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BIFURCATION OF POSITIVE SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS ROBIN AND NEUMANN PROBLEMS WITH COMPETING NONLINEARITIES

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Abstract. In this paper we deal with Robin and Neumann parametric elliptic equations driven by a nonhomogeneous differential operator and with a reaction that exhibits competing nonlinearities (concave-convex nonlinearities). For the Robin problem and without employing the Ambrosetti-Rabinowitz condition, we prove a bifurcation theorem for the positive solutions for small values of the parameter $\lambda > 0$. For the Neumann problem with a different geometry and using the Ambrosetti-Rabinowitz condition we prove bifurcation for large values of $\lambda > 0$.

1. **Introduction.**

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear, nonhomogeneous parametric Robin problem:

$$\left\{ \begin{array}{ll} -\operatorname{div} a(Du(z)) = f(z, u(z), \lambda) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a}(z) + \beta(z) u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ \quad u > 0, \ 1 < p < \infty. & \end{array} \right\} \tag{P_λ}$$

Hence $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone map, which satisfies certain other regularity and growth conditions, listed in hypotheses $H(a)$ below. These conditions are general enough, to incorporate in our setting various differential operators of interest, such as the p-Laplacian ($1 < p < \infty$). Also, $\dfrac{\partial u}{\partial n_a}$ denotes the conormal derivative defined by $\dfrac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$. The reaction $f(z, x, \lambda)$ is a parametric function with $\lambda > 0$ being the parameter and $(z, x) \to f(z, x, \lambda)$ is Carathéodory (that

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is, for all $x \in \mathbb{R}$ the mapping $z \longmapsto f(z,x,\lambda)$ is measurable and for a.a. $z \in \Omega$ the map $x \longmapsto f(z,x,\lambda)$ is continuous). We assume that $f(z,\cdot,\lambda)$ exhibits competing nonlinearities, namely near the origin, it has a "concave" term (that is, a strictly $(p-1)$- sublinear term), while near $+\infty$, the reaction is "convex" term (that is, $x \longmapsto f(z,x,\lambda)$ is $(p-1)$-superlinear). A special case of our reaction, is the following function:

$$f(z,x,\lambda) = f(x,\lambda) = \lambda x^{q-1} + x^{r-1} \text{ for all } x \geq 0,$$

with

$$1 < q < p < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p. \end{cases}$$

This reaction is encountered in the literature in the context of equations driven by the Laplacian (that is, $p=2$) or by the p-Laplacian ($1 < p < \infty$).

Our aim is to investigate the existence, nonexistence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. So, we prove two bifurcation type results, describing the set of positive solutions of (P_λ) as the parameter $\lambda > 0$ changes, when the reaction exhibits the competing effects of concave (that is, sublinear) and convex (that is, superlinear) nonlinearities. In the first theorem the bifurcation occurs near zero. More precisely, under general hypotheses we show that there exists $\lambda^* > 0$ such that the following properties hold:

(a) for all $\lambda \in (0,\lambda^*)$, problem ($P_\lambda$) has at least two positive solutions;
(b) for $\lambda = \lambda^*$ problem (P_{λ^*}) has at least one positive solution;
(c) for all $\lambda > \lambda^*$ problem (P_λ) has no positive solution.

In the second case, we assume that $\beta \equiv 0$ (Neumann boundary condition) and we consider the problem

$$\left\{ \begin{array}{ll} -\operatorname{div} a(Du(z)) = f_0(z,u(z)) - \lambda u(z)^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(z) = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega\,. \end{array} \right\} \qquad (S_\lambda)$$

We obtain a different geometry and we establish that the bifurcation occurs for large values of the parameter $\lambda > 0$. More precisely, under natural assumptions on f_0 we show that there exists $\lambda_* > 0$ such that

(a) for every $\lambda > \lambda_*$ problem (S_λ) has at least two positive solutions;
(b) for $\lambda = \lambda_*$ problem (S_{λ_*}) has at least one positive solution;
(c) for every $\lambda \in (0,\lambda_*)$ problem (S_λ) has no positive solution.

The first work concerning positive solutions for problems with concave and convex nonlinearities, was that of Ambrosetti, Brezis and Cerami [2]. They studied semilinear equations driven by the Dirichlet Laplacian and with a reaction of the form (1). Their work was extended to equations driven by the Dirichlet p-Laplacian by Garcia Azorero, Manfredi and Peral Alonso [10] and by Guo and Zhang [14]. We also refer to the contributions of de Figueiredo, Gossez and Ubilla [7], [8] to concave-convex type problems and general nonlinearities for the Laplacian, resp. p-Laplacian case. Extensions to equations involving more general reactions, were obtained by Gasinski and Papageorgiou [13], Hu and Papageorgiou [15] and Rădulescu and Repovš [22]. Other problems with competition phenomena, can be found in the works of Cîrstea, Ghergu and Rădulescu [4] (problems with singular terms) and of Kristaly and Moroşanu [16] (problems with oscillating reaction). Finally we mention the recent work of Papageorgiou and Rădulescu [20], who studied a Robin problem driven by the p-Laplacian and with a logistic reaction and proved multiplicity theorems for all large values of the parameter $\lambda > 0$, producing also nodal solutions.

We stress that the differential operator in (P_λ) is not homogeneous and this is a source of difficulties in the analysis of the problem, since many of the methods and techniques developed in the aforementioned papers do not work here. It appears that our results in the present paper are the first bifurcation-type theorems for nonhomogeneous elliptic equations.

2. **Mathematical background.** Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Cerami condition (the C-condition), if the following is true:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the function φ which compensates for the fact that the space X need not be locally compact (being in general infinite dimensional). It is more general than the more common Palais-Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of the critical values of φ. One of the main results in that theory, is the so-called mountain pass theorem of Ambrosetti and Rabinowitz [3]. Here we state it in a slightly more general form.

Theorem 2.1. *Let X be a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, u_0, $u_1 \in X$ with $||u_1 - u_0|| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geq m_\rho$ and c is a critical value of φ.

Let $\eta \in C^1(0, \infty)$ and assume that

$$0 < \hat{c} \leq \frac{t\eta'(t)}{\eta(t)} \leq c_0 \text{ and } c_1 t^{p-1} \leq \eta(t) \leq c_2(1 + t^{p-1}) \text{ for all } t > 0 \tag{1}$$
$$\text{with } c_1, c_2 > 0,\ 1 < p < \infty.$$

The hypotheses on the map $a(\cdot)$ are the following:

$H(a) : a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0, \infty)$, $a_0(t)t \to 0$ as $t \to 0^+$ and
$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(y)| \leq c_3 \dfrac{\eta(|y|)}{|y|}$ for some $c_3 > 0$, all $y \in \mathbb{R}^N \backslash \{0\}$;

(iii) $\dfrac{\eta(|y|)}{|y|}|\xi|^2 \leq (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \backslash \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)sds$ for all $t \geq 0$, then $pG_0(t) - a_0(t)t^2 \geq -\hat{\xi}$ for all $t \geq 0$, some $\hat{\xi} > 0$;

(v) there exists $\tau \in (1,p)$ such that $t \longmapsto G_0(t^{1/\tau})$ is convex on $(0,\infty)$, $\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0$ and

$$a_0(t)t^2 - \tau G_0(t) \geq \tilde{c}t^p \text{ for some } \tilde{c} > 0, \text{ all } t > 0.$$

Remark 1. These conditions on $a(\cdot)$ are motivated by the regularity results of Lieberman [17] and the nonlinear maximum principle of Pucci and Serrin [21]. According to the above conditions, the potential function $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then the function $y \longmapsto G(y)$ is convex and differentiable on $\mathbb{R}^N\backslash\{0\}$. We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N\backslash\{0\},\ \nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of the map $a(\cdot)$. Because $G(0) = 0$ and $y \longmapsto G(y)$ is convex, from the properties of convex functions, we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$. They follow easily from hypotheses $H(a)$ above.

Lemma 2.2. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then*

(a) $y \longmapsto a(y)$ is continuous and strictly monotone, hence maximal monotone too;
(b) $|a(y)| \leq c_4(1+|y|^{p-1})$ for some $c_4 > 0$, all $y \in \mathbb{R}^N$;
(c) $(a(y),y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.

Lemma 2.2 together with (1) and (2), lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 1. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1+|y|^p)$ for some $c_5 > 0$, all $y \in \mathbb{R}^N$.*

Example 1. The following maps $a(y)$, satisfy hypotheses $H(a)$ above:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.
This map corresponds to the p-Laplace operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ with $1 < q < p < \infty$ and $\mu > 0$.
This map corresponds to the (p,q)-differential operator defined by

$$\Delta_p u + \mu\Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such differential operators arise in many physical applications (see Papageorgiou and Rădulescu [18], [19] and the references therein).

(c) $a(y) = (1+|y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$.
This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}\left[(1+|Du|^2)^{\frac{p-2}{2}}Du\right] \text{ for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + \dfrac{|y|^{p-2}y}{1+|y|^p}$ with $1 < p < \infty$.

The hypotheses on the boundary weight map $\beta(\cdot)$ are the following:

$H(\beta)$: $\beta \in C^{1,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

In the analysis of problem (P_λ) in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space, with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In the Sobolev space $W^{1,p}(\Omega)$, we use the norm

$$||u|| = \left[||u||_p^p + ||Du||_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

To distinguish, we use $|\cdot|$ to denote the norm of $\mathbb{R}^N$.

If on $\partial\Omega$ we use the $(N-1)$-dimensional Hausdorff measure $\sigma(\cdot)$ (the surface measure on $\partial\Omega$), then we can define the Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$. We know that there exists a unique continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the trace map, such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. In fact γ_0 is compact. We have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)\ \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ and } \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0, with the understanding that all restrictions of elements of $W^{1,p}(\Omega)$ on $\partial\Omega$, are defined in the sense of traces.

Suppose $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth in the $x \in \mathbb{R}$ variable, that is

$$|f_0(z,x)| \leq a_0(z)(1 + |x|^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, $1 < r < p^*$. We set $F_0(z,x) = \displaystyle\int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega F_0(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The next proposition, was proved by Papageorgiou and Rădulescu [20] for $G(y) = \frac{1}{p}|y|^p$ for all $y \in \mathbb{R}^N$. The proof remains valid in the present more general setting, using Lemma 2.2, Corollary 1 and the regularity result of Lieberman [17] [p. 320].

Proposition 1. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exist $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } ||h||_{C^1(\overline{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$ and it is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } ||h|| \leq \rho_1.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega (a(Du), Dy)_{\mathbb{R}^N} dz \text{ for all } u, y \in W^{1,p}(\Omega) \tag{3}$$

The following, is a particular case of a more general result due to Gasinski and Papageorgiou [12].

Proposition 2. *If $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is defined by (3), then A is demicontinuous and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

In the sequel, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $x \in \mathbb{R}$, then we set $x^\pm = \max\{\pm x, 0\}$ and for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega) \text{ and } |u| = u^+ + u^-,\ u = u^+ - u^-.$$

Also, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega),$$

(the Nemytskii operator corresponding to the function h).

3. **Bifurcation near zero for the Robin problem.** In this section, we deal with competition phenomena that give rise to bifurcation of the problem solutions, when the parameter $\lambda > 0$ is near zero. This situation includes the classical equations with concave and convex nonlinearities.

The hypotheses on the reaction $f(z, x, \lambda)$ are the following:

$H_1 : f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a function such that for all $(z, x) \in \Omega \times [0, +\infty)$, $\lambda \longmapsto f(z, x, \lambda)$ is nondecreasing, for all $\lambda > 0$ $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and

(i) $(z, x) \longmapsto f(z, x, \lambda)$ is a Carathéodory function on $\Omega \times [0, +\infty)$;

(ii) $|f(z, x, \lambda)| \leq a_\lambda(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a_\lambda \in L^\infty(\Omega)_+$, $p < r < p^*$;

(iii) if $F(z, x, \lambda) = \displaystyle\int_0^x f(z, s, \lambda)ds$, then $\displaystyle\lim_{x\to+\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iv) there exists $\vartheta = \vartheta(\lambda) \in \left((r - p) \max\left\{\frac{N}{p}, 1\right\}, p^*\right)$ such that

$$0 < \gamma_0 \leq \liminf_{x\to+\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\vartheta} \text{ uniformly for a.a. } z \in \Omega;$$

(v) there exists $1 < \mu = \mu(\lambda) < q = q(\lambda) < \tau$ (see hypothesis $H(a)(v)$) and $\gamma = \gamma(\lambda) > \mu$, $\delta_0 = \delta_0(\lambda) \in (0, 1]$ such that

$$c_6 x^p \leq f(z, x, \lambda)x \leq qF(z, x, \lambda) \leq \xi_\lambda(z)x^\mu + \bar{c}x^\gamma \text{ for a.a. } z \in \Omega,\ \text{all } x \in [0, \delta_0],$$

with $c_6 = c_6(\lambda) > 0$, $\tau = \tau(\lambda) > 0$, $\xi_\lambda \in L^\infty(\Omega)_+$ and $||\xi_\lambda||_\infty \to 0$ as $\lambda \to 0^+$.

Remark 2. Since we are interested to find positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(z,x,\lambda)=0$ for a.a. $z\in\Omega$, all $x\leq 0$ and all $\lambda>0$. Note that hypotheses $H_1(ii),(iii)$ imply that

$$\lim_{x\to+\infty}\frac{f(z,x,\lambda)}{x^{p-1}}=+\infty \text{ uniformly for a.a. } z\in\Omega.$$

Therefore, $f(z,\cdot,\lambda)$ is $(p-1)$-superlinear near $+\infty$. However, we do not employ the AR-condition (unilateral version). We recall (see [3]), that $f(z,\cdot,\lambda)$ satisfies the (unilateral) AR-condition, if there exist $\eta=\eta(\lambda)>p$ and $M=M(\lambda)>0$ such that

$$\begin{array}{ll}(a) & 0<\eta F(z,x,\lambda)\leq f(z,x,\lambda)x \text{ for a.a. } z\in\Omega,\ \text{all } x\geq M,\\ (b) & \operatorname{ess\,inf}_\Omega\ F(\cdot,M,\lambda)>0.\end{array} \tag{4}$$

Integrating (4a) and using (4b), we obtain a weaker condition, namely that

$$c_7x^\eta\leq F(z,x,\lambda) \text{ for a.a. } z\in\Omega,\ \text{all } z\geq M \text{ and some } c_7>0. \tag{5}$$

Evidently (5) implies the much weaker hypothesis $H_1(iii)$. In (4) we may assume that $\eta>(r-p)\max\left\{\frac{N}{p},1\right\}$. Then we have

$$\begin{aligned}&\frac{f(z,x,\lambda)x-pF(z,x,\lambda)}{x^\eta}\\ =\ &\frac{f(z,x,\lambda)x-\eta F(z,x,\lambda)}{x^\eta}+\frac{(\eta-p)F(z,x,\lambda)}{x^\eta}\\ \geq\ &(\eta-p)c_7 \text{ for a.a. } z\in\Omega,\ \text{all } x\geq M \text{ (see (4a) and (5))}.\end{aligned}$$

So, we see that the AR-condition implies hypothesis $H_1(iv)$. This weaker "superlinearity" condition, incorporates in our setting $(p-1)$-superlinear nonlinearities with "slower" growth near $+\infty$, which fail to satisfy the AR-condition (see the examples below). Finally note that hypothesis $H_1(v)$ implies the presence of a concave nonlinearity near zero.

