DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **41**, Number **9**, September **2021**

doi:10.3934/dcds.2021036

pp. **4283–4296**

# SINGULAR DOUBLE-PHASE SYSTEMS WITH VARIABLE GROWTH FOR THE BAOUENDI-GRUSHIN OPERATOR

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(Communicated by Juncheng Wei)

Abstract. In this paper we study a class of singular systems with double-phase energy. The main feature is that the associated Euler equation is driven by the Baouendi-Grushin operator with variable coefficient. In such a way, we continue the analysis introduced in [6] to the case of lack of compactness corresponding to the whole Euclidean space. After establishing a related compactness property, we establish the existence of solutions for the Baouendi-Grushin singular system.

## 1. Introduction.

In this paper we study the following singular double-phase system

$$\begin{cases} -\Delta_{G(x,y)}u + |u|^{q(z)-2}u + |u|^{p(z)-2}u = a_1(z)u^{-\gamma_1(z)} - b(z)\alpha(z)|v|^{\beta(z)}|u|^{\alpha(z)-2}u, \\ -\Delta_{G(x,y)}v + |v|^{q(z)-2}v + |v|^{p(z)-2}v = a_2(z)v^{-\gamma_2(z)} - b(z)\beta(z)|u|^{\alpha(z)}|v|^{\beta(z)-2}v, \end{cases} \tag{1}$$

with $z = (x, y) \in \mathbb{R}^N$, $a_1, a_2, b, p, q, \alpha, \beta \in C(\mathbb{R}^N, \mathbb{R})$ and $\gamma_1, \gamma_2 : \mathbb{R}^N \to (0, 1)$ are continuous functions such that $\gamma_1 < \gamma_2$. Throughout this paper, we denote by $-\Delta_{G(x,y)}$ the Baouendi-Grushin operator with variable coefficient.

The novelty of our work is the fact that we combine several different phenomena in one problem. To be more precise, problem (1) includes the following features:

(1) the associated energy is double-phase functional with variable growth;
(2) the reaction contains a singular term;
(3) the domain is the whole space $\mathbb{R}^N$.

To the best of our knowledge, this is the first work which combines all these phenomena in one problem. First, we are going to prove a compactness result for the new needed function spaces related to problem (1). Precisely, we will extend some qualitative properties for the differential operator introduced recently by Bahrouni,

2020 *Mathematics Subject Classification.* Primary: 35J70, 35P30; Secondary: 76H05.
*Key words and phrases.* Baouendi-Grushin operator, double-phase problem, singular term, degenerate differential operator, existence of solutions.

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Rădulescu and Repovš [6] to the whole space $\mathbb{R}^N$. In the final part of this paper, by monotonicity arguments, we will prove that problem (1) has at least one solution.

We first recall the definition of the Baouendi-Grushin operator with variable growth. Consider the Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ and let $n, m$ be nonnegative integers such that $N = n+m$. This means that $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and so $z \in \mathbb{R}^N$ can be written as $z = (x, y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. In this paper, $G : \mathbb{R}^N \to (1, \infty)$ is supposed to be a continuous function and $\Delta_{G(x,y)}$ stands for the Baouendi-Grushin operator with variable coefficient, which is defined by

$$\begin{aligned}\Delta_{G(x,y)} u &= \operatorname{div}\left(\nabla_{G(x,y)} u\right) \\ &= \sum_{i=1}^{n} \left(|\nabla_x|^{G(x,y)-2} u_{x_i}\right)_{x_i} + |x|^{\gamma} \sum_{i=1}^{m} \left(|\nabla_y|^{G(x,y)-2} u_{y_i}\right)_{y_i},\end{aligned}$$

where

$$\nabla_{G(x,y)} u = \mathcal{A}(x) \begin{bmatrix} |\nabla_x|^{G(x,y)-2} & \nabla_x u \\ |x|^{\gamma} |\nabla_y|^{G(x,y)-2} & \nabla_y u \end{bmatrix}$$

and

$$\mathcal{A}(x) = \begin{bmatrix} I_n & 0_{n,m} \\ 0_{m,n} & |x|^{\gamma} I_m \end{bmatrix} \in \mathcal{M}_{N\times N}(\mathbb{R}),$$

with $I_n$ being the identity matrix of size $n \times n$, $O_{n,m}$ is the zero matrix of size $n \times m$ and $\mathcal{M}_{N\times N}$ stands for the class of $N \times N$-matrices with real-valued entries. From the representation above it is clear that $\Delta_{G(x,y)}$ is degenerate along the $m$-dimensional subspace $M := \{0\} \times \mathbb{R}^m$ of $\mathbb{R}^N$.

The differential operator $\Delta_{G(x,y)}$ generalizes the degenerate operator

$$\frac{\partial^2}{\partial x^2} + x^{2r} \frac{\partial^2}{\partial y^2} \quad (r \in \mathbb{N})$$

introduced by Baouendi [9] and Grushin [17]. The Baouendi–Grushin operator can be viewed as the Tricomi operator for transonic flow restricted to subsonic regions. On the other hand, a second-order differential operator $T$ in divergence form on the plane, can be written as an operator whose principal part is a Baouendi-Grushin-type operator, provided that the principal part of $T$ is nonnegative and its quadratic form does not vanish at any point, see Franchi & Tesi [16].

Also, the double-phase operator presented in system (1) is strongly related to the Caffarelli-Kohn-Nirenberg inequality. It is well known that this type of inequality is needed in several ways in the study of partial differential equations. We refer to the works of Adimurthi, Chaudhuri & Ramaswamy [2], Baroni, Colombo & Mingione [8], Colasuonno & Pucci [14], Colombo & Mingione [15] for relevant applications of the Caffarelli-Kohn-Nirenberg inequality. For recent contributions to the study of double-phase problems we refer to Ambrosio & Rădulescu [3], Beck & Mingione [10], Papageorgiou, Rădulescu & Repovš [22, 23, 24, 25], and Zhang & Rădulescu [30].

