, which contradicts to $v \not\equiv 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every $n$ large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t\geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned}c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla \tilde{v}|^2 + V(y_0)|\tilde{v}|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}|^2)dx \\ &\leq \liminf_n \Big(\frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla \tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}_n|^2)dx\Big) \\ &= \liminf_n \Big(\frac{t_n^2}{2}\int_{\mathbb{R}^3} (a|\nabla |u_n||^2 + V(\varepsilon_n z)|u_n|^2)dx + \frac{t_n^4 b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|t_n u_n|^2)dx\Big) \\ &\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0}\end{aligned}$$

which is impossible and the proof is complete. ☐

Let now

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Fixing $y \in M$, by Lemma 4.7, $|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

**Lemma 4.9.** *We have*

$$\lim_{\varepsilon\to 0^+}\sup_{u\in\tilde{\mathcal{N}}_\varepsilon}\operatorname{dist}(\beta_\varepsilon(u),M_\delta)=0.$$

*Proof.* Let $\varepsilon_n\to 0^+$ as $n\to+\infty$. For any $n\in\mathbb{N}$, there exists $u_n\in\tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}}\inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u)-y|=\inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u_n)-y|+o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n)\subset M_\delta$ such that

$$\lim_n|\beta_{\varepsilon_n}(u_n)-y_n|=0.$$

By the diamagnetic inequality (2.1), we can see that $I_{V_0}(t|u_n|)\le J_{\varepsilon_n}(tu_n)$ for any $t\ge 0$. Therefore, recalling that $\{u_n\}\subset\tilde{\mathcal{N}}_{\varepsilon_n}\subset\mathcal{N}_{\varepsilon_n}$, we can deduce that

$$c_{V_0}\le\max_{t\ge0}I_{V_0}(t|u_n|)\le\max_{t\ge0}J_{\varepsilon_n}(tu_n)=J_{\varepsilon_n}(u_n)\le c_{V_0}+h(\varepsilon_n)\tag{4.16}$$

which implies that $J_{\varepsilon_n}(u_n)\to c_{V_0}$ as $n\to+\infty$. Then Proposition 4.1 implies that there exists $\{\tilde{y}_n\}\subset\mathbb{R}^3$ such that $y_n=\varepsilon_n\tilde{y}_n\in M_\delta$ for $n$ large enough.

Thus, making the change of variable $z=x-\tilde{y}_n$, we get

$$\beta_{\varepsilon_n}(u_n)=y_n+\frac{\int_{\mathbb{R}^3}(\Upsilon(\varepsilon_n z+y_n)-y_n)|u_n(z+\tilde{y}_n)|^4dz}{\int_{\mathbb{R}^3}|u_n(z+\tilde{y}_n)|^4dz}.$$

Since, up to a subsequence, $|u_n|(\cdot+\tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^3,\mathbb{R})$ and $\varepsilon_n z+y_n\to y\in M$ for any $z\in\mathbb{R}^3$, we conclude. □

4.3. **Multiplicity of solutions for the problem (3.2).** Finally, we present a relation between the topology of $M$ and the number of solutions of the auxilliary problem (3.2).

**Theorem 4.1.** *For any $\delta>0$ such that $M_\delta\subset\Lambda$, there exists $\tilde{\varepsilon}_\delta>0$ such that, for any $\varepsilon\in(0,\tilde{\varepsilon}_\delta)$, the problem (3.2) has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

*Proof.* For any $\epsilon>0$, we define the function $\pi_\epsilon:M\to S_\varepsilon^+$ by

$$\pi_\epsilon(y)=m_\varepsilon^{-1}(\Phi_\epsilon(y)),\ \forall y\in M.$$

By Lemma 4.7 and Lemma 3.3$(B4)$, we obtain

$$\lim_{\epsilon\to0}\Psi_\epsilon(\pi_\epsilon(y))=\lim_{\epsilon\to0}J_\epsilon(\Phi_\epsilon(y))=c_{V_0},\ \text{uniformly in } y\in M.$$

Hence, there is a number $\hat{\epsilon}>0$ such that the set $\tilde{S}_\varepsilon^+:=\{u\in S_\varepsilon^+:\Psi_\varepsilon(u)\le c_{V_0}+h(\varepsilon)\}$ is nonempty, for all $\epsilon\in(0,\hat{\epsilon})$, since $\pi_\epsilon(M)\subset\tilde{S}_\varepsilon^+$. Here $h$ is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$.