Example 2. The following functions satisfy hypotheses H_1. For the sake of simplicity, we drop the z-dependence:

$$f_1(x,\lambda)=\lambda x^{q-1}+x^{r-1} \text{ for all } x\geq 0,\ \text{with } 1<q<p<r<p^*$$

$$f_2(x,\lambda)=\begin{cases}\lambda x^{q-1}-x^{\eta-1} & \text{if } x\in[0,1]\\ x^{p-1}\left(\ln x+\frac{1}{p}\right)+\left(\lambda-\frac{1}{p}\right)x^{\nu-1} & \text{if } 1<x\end{cases}$$

with $q,\nu\in(1,p)$ and $\eta>p$

$$f_3(x,\lambda)=\begin{cases}x^{q-1} & \text{if } x\in[0,\rho(\lambda)]\\ x^{r-1}+\eta(\lambda) & \text{if } \rho(\lambda)<x\end{cases}$$

with $1<q<p<r<p^*,\ \eta(\lambda)=\rho(\lambda)^{p-1}-\rho(\lambda)^{r-1}$ and $\rho(\lambda)\to 0^+$ as $\lambda\to 0^+$.

Note that $f_2(\cdot,\lambda)$ does not satisfy the AR-condition.

We introduce the following Carathéodory function

$$\hat f(z,x,\lambda)=f(z,x,\lambda)+(x^+)^{p-1} \text{ for all } (z,x,\lambda)\in\Omega\times\mathbb{R}\times(0,+\infty).$$

Let $\hat{F}(z,x,\lambda) = \int_0^x \hat{f}(z,s,\lambda)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}(z,u,\lambda)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 3. *If hypotheses $H(a)$, $H(\beta)$ and H_1 hold and $\lambda > 0$, then the functional $\hat{\varphi}_\lambda$ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\hat{\varphi}_\lambda(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{6}$$

$$(1 + ||u_n||)\hat{\varphi}'_\lambda(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{7}$$

From (7) we have

$$|\langle \hat{\varphi}'_\lambda(u_n), h\rangle| \leq \frac{\epsilon_n ||h||}{1 + ||u_n||} \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \geq 1, \quad \text{with } \epsilon_n \to 0^+ \text{ as } n \to \infty,$$

$$\Rightarrow \left| \langle A(u_n), h\rangle + \int_\Omega |u_n|^{p-2}u_n h dz + \int_{\partial\Omega}\beta(z)(u_n^+)^{p-1}h d\sigma - \int_\Omega \hat{f}(z,u_n,\lambda)h dz \right| \leq \frac{\epsilon_n ||h||}{1 + ||u_n||} \text{ for all } n \geq 1. \tag{8}$$

In (8), first we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Using Lemma 2.2, we have

$$\frac{c_1}{p-1}||Du_n^-||_p^p + ||u_n^-||_p^p \leq \epsilon_n \text{ for all } n \geq 1,$$

$$\Rightarrow \quad u_n^- \to 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{9}$$

From (6), (9) and hypothesis $H_1(i)$, we have

$$\int_\Omega pG(Du_n^+)dz + \int_{\partial\Omega}\beta(z)(u_n^+)^p d\sigma - \int_\Omega pF(z,u_n^+,\lambda)dz \leq M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geq 1. \tag{10}$$

Also, in (8) we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and obtain

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N} dz - \int_{\partial\Omega}\beta(z)(u_n^+)^p d\sigma + \int_\Omega f(z,u_n^+,\lambda)u_n^+ dz \leq \epsilon_n \quad \text{for all } n \geq 1. \tag{11}$$

Adding (10) and (11), we have

$$\int_\Omega \left[pG(Du_n^+) - (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\right] dz + \int_\Omega \left[f(z,u_n^+,\lambda)u_n^+ - pF(z,u_n^+,\lambda)\right] dz \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1,$$

$$\Rightarrow \int_\Omega \left[f(z,u_n^+,\lambda)u_n^+ - pF(z,u_n^+,\lambda)\right] dz \leq M_3 + \hat{\xi} \text{ for all } n \geq 1 \quad \text{(see hypothesis } H(a)(iv)). \tag{12}$$

By virtue of hypotheses $H_1(ii), (iv)$, we can find $\gamma_1 \in (0, \gamma_0)$ and $c_8 = c_8(\gamma_1, \lambda) > 0$ such that

$$f(z,x,\lambda)x - pF(z,x,\lambda) \geq \gamma_1 x^\vartheta - c_8 \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

We use this unilateral growth estimate in (12) and obtain

$$\begin{aligned} & \gamma_1 ||u_n^+||_\vartheta^\vartheta \le M_4 \text{ for some } M_4 > 0, \text{ all } n \ge 1, \\ \Rightarrow \quad & \{u_n^+\}_{n\ge 1} \subseteq L^\vartheta(\Omega) \text{ is bounded.} \end{aligned} \tag{13}$$

First assume that $N \neq p$. From hypothesis $H_1(iv)$ it is clear that without any loss of generality, we may assume that $\vartheta \le r < p^*$. Then we can find $t \in [0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\vartheta} + \frac{t}{p^*}. \tag{14}$$

From the interpolation inequality (see, for example, Gasinski and Papageorgiou [11] [p. 905]), we have

$$\begin{aligned} ||u_n^+||_r & \le ||u_n^+||_\vartheta^{1-t} ||u_n^+||_{p^*}^t \\ & \le c_9 ||u_n^+||^t \text{ for some } c_9 > 0, \text{ all } n \ge 1 \\ & \text{(see (13) and use the Sobolev embedding theorem),} \\ \Rightarrow \quad ||u_n^+||_r^r & \le c_{10} ||u_n^+||^{tr} \text{ for all } n \ge 1, \text{ with } c_{10} = c_9^p > 0. \end{aligned} \tag{15}$$

By virtue of hypothesis $H_1(ii)$ we have

$$f(z,x,\lambda)x \le a_\lambda(z)(x + x^r) \text{ for a.a } z \in \Omega, \text{ all } x \ge 0. \tag{16}$$

In (8) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & ||Du_n^+||_p^p + \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma - \int_\Omega f(z,u_n^+,\lambda) u_n^+ dz \le \epsilon_n \text{ for all } n \ge 1, \\ \Rightarrow \quad & ||Du_n^+||_p^p \le c_{11}(1 + ||u_n^+||_r^r) \text{ for some } c_{11} > 0, \text{ all } n \ge 1 \text{ (see (16) and } H(\beta)), \\ & \le c_{12}(1 + ||u_n^+||^{tr}) \text{ for some } c_{12} > 0, \text{ all } n \ge 1 \text{ (see (15))}, \\ \Rightarrow \quad & ||Du_n^+||_p^p + ||u_n^+||_\vartheta^p \le c_{13}(1 + ||u_n^+||^{tr}) \text{ for some } c_{13} > 0, \text{ all } n \ge 1 \\ & \text{(see (13))} \end{aligned} \tag{17}$$

Since $\vartheta \le r < p^*$, we know that

$$u \longmapsto ||u||_\vartheta + ||Du||_p$$

is an equivalent norm on $W^{1,p}(\Omega)$ (see, for example, Gasinski and Papageorgiou [11] [p. 227]). So, from (17) we obtain

$$||u_n^+||^p \le c_{14}(1 + ||u_n^+||^{tr}) \text{ for some } c_{14} > 0, \text{ all } n \ge 1. \tag{18}$$

The hypothesis on ϑ (see $H_1(iv)$) and (14), imply that $tr < p$. So, from (18) we infer that

$$\{u_n^+\}_{n\ge 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{19}$$

If $N = p$, then $p^* = \infty$, while from the Sobolev embedding theorem, we know that $W^{1,p}(\Omega)$ is embedded (compactly) in $L^s(\Omega)$ for all $s \in [1,\infty)$. So, in the above argument, we need to replace $p^* = \infty$ by $s > r$ large such that

$$tr = \frac{s(r-\mu)}{s-\mu} < p \text{ (see (14) with } p^* \text{ replaced by } s > r).$$

Then the previous argument works and leads again to (19).

From (9) and (19) it follows that $\{u_n\}_{n\ge 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{20}$$

In (8) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (20). Then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,\\
\Rightarrow\quad & u_n \to u \text{ in } W^{1,p}(\Omega),\\
\Rightarrow\quad & \hat{\varphi}_\lambda \text{ satisfies the } C-\text{condition.}
\end{aligned}$$

□

Proposition 4. *If hypotheses $H(a)$, $H(\beta)$ and H_1 hold, then there exists $\lambda_+ > 0$ such that for every $\lambda \in (0, \lambda_+)$ there exists $\rho_\lambda > 0$ for which we have*

$$\inf\left[\hat{\varphi}_\lambda(u) : ||u|| = \rho_\lambda\right] = \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0).$$

Proof. Hypotheses $H_1(ii), (v)$ imply that for every $\lambda > 0$, we can find $c_{15} = c_{15}(\lambda) > 0$ such that

$$F(z,x,\lambda) \le \frac{\xi_\lambda(z)}{\mu}(x^+)^\mu + c_{15}[(x^+)^\gamma + (x^+)^r] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{21}$$

Then for $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &= \int_\Omega G(Du)dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}(z,u,\lambda)dz\\
&\ge \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \frac{||\xi_\lambda||_\infty}{\mu}||u^+||_\mu^\mu -\\
&\quad -c_{15}||u^+||_r^r - c_{15}||u^+||_\gamma^\gamma - \frac{1}{p}||u^+||_p^p \text{ (see Corollary 1 and (21)).}
\end{aligned}\tag{22}$$

It is clear that in hypothesis $H_1(v)$ we can always assume $\gamma \le p^*$ and that $\mu > 1$ is small enough so that $\frac{p-1}{\mu-1}\mu \ge r$ and $\frac{\gamma-1}{\mu-1}\mu \ge r$. By Young's inequality with $\epsilon > 0$ (see, for example, Gasinski and Papageorgiou [11] [p. 913]), we have

$$\begin{aligned}
||u||^p = ||u||\,||u||^{p-1} &\le \frac{\epsilon}{\mu}||u||^\mu + \frac{1}{\epsilon\mu'}||u||^{\frac{(p-1)\mu}{\mu-1}} \quad \left(\frac{1}{\mu} + \frac{1}{\mu'} = 1\right)\\
&\le \frac{\epsilon}{\mu}||u||^\mu + \frac{\mu-1}{\epsilon\mu}||u||^r\\
||u||^\gamma = ||u||\,||u||^{\gamma-1} &\le \frac{\epsilon}{\mu}||u||^\mu + \frac{1}{\epsilon\mu'}||u||^{\frac{(\gamma-1)\mu}{\mu-1}}\\
&\le \frac{\epsilon}{\mu}||u||^\mu + \frac{\mu-1}{\epsilon\mu}||u||^r \text{ for all } u \in W^{1,p}(\Omega) \text{ with } ||u|| \le 1
\end{aligned}$$

(recall that $\mu < \gamma, p \le p^*$). Using these bounds in (22), we obtain

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &\ge c_{16}||u||^p - c_{17}\left[\left(||\xi_\lambda||_\infty + \epsilon\right)||u||^\mu + (1+c_\epsilon)||u||^r\right]\\
&\qquad\qquad \text{for some } c_{16}, c_{17}, c_\epsilon > 0\\
&= \left[c_{16} - c_{17}\left(\left(||\xi_\lambda||_\infty + \epsilon\right)||u||^{\mu-p} + (1+c_\epsilon)||u||^{r-p}\right)\right]||u||^p\\
&\qquad\qquad \text{for all } u \in W^{1,p}(\Omega) \text{ with } ||u|| \le 1.
\end{aligned}\tag{23}$$

Let $k_\epsilon^\lambda(t) = \left(||\xi_\lambda||_\infty + \epsilon\right)t^{\mu-p} + (1+c_\epsilon)t^{r-p}$. Evidently $k_\epsilon^\lambda \in C^1(0,\infty)$ and since $\mu < p < r$ we have

$$k_\epsilon^\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 > 0$ such that

$$\begin{aligned} & k_\epsilon^\lambda(t_0) = \min_{t>0} k_\epsilon^\lambda(t), \\ \Rightarrow\quad & (k_\epsilon^\lambda)'(t_0) = (\mu - p)(||\xi_\lambda||_\infty + \epsilon)t_0^{\mu-p-1} + (r-p)(1+c_\epsilon)t_0^{r-p-1}, \\ \Rightarrow\quad & t_0 = t_0(\lambda) = \left[\frac{(p-\mu)(||\xi_\lambda||_\infty + \epsilon)}{(r-p)(1+c_\epsilon)}\right]^{\frac{1}{r-\mu}}. \end{aligned}$$

Then we have

$$k_\epsilon^\lambda(t) \to \chi(\epsilon) \text{ as } \lambda \to 0^+ \text{ with } \chi(\epsilon) \to 0^+ \text{ as } \epsilon \to 0^+.$$

We choose $\epsilon > 0$ small such that $\chi(\epsilon) < \dfrac{1}{2}\,\dfrac{c_{16}}{c_{17}}$. Then for such an $\epsilon > 0$, we can find $\lambda_+ = \lambda_+(\epsilon) > 0$ such that

$$k_\epsilon^\lambda(t_0) < \frac{c_{16}}{c_{17}} \text{ and } t_0(\lambda) \le 1 \text{ for all } \lambda \in (0, \lambda_+) \text{ (see hypothesis } H_1(v))$$

Then by virtue of (23), we have

$$\hat{\varphi}_\lambda(u) \ge \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0) \text{ for all } u \in W^{1,p}(\Omega) \text{ with } ||u|| = \rho_\lambda = t_0(\lambda) \le 1.$$

□

Note that as a direct consequence of hypothesis $H_1(iii)$, we have:

Proposition 5. *If hypotheses $H(a), H(\beta)$ and H_1 hold, $\lambda > 0$ and $u \in \operatorname{int} C_+$, then $\hat{\varphi}_\lambda(tu) \to -\infty$ as $t \to \infty$.*

We introduce the following sets:

$$\begin{aligned} &\mathcal{S} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\}, \\ &S(\lambda) = \text{the set of positive solutions of } (P_\lambda). \end{aligned}$$

We can show that $\mathcal{S}$ is nonempty, as well as a useful structural property of the solution set $S(\lambda)$.

