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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Eigenvalue problems in anisotropic Orlicz–Sobolev spaces

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Abstract

We establish sufficient conditions for the existence of solutions to a class of nonlinear eigenvalue problems involving nonhomogeneous differential operators in Orlicz–Sobolev spaces. ***To cite this article: M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Problèmes de valeurs propres dans les espaces d'Orlicz–Sobolev anisotropes. On établit des conditions suffisantes pour l'existence des solutions pour une classe de problèmes non linéaires de valeurs propres avec des opérateurs différentiels non homogènes dans les espaces d'Orlicz–Sobolev. ***Pour citer cet article : M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Soit $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\sum_{i=1}^{N} \partial_i(\phi_i(\partial_i u)) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

où $\lambda > 0$ et q est une fonction continue telle que $q(x) > 1$ pour tout $x \in \overline{\Omega}$. Pour chaque $i \in \{1, \ldots, N\}$ on suppose qu'il existe deux constantes $(p_i)_0$ et $(p_i)^0$ telles que si $\Phi_i(t) = \int_0^t \phi_i(s)\,\mathrm{d}s$, alors

$$1 < (p_i)_0 \leqslant \frac{t\phi_i(t)}{\Phi_i(t)} \leqslant (p_i)^0 < \infty, \quad \forall t \geqslant 0.$$

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Soit $(P^0)_+ = \max\{(p_1)^0, \ldots, (p_N)^0\}$, $(P_0)_+ = \max\{(p_1)_0, \ldots, (p_N)_0\}$ et $(P_0)_- = \min\{(p_1)_0, \ldots, (p_N)_0\}$. On suppose que $\sum_{i=1}^N 1/(p_i)_0 > 1$ et on définit $(P_0)^\star = N/(\sum_{i=1}^N 1/[(p_i)_0 - 1])$ et $P_{0,\infty} = \max\{(P_0)_+, (P_0)^\star\}$.

Le résultat principal de cette Note est le suivant :

Théorème 0.1. a) *On suppose que*

$$\left(P^0\right)_+ < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < (P_0)^\star.$$

Alors, pour chaque $\lambda > 0$, *le problème* (1) *admet une solution faible non triviale.*

b) *On suppose que*

$$1 < \min_{x\in\overline{\Omega}} q(x) < (P_0)_- \quad \textit{et} \quad \max_{x\in\overline{\Omega}} q(x) < P_{0,\infty}.$$

Alors il existe $\lambda^\star > 0$ *tel que pour chaque* $\lambda \in (0, \lambda^\star)$ *le problème* (1) *admet une solution faible non triviale.*

c) *On suppose que*

$$1 < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < (P_0)_-.$$

Alors il existe $\lambda^\star > 0$ *et* $\lambda^{\star\star} > 0$ *tels que pour chaque* $\lambda \in (0, \lambda^\star)$ *et pour chaque* $\lambda > \lambda^{\star\star}$ *le problème* (1) *admet une solution faible non triviale.*

1. The main result

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that for each $i \in \{1, \ldots, N\}$, ϕ_i are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, λ is a positive real and $q : \overline{\Omega} \to (1, \infty)$ is a continuous function. In this Note we study the following anisotropic eigenvalue problem:

$$\begin{cases} -\sum_{i=1}^N \partial_i(\phi_i(\partial_i u)) = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For all $t \in \mathbb{R}$ and $i \in \{1, \ldots, N\}$, define $\Phi_i(t) = \int_0^t \phi_i(s)\,\mathrm{d}s$. Let $L_{\Phi_i}(\Omega)$ $(i \in \{1, \ldots, N\})$ be the corresponding Orlicz spaces (see [1,2]), which are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{\Phi_i} := \inf\left\{k > 0;\ \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)\mathrm{d}x \leqslant 1\right\} < \infty.$$

The Orlicz space $L_{\Phi_i}(\Omega)$ endowed with the norm $\|u\|_{\Phi_i}$ is a Banach space.

Define

$$(p_i)_0 := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)}, \quad i \in \{1, \ldots, N\}.$$

In this Note we assume that for each $i \in \{1, \ldots, N\}$ we have

$$1 < (p_i)_0 \leqslant \frac{t\phi_i(t)}{\Phi_i(t)} \leqslant (p_i)^0 < \infty, \quad \forall t \geqslant 0. \tag{3}$$

The above relation implies that each Φ_i, $i \in \{1, \ldots, N\}$, satisfies the Δ_2-condition, that is,

$$\Phi_i(2t) \leqslant K\Phi_i(t), \quad \forall t \geqslant 0, \tag{4}$$

where K is a positive constant (see [7, Proposition 2.3]).