Let $\Omega \subset \mathbb{R}^N$ be an open set. The following Caffarelli-Kohn-Nirenberg inequality [11] establishes that for given $p \in (1, N)$ and real numbers $a$, $b$ and $q$ such that

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = \frac{Np}{N - p(1+a-b)},$$

there exists a positive constant $C_{a,b}$ such that for all $u \in C_c^1(\Omega)$

$$\left(\int_\Omega |x|^{-bq}|u|^q\,dx\right)^{p/q} \le C_{a,b}\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\,.$$

This inequality was extensively studied, see for example Abdellaoui & Peral [1], Adimurthi, Chaudhuri & Ramaswamy [2], Bahrouni, Rădulescu & Repovš [5], Bahrouni, Rădulescu & Repovš [6], and the references therein. In particular, Bahrouni, Rădulescu & Repovš [6] proved a new version of a Caffarelli-Kohn-Nirenberg inequality with variable exponent for the Baouendi-Grushin operator $\Delta_G$. More precisely, the following weighted inequality has been proved.

**Theorem 1.1.** *Assume that $G$ is a function of class $C^1$ and that $G(x,y) \in (2,N)$ for all $(x,y) \in \Omega$. Then there exists a positive constant $\beta$ such that for all $u \in C_c^1(\Omega)$*

$$\begin{aligned}&\int_\Omega (1+|x|^\gamma)\,|u|^{G(x,y)}\,dx\,dy\\ &\le \beta\int_\Omega \left(|\nabla_x u|^{G(x,y)} + |x|^\gamma|\nabla_y u|^{G(x,y)}\right)\,dx\,dy\\ &\quad + \beta\int_\Omega |u|^{G(x,y)-1}\left(1+u^2\right)\left(|\nabla_x G(x,y)| + |x|^\gamma|\nabla_y G(x,y)|\right)\,dx\,dy.\end{aligned}$$

Using the above theorem, Bahrouni, Rădulescu & Repovš [6] introduced a new Bouendi-Grushin-type operator and a suitable functions space (see section 3).

Contributions related with the content of this paper and dealing with certain types of double-phase problems are due to Cencelj, Rădulescu & Repovš [12] (problems with variable growth), Colasuonno & Squassina [13], and Liu & Dai [19] (problems with a differential operator which exhibits unbalanced growth). We also mention the recent works on $(p,q)-$equations (equations driven by the sum of a $p-$Laplacian and of a $q-$Laplacian) with singular terms of Papageorgiou, Rădulescu & Repovš [21] and Papageorgiou, Vetro & Vetro [26].

The paper is organized as follows. In Section 2 we present the basic properties of variable Lebesgue space and state the main tools which will be used later. New properties concerning the Baouendi-Grushin operator will be discussed in Section 3 and in the last section we state and prove our main result concerning the existence of a weak solution to problem (1).

2. **Terminology and the abstract setting.** In this section we recall some necessary definitions and properties of of variable exponent spaces. We refer to the papers of Bahrouni & Repovš [4], Hájek, Montesinos Santalucía, Vanderwerff & Zizler [18], Musielak [20], Rădulescu [27, 28], Rădulescu & Repovš [29] and the references therein.

Consider the set

$$C_+(\Omega) = \left\{ p \in C(\Omega) \;\middle|\; p(x) > 1 \text{ for all } x \in \Omega \right\}$$

and define for any $p \in C_+(\Omega)$

$$p^+ := \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- := \inf_{x\in\Omega} p(x).$$

Then $1 < p^- \le p^+ < \infty$ for each $p \in C_+(\Omega)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \inf\left\{ \mu > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1 \right\}.$$

If $\Omega = \mathbb{R}^N$, we denote $\|u\|_{p(\cdot),\Omega} = \|u\|_{p(\cdot)}$.

It is known that $L^{p(\cdot)}(\Omega)$ is a reflexive Banach space.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left| \int_\Omega uv\,dx \right| \le \left( \frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.$$

Moreover, for $p_1 \le p_2$ in $\Omega$, then there exists the continuous embedding

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega). \tag{2}$$

The following two propositions will be useful in the sequel.

**Proposition 1.** *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)}\,dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

*Then the following holds:*

*(i)* $\|u\|_{p(\cdot),\Omega} < 1$ *(resp.*, $= 1; > 1$*) if and only if* $\rho_1(u) < 1$ *(resp.*, $= 1; > 1$*);*

*(ii)* $\|u\|_{p(\cdot),\Omega} > 1$ *implies* $\|u\|^{p^-}_{p(\cdot),\Omega} \le \rho_1(u) \le \|u\|^{p^+}_{p(\cdot),\Omega}$*;*

*(iii)* $\|u\|_{p(\cdot),\Omega} < 1$ *implies* $\|u\|^{p^+}_{p(\cdot),\Omega} \le \rho_1(u) \le \|u\|^{p^-}_{p(\cdot),\Omega}$.

**Proposition 2.** *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)}\,dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

*If* $u, u_n \in L^{p(\cdot)}(\Omega)$ *and* $n \in \mathbb{N}$*, then the following statements are equivalent:*

*(i)* $\lim\limits_{n\to+\infty} \|u_n - u\|_{p(\cdot),\Omega} = 0$*;*

*(ii)* $\lim\limits_{n\to+\infty} \rho_1(u_n - u) = 0$*;*

*(iii)* $u_n(x) \to u(x)$ *in* $\Omega$ *and* $\lim\limits_{n\to+\infty} \rho_1(u_n) = \rho_1(u)$.

3. **Compactness results related to double-phase operator in $\mathbb{R}^N$.** In this section we recall and prove new results concerning the Baouendi-Grushin operator in $\mathbb{R}^N$.

We will start by recalling some results proved in [6]. Based on Theorem 1.1, we denote by $\mathcal{W}$ the closure of $C^1_c(\Omega)$ with respect to the norm

$$\begin{aligned}
\|u\|_1 = \|\nabla_x u\|_{G(\cdot,\cdot)} &+ \left\| |x|^{\frac{\gamma}{G(\cdot,\cdot)}} \nabla_y u \right\|_{G(\cdot,\cdot)} \\
&+ \left\| u \left( |\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)| \right)^{\frac{1}{G(x,y)+1}} \right\|_{G(\cdot,\cdot)+1} \\
&+ \left\| u \left( |\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)| \right)^{\frac{1}{G(x,y)-1}} \right\|_{G(\cdot,\cdot)-1}.
\end{aligned}$$