Proposition 6. *If hypotheses $H(a), H(\beta)$ and H_1 hold, then $\mathcal{S} \neq \varnothing$ and for every $\lambda \in \mathcal{S}$ $\varnothing \neq S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. Let $\lambda_+ > 0$ be as postulated by Proposition 4 and let $\lambda \in (0, \lambda_+)$. Propositions 3, 4 and 5 permit the use of Theorem 2.1 (the mountain pass theorem) on the functional $\hat{\varphi}_\lambda$. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda'(u_0) = 0 \text{ and } \hat{\varphi}_\lambda(0) = 0 < \hat{m}_\lambda \le \hat{\varphi}_\lambda(u_0). \tag{24}$$

From the inequality in (24) we see that $u_0 \neq 0$. From the inequality in (24), we have

$$\langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1} h d\sigma = \int_\Omega \hat{f}(z, u_0, \lambda) h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{25}$$

In (25) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Using Lemma 2.2, we have

$$\begin{aligned} & \frac{c_1}{p-1}||Du_0^-||_p^p + ||u_0^-||_p^p \le 0, \\ \Rightarrow\quad & u_0 \ge 0,\ u_0 \neq 0. \end{aligned}$$

Therefore (25) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma = \int_\Omega f(z, u_0, \lambda) dz \text{ for all } h \in W^{1,p}(\Omega). \tag{26}$$

In what follows by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$ (recall that $\frac{1}{p} + \frac{1}{p'} = 1$ and $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$). From the representation theorem for the elements of the dual space $W^{-1,p'}(\Omega)$ (see, for example, Gasinski and Papageorgiou [11] [p. 212]), we have

$$\operatorname{div} a(Du_0) \in W^{-1,p'}(\Omega) \text{ (see Lemma 2.2).}$$

Performing integration by parts, we have

$$\langle A(u_0), h\rangle = \langle -\operatorname{div} a(Du_0), h\rangle_0 \text{ for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

Using this equation in (26) and recalling that $h|_{\partial\Omega} = 0$ for all $h \in W_0^{1,p}(\Omega)$, we obtain

$$\langle -\operatorname{div} a(Du_0), h\rangle = \int_\Omega f(z, u_0, \lambda) h dz \text{ for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega),$$

$$\Rightarrow \quad -\operatorname{div} a(Du_0(z)) = f(z, u_0(z), \lambda) \text{ for a.a. } z \in \Omega. \tag{27}$$

Note that $f(\cdot, u_0(\cdot), \lambda) \in L^{r'}(\Omega)$ where $\frac{1}{r} + \frac{1}{r'} = 1$ (see hypothesis $H_1(ii)$). Since $p < r$, we have $W_0^{1,r}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ continuously and densely. Then $W^{-1,p'}(\Omega) \hookrightarrow W^{-1,r'}(\Omega)$ continuously and densely (see, for example, Gasinski and Papageorgiou [11] [p. 141]). Then from (27) we see that we can apply the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [11] [p. 210]) and have

$$\langle A(u_0), h\rangle + \int_\Omega (\operatorname{div} a(Du_0)) h dz = \left\langle \frac{\partial u_0}{\partial n_a}, h \right\rangle_{\partial\Omega} \tag{28}$$
$$\text{for all } h \in W^{1,r}(\Omega) \subseteq W^{1,p}(\Omega).$$

Here by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair $(W^{-\frac{1}{r'},r'}(\partial\Omega), W^{\frac{1}{r'},r}(\partial\Omega))$. Returning to (26) and using (28), we obtain

$$\langle -\operatorname{div} a(Du_0), h\rangle + \left\langle \frac{\partial u_0}{\partial n_a}, h \right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma = \int_\Omega f(z, u_0, \lambda) h dz$$
$$\text{for all } h \in W^{1,r}(\Omega)$$

$$\Rightarrow \quad \left\langle \frac{\partial u_0}{\partial n_a}, h \right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma = 0 \text{ for all } h \in W^{1,r}(\Omega) \text{ (see (27)).} \tag{29}$$

But we know that if γ_0 is the trace map on $W^{1,p}(\Omega)$, then $\operatorname{im}\left(\gamma_0|_{W^{1,r}(\Omega)}\right) = W^{\frac{1}{r'},r}(\partial\Omega)$. So, from (29), it follows that

$$\frac{\partial u_0}{\partial n_a} + \beta(z) u_0^{p-1} = 0 \text{ on } \partial\Omega. \tag{30}$$

From (27) and (30) it follows that $u_0 \in S(\lambda)$ and so $(0, \lambda_+) \subseteq \mathcal{S}$.

From Winkert [23] we have that $u_0 \in L^\infty(\Omega)$. Then we can apply the regularity result of Lieberman [17] [p. 320] and infer that $u_0 \in C_+,\ u_0 \neq 0$.

Hypotheses $H_1(ii), (v)$ imply that given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z, x, \lambda) + \xi_\rho x^{p-1} \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } x \in [0, \rho]. \tag{31}$$

Let $\rho = ||u_0||_\infty$ and let $\xi_\rho > 0$ be as postulated by (31). Then

$$\begin{aligned} & -\mathrm{div}\, a(Du_0(z)) + \xi_\rho u_0(z)^{p-1} \\ & = f(z, u_0(z), \lambda) + \xi_\rho u_0(z)^{p-1} \geq 0 \text{ for a.a. } z \in \Omega \text{ (see (27) and (31))}, \\ \Rightarrow \quad & \mathrm{div}\, a(Du_0(z)) \leq \xi_\rho u_0(z)^{p-1} \text{ a.e. in } \Omega, \\ \Rightarrow \quad & u_0 \in \mathrm{int}\, C_+ \text{ (see Pucci and Serrin [21] [pp. 111, 120])} \\ \Rightarrow \quad & S(\lambda) \subseteq \mathrm{int}\, C_+. \end{aligned}$$

□

The next proposition establishes a useful property of the set $\mathcal{S}$.

Proposition 7. *If hypotheses $H(a)$, $H(\beta)$ and H_1 hold and $\lambda \in \mathcal{S}$, then $(0, \lambda] \subseteq \mathcal{S}$.*

Proof. Since $\lambda \in \mathcal{S}$, we can find $u_\lambda \in S(\lambda) \subseteq \mathrm{int}\, C_+$. Let $\eta \in (0, \lambda)$ and consider the following truncation-perturbation of the reaction in problem (P_η):

$$k_\eta(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z, x, \eta) + x^{p-1} & \text{if } 0 \leq x \leq u_\lambda(z) \\ f(z, u_\lambda(z), \eta) + u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{32}$$

This is a Carathéodory function. We set $K_\eta(z, x) = \int_0^x k_\eta(z, s) ds$ and consider the C^1-functional $\hat{\psi}_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\eta(u) = \int_\Omega G(Du) dz + \frac{1}{p} ||u||_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega K_\eta(z, u) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From Corollary 1, hypothesis $H(\beta)$ and (32), it is clear that $\hat{\psi}_\eta$ is coercive. Also, from the Sobolev embedding theorem and the compactness of the trace map γ_0 into $L^p(\partial\Omega)$, we see that $\hat{\psi}_\eta$ is sequentially weakly lower semicontinuous. So, from the Weierstrass theorem we can find $u_\eta \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\eta(u_\eta) = \inf \left[\hat{\psi}(u) : u \in W^{1,p}(\Omega) \right]. \tag{33}$$

Let $\xi \in (0, \delta_0(\eta)]$ and $\xi \leq \min_{\overline{\Omega}} u_\lambda$ (see hypothesis $H_1(v)$ and recall that $u_\lambda \in \mathrm{int}\, C_+$). Then

$$\hat{\psi}_\eta(\xi) \leq \frac{\xi^p}{p} ||\beta||_{L^\infty(\partial\Omega)} - \frac{\xi^q c_6}{q} |\Omega|_N \text{ (see (32))}.$$

Since $q < p$ (see hypothesis $H_1(v)$), by taking $\xi \in (0, 1)$ even smaller if necessary, we will have

$$\begin{aligned} & \hat{\psi}_\eta(\xi) < 0 \\ \Rightarrow \quad & \hat{\psi}_\eta(u_\eta) < 0 = \hat{\psi}_\eta(0) \text{ (see (33))}, \text{ hence } u_\eta \neq 0. \end{aligned}$$

From (33) we have

$$\hat{\psi}'_\eta(u_\eta) = 0, \Rightarrow$$

$$\langle A(u_\eta), h \rangle + \int_\Omega |u_\eta|^{p-2} u_\eta h dz + \int_{\partial\Omega} \beta(z)(u_\eta^+)^{p-1} h d\sigma = \int_\Omega k_\eta(z, u_\eta) h dz \tag{34}$$
$$\text{for all } h \in W^{1,p}(\Omega).$$

In (34), first we choose $h = -u_\eta^- \in W^{1,p}(\Omega)$. Using Lemma 2.2 and (32), we have

$$\begin{aligned} &\frac{c_1}{p-1}||Du_\eta^-||_p^p + ||u_\eta^-||_p^p \leq 0, \\ \Rightarrow\quad & u_\eta \geq 0,\ u_\eta \neq 0. \end{aligned}$$

Next, in (34), we choose $h = (u_\eta - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(u_\eta), (u_\eta - u_\lambda)^+\rangle + \int_\Omega u_\eta^{p-1}(u_\eta - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z) u_\eta^{p-1}(u_\eta - u_\lambda)^+ d\sigma \\ = \ & \int_\Omega [f(z, u_\lambda, \eta) + u_\lambda^{p-1}](u_\eta - u_\lambda)^+ dz \text{ (see (32))} \\ \leq \ & \int_\Omega [f(z, u_\lambda, \lambda) + u_\lambda^{p-1}](u_\eta - u_\lambda)^+ dz \text{ (since } f(z, u_\lambda(z), \cdot) \text{ is nondecreasing)} \\ = \ & \langle A(u_\lambda), (u_\eta, u_\lambda)^+\rangle + \int_\Omega u_\lambda^{p-1}(u_\eta - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\eta - u_\lambda)^+ d\sigma \\ \Rightarrow \ & \langle A(u_\eta) - A(u_\lambda), (u_\eta - u_\lambda)^+\rangle + \int_\Omega (u_\eta^{p-1} - u_\lambda^{p-1})(u_\eta - u_\lambda)^+ dz \leq 0 \\ & \qquad\qquad\qquad\qquad \text{(see hypothesis } H(\beta)) \\ \Rightarrow \ & |\{u_\eta > u_\lambda\}|_N = 0, \text{ hence } u_\eta \leq u_\lambda. \end{aligned}$$

So, we have proved that

$$u_\eta \in [0, u_\lambda] = \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq u_\lambda(z) \text{ for a.a. } z \in \Omega\},\ u_\eta \neq 0.$$

Then because of (32), equation (34) becomes

$$\langle A(u_\eta), h\rangle + \int_{\partial\Omega} \beta(z) u_\eta^{p-1} h d\sigma = \int_\Omega f(z, u_\eta, \eta) h dz \text{ for all } h \in W^{1,p}(\Omega).$$

From this, as in the proof of Proposition 6, using the nonlinear Green's identity, we infer that

$$\begin{aligned} & u_\eta \in S(\eta) \subseteq \operatorname{int} C_+, \text{ hence } \eta \in \mathcal{S}, \\ \Rightarrow\quad & (0, \lambda] \subseteq \mathcal{S}. \end{aligned}$$

□

Let $\lambda^* = \sup \mathcal{S}$. We show that λ^* is finite by strengthening the conditions on the reaction $f(z, x, \lambda)$. So, the new stronger hypotheses on f are the following:

$H_2 : f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a function such that for a.a. $z \in \Omega$ and all $\lambda > 0$ $f(z, 0, \lambda) = 0$ and

(i) for all $(x, \lambda) \in \mathbb{R} \times (0, \infty)$, $z \longmapsto f(z, x, \lambda)$ is measurable, while for a.a. $z \in \Omega$, $(x, \lambda) \longmapsto f(z, x, \lambda)$ is continuous;

(ii) $|f(z, x, \lambda)| \leq a_\lambda(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, all $\lambda > 0$, with $a_\lambda \in L^\infty(\Omega)$, $\lambda \longmapsto ||a_\lambda||_\infty$ bounded on bounded sets in $(0, \infty)$ and $p < r < p^*$;

(iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds$, then $\lim_{x \to +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iv) there exists $\vartheta = \vartheta(\lambda) \in \left((r-p)\max\left\{\frac{N}{p}, 1\right\}, p^*\right)$ such that

$$0 < \gamma_0 \leq \liminf_{x \to +\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\vartheta} \text{ uniformly for a.a. } z \in \Omega;$$

(v) there exists $1 < \mu = \mu(\lambda) < q = q(\lambda) < \tau$ (see hypothesis $H(a)(v)$) and $\gamma = \gamma(\lambda) > \mu$, $\delta_0 = \delta_0(\lambda) \in (0,1)$ such that

$$c_6 x^q \leq f(z,x,\lambda)x \leq qF(z,x,\lambda) \leq \xi_\lambda(z)x^\mu + \tau x^\gamma \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta_0$$

with $c_6 = c_6(\lambda) > 0$, $c_6(\lambda) \to +\infty$ as $\lambda \to +\infty$, $\bar{c} = \bar{c}(\lambda) > 0$, $\xi_\lambda \in L^\infty(\Omega)_+$ with $||\xi_\lambda||_\infty \to 0$ as $\lambda \to 0^+$;

(vi) for every $\rho > 0$, there exists $\xi_\rho = \xi_\rho(\lambda) > 0$ such that for a.a. $z \in \Omega$, $x \longmapsto f(z,x,\lambda) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$;

(vii) for every interval $K = [x_0, \hat{x}]$ with $x_0 > 0$ and every $\lambda > \lambda' > 0$, there exists $d_K(x_0,\lambda)$ nondecreasing in λ with $d_K(x_0,\lambda) \to +\infty$ as $\lambda \to +\infty$ and $\hat{d}_K(x_0,\lambda,\lambda')$ such that

$$f(z,x,\lambda) \geq d_K(x_0,\lambda) \text{ for a.a. } z \in \Omega, \text{ all } x \in K$$
$$f(z,x,\lambda) - f(z,x,\lambda') \geq \hat{d}_K(x_0,\lambda,\lambda') \text{ for a.a. } z \in \Omega, \text{ all } x \in K.$$

Remark 3. Suppose that $f(z,x,\lambda) = \lambda g(x) + h(z,x)$ with $g(\cdot)$ continuous, nondecreasing, positive on $(0,\infty)$ and $h \geq 0$, $h(z,\cdot) \in C^1(\mathbb{R})$ for a.a. $z \in \Omega$ and $h'_x(z,x) \geq -\xi^* x^{\eta-2}$ for a.a. $z \in \Omega$, all $x > 0$ and some $\xi^* > 0$, $\eta \geq p$. Then hypotheses $H_2(vi),(vii)$ are satisfied. Also, the examples presented after hypotheses H_1, satisfy also the new conditions.