Furthermore, in this Note we assume that for each $i \in \{1, \ldots, N\}$ the function Φ_i satisfies the following condition:

$$\text{the function } [0, \infty) \ni t \to \Phi_i(\sqrt{t}) \text{ is convex.} \tag{5}$$

Next, for each $i \in \{1, \ldots, N\}$ we build upon $L_{\Phi_i}(\Omega)$ the Orlicz–Sobolev space $W^1L_{\Phi_i}(\Omega)$ as the space of those weakly differentiable functions in Ω for which the weak derivatives belong to $L_{\Phi_i}(\Omega)$. These are Banach spaces with respect to the norms $\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}$, for $i \in \{1, \ldots, N\}$. We also define the Orlicz–Sobolev spaces

$W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, \ldots, N\}$, as the closures of $C_0^1(\Omega)$ in $W^1 L_{\Phi_i}(\Omega)$. On $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, \ldots, N\}$, we may consider the equivalent norm $\|u\|_i := \||\nabla u|\|_{\Phi_i}$. Moreover, the above norm is equivalent to the norm $\|u\|_{i,1} = \sum_{j=1}^N \|\partial_j u\|_{\Phi_i}$. Conditions (4) and (5) assure that for each $i \in \{1, \ldots, N\}$ the Orlicz spaces $L_{\Phi_i}(\Omega)$ are uniformly convex spaces and thus, reflexive Banach spaces (see [7, Proposition 2.2]). That fact implies that also the Orlicz–Sobolev spaces $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, \ldots, N\}$, are reflexive Banach spaces.

Remark 1. We point out certain examples of functions $\phi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and satisfy conditions (3) and (5):

1) $\phi(t) = |t|^{p-2}t$, for all $t \in \mathbb{R}$, with $p > 1$. For this function it can be proved that $(\phi)_0 = (\phi)^0 = p$.
2) $\phi(t) = \log(1 + |t|^r)|t|^{p-2}t$, for all $t \in \mathbb{R}$, with $p, r > 1$. In this case it can be proved that $(\phi)_0 = p$ and $(\phi)^0 = p + r$.
3) $\phi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$, if $t \neq 0$ and $\phi(0) = 0$, with $p > 2$. In this case we have $(\phi)_0 = p - 1$ and $(\phi)^0 = p$.

For more details the reader can consult [3, Examples 1–3, p. 243].

Finally, we introduce a natural generalization of the Orlicz–Sobolev spaces $W_0^1 L_{\Phi_i}(\Omega)$ that will enable us to study with sufficient accuracy problem (2). For this purpose, let us denote by $\overline{\Phi} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overline{\Phi} = (\Phi_1, \ldots, \Phi_N)$. We define $W_0^1 L_{\overline{\Phi}}(\Omega)$, the *anisotropic Orlicz–Sobolev space*, as the closure of $C_0^1(\Omega)$ with respect to the norm $\|u\|_{\overline{\Phi}} = \sum_{i=1}^N |\partial_i u|_{\Phi_i}$.

In the case when $\Phi_i(t) = |t|^{\theta_i}$, where θ_i are constants for any $i \in \{1, \ldots, N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\bar\theta}(\Omega)$, where $\bar\theta$ is the constant vector $(\theta_1, \ldots, \theta_N)$. The theory of such spaces was developed in [4, 9,10,12,8]. It was proved that $W_0^{1,\bar\theta}(\Omega)$ is a reflexive Banach space for any $\bar\theta \in \mathbb{R}^N$ with $\theta_i > 1$ for all $i \in \{1, \ldots, N\}$. This result can be easily extended, and thus, we can show that $W_0^1 L_{\overline{\Phi}}(\Omega)$ is a reflexive Banach space.

On the other hand, in order to facilitate the manipulation of the space $W_0^1 L_{\overline{\Phi}}(\Omega)$ we introduce $\overline{P^0}$, $\overline{P_0} \in \mathbb{R}^N$ as

$$\overline{P^0} = \big((p_1)^0, \ldots, (p_N)^0\big), \quad \overline{P_0} = \big((p_1)_0, \ldots, (p_N)_0\big),$$

and $(P^0)_+$, $(P_0)_+$, $(P_0)_- \in \mathbb{R}^+$ as

$$\big(P^0\big)_+ = \max\big\{(p_1)^0, \ldots, (p_N)^0\big\}, \quad (P_0)_+ = \max\big\{(p_1)_0, \ldots, (p_N)_0\big\}, \quad (P_0)_- = \min\big\{(p_1)_0, \ldots, (p_N)_0\big\}.$$

Throughout this Note we assume that

$$\sum_{i=1}^N \frac{1}{(p_i)_0} > 1, \tag{6}$$

and define $(P_0)^\star \in \mathbb{R}^+$ and $P_{0,\infty} \in \mathbb{R}^+$ by

$$(P_0)^\star = \frac{N}{\sum_{i=1}^N 1/(p_i)_0 - 1}, \quad P_{0,\infty} = \max\big\{(P_0)_+, (P_0)^\star\big\}.$$