Note that the norm $\|\cdot\|_1$ on $\mathcal{W}$ is equivalent to

$$\begin{aligned}
&\|u\|_{\mathcal{W}}\\
&= \inf\left\{\mu \geq 0 \;\middle|\; \rho\left(\frac{u}{\mu}\right) \leq 1\right\}\\
&= \inf\left\{\mu \geq 0 \;\middle|\; \int_\Omega \frac{1}{G(x,y)}\left[\left|\nabla_x\left(\frac{u}{\mu}\right)\right|^{G(x,y)} + |x|^\gamma \left|\nabla_y\left(\frac{u}{\mu}\right)\right|^{G(x,y)}\right] dx\, dy \right.\\
&\qquad\left. + \int_\Omega A(x,y)\left[\frac{\left|\frac{u}{\mu}\right|^{G(x,y)+1}}{G(x,y)+1} + \frac{\left|\frac{u}{\mu}\right|^{G(x,y)-1}}{G(x,y)-1}\right] dx\, dy \leq 1\right\},
\end{aligned} \tag{3}$$

with

$$A(x,y) = |\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)| \quad \text{for all } (x,y) \in \Omega.$$

The following compactness property was proved by Bahrouni, Rădulescu & Repovš [6].

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^N$, be a bounded domain with smooth boundary $\partial\Omega$. We suppose that the domain $\Omega$ intersects the degeneracy set $[x = 0]$, that is,*

$$\Omega \cap \{(0,y) : \; y \in \mathbb{R}^m\} \neq \emptyset.$$

*Assume that $G$ is a function of class $C^1$ and that $G(x,y) \in (2,N)$ for all $(x,y) \in \overline{\Omega}$. Furthermore, suppose that $s \in (1, G^-)$ and $0 < \gamma < \frac{N(G^- - s)}{s}$. Then $\mathcal{W}$ is compactly embedded in $L^s(\Omega)$.*

Now, we will try to extend the above Lemma to the whole space $\mathbb{R}^N$. First, we give the hypotheses on continuous functions $K, p, q : \mathbb{R}^N \to \mathbb{R}$.

$(K)$ $K \in L^\infty(\mathbb{R}^N)$, $K(x) > 0$ for all $x \in \mathbb{R}^N$ and if $(A_n) \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n| \leq R$, for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{n\to+\infty} \int_{A_n \frown B_r^c(0)} K(x)dx = 0.$$

$(PQ)$ $p, q \in C_+(\mathbb{R}^N)$, $q^+ < G^-$ and $G(x) + 1 \leq p^+ < \infty$, for all $x \in \mathbb{R}^N$.

In order to treat problem (1), let us consider the space:

$$\begin{aligned}
X = \{u : \mathbb{R}^N \to \mathbb{R},\ &\text{ u is measeurable and}\\
&\int_{\mathbb{R}^N} (|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)})dxdy + \int_{\mathbb{R}^N} |u|^{q(z)}dz\\
&+ \int_{\mathbb{R}^N} |u|^{p(z)}dz < +\infty\}
\end{aligned}$$

endowed with the norm

$$\begin{aligned}
\|u\|_X = \|\nabla_x u\|_{G(\cdot,\cdot)} &+ \left\| |x|^{\frac{\gamma}{G(\cdot,\cdot)}} \nabla_y u\right\|_{G(\cdot,\cdot)}\\
&+ \|u\|_{q(\cdot)} + \|u\|_{p(\cdot)}, \quad \forall u \in X.
\end{aligned}$$

**Lemma 3.2.** *Assume that assumptions of Lemma 3.1 are fulfilled. Moreover, suppose that $(K)$, $(PQ)$ are satisfied. Then $X$ is compactly embedded in $L^s(\Omega)$.*

*Proof.* Let $(u_n)$ be an arbitrary bounded sequence in $X$. Then, using Proposition 1 and (2), we get

$$\begin{aligned}
\rho_{\mathcal{W}}(u_n) = &= \int_\Omega \frac{1}{G(x,y)} \left[|\nabla_x(u_n)|^{G(x,y)} + |x|^\gamma |\nabla_y(u_n)|^{G(x,y)}\right] dx\, dy \\
&+ \int_\Omega A(x,y) \left[\frac{|u_n|^{G(x,y)+1}}{G(x,y)+1} + \frac{|u_n|^{G(x,y)-1}}{G(x,y)-1}\right] dx\, dy \\
&\leq C(\|\nabla_x u_n\|^{G^+}_{G(\cdot,\cdot)} + \|\nabla_x u_n\|^{G^-}_{G(\cdot,\cdot)} + \left\||x|^{\frac{\gamma}{G(\cdot,\cdot)}} \nabla_y u_n\right\|^{G^+}_{G(\cdot,\cdot)} + \left\||x|^{\frac{\gamma}{G(\cdot,\cdot)}} \nabla_y u_n\right\|^{G^-}_{G(\cdot,\cdot)} \\
&+ \|u_n\|^{G^- -1}_{G(\cdot,\cdot)-1,\Omega} + \|u_n\|^{G^+ -1}_{G(\cdot,\cdot)-1,\Omega} + \|u_n\|^{G^- +1}_{G(\cdot,\cdot)+1,\Omega} + \|u_n\|^{G^+ +1}_{G(\cdot,\cdot)+1,\Omega}) \\
&\leq C(\|u_n\|^{G^+}_X + \|u_n\|^{G^-}_X + \|u_n\|^{G^- -1}_{p(\cdot),\Omega} + \|u_n\|^{G^+ -1}_{p(\cdot),\Omega} + \|u_n\|^{G^- +1}_{p(\cdot),\Omega} + \|u_n\|^{G^+ +1}_{p(\cdot),\Omega}) \\
&\leq C(\|u_n\|^{G^+}_X + \|u_n\|^{G^-}_X + \|u_n\|^{G^- -1}_X + \|u_n\|^{G^+ -1}_X + \|u_n\|^{G^- +1}_X + \|u_n\|^{G^+ +1}_X)
\end{aligned}$$

which implies that $(u_n|_\Omega)$ is bounded in $\mathcal{W}$. Thus, in light of Lemma 3.1, we conclude the proof of our lemma. □

Note that the norm $\|\cdot\|_X$ on $X$ is equivalent to

$$\begin{aligned}
\|u\| &= \inf\left\{\mu \geq 0 \;\middle|\; \rho\left(\frac{u}{\mu}\right) \leq 1\right\} \\
&= \inf\left\{\mu \geq 0 \;\middle|\; \int_{\mathbb{R}^N} \left[\left|\nabla_x\left(\frac{u}{\mu}\right)\right|^{G(x,y)} + |x|^\gamma \left|\nabla_y\left(\frac{u}{\mu}\right)\right|^{G(x,y)}\right] dx\, dy \right. \\
&\quad \left. + \int_{\mathbb{R}^N} \left[\left|\frac{u}{\mu}\right|^{q(x,y)} + \left|\frac{u}{\mu}\right|^{p(x,y)}\right] dx\, dy \leq 1\right\}.
\end{aligned} \tag{4}$$

From now on we denote the duality pairing between $X$ and its dual space $X^*$ by $\langle \cdot, \cdot \rangle_X$. The following lemma will be helpful in later treatments.