Proposition 8. *If hypotheses $H(a)$, $H(\beta)$ and H_2 hold, then $\lambda^* < \infty$.*

Proof. We claim that there exists $\overline{\lambda} > 0$ such that

$$f(z,x,\overline{\lambda}) \geq x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{35}$$

Indeed by virtue of hypothesis $H_2(v)$, we have $f(z,x,\lambda) \geq c_6(\lambda)x^{q-1}$ for a.a. $z \in \Omega$ all $x \in [0,\delta_0(\lambda)]$. The hypothesis on $c_6(\cdot)$ implies that we can find $\lambda_0 > 0$ and $0 < \delta_1 \leq \delta_0(\lambda_0)$ such that

$$f(z,x,\lambda_0) \geq c_6(\lambda_0)x^{q-1} \geq x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta_1] \tag{36}$$

Hypotheses $H_1(iii),(iv)$ imply that we can find $M_5 > 0$ such that

$$f(z,x,\lambda_0) \geq x^{p-1} \text{ for all a.a. } z \in \Omega, \text{ all } x \geq M_5. \tag{37}$$

Finally, from hypothesis $H_2(vii)$, for $K = [\delta_1, M_5]$ we have

$$f(z,x,\lambda) \geq d_K(z,\lambda) \text{ for a.a. } z \in \Omega, \text{ all } x \in [\delta_1, M_5], \text{ all } \lambda > 0.$$

Since $d_K(x,\lambda) \to +\infty$ as $\lambda \to +\infty$, we can find $\overline{\lambda} \geq \lambda_0$ such that

$$f(z,x,\overline{\lambda}) \geq d_K(x,\overline{\lambda}) \geq M_5^{p-1} \geq x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in [\delta_1, M_5]. \tag{38}$$

Recalling that $f(z,x,\cdot)$ and $c_6(\cdot)$ are nondecreasing in $\lambda > 0$, from (36), (37) and (38) we conclude that (35) is true.

Now, let $\lambda > \overline{\lambda}$ and assume that $\lambda \in \mathcal{S}$. Then we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_1$ (see Proposition 6). Let $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$. For $\delta > 0$ we set $m_\lambda^\delta = m_\lambda + \delta \in \operatorname{int} C_+$. Also, let $\rho = ||u_\lambda||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H_2(vi)$. We have

$$\begin{aligned}
& -\operatorname{div} a(Dm_\lambda^\delta) + \xi_\rho (m_\lambda^\delta)^{p-1} \\
\leq\ & \xi_\rho m_\lambda^{p-1} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
\leq\ & (1+\xi_\rho) m_\lambda^{p-1} + \chi(\delta) \\
\leq\ & f(z, m_\lambda, \overline{\lambda}) + \xi_\rho m_\lambda^{p-1} + \chi(\delta) \text{ (see (35))} \\
=\ & f(z, m_\lambda, \lambda) + \xi_\rho m_\lambda^{p-1} + [f(z, m_\lambda, \overline{\lambda}) - f(z, m_\lambda, \lambda)] + \chi(\delta) \\
\leq\ & f(z, m_\lambda, \lambda) + \xi_\rho m_\lambda^{p-1} - \hat{d}_K(m_\lambda, \lambda, \overline{\lambda}) + \chi(\delta) \text{ with } K = \{m_\lambda\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(see hypothesis } H_2(vii)) \\
\leq\ & f(z, m_\lambda, \lambda) + \xi_\rho m_\lambda^{p-1} \text{ for all } \delta > 0 \text{ small} \\
\leq\ & f(z, u_\lambda, \lambda) + \xi_\rho u_\lambda(z)^{p-1} \text{ (since } m_\lambda \leq u_\lambda(z) \text{ for all } z \in \overline{\Omega}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{see hypothesis } H_2(vii)) \\
=\ & -\operatorname{div} a(Du_\lambda(z)) + \xi_\rho u_\lambda(z)^{p-1} \text{ for a.a. } z \in \Omega \text{ (since } u_\lambda \in S(\lambda)), \\
\Rightarrow\ & m_\lambda^\delta \leq u_\lambda(z) \text{ for all } z \in \overline{\Omega}, \text{ all } \delta > 0 \text{ small, a contradiction.}
\end{aligned}$$

This means that $\lambda \notin \mathcal{S}$ and so $\lambda^* \leq \overline{\lambda} < \infty$. $\square$

Proposition 9. *If hypotheses $H(a)$, $H(\beta)$ and H_2 hold and $\eta \in (0, \lambda^*)$, then problem (P_η) admits at least two distinct positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \leq \hat{u}.$$

Proof. Let $\eta, \lambda \in (0, \lambda^*)$ with $\eta < \lambda$ and let $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$. From the proof of Proposition 7, we know that by using a suitable truncation-perturbation of the reaction of problem (P_η) (see (32)), we can find $u_0 \in [0, u_\lambda] \cap S(\eta)$, which is a minimizer of the corresponding truncated energy functional $\hat{\psi}_\eta$ (see the proof of Proposition 7).

For $\delta > 0$, let $u_0^\delta = u_0 + \delta \in \operatorname{int} C_+$ and for $\rho = ||u_\lambda||_\infty$, let $\xi_\rho > 0$ be as postulated by hypothesis $H_2(vi)$. We have

$$\begin{aligned}
& -\operatorname{div} a(Du_0^\delta(z)) + \xi_\rho u_0^\delta(z)^{p-1} \\
\leq\ & -\operatorname{div} a(Du_0(z)) + \xi_\rho u_0(z)^{p-1} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
=\ & f(z, u_0(z), \eta) + \xi_\rho u_0(z)^{p-1} + \chi(\delta) \text{ (since } u_0 \in S(\eta)) \\
=\ & f(z, u_0(z), \lambda) + \xi_\rho u_0(z)^{p-1} + [f(z, u_0(z), \eta) - f(z, u_0(z), \lambda)] + \chi(\delta) \\
\leq\ & f(z, u_\lambda(z), \lambda) + \xi_\rho u_\lambda(z)^{p-1} - \hat{d}_K(m_0, \lambda, \eta) + \chi(\delta) \\
& \text{(since } u_0 \leq u_\lambda, \text{ see hypothesis } H_2(vi) \text{ and with } K = u_0(\overline{\Omega}), \ m_0 = \inf K) \\
\leq\ & f(z, u_\lambda(z), \lambda) + \xi_\rho u_\lambda(z)^{p-1} \text{ for } \delta > 0 \text{ small,} \\
=\ & -\operatorname{div} a(Du_\lambda(z)) + \xi_\rho u_\lambda(z)^{p-1} \text{ a.e. in } \Omega \text{ (since } u_\lambda \in S(\lambda)), \\
\Rightarrow\ & u_0^\delta \leq u_\lambda \text{ for } \delta > 0 \text{ small,} \\
\Rightarrow\ & u_\lambda - u_0 \in \operatorname{int} C_+.
\end{aligned}$$

So, we have proved that

$$u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, u_\lambda]. \tag{39}$$

Recall that u_0 is a minimizer of the functional $\hat{\psi}_\lambda$ (see the proof of Proposition 7). Note that

$$\begin{aligned}
& \hat{\psi}_\lambda|_{[0,u_\lambda]} = \hat{\varphi}_\lambda|_{[0,u_\lambda]} \text{ (see (32))} \\
\Rightarrow\quad & u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \hat{\varphi}_\lambda \text{ (see (39))}, \\
\Rightarrow\quad & u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\lambda \text{ (see Proposition 1)}.
\end{aligned}$$

Next, we consider the following truncation-perturbation of the reaction in problem (P_η):

$$\gamma_\eta(z,x) = \begin{cases} f(z,u_0(z),\eta) + u_0(z)^{p-1} & \text{if } x \le u_0(z) \\ f(z,x,\eta) + x^{p-1} & \text{if } u_0(z) < x. \end{cases} \tag{40}$$

This is a Carathéodory function. Let $\Gamma_\eta(z,x) = \int_0^x \gamma_\eta(z,s)ds$ and consider the C^1-functional $\sigma_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\eta(u) = \int_\Omega G(Du)dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma = \int_\Omega \Gamma_\eta(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Note that

$$\begin{aligned}
& \sigma_\eta = \hat{\varphi}_\eta + \hat{\xi}_\eta \text{ with } \hat{\xi}_\eta \in \mathbb{R} \text{ (see (40))}, \\
\Rightarrow\quad & \sigma_\eta \text{ satisfies the } C-\text{condition (see Proposition 3)}.
\end{aligned} \tag{41}$$

Moreover, Proposition 5 implies that if $u \in \operatorname{int} C_+$, then

$$\sigma_\eta(tu) \to -\infty \text{ as } t \to +\infty. \tag{42}$$

Claim 1. *We may assume that u_0 is a local minimizer of σ_η.*

Recall that $u_0 \le u_\lambda$. Then using u_λ, we truncate $\gamma_\eta(z,\cdot)$ as follows:

$$\hat{\gamma}_\eta(z,x) = \begin{cases} \gamma_\eta(z,x) & \text{if } x \le u_\lambda(z) \\ \gamma_\eta(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x \end{cases} \tag{43}$$

This is a Carathéodory function. We set $\hat{\Gamma}_\eta(z,x) = \int_0^x \hat{\gamma}_\eta(z,s)ds$ and consider the C^1-functional $\hat{\sigma}_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\eta(u) = \int_\Omega G(Du)dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{\Gamma}_\eta(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (43), Corollary 1 and hypothesis $H(\beta)$, we see that the functional $\hat{\sigma}_\eta$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\hat{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \hat{\sigma}_\eta(\hat{u}_0) = \inf[\hat{\sigma}_\eta(u) : u \in W^{1,p}(\Omega)], \\
\Rightarrow\; & \hat{\sigma}'_\eta(\hat{u}_0) = 0, \\
\Rightarrow\; & \langle A(\hat{u}_0), h\rangle + \int_\Omega |\hat{u}_0|^{p-2}\hat{u}_0 h dz + \int_{\partial\Omega} \beta(z)(\hat{u}_0^+)^{p-1} h d\sigma = \int_\Omega \hat{\gamma}_\eta(z,\hat{u}_0) h dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned} \tag{44}$$

In (44), first we choose $h = (u_0 - \hat{u}_0)^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
& \left\langle A(\hat{u}_0), (u_0 - \hat{u}_0)^+ \right\rangle + \int_\Omega |\hat{u}_0|^{p-2}\hat{u}_0 (u_0 - \hat{u}_0)^+ dz + \\
& \qquad\qquad + \int_{\partial\Omega} \beta(z)(\hat{u}_0^+)^{p-1}(u_0 - \hat{u}_0)^+ d\sigma \\
= \; & \int_\Omega \left[f(z, u_0, \eta) + u_0^{p-1} \right] (u_0 - \hat{u}_0)^+ dz \text{ (recall that } u_0 \le u_\lambda \\
& \qquad\qquad \text{and see (43) and (40))} \\
= \; & \left\langle A(u_0), (u_0 - \hat{u}_0)^+ \right\rangle + \int_\Omega u_0^{p-1}(u_0 - \hat{u}_0)^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - \hat{u}_0)^+ d\sigma \\
& \qquad\qquad \text{(since } u_0 \in S(\eta)), \\
\Rightarrow \; & \left\langle A(u_0) - A(\hat{u}_0), (u_0 - \hat{u}_0)^+ \right\rangle + \int_\Omega (u_0^{p-1} - |\hat{u}_0|^{p-2}\hat{u}_0)\, (u_0 - \hat{u}_0)^+ dz \le 0 \\
& \qquad\qquad \text{(see hypothesis } H(\beta)), \\
\Rightarrow \; & |\{u_0 > \hat{u}_0\}_N| = 0, \text{ hence } u_0 \le \hat{u}_0.
\end{aligned}
$$

Next in (44) we choose $(\hat{u}_0 - u_\lambda)^+ \in W^{1,p}(\Omega)$. We obtain

$$
\begin{aligned}
& \left\langle A(\hat{u}_0), (\hat{u}_0 - u_\lambda)^+ \right\rangle + \int_\Omega \hat{u}_0^{p-1}(\hat{u}_0 - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z)\hat{u}_0^{p-1}(\hat{u}_0 - u_\lambda)^+ d\sigma \\
= \; & \int_\Omega [f(z, u_\lambda, \eta) + u_\lambda^{p-1}](\hat{u}_0 - u_\lambda)^+ dz \text{ (see (43) and (40))} \\
\le \; & \int_\Omega \left[f(z, u_\lambda, \lambda) + u_\lambda^{p-1} \right] (\hat{u}_0 - u_\lambda)^+ dz \text{ (see hypothesis } H_2(vii)) \\
= \; & \left\langle A(u_\lambda), (\hat{u}_0 - u_\lambda)^+ \right\rangle + \int_\Omega u_\lambda^{p-1}(\hat{u}_0 - u_\lambda)^+ dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(\hat{u}_0 - u_\lambda)^+ d\sigma \\
& \qquad\qquad \text{(since } u_\lambda \in S(\lambda)), \\
\Rightarrow \; & \left\langle A(\hat{u}_0) - A(u_\lambda), (\hat{u}_0 - u_\lambda)^+ \right\rangle + \int_\Omega (\hat{u}_0^{p-1} - u_\lambda^{p-1})(\hat{u}_0 - u_\lambda)^+ dz \le 0 \\
& \qquad\qquad \text{(see hypothesis } H(\beta)), \\
\Rightarrow \; & |\{\hat{u}_0 > u_\lambda\}|_N = 0, \text{ hence } \hat{u}_0 \le u_\lambda.
\end{aligned}
$$

So, we have proved that

$$\hat{u}_0 \in [u_0, u_\lambda] = \{u \in W^{1,p}(\Omega) : u_0(z) \le u(z) \le u_\lambda(z) \text{ a.e. in } \Omega\}.$$

If $\hat{u}_0 \neq u_0$, then by virtue of (43) and (40), we see that

$$\hat{u}_0 \in S(\eta) \subseteq \operatorname{int} C_+, \; u_0 \le \hat{u}_0, \; u_0 \neq \hat{u}_0$$

and so we are done, since this is the desired second positive solution of problem (P_η).