Given $\delta>0$, by Lemma 4.7, Lemma 3.2$(A3)$, Lemma 4.8, and Lemma 4.9, we can find $\tilde{\varepsilon}_\delta>0$ such that for any $\varepsilon\in(0,\tilde{\varepsilon}_\delta)$, the following diagram

$$M\xrightarrow{\Phi_\varepsilon}\Phi_\varepsilon(M)\xrightarrow{m_\varepsilon^{-1}}\pi_\epsilon(M)\xrightarrow{m_\varepsilon}\Phi_\varepsilon(M)\xrightarrow{\beta_\varepsilon}M_\delta$$

is well defined and continuous. From Lemma 4.8, we can choose a function $\Theta(\epsilon,z)$ with $|\Theta(\epsilon,z)|<\frac{\delta}{2}$ uniformly in $z\in M$, for all $\epsilon\in(0,\hat{\epsilon})$ such that $\beta_\varepsilon(\Phi_\varepsilon(z))=z+\Theta(\epsilon,z)$ for all $z\in M$. Define $H(t,z)=z+(1-t)\Theta(\epsilon,z)$. Then $H:[0,1]\times M\to M_\delta$ is continuous. Clearly, $H(0,z)=\beta_\varepsilon(\Phi_\varepsilon(z))$, $H(1,z)=z$ for all $z\in M$. That is, $H(t,z)$ is a homotopy between $\beta_\varepsilon\circ\Phi_\varepsilon=(\beta_\varepsilon\circ m_\varepsilon)\circ\pi_\epsilon$ and the embedding $\iota:M\to M_\delta$. Thus, this fact implies that

$$\operatorname{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))\ge\operatorname{cat}_{M_\delta}(M).\tag{4.17}$$

By Corollary 3.1 and the abstract category theorem [26], $\Psi_\varepsilon$ has at least $\text{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on $S_\varepsilon^+$. Therefore, from Lemma 3.3($B4$) and (4.17), we have that $J_\varepsilon$ has at least $\text{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\varepsilon$ which implies that the problem (3.2) has at least $\text{cat}_{M_\delta}(M)$ solutions. $\square$

5. **Proof of Theorem 1.1.** In this section we prove our main result. The idea is to show that the solutions $u_\varepsilon$ obtained in Theorem 4.1 satisfy

$$|u_\varepsilon(x)|^2 \le a_0 \text{ for } x \in \Lambda_\varepsilon^c$$

for $\varepsilon$ small. The key ingredient is the following result.

**Lemma 5.1.** *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of the problem (3.2) for $\varepsilon = \varepsilon_n$. Then $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^N, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

We use the Moser iteration method to prove the above lemma. For our problem (3.2), there is one more nonlocal term and the nonlinear term has the critical growth, arguing as in [15, Lemma 5.1], just make some appropriate changes, it is easy to prove it and we omit it here.

Now, we are ready to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ solution of the problem (3.2), it holds

$$\|u_\varepsilon\|^2_{L^\infty(\Lambda_\varepsilon^c)} \le a_0. \tag{5.1}$$

We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ such that for every $n$ there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ which satisfies $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda^c_{\varepsilon_n})} > a_0. \tag{5.2}$$

Arguing as in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n := \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all $n$ large enough.

Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \le a_0$ in $B_R^c(0)$ and $n$ large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \le a_0$ in $B_R^c(\tilde{y}_n)$ and $n$ large enough. Moreover, if $n$ is so large that $r/\varepsilon_n > R$, then $\Lambda^c_{\varepsilon_n} \subset B^c_{r/\varepsilon_n}(\tilde{y}_n) \subset B_R^c(\tilde{y}_n)$, which gives $|u_n|^2 \le a$ for any $x \in \Lambda^c_{\varepsilon_n}$. This contradicts (5.2) and proves the claim.

Let now $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$, where $\tilde{\varepsilon}_\delta > 0$ is given by Theorem 4.1. Then we have $\text{cat}_{M_\delta}(M)$ nontrivial solutions to the problem (3.2). If $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is one of these solutions, then, by (5.1) and the definition of $g$, we conclude that $u_\varepsilon$ is also a solution to the problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\varepsilon|$, where $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a solution to the problem (1.1), as $\varepsilon \to 0^+$.

Take $\varepsilon_n \to 0^+$ and the sequence $(u_n)$ where each $u_n$ is a solution of (3.2) for $\varepsilon = \varepsilon_n$. From the definition of $g$, there exists $\gamma \in (0, a_0)$ such that

$$g(\varepsilon x, t^2)t^2 \le \frac{V_0}{K} t^2, \quad \text{for all } x \in \mathbb{R}^N,\ |t| \le \gamma.$$

Arguing as above we can take $R > 0$ such that, for $n$ large enough,

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{5.3}$$

Up to a subsequence, we may also assume that for $n$ large enough

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.4}$$

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < \gamma$. Thus, since $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, using $(g_5)$ and the diamagnetic inequality (2.1) that

$$\begin{aligned}\int_{\mathbb{R}^3}(a|\nabla|u_n||^2 + V_0|u_n|^2)dx + b\Big(\int_{\mathbb{R}^3}(|\nabla|u_n||^2dx\Big)^2 &\leq \int_{\mathbb{R}^3} g(\varepsilon_n x, |u_n|^2)|u_n|^2dx \\ &\leq \frac{V_0}{K}\int_{\mathbb{R}^3}|u_n|^2dx\end{aligned}$$

and, since $K > 2$, we get $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points $p_n$ of $|u_{\varepsilon_n}|$ belongs to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of the problem (1.1) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point $\eta_{\varepsilon_n}$ of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n\tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n\tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of $V$ allows to conclude that

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

The proof is now complete. □

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