**Lemma 3.3.** *Suppose that conditions of Lemma 3.2 are satisfied. Let $u \in X$, then the following holds:*

*(i) For $u \neq 0$ we have: $\|u\| = a$ if and only if $\rho(\frac{u}{a}) = 1$;*

*(ii) $\|u\| < 1$ implies $\frac{\|u\|^{p^+}}{4^{\frac{1}{p^+-1}}} \leq \rho(u) \leq 4\|u\|^{q^-}$;*

*(iii) $\|u\| > 1$ implies $\|u\|^{q^-} \leq \rho(u)$.*

*Proof.* (i) For every fixed $u \in X$, the mapping $\lambda \mapsto \rho(\lambda u)$ is a continuous, convex, even function, which is strictly increasing in $[0, +\infty)$. Thus, by the definition of $\rho$ and the equivalent norm given in (4), we have

$$\|u\| = a \quad \Longleftrightarrow \quad \rho\left(\frac{u}{a}\right) = 1.$$

(ii) Let $u \in X$ be such that $\|u\| < 1$, then

$$\begin{aligned}
&\|\nabla_x u\|_{G(\cdot,\cdot)} < 1, \left\||x|^{\frac{\gamma}{G(x,y)}} \nabla_y u\right\|_{G(\cdot,\cdot)} < 1, \\
&\|u\|_{q(\cdot)} < 1, \|u\|_{p(\cdot)} < 1.
\end{aligned}$$

So, by Proposition 1, we get the desired result.
(iii) Let $u \in X$ be such that $\|u\| > 1$. By (i), we obtain

$$\rho\left(\frac{u}{\|u\|}\right) = \int_{\mathbb{R}^N} \left[\left|\nabla_x\left(\frac{u}{\|u\|}\right)\right|^{G(x,y)} + |x|^\gamma \left|\nabla_y\left(\frac{u}{\|u\|}\right)\right|^{G(x,y)}\right] dx\, dy$$

$$+\int_{\mathbb{R}^N}\left[\left|\frac{u}{\|u\|}\right|^{q(x,y)}+\left|\frac{u}{\|u\|}\right|^{p(x,y)}\right]dx\,dy=1.$$

Then, by the mean value theorem, there exist $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in\mathbb{R}^N$ depending on $u,G$ such that

$$\begin{aligned}1=&\frac{1}{\|u\|^{G(x_1,y_1)}}\int_{\mathbb{R}^N}\left[|\nabla_x u|^{G(x,y)}+|x|^{\gamma}\,|\nabla_y u|^{G(x,y)}\right]dx\,dy\\&+\frac{1}{\|u\|^{q(x_2,y_2)}}\int_{\Omega}|u|^{q(x,y)}\,dx\,dy+\frac{1}{\|u\|^{p(x_3,y_3)}}\int_{\Omega}|u|^{p(x,y)}\,dx\,dy.\end{aligned}$$

Since $\|u\|>1$, it follows that

$$\begin{aligned}1\le&\frac{1}{\|u\|^{q^-}}\left[\int_{\Omega}\left[|\nabla_x u|^{G(x,y)}+|x|^{\gamma}\,|\nabla_y u|^{G(x,y)}\right]dx\,dy\right]\\&+\frac{1}{\|u\|^{q^-}}\left[\int_{\mathbb{R}^N}\left[|u|^{q(x,y)}+|u|^{p(x,y)}\right]dx\,dy\right].\end{aligned}$$

This finishes the proof. □

Now, we are ready to prove our compact embedding result in whole space $\mathbb{R}^N$. Let us define, for every $s(\cdot)\in C_+(\mathbb{R}^N)$, the following Lebesgue space

$$L_K^{s(\cdot)}(\mathbb{R}^N)=\left\{u:\mathbb{R}^N\to\mathbb{R},\ u\text{ is measurable and }\int_{\mathbb{R}^N}K(z)|u|^{s(z)}dz<+\infty\right\}.$$

**Proposition 3.** *Assume that $G$ is a function of class $C^1$ and that $G(x,y)\in(2,N)$ for all $(x,y)\in\mathbb{R}^N$. Let $(K)$ and $(PQ)$ be satisfied. Furthermore, suppose that $s(\cdot)\in(q^+,G^-)$ and $0<\gamma<\frac{N(G^- - s^+)}{s^+}$. Then $X$ is compactly embedded in $L_K^{s(\cdot)}(\mathbb{R}^N)$.*

*Proof.* Fix $s(\cdot)\in(q^+,G^-)$ and $\epsilon>0$. Using condition $(PQ)$, we conclude that

$$\lim_{t\to0}\frac{|t|^{s(z)}}{|t|^{q(z)}}=\lim_{t\to+\infty}\frac{|t|^{s(z)}}{|t|^{p(z)}}=0\ \text{ uniformly for }\ z\in\mathbb{R}^N.$$

Thus, there exists $0<t_0<t_1$ and a positive constant $C>0$ such that

$$K(z)|t|^{s(z)}\le\epsilon C(|t|^{q(z)}+|t|^{p(z)})+\chi_{[t_0,t_1]}(z)K(z)|t|^{p(z)},\quad\forall t\in\mathbb{R}\ \text{ and }\ z\in\mathbb{R}^N.$$