Hence, we may assume that $\hat{u}_0 = u_0 \in \operatorname{int} C_+$. Recall that $u_\lambda - u_0 \in \operatorname{int} C_+$ (see (39)) and $\hat{\sigma}_\eta|_{[0,u_\lambda]} = \sigma_\eta|_{[0,u_\lambda]}$ (see (43)). Therefore

$$
\begin{aligned}
& u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \sigma_\eta, \\
\Rightarrow \; & u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \sigma_\eta \text{ (see Proposition 1).}
\end{aligned}
$$

This proves the Claim.

Reasoning as above, we can show that

$$K_{\sigma_\eta} \subseteq [u_0, \infty) = \{u \in W^{1,p}(\Omega) : u_0(z) \le u(z) \text{ a.e. in } \Omega\}. \tag{45}$$

Then from (40) we see that the elements of K_{σ_η} are positive solutions of problem (P_η). Therefore, we may assume that K_{σ_η} is finite of otherwise we already have an infinity of positive solutions for problem (P_η).

The finiteness of K_{σ_η} and the Claim imply that we can find $\rho \in (0,1)$ small such that

$$\sigma_\eta(u_0) < \inf[\sigma_\eta(u) : ||u-u_0|| = \rho] = m_\rho^\eta \tag{46}$$

(see Aizicovici, Papageorgiou and Staicu [1] (proof of Proposition 29)). Then (41), (42) and (46) imply that we can use Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_\eta} \text{ and } \sigma_\eta(u_0) < m_\rho^\eta \le \sigma_\eta(\hat{u}). \tag{47}$$

From (47) it follows that $\hat{u} \neq u_0$ and $\hat{u} \in S(\eta) \subseteq \operatorname{int} C_+,\ u_0 \le \hat{u}$ (see (45)). $\square$

Next we examine what happens in the critical case $\lambda = \lambda^*$. To this end, note that hypotheses $H_2(ii), (v)$ imply that we can find $c_{18} = c_{18}(\lambda) > 0$ such that

$$f(z,x,\lambda) \ge c_6 x^{q-1} - c_{18} x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } z \ge 0. \tag{48}$$

This unilateral growth estimate on the reaction $f(z,\cdot,\lambda)$ leads to the following auxiliary Robin problem:

$$\left\{ \begin{array}{ll} -\operatorname{div} a(Du(z)) = c_6 u(z)^{q-1} - c_{18} u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_0}(z) + \beta(z) u(z)^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0. \end{array} \right\} \tag{49}$$

For this problem we have the following existence and uniqueness result.

Proposition 10. *If hypotheses $H(a)$ and $H(\beta)$ hold, the problem (49) admits a unique positive solution $\bar{u} \in \operatorname{int} C_+$.*

Proof. First we show the existence of a positive solution for problem (49). To this end let $\xi_+ : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\xi_+(u) = \int_\Omega G(Du)dz + \frac{1}{p}||u^-||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma + \frac{c_{18}}{r}||u^+||_r^r - \frac{c_6}{q}||u^+||_q^q$$
$$\text{for all } u \in W^{1,p}(\Omega).$$

Using Corollary 1 and hypothesis $H(\beta)$, we have

$$\xi_+(u) \ge \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{c_{18}}{r}||u^+||_r^r - c_{19}(||u^+||_r^q + ||u^+||_r^p)$$
$$\text{for some } c_{19} > 0 \text{ (recall } q < p < r).$$

Because $q < p < r$, it follows that ξ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$\xi_+(\bar{u}) = \inf[\xi_+(u) : u \in W^{1,p}(\Omega)]. \tag{50}$$

Exploiting the fact $q < p < r$, by choosing $\xi^* \in (0,1)$ small, we have

$$\begin{aligned} & \xi_+(\xi^*) < 0, \\ \Rightarrow\quad & \xi_+(\bar{u}) < 0 = \xi_+(0) \text{ (see (50)), hence } \bar{u} \neq 0. \end{aligned}$$

From (50) we have

$$
\begin{aligned}
& \xi_+'(\bar{u}) = 0, \\
\Rightarrow\quad & \langle A(\bar{u}), h\rangle - \int_\Omega (\bar{u}^-)^{p-1} h dz + \int_{\partial\Omega} \beta(z)(\bar{u}^+)^{p-1} h d\sigma = c_6 \int_\Omega (\bar{u}^+)^{q-1} h dz - \\
& \qquad - c_{18} \int_\Omega (\bar{u}^+)^{r-1} h dz \text{ for all } u \in W^{1,p}(\Omega).
\end{aligned} \tag{51}
$$

Let, $h = -\bar{u}^- \in W^{1,p}(\Omega)$ in (51). Then, we see that $\bar{u} \geq 0$, $\bar{u} \neq 0$. So, (51) becomes

$$
\begin{aligned}
& \langle A(\bar{u}), h\rangle + \int_{\partial\Omega} \beta(z) \bar{u}^{p-1} h d\sigma = c_6 \int_\Omega \bar{u}^{q-1} h dz - c_{18} \int_\Omega \bar{u}^{r-1} h dz \\
& \qquad \text{for all } h \in W^{1,p}(\Omega), \\
\Rightarrow\quad & \bar{u} \in \operatorname{int} C_+ \text{ is a solution of (49) (see the proof of Proposition 6).}
\end{aligned}
$$

So, we have established the existence of positive solutions for problem (49).

Next we show the uniqueness of this positive solution.

To this end, let $e : L^\tau(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$
e(u) = \begin{cases} \displaystyle\int_\Omega G(Du^{1/\tau}) dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) u^{p/\tau} d\sigma & \text{if } u \geq 0,\ u^{1/\tau} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}
$$

Let $u_1, u_2 \in \operatorname{dom} e = \{u \in W^{1,p}(\Omega) : e(u) < \infty\}$ (the effective domain of the functional e) and let $t \in [0,1]$. We define

$$
y = ((1-t)u_1 + tu_2)^{1/\tau},\ v_1 = u_1^{1/\tau},\ v_2 = u_2^{1/\tau}.
$$

Using Lemma 1 of Diaz and Saa [5], we have

$$
\begin{aligned}
& |Dy(z)| && \leq [(1-t)|Dv_1(z)|^\tau + t|Dv_2(z)|^\tau]^{1/\tau} \text{ for a.a. } z \in \Omega, \\
\Rightarrow\quad & G_0(|Dy(z)|) && \leq G_0\left(((1-t)|Dv_1(z)|^\tau + t|Dv_2(z)|^\tau)\right) \\
& && \qquad \text{(since } G_0 \text{ is increasing)} \\
& && \leq (1-t)G_0(|Dv_1(z)|) + tG_0(|Dv_2(z)|) \text{ for a.a. } z \in \Omega \\
& && \qquad \text{(see hypothesis } H(a)(v)), \\
\Rightarrow\quad & G(Dy(z)) && \leq (1-t)G(Du_1(z)^{1/\tau}) + tG(Du_2(z)^{1/\tau}) \text{ for a.a. } z \in \Omega, \\
\Rightarrow\quad & && u \longmapsto \int_\Omega G(Du^{1/\tau}) dz \text{ is convex.}
\end{aligned}
$$

Since $p > \tau$ and $\beta \geq 0$ (see hypothesis $H(\beta)$), we see that $u \longmapsto \dfrac{1}{p} \displaystyle\int_{\partial\Omega} \beta(z) u^{p/\tau} d\sigma$ is a convex functional. Therefore, e is convex and also via Fatou's lemma, we have that e is lower semicontinuous.

We already have $\bar{u} \in \operatorname{int} C_+$ a positive solution of problem (49). Let $\bar{y} \in W^{1,p}(\Omega)$ be another positive solution. As above, we can show that $\bar{y} \in \operatorname{int} C_+$. Then for all $h \in C^1(\overline{\Omega})$ and for $|t|$ small, we have

$$
\bar{u}^\tau + th,\ \bar{y}^\tau + th \in \operatorname{dom} e.
$$

Then $e(\cdot)$ is Gâteaux differentiable at $\bar{u}^\tau$ and $\bar{y}^\tau$ in the direction h. Moreover, via the chain rule and the nonlinear Green's identity, we obtain

$$e'(\bar{u}^p)(h) = \int_\Omega \frac{-\mathrm{div}\, a(D\bar{u})}{\bar{u}^{\tau-1}} h dz$$
$$e'(\bar{y}^p)(h) = \int_\Omega \frac{-\mathrm{div}\, a(D\bar{y})}{\bar{y}^{\tau-1}} h dz \text{ for all } h \in W^{1,p}(\Omega)$$

(recall that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$). The convexity of e implies the monotonicity of e'. Then

$$\begin{aligned} 0 &\le \int_\Omega -\mathrm{div}\, a(D\bar{u}) \left(\frac{\bar{u}^p - \bar{y}^p}{\bar{u}^{\tau-1}}\right) dz - \int_\Omega (-\mathrm{div}\, a(D\bar{y})) \left(\frac{\bar{u}^p - \bar{y}^p}{\bar{y}^{p-1}}\right) dz \\ &= \int_\Omega \frac{c_6 \bar{u}^{q-1} - c_{18}\bar{u}^{r-1}}{\bar{u}^{\tau-1}} (\bar{u}^p - \bar{y}^p) dz - \int_\Omega \frac{c_6 \bar{y}^{q-1} - c_{18}\bar{y}^{r-1}}{\bar{y}^{\tau-1}} (\bar{u}^p - \bar{y}^p) dz \\ &= \int_\Omega c_6 (\bar{u}^{q-\tau} - \bar{y}^{q-\tau})(\bar{u}^p - \bar{y}^p) dz - \int_\Omega c_{18} (\bar{u}^{r-\tau} - \bar{y}^{r-\tau})(\bar{u}^p - \bar{y}^p) dz \\ &\le 0 \text{ (since } q < \tau < p < r) \\ \Rightarrow\quad & \bar{u} = \bar{y}. \end{aligned}$$

This proves the uniqueness of the positive solution $\bar{u} \in \mathrm{int}\, C_+$ of problem (49).□

Proposition 11. *If hypotheses $H(a)$, $H(\beta)$ and H_2 hold and $\lambda \in \mathcal{S}$, then $\bar{u} \le u$ for all $u \in S(\lambda)$.*

Proof. Let $u \in S(\lambda)$ and consider the following Carathéodory function:

$$w(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_6 x^{q-1} - c_{18} x^{r-1} + x^{p-1} & \text{if } 0 \le x \le u(z) \\ c_6 u(z)^{q-1} - c_{18} u(z)^{r-1} + u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{52}$$

Let $W(z,x) = \displaystyle\int_0^x w(z,s) ds$ and consider the C^1-functional $\hat{\gamma} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\gamma}(u) = \int_\Omega G(Du) dz + \frac{1}{p} ||u||_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega W(z,u) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From hypothesis $H(\beta)$ and (52) it is clear that $\hat{\gamma}$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $\bar{u}_* \in W^{1,p}(\Omega)$ such that

$$\hat{\gamma}(\bar{u}_*) = \inf[\hat{\gamma}(u) : u \in W^{1,p}(\Omega)]. \tag{53}$$

Since $q < p < r$, for $\xi \in (0, \min_{\overline{\Omega}} u)$ (recall that $u \in \mathrm{int}\, C_+$) small, we have

$$\begin{aligned} & \hat{\gamma}(\xi) < 0, \\ \Rightarrow\quad & \hat{\gamma}(\bar{u}_*) < 0 = \hat{\gamma}(0) \text{ (see (53)), hence } \bar{u}_* \ne 0. \end{aligned}$$

From (53) we have

$$
\begin{aligned}
&\hat{\gamma}'(\bar{u}_*) = 0,\\
\Rightarrow\quad &\langle A(\bar{u}_*), h\rangle + \int_\Omega |\bar{u}_*|^{p-2}\bar{u}_* h dz + \int_{\partial\Omega}\beta(z)(\bar{u}_*^+)^{p-1} h d\sigma = \int_\Omega w(z,\bar{u}_*) h dz \quad (54)\\
&\text{for all } h \in W^{1,p}(\Omega).
\end{aligned}
$$

In (54) we choose first $h = -\bar{u}_*^- \in W^{1,p}(\Omega)$ and then $h = (\bar{u}_* - u)^+ \in W^{1,p}(\Omega)$ and as in the proof of Proposition 7, we show that

$$\bar{u}_* \in [0, u],\ \bar{u}_* \neq 0$$

So, (54) becomes

$$
\begin{aligned}
&\langle A(\bar{u}_*), h\rangle + \int_{\partial\Omega}\beta(z)\bar{u}_*^{p-1} h d\sigma = c_6\int_\Omega \bar{u}_*^{q-1} h dz - c_{18}\int_\Omega \bar{u}_*^{r-1} h dz\\
&\qquad\text{for all } h \in W^{1,p}(\Omega),\\
\Rightarrow\quad &\bar{u}_* \text{ is a positive solution of the auxiliary problem (49)},\\
\Rightarrow\quad &\bar{u}_* = \bar{u} \in \operatorname{int} C_+ \text{ (see Proposition 10)}\\
\Rightarrow\quad &\bar{u} \leq u \text{ for all } u \in S(\lambda).
\end{aligned}
$$

∎

Now we can show that the critical value λ^* is admissible, that is $\lambda^* \in \mathcal{S}$.