Next, we recall some background facts concerning the variable exponent Lebesgue spaces. For any $h \in C_+(\overline{\Omega}) := \{h;\, h \in C(\overline{\Omega}),\, h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$ we define $h^+ := \sup_{x\in\Omega} h(x)$ and $h^- := \inf_{x\in\Omega} h(x)$. For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u(x)|^{q(x)}\,\mathrm{d}x < \infty$, where

$$|u|_{q(x)} := \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{q(x)} \mathrm{d}x \leqslant 1\right\}.$$

An important role is played by the *modular* of the $L^{q(x)}(\Omega)$ space, which is defined by $\rho_{q(x)}(u) = \int_\Omega |u|^{q(x)}\,\mathrm{d}x$, for any $u \in L^{q(x)}(\Omega)$. If $u_n, u \in L^{q(x)}(\Omega)$ then the following relations hold true:

$$|u|_{q(x)} > 1 \quad \Rightarrow \quad |u|_{q(x)}^{q^-} \leqslant \int_\Omega |u|^{q(x)}\,\mathrm{d}x \leqslant |u|_{q(x)}^{q^+}, \tag{7}$$

$$|u|_{q(x)} < 1 \quad \Rightarrow \quad |u|_{q(x)}^{q^+} \leqslant \int_\Omega |u|^{q(x)}\,\mathrm{d}x \leqslant |u|_{q(x)}^{q^-}. \tag{8}$$

In the following, for each $i \in \{1, \dots, N\}$ we define $a_i : [0,\infty) \to \mathbb{R}$ by $a_i(t) = \frac{\phi_i(t)}{t}$, for $t > 0$ and $a_i(0) = 0$. Since ϕ_i are odd we deduce that actually, $\phi_i(t) = a_i(|t|)t$ for each $t \in \mathbb{R}$ and each $i \in \{1, \dots, N\}$.

We say that $u \in W_0^1 L_{\overline{\Phi}}(\Omega)$ is a *weak solution* of problem (2) if

$$\int_\Omega \left\{ \sum_{i=1}^N a_i\big(|\partial_i u|\big)\partial_i u \partial_i w - \lambda |u|^{q(x)-2} u w \right\} \mathrm{d}x = 0$$

for all $w \in W_0^1 L_{\overline{\Phi}}(\Omega)$.

The main result of this Note is stated in the following theorem:

Theorem 1.1. a) *Assume that the function $q(x) \in C(\overline{\Omega})$ verifies the hypothesis*

$$\big(P^0\big)_+ < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < (P_0)^\star. \tag{9}$$

Then for any $\lambda > 0$ problem (2) *has a nontrivial solution in $W_0^1 L_{\overline{\Phi}}(\Omega)$.*

b) *Assume that the function $q(x) \in C(\overline{\Omega})$ verifies the hypothesis*

$$1 < \min_{x\in\overline{\Omega}} q(x) < (P_0)_- \quad \textit{and} \quad \max_{x\in\overline{\Omega}} q(x) < P_{0,\infty}. \tag{10}$$

Then there exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ problem (2) *has a nontrivial solution in $W_0^1 L_{\overline{\Phi}}(\Omega)$.*

c) *Assume that the function $q(x) \in C(\overline{\Omega})$ verifies the hypothesis*

$$1 < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < (P_0)_-. \tag{11}$$

Then there exist $\lambda^\star > 0$ and $\lambda^{\star\star} > 0$ such that for any $\lambda \in (0, \lambda^\star)$ and any $\lambda > \lambda^{\star\star}$ problem (2) *has a nontrivial solution in $W_0^1 L_{\overline{\Phi}}(\Omega)$.*

2. Proof of Theorem 1.1

The following result extends Theorem 1 in [4]:

Proposition 2.1. *Assume $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) is a bounded domain with smooth boundary. Assume relation* (6) *is fulfilled. Assume that $q \in C(\overline{\Omega})$ verifies $1 < q(x) < P_{0,\infty}$, for all $x \in \overline{\Omega}$. Then the embedding $W_0^1 L_{\overline{\Phi}}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

From now on E denotes the anisotropic Orlicz–Sobolev space $W_0^1 L_{\overline{\Phi}}(\Omega)$.

For any $\lambda > 0$ the energy functional corresponding to problem (2) is defined as $T_\lambda : E \to \mathbb{R}$,

$$T_\lambda(u) = \int_\Omega \sum_{i=1}^N \Phi_i\big(|\partial_i u|\big)\,\mathrm{d}x - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\,\mathrm{d}x.$$

Proposition 2.1 implies that $T_\lambda \in C^1(E, \mathbb{R})$ and

$$\big\langle T_\lambda'(u), v\big\rangle = \int_\Omega \sum_{i=1}^N a_i\big(|\partial_i u|\big)\partial_i u \partial_i v\,\mathrm{d}x - \lambda \int_\Omega |u|^{q(x)-2} u v\,\mathrm{d}x,$$

for all $u, v \in E$. Thus, the weak solutions of (2) coincides with the critical points of T_λ.