Set

$$A(u)=\int_{\mathbb{R}^N}|u|^{p(z)}dz+\int_{\mathbb{R}^N}|u|^{q(z)}dz$$

and

$$R=\{z\in\mathbb{R}^N,\quad t_0<|u(z)|<t_1\}.$$

Therefore

$$\begin{aligned}\int_{B_r^c(0)}K(z)|u|^{s(z)}dz&\le\epsilon CA(u)+\int_{B_r^c(0)}\chi_{[t_0,t_1]}(z)K(z)|u|^{p(z)}dz\\&\le\epsilon CA(u)+(t_1^{p^-}+t_1^{p^+})\int_{B_r^c(0)\frown R}K(z)dz.\end{aligned}\tag{5}$$

Let $(u_n)\in X$ be a sequence such that $u_n\rightharpoonup u$ in $X$. It is easy to se that $(A(u_n))_n$ is bounded in $\mathbb{R}$. Denoting $R_n=\{x\in\mathbb{R}^N,\quad t_0<|u_n(x)|<t_1\}$, it follows that $\sup_{n\in\mathbb{N}}|A_n|<+\infty$. Hence, from $(K)$ and (5), there exists a positive radius $r>0$ such that

$$\int_{B_r^c(0)}K(z)|u_n|^{s(z)}dz\le\epsilon CA(u_n)+\int_{B_r^c(0)}\chi_{[t_0,t_1]}(z)K(z)|u_n|^{p(z)}dz$$

$$\begin{aligned} &\leq \epsilon C A(u_n) + (t_1^{p^-} + t_1^{p^+}) \int_{B_r^c(0) \frown R_n} K(z)dz \\ &\leq (C' + t_1^{p^-} + t_1^{p^+})\epsilon, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{6}$$

Now, since $s(\cdot) \in (q^+, G^-)$ and $K \in L^\infty(\mathbb{R}^N)$, we deduce, that

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)|u_n|^{s(z)}dz = \int_{B_r(0)} K(x)|u|^{s(z)}dz. \tag{7}$$

Here we used Lemma 3.2. Combining (6) and (7), we conclude for $\epsilon > 0$ small enough, that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(z)|u_n|^{s(z)}dz = \int_{\mathbb{R}^N} K(z)|u|^{s(z)}dz.$$

Consequently, using Proposition 1, we infer that

$$u_n \to u \text{ in } L_K^{s(\cdot)}(\mathbb{R}^N), \quad \forall s(\cdot) \in (q^+, G^-).$$

This ends the proof. □

Now, we prove the compactness result related to the nonlinear term defined in problem (1).

**Proposition 4.** *Assume that $G$ is a function of class $C^1$ and that $G(x,y) \in (2, N)$ for all $(x,y) \in \mathbb{R}^N$. Let $(K)$ and $(PQ)$ be satisfied. Furthermore, suppose that $s(\cdot), r(\cdot) \in (q^+, G^-)$ and $0 < \gamma < \min(\frac{N(G^- - s^+)}{s^+}, \frac{N(G^- - r^+)}{r^+})$.*
*Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{t\to 0} \frac{f(z,t)}{t^{s(z)-1}} = \lim_{t\to+\infty} \frac{f(z,t)}{t^{r(z)-1}} = 0, \quad \textit{for } z \in \mathbb{R}^N \quad \textit{uniformly.} \tag{8}$$

*If $(u_n)_n$ is a sequence such that $u_n \rightharpoonup u$ in $X$, then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(z)F(z,u_n)dz = \int_{\mathbb{R}^N} K(z)F(z,u)dz.$$

*Proof.* By Proposition 3, there exists a subsequence $u_n \to u$ a.e in $\mathbb{R}^N$. It follows that

$$K(z)F(z,u_n(z)) \to K(z)F(z,u(z)) \text{ a.e in } \mathbb{R}^N. \tag{9}$$

Due to (8), we obtain

$$K(z)F(z,u(z)) \leq \epsilon C(K(z)|t|^{r(z)} + K(z)|t|^{s(z)}) + K(z)|t|^{s(z)}, \quad \forall \ z \in \mathbb{R}^N. \tag{10}$$

By Proposition 3, $u_n \to u$ in $L_K^{r(\cdot)}(\mathbb{R}^N)$ and $u_n \to u$ in $L_K^{s(\cdot)}(\mathbb{R}^N)$. Thus, there exist $f_1, f_2, f_3 \in L^1(\mathbb{R}^N)$ such that

$$K(z)F(z,u(z)) \leq f_1(z) + f_2(z) + f_3(z), \quad \forall z \in \mathbb{R}^N. \tag{11}$$

Here we used inequality (10). Then using (9), (11) and the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(z)F(z,u_n)dz = \int_{\mathbb{R}^N} K(z)F(z,u)dz.$$

The proof is now complete. □

4. **A singular system driven by $\Delta_G$.** In this section, we work under conditions introduced in Proposition 3. Using the previous abstract results, we investigate the existence of solutions for the singular system (1).

The hypotheses on functions $a_1, a_2, b_1$ and $b_2$ are the following:
$(H_1)$ $\frac{a_1}{K} \in L^\infty(\mathbb{R}^N)$ and $a_1 \in L^{\frac{t_1(z)}{t_1(z)+\gamma_1(z)-1}}(\mathbb{R}^N)$ where $t_1(\cdot) \in C_+(\mathbb{R}^N)$ and $t_1(\cdot) \in (q^+, G^-)$. Moreover, we assume that there are $R_0 > 0$ and $x_0 \in \mathbb{R}^N$ such that

$$a_1(x) > 0, \quad \forall x \in B(x_0, R_0).$$

$(H_2)$ $\frac{a_2}{K} \in L^\infty(\mathbb{R}^N)$ and $a_2 \in L^{\frac{t_2(z)}{t_2(z)+\gamma_2(z)-1}}(\mathbb{R}^N)$ where $t_2(\cdot) \in C_+(\mathbb{R}^N)$ and $t_2(\cdot) \in (q^+, G^-)$.
$(H_3)$ $\alpha, \beta \in C_+(\mathbb{R}^N)$ such that $\alpha + \beta \in (q+, G^-)$.
$(H_4)$ $b \geq 0$ and $\frac{b}{K} \in L^\infty(\mathbb{R}^N)$.