Proposition 12. *If hypotheses $H(a)$, $H(\beta)$ and H_2 hold, then $\lambda^* \in \mathcal{S}$ and so $\mathcal{S} = (0, \lambda^*]$.*

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{S}$ such that $\lambda_n \to (\lambda^*)^-$. Then we can find $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ and from the proof of Proposition 7 we know that we can assume that

$$
\begin{aligned}
&\hat{\varphi}_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1,\\
\Rightarrow\quad &\int_\Omega pG(Du_n)dz + \int_{\partial\Omega}\beta(z)u_n^p d\sigma - \int_\Omega pF(z,u_n,\lambda_n)dz < 0 \quad (55)\\
&\qquad \text{for all } n \geq 1.
\end{aligned}
$$

Also, we have

$$-\langle A(u_n), u_n\rangle - \int_{\partial\Omega}\beta(z)u_n^p d\sigma + \int_\Omega f(z,u_n,\lambda_n)u_n dz = 0 \text{ for all } n \geq 1. \quad (56)$$

Adding (55) and (56), we obtain

$$
\begin{aligned}
&\int_\Omega [pG(Du_n) - (a(Du_n), Du_n)_{\mathbb{R}^N}]\, dz + \int_\Omega [f(z,u_n,\lambda_n)u_n -\\
&\qquad - pF(z,u_n,\lambda_n)]\, dz \leq \xi_0 \quad \text{for all } n \geq 1,\ \text{some } \xi_0 > 0,\\
\Rightarrow\quad &\int_\Omega [f(z,u_n,\lambda_n)u_n - pF(z,u_n,\lambda_n)]\, dz < 0 \text{ for all } n \geq 1 \quad (57)\\
&\qquad \text{(see hypothesis } H(a)(iv)).
\end{aligned}
$$

From (57), as in the proof of Proposition 3, using hypothesis $H_2(iv)$, we show that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \quad (58)$$

Since $u_n \in S(\lambda)$ for all $n \geq 1$, we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma - \int_\Omega f(z, u_n, \lambda_n) h dz = 0 \text{ for all } h \in W^{1,p}(\Omega). \quad (59)$$

Choosing $h = u_n - u_* \in W^{1,p}(\Omega)$ in (59), passing to the limit as $n \to \infty$ and using (58), we obtain

$$\begin{aligned} & \lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0, \\ \Rightarrow \quad & u_n \to u_* \text{ in } W^{1,p}(\Omega). \end{aligned} \quad (60)$$

So, if in (59) we pass to the limit as $n \to \infty$ and use (60), then

$$\langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma = \int_\Omega f(z, u_*, \lambda_*) h dz \text{ for all } h \in W^{1,p}(\Omega),$$
$$\Rightarrow \quad u_* \geq 0 \text{ is a solution of problem } (P_{\lambda^*}).$$

From Proposition 11 we have $\bar{u} \leq u_n$ for all $n \geq 1$. Hence $\bar{u} \leq u_*$ and so $u_* \in S(\lambda^*) \subseteq \operatorname{int} C_+$. Therefore $\lambda^* \in \mathcal{S}$ and so $\mathcal{S} = (0, \lambda^*]$. ☐

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type result.

Theorem 3.1. *If hypotheses $H(a)$, $H(\beta)$ and H_2 hold, then there exists $\lambda^* > 0$ such that*

(a) for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \leq \hat{u}$, $u_0 \neq \hat{u}$;
(b) for $\lambda = \lambda^*$ problem (P_{λ^*}) has at least one positive solution $u_* \in \operatorname{int} C_+$;
(c) for all $\lambda > \lambda^*$ problem (P_λ) has no positive solution.

4. **Bifurcation near infinity for the Neumann problem.** In this section we deal with the Neumann problem (that is, $\beta \equiv 0$) and with a parametric reaction of the form

$$f(z, x, \lambda) = f_0(z, x) - \lambda x^{p-1} \text{ for all } (z, x) \in \Omega \times [0, \infty).$$

Here f_0 is a Carathéodory function which as before exhibits competing nonlinearities, namely it is $(p-1)$-superlinear near $+\infty$ and admits a concave term near zero. This time the superlinearity of $f(z, \cdot)$ is expressed via the AR-condition. The presence of the term $-\lambda x^{p-1}$ changes the geometry of the problem and hypotheses H_1 and H_2 do not hold anymore. In fact, we will show that in this case the bifurcation occurs at large values of the parameter $\lambda > 0$ (bifurcation near infinity).

The problem under consideration, is the following:

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) = f_0(z, u(z)) - \lambda u(z)^{p-1} \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(z) = 0 \text{ on } \partial\Omega, \ u > 0 \end{array} \right\} \quad (S_\lambda)$$

For the differential operator, we keep hypotheses $H(a)$ as in Section 3. On the nonparametric nonlinearity $f_0(z, x)$, we impose the following conditions:

H_3 : $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f_0(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f_0(z, x)| \leq a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) if $F_0(z,x) = \int_0^x f_0(z,s)ds$, then there exist $c_{19} > 0$ and $\eta > p$ such that

$$c_{19}x^\eta \leq \eta F_0(z,x) \leq f_0(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0;$$

(iii) there exists $q \in (1,\tau)$ (see hypothesis $H(a)(v)$) such that

$$0 < c_{20} \leq \liminf_{x\to 0^+} \frac{f_0(z,x)}{x^{q-1}} \leq \limsup_{x\to 0^+} \frac{f_0(z,x)}{x^{q-1}} \leq c_{21} < \infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, $x \longmapsto f_0(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 4. As in Section 3, without any loss of generality, we may assume that $f_0(z,x) = 0$ for all $(z,x) \in \Omega \times (-\infty, 0]$. Hypotheses $H_3(ii),(iii)$ reveal the competing nonlinearities (concave-convex nonlinearities). Observe that in this case the superlinearity of $f_0(z,\cdot)$ is expressed using a global version of the unilateral AR-condition.

Example 3. The model for the nonlinearity $f_0(z,\cdot)$, is the function

$$f_0(z,x) = f_0(x) = x^{q-1} + x^{r-1} \text{ for all } x \geq 0$$

with $1 < q < \tau < p < r < p^*$.

As before, we introduce the following two sets

$$\begin{aligned} \mathcal{S}_0 &= \{\lambda > 0 : \text{problem } (S_\lambda) \text{ admits a positive solution}\} \\ S_0(\lambda) &= \text{the set of positive solutions of problem } (S_\lambda). \end{aligned}$$

Proposition 13. *Assume that hypotheses $H(a)$ and H_3 hold. Then $\mathcal{S}_0 \neq \varnothing$ and for all $\lambda > 0$, $S_0(\lambda) \subseteq \operatorname{int} C_+$ and for $\lambda \in \mathcal{S}_0$, we have $[\lambda, +\infty) \subseteq \mathcal{S}_0$.*

Proof. We consider the following auxiliary Neumann problem

$$-\operatorname{div} a(Du(z)) + |u(z)|^{p-2}u(z) = 1 \text{ in } \Omega, \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{61}$$

Let $K_p : L^p(\Omega) \to L^{p'}(\Omega)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be the nonlinear map defined by

$$K_p(u)(\cdot) = |u(\cdot)|^{p-2}u(\cdot)$$

This is bounded (maps bounded sets to bounded sets) and continuous. Moreover, by the Sobolev embedding theorem $\hat{K}_p = K_p|_{W^{1,p}(\Omega)}$ is completely continuous (that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$, then $\hat{K}_p(u_n) \to K_p(u)$ in $L^{p'}(\Omega)$). So, the map $u \longmapsto V(u) = A(u) + \hat{K}_p(u)$ is demicontinuous and of type $(S)_+$, hence pseudomonotone (see [11]). Moreover, we have

$$\begin{aligned} &\langle V(u), u\rangle \geq \frac{c_1}{p-1}||Du||_p^p + ||u||_p^p \text{ for all } u \in W^{1,p}(\Omega) \text{ (see Lemma 2.2)}, \\ \Rightarrow \quad & V \text{ is coercive.} \end{aligned}$$

But a pseudomonotone coercive operator is surjective (see, for example, Gasinski and Papageorgiou [11] [p. 336]). So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$V(\bar{u}) + \hat{K}_p(\bar{u}) = 1. \tag{62}$$

In fact, using Lemma 2.2 and the strict monotonicity of the map $x \longmapsto |x|^{p-2}x$, $x \in \mathbb{R}$, we see that V is strictly monotone and so $\bar{u} \in W^{1,p}(\Omega)$ is unique. Acting on (62) with $-\bar{u}^- \in W^{1,p}(\Omega)$ and using Lemma 2.2, we have

$$\begin{aligned} & \frac{c_1}{p-1}||D\bar{u}^-||_p^p + ||\bar{u}^-||_p^p \leq 0, \\ \Rightarrow \quad & \bar{u} \geq 0,\ \bar{u} \neq 0 \end{aligned}$$

So, $\bar{u} \geq 0$ is the unique solution of the auxiliary problem (61) and as before the nonlinear regularity theory (see [17]) and the nonlinear maximum principle (see [21]), imply $\bar{u} \in \operatorname{int} C_+$.

Let $0 < \bar{m} = \min\limits_{\overline{\Omega}} \bar{u}$ and let $\lambda_0 = 1 + \dfrac{||N_{f_0}(\bar{u})||_\infty}{\bar{m}^{p-1}}$. We have

$$\begin{aligned} & \langle A(\bar{u}), h\rangle + \lambda_0 \int_\Omega \bar{u}^{p-1} h dz \\ = \quad & \langle A(\bar{u}), h\rangle + \int_\Omega \bar{u}^{p-1} h dz + ||N_{f_0}(\bar{u})||_\infty \int_\Omega \frac{\bar{u}^{p-1}}{\bar{m}^{p-1}} h dz \\ = \quad & \int_\Omega h dz + \int_\Omega f_0(z, \bar{u}) h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0 \qquad (63) \\ & \qquad \text{(see (61) and recall } \bar{m} = \min_{\overline{\Omega}} \bar{u} > 0). \end{aligned}$$

We introduce the following truncation of $f_0(z, \cdot)$:

$$\hat{f}_0(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f_0(z,x) & \text{if } 0 \leq x \leq \bar{x}(z) \\ f_0(z,\bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases} \qquad (64)$$

This is a Carathéodory function. We set $\hat{F}_0(z,x) = \displaystyle\int_0^x \hat{f}_0(z,s) ds$ and consider the C^1-conditional $\hat{\varphi}_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_0(u) = \int_\Omega G(Du) dz + \frac{\lambda_0}{p}||u||_p^p - \int_\Omega \hat{F}_0(z,u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (64) it is clear that $\hat{\varphi}_0$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_0(u_0) = \inf[\hat{\varphi}_0(u) : u \in W^{1,p}(\Omega)]. \qquad (65)$$

Using hypothesis $H_3(iii)$ and since $1 < q < p$, we have that for $\xi \in (0, \min\{1, \bar{m}\})$ small

$$\begin{aligned} & \hat{\varphi}_0(\xi) < 0 = \hat{\varphi}_0(0), \\ \Rightarrow \quad & \hat{\varphi}_0(u_0) < 0 = \hat{\varphi}_0(0) \text{ (see (65)), hence } u_0 \neq 0. \end{aligned}$$

From (65) we have

$$\begin{aligned} & \hat{\varphi}_0'(u_0) = 0, \\ \Rightarrow \quad & A(u_0) + \lambda_0 |u_0|^{p-2} u_0 = N_{\hat{f}_0}(u_0) \text{ in } W^{1,p}(\Omega)^*. \qquad (66) \end{aligned}$$

On (66) we act with $-u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \frac{c_1}{p-1}||Du_0^-||_p^p + \lambda_0 ||u_0^-||_p^p \leq 0 \text{ (see Lemma 2.2 and (64))}, \\ \Rightarrow \quad & u_0 \geq 0,\ u_0 \neq 0. \end{aligned}$$

Next on (66) we act with $(u_0 - \bar{u})^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
& \left\langle A(u_0), (u_0-\bar{u})^+ \right\rangle + \lambda_0 \int_\Omega u_0^{p-1}(u_0-\bar{u})^+ dz \\
= \ & \int_\Omega f_0(z,\bar{u})(u_0-\bar{u})^+ dz \text{ (see (64))} \\
\leq \ & \left\langle A(\bar{u}), (u_0-\bar{u})^+ \right\rangle + \lambda_0 \int_\Omega \bar{u}^{p-1}(u_0-\bar{u})^+ dz \text{ (see (63))} \\
\Rightarrow \ & \left\langle A(u_0) - A(\bar{u}), (u_0-\bar{u})^+ \right\rangle + \lambda_0 \int_\Omega (u_0^{p-1} - \bar{u}^{p-1})(u_0-\bar{u})^+ dz \leq 0, \\
\Rightarrow \ & |\{u_0 > \bar{u}\}|_N = 0, \text{ hence } u_0 \leq \bar{u}.
\end{aligned}
$$

So, we have proved that

$$u_0 \in [0, \bar{u}] = \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \bar{u}(z) \text{ a.e. in } \Omega\}.$$

Then by virtue of (64), equation (66) becomes

$$
\begin{aligned}
& A(u_0) + \lambda_0 u_0^{p-1} = N_{f_0}(u_0), \\
\Rightarrow \ & u_0 \text{ is a positive solution of } (P_{\lambda_0}) \text{ and so } \lambda_0 \in \mathcal{S}_0 \neq \varnothing
\end{aligned}
$$

Moreover, as before the nonlinear regularity theory and the nonlinear maximum principle imply that $u_0 \in \operatorname{int} C_+$. Therefore, for every $\lambda \in \mathcal{S}_0$ $S_0(\lambda) \subseteq \operatorname{int} C_+$.