We say that $(u, v) \in X \times X$ is a weak solution of problem (1) if

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[ |\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x \phi + |x|^\gamma |\nabla_y u|^{G(x,y)-2} \nabla_y u \nabla_y \phi \right] dx\, dy \\
&\quad + \int_{\mathbb{R}^N} \left[ |\nabla_x v|^{G(x,y)-2} \nabla_x v \nabla_x \psi + |x|^\gamma |\nabla_y v|^{G(x,y)-2} \nabla_y v \nabla_y \psi \right] dx\, dy \\
&\quad + \int_{\mathbb{R}^N} |u|^{q(z)-2} u\phi\, dz + \int_{\mathbb{R}^N} |u|^{p(z)-2} u\phi\, dz \\
&\quad + \int_{\mathbb{R}^N} |v|^{q(z)-2} u\psi\, dz + \int_{\mathbb{R}^N} |v|^{p(z)-2} v\psi\, dz \\
&= \int_{\mathbb{R}^N} a_1(z) u^{-\gamma_1(z)} \phi\, dx\, dy + \int_{\mathbb{R}^N} a_2(z) v^{-\gamma_2(z)} \psi\, dz \\
&- \int_{\mathbb{R}^N} b(z)\alpha(z) |u|^{\alpha(z)-2} u |v|^{\beta(z)} \phi dz - \int_{\mathbb{R}^N} b(z)\beta(z) |u|^{\alpha(z)} |v|^{\beta(z)-2} v\psi\, dz.
\end{aligned}$$

is satisfied for all $\phi, \psi \in X \setminus \{0\}$.

We associate to the problem (1), the singular functional $I : X \times X \to \mathbb{R}$, as follows:

$$\begin{aligned}
I(u, v) = &\int_{\mathbb{R}^N} \frac{1}{G(x,y)} \left[ |\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)} \right] dx\, dy \\
&+ \int_{\mathbb{R}^N} \frac{1}{G(x,y)} \left[ |\nabla_x v|^{G(x,y)} + |x|^\gamma |\nabla_y v|^{G(x,y)} \right] dx\, dy \\
&+ \int_{\mathbb{R}^N} \frac{|u|^{q(z)}}{q(z)} dz + \int_{\mathbb{R}^N} \frac{|u|^{p(z)}}{p(z)} dz + \int_{\mathbb{R}^N} \frac{|v|^{q(z)}}{q(z)} dz + \int_{\mathbb{R}^N} \frac{|v|^{p(z)}}{p(z)} dz \\
&- \int_{\mathbb{R}^N} \frac{a_1(z) u^{1-\gamma_1(z)}}{1-\gamma_1(z)} dz - \int_{\mathbb{R}^N} \frac{a_2(z) v^{1-\gamma_2(z)}}{1-\gamma_2(z)} dz - \int_{\mathbb{R}^N} b(z) |u|^{\alpha(z)} |v|^{\beta(z)} dz,
\end{aligned}$$

for all $(u, v) \in X \times X$.

Now, we are ready to state our main result.

**Theorem 4.1.** *Suppose that conditions of Proposition 3 are fulfilled. Moreover, we assume that $(H_1) - (H_4)$ are satisfied. Then there exists $m > 0$, such that if $0 < \gamma < m$, problem (1) admits at least one nonnegative weak solution.*

We give some useful remarks.

**Remark 1.** $(i)$ Under conditions of Theorem 4.1, we have

$$I(u,v)\in\mathbb{R} \ \text{ for all } \ (u,v)\in X\times X.$$

Indeed, it is sufficient to prove that $\int_{\mathbb{R}^N} b(z)|u|^{\alpha(z)}|v|^{\beta(z)}\,dz < +\infty, \quad \forall (u,v)\in X\times X$. For $(u,v)\in X\times X$, using Young's inequality and condition $(H_4)$ , we get

$$\begin{aligned} b(z)|u|^{\alpha(z)}|v|^{\beta(z)} &\leq \frac{\alpha(z)b(z)}{\alpha(z)+\beta(z)}|u|^{\alpha(z)+\beta(z)} + \frac{\beta(z)b(z)}{\alpha(z)+\beta(z)}|v|^{\alpha(z)+\beta(z)} \\ &\leq C(K(z)|u|^{\alpha(z)+\beta(z)} + K(z)|v|^{\alpha(z)+\beta(z)}), \quad \forall z\in\mathbb{R}^N. \end{aligned} \tag{12}$$

Set $f(z,t)=|t|^{\alpha(z)+\beta(z)}$ for $z\in\mathbb{R}^N$ and $t\in\mathbb{R}$. In light of $(H_3)$, there exist $r(\cdot),s(\cdot)\in(q^+,G^-)$ such that

$$\lim_{t\to 0}\frac{f(z,t)}{|t|^{s(z)}} = \lim_{t\to+\infty}\frac{f(z,t)}{|t|^{r(z)}} = 0.$$

Let $\gamma\in\mathbb{R}$ such that

$$0<\gamma<\min\Big(\frac{N(G^- - s^+)}{s^+},\frac{N(G^- - r^+)}{r^+}\Big) = m_1. \tag{13}$$

Hence, by (10) and Proposition 3, we prove that $\int_{\mathbb{R}^N} K(z)|u|^{\alpha(z)+\beta(z)}\,dz < +\infty$. It follows, by (12), that

$$\int_{\mathbb{R}^N} b(z)|u|^{\alpha(z)}|v|^{\beta(z)}\,dz < +\infty.$$

$(ii)$ It is important to mention that the energy functional $I$ is well defined (by $(i)$) but not differentiable due to the singular term.
$(iii)$ Lemma 3.3 still holds true if we replace $X$ by $X\times X$.

**Lemma 4.2.** *Suppose that assumptions of Theorem 4.1 are fulfilled. Then the functional $I$ is coercive.*

*Proof.* Let $(u,v)\in X\times X$ such that $\|(u,v)\|>1$. Then, in view of Hölder inequality and Lemma 3.3, we infer that

$$\begin{aligned} I(u,v) \geq C\Big(&\int_{\mathbb{R}^N}\Big[|\nabla_x u|^{G(x,y)} + |x|^{\gamma}\,|\nabla_y u|^{G(x,y)}\Big]\,dx\,dy \\ &+\int_{\mathbb{R}^N}\Big[|\nabla_x v|^{G(x,y)} + |x|^{\gamma}\,|\nabla_y v|^{G(x,y)}\Big]\,dx\,dy \\ &+\int_{\mathbb{R}^N}|u|^{q(z)}\,dz + \int_{\mathbb{R}^N}|u|^{p(z)}\,dz + \int_{\mathbb{R}^N}|v|^{q(z)}\,dz + \int_{\mathbb{R}^N}|v|^{p(z)}\,dz \\ -&\int_{\mathbb{R}^N}\frac{a_1(z)u^{1-\gamma_1(z)}}{1-\gamma_1(z)}\,dz - \int_{\mathbb{R}^N}\frac{a_2(z)v^{1-\gamma_2(z)}}{1-\gamma_2(z)}\,dz\Big) \\ \geq C\big(&\|u,v\|^{q^-} - \|(u,v)\|^{1-\gamma_1^-} - \|(u,v)\|^{1-\gamma_1^+} - \|(u,v)\|^{1-\gamma_2^-} - \|(u,v)\|^{1-\gamma_2^+}\big). \end{aligned}$$

Since $1-\gamma_1^+, 1-\gamma_2^+ < q^-$, it follows that $I$ is coercive. This ends the proof. $\square$

**Lemma 4.3.** *Under conditions of Theorem 4.1, If $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $X\times X$, then there exist $m_2>0$ and a subsequence of $(u_n,v_n)$ satisfies*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} b(z)|u_n|^{\alpha(z)}|v_n|^{\beta(z)}\,dz = \int_{\mathbb{R}^N} b(z)|u|^{\alpha(z)}|v|^{\beta(z)}\,dz, \tag{14}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} a_1(z)|u_n|^{1-\gamma_1(z)}dz = \int_{\mathbb{R}^N}|u|^{1-\gamma_1(z)}dz, \tag{15}$$

*and*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} a_1(z)|v_n|^{1-\gamma_1(z)}dz = \int_{\mathbb{R}^N} |v|^{1-\gamma_1(z)}dz, \tag{16}$$

*where* $0 < \gamma < m_2$.

*Proof.* By the same argument used in (12), we have

$$b(z)|u_n|^{\alpha(z)}|v_n|^{\beta(z)} \le C(K(z)|u_n|^{\alpha(z)+\beta(z)} + K(z)|v_n|^{\alpha(z)+\beta(z)}), \forall z \in \mathbb{R}^N.$$

Therefore, by the same method used in the proof of Proposition 4, we show that there exists $m_1^{'} \in \mathbb{R}$ such that if $0 < \gamma < m_1^{'}$ and (14) still holds true.

Invoking Hölder's inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} a_1(z)|u_n - u|^{1-\gamma_1(z)}dz \le \\ &\int_{\mathbb{R}^N} (a_1(z)K^{\frac{t_1(z)+\gamma_1(z)-1}{t_1(z)}}(z))(K^{\frac{1-\gamma_1(z)}{t_1(z)}}(z)|u_n - u|^{1-\gamma_1(z)})dz \le \\ &C\|a_1(z)K^{\frac{t_1(z)+\gamma_1(z)-1}{t_1(z)}}(z)\|_{\frac{t_1(z)}{t_1(z)+\gamma_1(z)-1}} \|K^{\frac{1-\gamma_1(z)}{t_1(z)}}(z)|u_n - u|^{1-\gamma_1(z)}\|_{\frac{t_1(z)}{1-\gamma_1(z)}},\end{aligned} \tag{17}$$

with $0 < \gamma < m_2^{'}$ for some positive $m_2^{'}$. Thus, combining (17), Propositions 1 and Proposition 3, we prove (15) for $0 < \gamma < \min(m_1^{'}, m_1^{'})$. By the same idea, we conclude assertion (16). This finishes the proof. □

We consider the following minimization problem

$$m = \inf_{(u,v)\in X\times X} I(u, v).$$

**Lemma 4.4.** *Suppose that hypotheses of Theorem 4.1 are satisfied. Then there exists* $m_3 > 0$ *such that if* $0 < \gamma < m_3$*, the functional* $I$ *reaches its global minimizer in* $X \times X$*, that is, there exists* $(u, v) \in X \times X \setminus \{(0, 0)\}$ *such that* $I(u, v) = m$ *and* $u, v \ge 0$ *in* $\mathbb{R}^N$.

*Proof.* In light of Lemma 4.2, we deduce that $-\infty < m$. Thus, there is a minimizing sequence $(u_n, v_n) \in X \times X$ such that

$$I(u_n, v_n) \to m \quad \text{as} \quad n \to +\infty. \tag{18}$$

It follows, again by Lemma 4.2, that $(u_n, v_n)$ is bounded in $X \times X$. This fact combined with Proposition 3 implies, up to a subsequence, that $u_n \to u$ a.e. in $\mathbb{R}^N$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. Hence, by Fatou's lemma, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}\right] dx\, dy \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x v|^{G(x,y)} + |x|^\gamma |\nabla_y v|^{G(x,y)}\right] dx\, dy \\ &\quad + \int_{\mathbb{R}^N} \frac{|u|^{q(z)}}{q(z)}dz + \int_{\mathbb{R}^N} \frac{|u|^{p(z)}}{p(z)}dz + \int_{\mathbb{R}^N} \frac{|v|^{q(z)}}{q(z)}dz + \int_{\mathbb{R}^N} \frac{|v|^{p(z)}}{p(z)}dz \\ &\le \liminf_{n\to+\infty}\Big(\int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)}\right] dx\, dy \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x v_n|^{G(x,y)} + |x|^\gamma |\nabla_y v_n|^{G(x,y)}\right] dx\, dy \\ &\quad + \int_{\mathbb{R}^N} \frac{|u_n|^{q(z)}}{q(z)}dz + \int_{\mathbb{R}^N} \frac{|u_n|^{p(z)}}{p(z)}dz + \int_{\mathbb{R}^N} \frac{|v_n|^{q(z)}}{q(z)}dz + \int_{\mathbb{R}^N} \frac{|v_n|^{p(z)}}{p(z)}dz\Big).\end{aligned} \tag{19}$$