Next, let $\lambda \in \mathcal{S}_0$ and $\mu > \lambda$. Let $u_\lambda \in S_0(\lambda) \subseteq \operatorname{int} C_+$. Then

$$\langle A(u_\lambda), h\rangle + \mu \int_\Omega u_\lambda^{p-1} h dz \geq \langle A(u_\lambda), h\rangle + \lambda \int_\Omega u_\lambda^{p-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0. \tag{67}$$

Then truncating $f_0(z,\cdot)$ at $\{0, u_\lambda(z)\}$ and reasoning as above, via the direct method and using this time (67), we obtain $u_\mu \in [0, u_\lambda] \cap S_0(\mu)$, hence $\mu \in \mathcal{S}_0$. Therefore we infer that $[\lambda, +\infty) \subseteq \mathcal{S}_0$. ☐

Remark 5. Note that in the above proof we have proved that, if $\lambda \in \mathcal{S}_0$, $u_\lambda \in S_0(\lambda) \subseteq \operatorname{int} C_+$ and $\mu > \lambda$, then $\mu \in \mathcal{S}_0$ and we can find $u_\mu \in S_0(\mu) \subseteq \operatorname{int} C_+$ such that $u_\mu \leq u_\lambda$. In fact in the next proposition, we improve this conclusion.

Proposition 14. *If hypotheses $H(a)$ and H_3 hold, $\lambda \in \mathcal{S}_0$, $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and $\mu > \lambda$, then we can find $u_\mu \in S_0(\mu) \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Proposition 13, we already know that we can find $u_\mu \in S_0(\mu) \subseteq \operatorname{int} C_+$ such that $u_\mu \leq u_\lambda$.

Let $m_\mu = \min_{\overline{\Omega}} u_\mu > 0$ and let $\delta \in \left(0, \frac{m_\mu}{2}\right)$. We set $u_\lambda^\delta = u_\lambda - \delta \in \operatorname{int} C_+$. Also, for $\rho = ||u_\lambda||_\infty$, we let $\xi_\rho > 0$ be as postulated by hypothesis $H_3(iv)$. Then

$$
\begin{aligned}
& -\operatorname{div} a(Du_\lambda^\delta) + (\mu + \xi_\rho)(u_\lambda^\delta)^{p-1} \\
\geq \ & -\operatorname{div} a(Du_\lambda) + (\mu + \xi_\rho) u_\lambda^{p-1} - \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
= \ & f_0(z, u_\lambda) + (\mu - \lambda) u_\lambda^{p-1} + \xi_\rho u_\lambda^{p-1} - \chi(\delta) \text{ (since } u_\lambda \in S_0(\lambda)) \\
\geq \ & f_0(z, u_\mu) + \xi_\rho u_\mu^{p-1} + (\mu - \lambda) m_\mu - \chi(\delta) \\
& \qquad \text{(since } m_\mu \leq u_\mu \leq u_\lambda \text{ and use hypothesis } H_3(iv)) \\
\geq \ & f_0(z, u_\mu) + \xi_\rho u_\mu^{p-1} \text{ for } \delta > 0 \text{ small}
\end{aligned}
$$

$$
\begin{aligned}
&= && -\operatorname{div} a(Du_\mu) + (\mu+\xi_\rho)u_\mu^{p-1} \text{ (since } u_\mu \in S_0(\mu)),\\
\Rightarrow & && u_\mu \le u_\lambda^\delta, \text{ for all } \delta > 0 \text{ small},\\
\Rightarrow & && u_\lambda - u_\mu \in \operatorname{int} C_+.
\end{aligned}
$$

□

Let $\lambda_* = \inf \mathcal{S}_0$.

Proposition 15. *If hypotheses $H(a)$ and H_3 hold, then $\lambda_* > 0$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\geq} \subseteq \mathcal{S}_0$ such that $\lambda_n \downarrow \lambda_*$. We can find a corresponding sequence $\{u_n\}_{n\geq 1}$ such that $u_n \in S_0(\lambda_n) \subseteq \operatorname{int} C_+$ for all $n \geq 1$. We claim that $\{u_n\}_{n\geq 1}$ can be chosen to be increasing. To see this, note that since $\lambda_2 < \lambda_1$, the function $u_{\lambda_2} \in S_0(\lambda_2) \subseteq \operatorname{int} C_+$ satisfies

$$
\langle A(u_2), h\rangle + \lambda_1 \int_\Omega u_2^{p-1} h dz \geq \langle A(u_2), h\rangle + \lambda_2 \int_\Omega u_2^{p-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0 \tag{68}
$$

Considering problem (P_{λ_1}) and truncating $f_0(z,\cdot)$ at $\{0, u_2(z)\}$, via the direct method and using (68), as in the proof of Proposition 13, we obtain $u_1 \in [0, u_2] \cap S_0(\lambda_1)$.

Then we have

$$
\begin{aligned}
&\langle A(u_1), h\rangle + \lambda_2 \int_\Omega u_1^{p-1} h dz\\
= & \int_\Omega f_0(z,u_1) h dz + (\lambda_2 - \lambda_1)\int_\Omega u_1^{p-1} h dz \text{ (since } u_1 \in S_0(\lambda_1))\\
\leq & \int_\Omega f_0(z,u_1) h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0 \text{ (recall } \lambda_1 > \lambda_2)
\end{aligned} \tag{69}
$$

Also, we have

$$
\langle A(u_3), h\rangle + \lambda_2 \int_\Omega u_3^{p-1} h dz \geq \langle A(u_3), h\rangle + \lambda_3 \int_\Omega u_3^{p-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0. \tag{70}
$$

Truncating $f_0(z,\cdot)$ at $\{u_1(z), u_3(z)\}$ and using the direct method and (69), (70), we produce $u_2 \in [u_1, u_3] \cap S_0(\lambda_2)$. Continuing this way, we see that we can choose $\{u_n\}_{n\geq 1}$ to be increasing.

We have

$$
\langle A(u_n), h\rangle + \lambda_n \int_\lambda u_n^{p-1} h dz = \int_\Omega f_0(z,u_n) h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \geq 1.
$$

Choose $h \equiv 1 \in W^{1,p}(\Omega)$. Then for all $n \geq 1$ we have

$$
\begin{aligned}
& \lambda_n \int_\Omega u_n^{p-1} dz = \int_\Omega f_0(z,u_n) dz,\\
\Rightarrow\ & \lambda_n ||u_n||_{p-1}^{p-1} \geq c_{22} ||u_n||_{p-1}^{\eta-1} \text{ for some } c_{22} > 0,\\
& \qquad \text{(see hypothesis } H_3(ii) \text{ and recall } p < \eta),\\
\Rightarrow\ & \lambda_n \geq c_{22} ||u_1||_{p-1}^{\eta-p} \text{ (recall } u_n \geq u_1, \text{ for all } n \geq 1),\\
\Rightarrow\ & \lambda_* \geq c_{22} ||u_1||_{p-1}^{\eta-p} > 0.
\end{aligned}
$$

□

Proposition 16. *If hypotheses $H(a)$ and H_3 hold and $\lambda > \lambda_*$ then problem (S_λ) admits at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in \operatorname{int} C_+,\ u_\lambda \neq \hat{u}_\lambda.$$

Proof. Let $\lambda_* < \lambda_1 < \lambda < \lambda_2$. We know that we can find $u_{\lambda_1} \in S_0(\lambda_1) \subseteq \operatorname{int} C_+$ and $u_{\lambda_2} \in S_0(\lambda_2) \subseteq \operatorname{int} C_+$ such that $u_{\lambda_1} - u_{\lambda_2} \in \operatorname{int} C_+$ (see Proposition 14). We introduce the Carathéodory function $e(z,x)$ defined by

$$e(z,x) = \begin{cases} f_0(z, u_{\lambda_2}(z)) & \text{if } x < u_{\lambda_2}(z) \\ f_0(z,x) & \text{if } u_{\lambda_2}(z) \leq x \leq u_{\lambda_1}(z) \\ f_0(z, u_{\lambda_1}(z)) & \text{if } u_{\lambda_1}(z) < x. \end{cases} \tag{71}$$

We set $E(z,x) = \int_0^x e(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega G(Du)dz + \frac{\lambda}{p}||u||_p^p - \int_\Omega E(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently ψ_λ is coercive (see (71)) and sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \psi_\lambda(u_\lambda) = \inf[\psi_\lambda(u) : u \in W^{1,p}(\Omega)], \\ \Rightarrow\ & \psi'_\lambda(u_\lambda) = 0, \\ \Rightarrow\ & \langle A(u_\lambda), h\rangle + \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda h dz = \int_\Omega e(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega) \end{aligned} \tag{72}$$

If in (72) we use $h = (u_\lambda - u_{\lambda_1})^+ \in W^{1,p}(\Omega)$ and $h = (u_{\lambda_2} - u_\lambda)^+ \in W^{1,p}(\Omega)$, we show that $u_\lambda \in [u_{\lambda_2}, u_{\lambda_1}] = \{u \in W^{1,p}(\Omega) : u_{\lambda_2}(z) \leq u(z) \leq u_{\lambda_1}(z) \text{ a.e. in } \Omega\}$. In fact reasoning as in the proof of Proposition 14, we obtain

$$u_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[u_{\lambda_2}, u_{\lambda_1}]. \tag{73}$$

From (71) and (72) we have

$$\begin{aligned} & \langle A(u_\lambda), h\rangle + \lambda \int_\Omega u_\lambda^{p-1} h dz = \int_\Omega f_0(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega), \\ \Rightarrow\ & u_\lambda \in S_0(\lambda) \subseteq \operatorname{int} C_+. \end{aligned}$$

Next, using u_{λ_2}, we introduce the following truncation of $f_0(z,\cdot)$:

$$\gamma(z,x) = \begin{cases} f_0(z, u_{\lambda_2}(z)) & \text{if } x \leq u_{\lambda_2}(z) \\ f_0(z,x) & \text{if } u_{\lambda_2}(z) < x. \end{cases} \tag{74}$$

This is a Carathéodory function. Let $\Gamma(z,x) = \int_0^x \gamma(z,s)ds$ and consider the C^1-functional $\sigma_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \int_\Omega G(Du)dz + \frac{\lambda}{p}||u||_p^p - \int_\Omega \Gamma(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypothesis $H_3(ii)$ implies that the AR-condition (see (5)) is satisfied by $\gamma(z,x)$. It follows that

$$\sigma_\lambda \text{ satisfied the } C-\text{condition}. \tag{75}$$

Note that

$$\psi_\lambda|_{[u_{\lambda_2}, u_{\lambda_1}]} = \sigma_\lambda|_{[u_{\lambda_2}, u_{\lambda_1}]} \text{ (see (71) and (74))}.$$

From this equality and (73), we infer that u_λ is a local $C^1(\overline{\Omega})$ minimizer of σ_λ, hence u_λ is a local $W^{1,p}(\Omega)$-minimizer of σ_λ (see Proposition 1). We can easily check that the critical points of σ_λ are in $[u_{\lambda_2}) = \{u \in W^{1,p}(\Omega) : u_{\lambda_2}(z) \le u(z)$ for a.a $z \in \Omega\}$. So, we may assume that the critical points are finite or otherwise we already have an infinity of positive solutions for problem (S_λ) (see (74)) and so we are done. Then we can find $\rho \in (0,1]$ small such that

$$\sigma_\lambda(u_\lambda) < \inf\left[\sigma_\lambda(u) : ||u - u_\lambda|| = \rho\right] = m_\rho^\lambda \tag{76}$$

Hypothesis $H_3(ii)$ and (74) imply that

$$\sigma_\lambda(\xi) \to -\infty \text{ as } \xi \to +\infty. \tag{77}$$

Then (75), (76) and (77) permit the use of Theorem 2.1 (the mountain pass theorem) and so we can find $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\sigma_\lambda(u_\lambda) < m_\rho^\lambda \le \sigma_\lambda(\hat{u}_\lambda), \text{ hence } \hat{u}_\lambda \ne u_\lambda,\\
&\sigma'_\lambda(\hat{u}_\lambda) = 0.
\end{aligned}$$

From the last equality and some $\hat{u}_\lambda \ge u_{\lambda_2}$, we infer that $\hat{u}_\lambda \in S_0(\lambda) \subseteq \operatorname{int} C_+$ (see (74)). ☐

Note that hypotheses $H_3(i), (iii)$ imply that we can find $c_{23}, c_{24} > 0$ such that

$$f_0(z,x) \ge c_{23}x^{q-1} - c_{24}x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{78}$$

This leads to the following auxiliary Neumann problem

$$-\operatorname{div} a(Du(z)) + \lambda u(z)^{p-1} = c_{23}u(z)^{q-1} - c_{24}u(z)^{r-1} \text{ in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ u > 0 \tag{Q_λ}$$

Reasoning as in the proof of Proposition 10, we have the following existence and uniqueness result for problem (Q_λ).

Proposition 17. *If hypotheses $H(a)$ hold and $\lambda > 0$ then problem (Q_λ) admits a unique positive solution $u_\lambda^* \in \operatorname{int} C_+$.*

In fact the map $\lambda \to u_\lambda^*$ has useful monotonicity and continuity properties.

Proposition 18. *If hypotheses $H(a)$ hold, then $\lambda \to u_\lambda^*$ is nonincreasing and continuous from $(0,\infty)$ into $C^1(\overline{\Omega})$.*

Proof. The monotonicity of $\lambda \to u_\lambda^*$ is established as in the first part of the proof of Proposition 15, by exploiting the uniqueness result of Proposition 17.