Consequently, using (19) and Lemma 4.3, we conclude that

$$I(u,v) \leq \liminf_{n\to+\infty} I(u_n,v_n). \tag{20}$$

Then, by (18) and (20), we prove that $I(u,v) = m$. It remains to show that $(u,v) \in X\times X\setminus\{(0,0)\}$. Indeed: Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\overline{B}(x_0,\frac{R_0}{2}) \subset supp(\varphi)$, $\varphi = 1$ for all $x \in \overline{B}(x_0,\frac{R_0}{2})$ and $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$. It then follows that for $t \in (0,1)$,

$$\begin{aligned}
I(t\varphi,t\psi) \leq\ & t^{G^-}\Big(\int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x\varphi|^{G(x,y)} + |x|^\gamma\,|\nabla_y\varphi|^{G(x,y)}\right] dx\,dy \\
&+ \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x\psi|^{G(x,y)} + |x|^\gamma\,|\nabla_y\psi|^{G(x,y)}\right] dx\,dy\Big) \\
&+ t^{q^-}\int_{\mathbb{R}^N} \frac{|\varphi|^{q(z)}}{q(z)} dz + t^{p^-}\int_{\mathbb{R}^N} \frac{|\varphi|^{p(z)}}{p(z)}\,dz + t^{q^-}\int_{\mathbb{R}^N} \frac{|\psi|^{q(z)}}{q(z)}\,dz \\
&+ t^{p^-}\int_{\mathbb{R}^N} \frac{|\psi|^{p(z)}}{p(z)}\,dz \\
&- t^{1-\gamma_1^-}\int_{\mathbb{R}^N} \frac{a_1(z)\varphi^{1-\gamma_1(z)}}{1-\gamma_1(z)}\,dz - t^{1-\gamma_2^-}\int_{\mathbb{R}^N} \frac{a_2(z)\psi^{1-\gamma_2(z)}}{1-\gamma_2(z)}\,dz \\
&- t^{\alpha^++\beta^+}\int_{\mathbb{R}^N} b(z)|\varphi|^{\alpha(z)}|\psi|^{\beta(z)}\,dz < 0 \ \text{ for t small enough,}
\end{aligned}$$

which implies that $(u,v) \neq (0,0)$. Moreover, it is easy to see that $(|u_n|,|v_n|)$ is also a minimizing sequence of $I$. Then, $u$ and $v$ are nonnegative functions. This ends the proof. □

4.1. **Proof of Theorem 4.1 completed.** We need to prove that $(u,v)$ defined in Lemma 4.4 is a weak solution of problem (1). Let $\phi,\psi \in X$ and $t > 0$ and choose $0 < \gamma < m = \min(m_1,m_2,m_3)$. Invoking lemma 4.4 we have

$$0 \leq I[(u,v)+t(\phi,\psi)] - I(u,v),$$

$$\begin{aligned}
\Rightarrow 0 \leq\ & \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x u + t\nabla_x\phi|^{G(x,y)} - |\nabla_x u|^{G(x,y)}\right] dx\,dy \\
&+ \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|x|^\gamma\,|\nabla_y u + t\nabla_y\phi|^{G(x,y)} - |x|^\gamma\,|\nabla_y u|^{G(x,y)}\right] dx\,dy \\
&+ \int_{\mathbb{R}^N} \frac{1}{G(x,y)}\left[|\nabla_x v + t\nabla_x\psi|^{G(x,y)} - |\nabla_x v|^{G(x,y)}\right] dx\,dy \\
&+ \int_{\mathbb{R}^N} \left[|x|^\gamma\,|\nabla_y v + t\nabla_y v|^{G(x,y)} - |x|^\gamma\,|\nabla_y v|^{G(x,y)}\right] dx\,dy \\
&+ \int_{\mathbb{R}^N} \frac{|u+t\phi|^{q(z)} - |u|^{q(z)}}{q(z)} dz + \int_{\mathbb{R}^N} \frac{|u+t\phi|^{p(z)} - |u|^{p(z)}}{p(z)}\,dz \\
&+ \int_{\mathbb{R}^N} \frac{|v+t\psi|^{q(z)} - |v|^{q(z)}}{q(z)}\,dz + \int_{\mathbb{R}^N} \frac{|v+t\psi|^{p(z)} - |v|^{p(z)}}{p(z)}\,dz \\
&- \int_{\mathbb{R}^N} \frac{a_1(z)\left[|u+t\phi|^{1-\gamma_1(z)} - u^{1-\gamma_1(z)}\right]}{1-\gamma_1(z)}\,dz \\
&- \int_{\mathbb{R}^N} \frac{a_2(z)\left[|v+t\psi|^{1-\gamma_2(z)} - v^{1-\gamma_2(z)}\right]}{1-\gamma_2(z)}\,dz
\end{aligned}$$

$$- \int_{\mathbb{R}^N} b(z) \left[ |u + t\phi|^{\alpha(z)} |v + t\psi|^{\beta(z)} - |u|^{\alpha(z)} |v|^{\beta(z)} \right] dz.$$

We divide by $t > 0$ and then let $t \to 0^+$. We obtain

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^N} \left[ |\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x \phi + |x|^\gamma |\nabla_y u|^{G(x,y)-2} \nabla_y u \nabla_y \phi \right] dx\, dy \\ & + \int_{\mathbb{R}^N} \left[ |\nabla_x v|^{G(x,y)-2} \nabla_x v \nabla_x \psi + |x|^\gamma |\nabla_y v|^{G(x,y)-2} \nabla_y v \nabla_y \psi \right] dx\, dy \\ & + \int_{\mathbb{R}^N} |u|^{q(z)-2} u\phi \, dx\, dy + \int_{\mathbb{R}^N} |u|^{p(z)-2} u\phi \, dz \\ & + \int_{\mathbb{R}^N} |v|^{q(z)-2} v\psi \, dz + \int_{\mathbb{R}^N} |v|^{p(z)-2} v\psi \, dz \\ & - \int_{\mathbb{R}^N} a_1(z) u^{-\gamma_1(z)} \phi \, dz - \int_{\mathbb{R}^N} a_2(z) v^{-\gamma_2(z)} \psi \, dz \\ & - \int_{\mathbb{R}^N} b(z)\alpha(z) |u|^{\alpha(z)-2} u |v|^{\beta(z)} \phi dz - \int_{\mathbb{R}^N} b(z)\beta(z) |u|^{\alpha(z)} |v|^{\beta(z)-2} v\psi \, dz. \end{aligned}$$

Since $(\phi, \psi) \in X \times X$ is arbitrary, the equality must hold and so $(u, v)$ is a nonnegative weak solution of problem (1). The proof is now complete. $\square$

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Received for publication November 2020.

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