Next, let $\lambda_n \to \lambda > 0$ and let $0 < \hat{\lambda} < \lambda_n$ for all $n \ge 1$. Then $u_{\lambda_n}^* \le u_{\hat{\lambda}}^*$ for all $n \ge 1$. Also, we have

$$A(u_{\lambda_n}^*) + \lambda_n (u_{\lambda_n}^*)^{p-1} = c_{23}(u_{\lambda_n}^*)^{q-1} - c_{24}(u_{\lambda_n}^*)^{r-1} \text{ for all } n \ge 1. \tag{79}$$

Since $u_{\lambda_n}^* \le u_{\hat{\lambda}}^* \in \operatorname{int} C_+$, from Lieberman [17], we know that we can find $\vartheta \in (0,1)$ and $M_6 > 0$ such that

$$u_{\lambda_n}^* \in C^{1,\vartheta}(\overline{\Omega}),\ ||u_{\lambda_n}^*||_{C^{1,\vartheta}(\overline{\Omega})} \le M_6 \text{ for all } n \ge 1$$

Exploiting the compact embedding of $C^{1,\vartheta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we may assume that

$$u_{\lambda_n}^* \to u^* \text{ in } C^1(\overline{\Omega}). \tag{80}$$

So, if in (79) we pass to the limit as $n \to \infty$ and we use (80), then

$$\begin{aligned} & A(u^*) + \lambda(u^*)^{p-1} = c_{23}(u^*)^{p-1} - c_{24}(u^*)^{r-1}, \\ \Rightarrow \quad & u^* = u^*_\lambda \text{ (see Proposition 17).} \end{aligned}$$

This proves the desired continuity of $\lambda \to u^*_\lambda$. ∎

Moreover, as in Proposition 11, we have:

Proposition 19. *If hypotheses $H(a)$ hold and $\lambda > 0$, then $u^*_\lambda \leq u$ for all $u \in S_0(\lambda)$.*

Using these facts we can treat the critical case $\lambda = \lambda_* > 0$. In what follows for $\lambda > 0$, $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional for problem (S_λ) defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)dz + \frac{\lambda}{p}||u||_p^p - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

Proposition 20. *If hypotheses $H(a)$ and H_3 hold, then $\lambda_* \in \mathcal{S}_0$ (that is, $\mathcal{S} = [\lambda_*, \infty)$).*

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{S}_0$ such that $\lambda_N \downarrow \lambda_*$. There exists a corresponding sequence $\{u_n\}_{n\geq 1}$ such that $u_n \in S_0(\lambda_n)$ for all $n \geq 1$. We claim that this sequence of solutions can be chosen so that

$$\varphi_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1. \tag{81}$$

To see this note that

$$\begin{aligned} & \langle A(u_2), h\rangle + \lambda_1 \int_\Omega u_2^{p-1} h dz \\ \geq \quad & \langle A(u_2), h\rangle + \lambda_2 \int_\Omega u_2^{p-1} h dz = \int_\Omega f_0(z,u_2) h dz \text{ for all } h \in W^{1,p}(\Omega) \\ & \text{with } h \geq 0. \end{aligned} \tag{82}$$

Truncating $f_0(z,\cdot)$ at $\{0, u_2(z)\}$ and reasoning as in the proof of Proposition 13 via the direct method and using hypothesis $H_2(iii)$ and (82), we obtain $u_1 \in S_0(\lambda_1)$ such that

$$\varphi_{\lambda_1}(u_1) < 0$$

Next note that

$$\begin{aligned} & \langle A(u_3), h\rangle + \lambda_2 \int_\Omega u_3^{p-1} h dz \\ \geq \quad & \langle A(u_3), h\rangle + \lambda_3 \int_\Omega u_3^{p-1} h dz = \int_\Omega f_0(z,u_3) h dz \text{ for all } h \in W^{1,p}(\Omega) \\ & \text{with } h \geq 0. \end{aligned} \tag{83}$$

As above truncating $f_0(z,\cdot)$ at $\{0, u_3(z)\}$ and using this time (83), we produce $u_2 \in S_0(\lambda_2)$ such that

$$\varphi_\lambda(u_2) < 0.$$

So, continuing this way, we see that we can have $u_n \in S_0(\lambda_n)$ $n \geq 1$ such that (81) holds. Then it follows that

$$\int_\Omega \eta G(Du_n)dz + \frac{\lambda_n \eta}{p}||u_n||_p^p \leq \int_\Omega \eta F_0(z,u_n)dz \text{ for all } n \geq 1. \tag{84}$$

Also, since $u_n \in S_0(\lambda_n)$ $n \geq 1$, we have

$$-\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N} dz - \lambda_n ||u_n||_p^p = -\int_\Omega f_0(z, u_n)u_n dz \text{ for all } n \geq 1. \quad (85)$$

Adding (84) and (85) and using hypothesis $H(a)(iv)$, we obtain

$$(\eta - p)\int_\Omega G(Du_n)dz + \int_\Omega [f_0(z,u_n)u_n - pF_0(z,u_n)]\, dz +$$
$$+\lambda_n \left(\frac{\eta}{p} - 1\right) ||u_n||_p^p \leq 0 \ \text{ for all } n \geq 1,$$
$$\Rightarrow \quad \frac{(\eta - p)c_1}{p(p-1)} ||Du_n||_p^p + \lambda_* \left(\frac{\eta}{p} - 1\right) ||u_n||_p^p \leq 0 \text{ for all } n \geq 1$$

(see Corollary 1, hypothesis $H_3(ii)$ and recall $\lambda_* \leq \lambda_n$ for all $n \geq 1$)

$$\Rightarrow \quad \{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (recall } \eta > p).$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega). \quad (86)$$

Recall that

$$\langle A(u_n), h\rangle + \lambda_n \int_\Omega u_n^{p-1} h dz = \int_\Omega f_0(z, u_n) h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \geq 1. \quad (87)$$

If in (87) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (86), then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0,$$
$$\Rightarrow \quad u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 2 and (86))}. \quad (88)$$

Passing to the limit as $n \to \infty$ in (87) and using (88), we obtain

$$\langle A(u_*), h\rangle + \lambda_* \int_\Omega u_*^{p-1} h dz = \int_\Omega f_0(z, u_*) h dz \text{ for all } h \in W^{1,p}(\Omega). \quad (89)$$

From Proposition 19, we have

$$u^*_{\lambda_n} \leq u_n \text{ for all } n \geq 1,$$
$$\Rightarrow \quad u^*_{\lambda_*} \leq u_* \text{ (see Proposition 18 and (88))}. \quad (90)$$

From (89) and (90) it follows that $u_* \in S_0(\lambda_*)$, hence $\lambda_* \in \mathcal{S}_0$. ☐

Summarizing, for problem (S_λ) we have the following bifurcation-type result.

Theorem 4.1. *If hypotheses $H(a)$ and H_3 hold, then exists $\lambda_* > 0$ such that*

(a) for every $\lambda > \lambda_*$ problem (S_λ) has at least two positive solutions

$$u_\lambda, \hat{u}_\lambda \in \operatorname{int} C_+, \ u_\lambda \neq \hat{u}_\lambda;$$

(b) for $\lambda = \lambda_*$ problem (S_{λ_*}) has at least one positive solution $u_* \in \operatorname{int} C_+$;

(c) for every $\lambda \in (0, \lambda_*)$ problem (S_λ) has no positive solution.

Remark 6. This bifurcation-type theorem leaves open two interesting questions:

(a) Is it possible in hypothesis $H_3(ii)$ to replace the global *AR*-condition by the usual local one (see (5)) or even better by the more general "superlinearity" condition used in Section 3? The difficulties can be traced in Proposition 15, which shows that $\lambda_* > 0$. It is not clear how this can be proved in the aforementioned two more general settings.

(b) Can we have Theorem 4.1 for the Robin problem (that is, for $\beta \neq 0,\ \beta \geq 0$)? Again the difficulty is in Proposition 15. The proof of that proposition fails since we can not control the boundary term $\int_{\partial\Omega} \beta(z) u_n^{p-1} d\sigma$.

Next, we carry the study of problem (S_λ) ($\lambda \geq \lambda_*$) a little further and produce a smallest positive solution $\hat{w}_\lambda \in \operatorname{int} C_+$ and show that the map $\lambda \longmapsto \hat{w}_\lambda$ is strictly decreasing from $(0, \infty)$ into $C^1(\overline{\Omega})$ and $||\hat{w}_\lambda|| \to 0$ as $\lambda \to +\infty$.

Proposition 21. *If hypotheses $H(a)$ and H_3 hold and $\lambda \geq \lambda_*$, then problem (S_λ) admits a smallest positive solution $\hat{w}_\lambda \in \operatorname{int} C_+$.*

Proof. As in Filippakis, Kristaly and Papageorgiou [9], exploiting the monotonicity of A (see Lemma 2.2), we show that $S_0(\lambda)$ is downward directed (that is, if $u_1, u_2 \in S_0(\lambda)$), then we can find $u \in S_0(\lambda)$ such that $u \leq u_1,\ u \leq u_2$. Therefore, since we are looking for the smallest positive solution, without any loss of generality, we may assume that

$u_\lambda^*(z) \leq u(z) \leq c_{25}$ for some $c_{25} > 0$, all $u \in S_0(\lambda)$, all $z \in \overline{\Omega}$ (see Proposition 19)

Then from Dunford and Schwartz [3] [p. 336], we know that we can find $\{u_n\}_{n\geq 1} \subseteq S_0(\lambda)$ such that

$$\inf S_0(\lambda) = \inf_{n\geq 1} u_n.$$

We have

$$\begin{aligned} & A(u_n) + \lambda u_n^{p-1} = N_f(u_n) \text{ and } u_\lambda^* \leq u_n \leq c_{25} \text{ for all } n \geq 1, \qquad (91)\\ \Rightarrow\quad & \{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

Thus we may assume that

$$u_n \xrightarrow{w} \hat{w}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to \hat{w}_\Lambda \text{ in } L^r(\Omega). \qquad (92)$$

On (91) we act with $u_n - \hat{w}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (92). Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(u_n), u_n - \hat{w}_\lambda \rangle = 0,\\ \Rightarrow\quad & u_n \to \hat{w}_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2)},\\ \Rightarrow\quad & A(\hat{w}_\lambda) + \lambda \hat{w}_\lambda^{p-1} = N_f(\hat{w}_\lambda),\ u_\lambda^* \leq \hat{w}_\lambda \leq c_{25} \text{ (see (91))},\\ \Rightarrow\quad & \hat{w}_\lambda \in S(\lambda) \text{ and } \hat{w}_\lambda = \inf S_0(\lambda.) \end{aligned}$$

□

We examine the map $\lambda \longmapsto \hat{w}_\lambda$.

Proposition 22. *If hypotheses $H(a)$ and H_3 hold then $\lambda \longmapsto \hat{w}_\lambda$ is strictly decreasing from $[\lambda_*, \infty)$ into $C^1(\overline{\Omega})$, that is, if $\lambda_* \leq \lambda < \mu$, then $\hat{w}_\lambda - \hat{w}_\mu \in \operatorname{int} C_+$.*

Proof. Note that

$$\begin{aligned} & \langle A(\hat{w}_\lambda), h\rangle + \mu \int_\Omega \hat{w}_\lambda^{p-1} h dz \geq \langle A(\hat{w}_\lambda), h\rangle + \lambda \int_\Omega \hat{w}_\lambda^{p-1} h dz =\\ & = \int_\Omega f_0(z, \hat{w}_\lambda) h dz \ \text{ for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0. \qquad (93) \end{aligned}$$

We consider the problem (S_μ) and truncate the reaction $f_0(z, \cdot)$ at $\{0, \hat{w}_\lambda(z)\}$. Then reasoning as in the proof of Proposition 13, via the direct method and using

(93), we produce $u_\mu \in [0, \hat{w}_\lambda] \cap S_0(\mu)$, hence $\hat{w}_\mu \le u_\mu \le \hat{w}_\lambda$. Moreover, from Proposition 14, we have $\hat{w}_\lambda - \hat{w}_\mu \in \operatorname{int} C_+$. $\square$

Proposition 23. *If hypotheses $H(a)$ and H_3 hold, then $\hat{w}_\lambda \to 0$ in $W^{1,p}(\Omega)$ as $\lambda \uparrow +\infty$.*

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq (\lambda_*, \infty)$ such that $\lambda_n \uparrow +\infty$. Let $\hat{w}_{\lambda_n} \in S_0(\lambda_n) \subseteq \operatorname{int} C_+$ be the smallest positive solution of (S_{λ_n}) $n \ge 1$. From Proposition 21, we know that $\{\hat{w}_{\lambda_n}\}_{n\ge1}$ is (strictly) decreasing. We have

$$A(\hat{w}_{\lambda_n}) + \lambda_n \hat{w}_{\lambda_n}^{p-1} = N_{f_0}(\hat{w}_{\lambda_n}),\ \hat{w}_{\lambda_n} \le \hat{w}_1 \text{ for all } n \ge 1, \tag{94}$$

$$\Rightarrow \quad \frac{c_1}{p-1}||D\hat{w}_{\lambda_n}||_p^p + \lambda_n||\hat{w}_{\lambda_n}|| \le \int_\Omega f_0(z, \hat{w}_{\lambda_n})\hat{w}_{\lambda_n} dz \le c_{26}$$

$$\text{for some } c_{26} > 0,\ \text{all } n \ge 1 \text{ (see Lemma 2.2)},$$

$$\Rightarrow \quad \{\hat{w}_{\lambda_n}\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{95}$$

From (94) and (95) is follows that

$$\{A(\hat{w}_{\lambda_n}) - N_{f_0}(\hat{w}_{\lambda_n})\}_{n\ge1} = \{\lambda_n \hat{w}_{\lambda_n}^{p-1}\}_{n\ge1} \subseteq W^{1,p}(\Omega)^* \text{ is bounded.}$$

So, for every $h \in W^{1,p}(\Omega)$ we have

$$\left|\lambda_n \int_\Omega \hat{w}_{\lambda_n}^{p-1} h dz\right| \le c_{27}||h|| \text{ for some } c_{27} > 0,\ \text{all } n \ge 1. \tag{96}$$

In (96) we choose $h - \hat{w}_{\lambda_n}$. Then

$$\lambda_n||\hat{w}_{\lambda_n}||_p^p \le c_{27}||\hat{w}_{\lambda_n}|| \le c_{28} \text{ for some } c_{28} > 0 \text{ all } n \ge 1 \text{ (see (95))},$$

$$\Rightarrow \quad ||\hat{w}_{\lambda_n}||_p \to 0 \text{ as } n \to \infty \text{ (recall } \lambda_n \uparrow \infty). \tag{97}$$

Recall that

$$\frac{c_1}{p-1}||D\hat{w}_{\lambda_n}||_p^p + \lambda_n||\hat{w}_{\lambda_n}||_p^p \le \int_\Omega f_0(z, \hat{w}_{\lambda_n})\hat{w}_{\lambda_n} dz \to 0 \text{ as } n \to \infty \text{ (see (97))}.$$

We conclude that $\hat{w}_{\lambda_n} \to 0$ in $W^{1,p}(\Omega)$. $\square$

Remark 7. An interesting open question is whether $\lambda \longmapsto \hat{w}_\lambda$ is continuous from $[\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$.

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