The following auxiliary results show that T_λ has a mountain-pass geometry:

Lemma 2.2. *Assume that the hypothesis* (9) *of Theorem* 1.1 *is fulfilled. Then there exist $\eta > 0$ and $\alpha > 0$ such that $T_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\|_{\overline{\Phi}} = \eta$.*

Lemma 2.3. *Assume that the hypothesis* (9) *of Theorem* 1.1 *is fulfilled. Then there exists $e \in E$ with $\|e\|_{\overline{\Phi}} > \eta$ (where η is given in Lemma* 2.2) *such that $T_\lambda(e) < 0$.*

Proof of Theorem 1.1 a). By Lemmas 2.2 and 2.3 and the mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of a sequence $\{u_n\} \subset E$ such that

$$T_\lambda(u_n) \to \bar{c} > 0 \quad \text{and} \quad T'_\lambda(u_n) \to 0 \text{ (in } E^\star) \quad \text{as } n \to \infty. \tag{12}$$

We prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, there exists a subsequence (still denoted by $\{u_n\}$) such that $\|u_n\|_{\overline{\Phi}} \to \infty$. Thus, we may assume that for n large enough we have $\|u_n\|_{\overline{\Phi}} > 1$.

For each $i \in \{1, \ldots, N\}$ and any positive integer n we define

$$\alpha_{i,n} = \begin{cases} (P^0)_+, & \text{if } \|\partial_i u_n\|_{\Phi_i} < 1, \\ (P_0)_-, & \text{if } \|\partial_i u_n\|_{\Phi_i} > 1. \end{cases}$$

So, by the above considerations (combined with inequalities (C.9) and (C.10) in [3], see also [6, Lemma 1]) we deduce that for n large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\|_{\overline{\Phi}} &\geqslant T_\lambda(u_n) - \frac{1}{q^-}\langle T'_\lambda(u_n), u_n\rangle \geqslant \sum_{i=1}^N \int_\Omega \left(\Phi_i\big(|\partial_i u_n|\big) - \frac{1}{q^-}\phi_i\big(|\partial_i u_n|\big)|\partial_i u_n|\right) \mathrm{d}x \\ &\geqslant \left(1 - \frac{(P^0)_+}{q^-}\right) \sum_{i=1}^N \int_\Omega \Phi_i\big(|\partial_i u_n|\big)\,\mathrm{d}x \geqslant \left(1 - \frac{(P^0)_+}{q^-}\right) \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{\alpha_{i,n}} \\ &\geqslant \left(1 - \frac{(P^0)_+}{q^-}\right)\left[\frac{1}{N^{(P_0)_- - 1}}\|u_n\|_{\overline{\Phi}}^{(P_0)_-} - N\right]. \end{aligned} \tag{13}$$

Dividing by $\|u_n\|_{\overline{\Phi}}^{(P_0)_-}$ in (13) and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Since E is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. So, by Proposition 2.1, $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$. The above considerations and relations (12) and (5) imply that actually, $\{u_n\}$ converges strongly to u_0 in E. Then, by relation (12) we have $T_\lambda(u_0) = \bar{c} > 0$ and $T'_\lambda(u_0) = 0$, that is, u_0 is a nontrivial weak solution of equation (2). The proof of Theorem 1.1 a) is complete. □

Proof of Theorem 1.1 b). We start with the following auxiliary result:

Lemma 2.4. *Assume that the hypothesis* (10) *of Theorem* 1.1 *is fulfilled. Then there exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there are ρ, $a > 0$ such that $T_\lambda(u) \geqslant a > 0$ for any $u \in E$ with $\|u\|_{\overline{\Phi}} = \rho$.*

Let $\lambda^\star > 0$ be defined as above and fix $\lambda \in (0, \lambda^\star)$. By Lemma 2.4 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} J_\lambda > 0$. Standard arguments show that there exists $\phi \in E$, $\phi \geqslant 0$, such that $T_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, we can show that for any $u \in B_\rho(0)$ we have

$$T_\lambda(u) \geqslant C_1 \cdot \|u\|_{\overline{\Phi}}^{(P^0)_+} - C_2 \cdot \|u\|_{\overline{\Phi}}^{q^-},$$

where $C_1, C_2 > 0$. It follows that $-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} T_\lambda < 0$. Fix $0 < \epsilon < \inf_{\partial B_\rho(0)} T_\lambda - \inf_{B_\rho(0)} T_\lambda$. Applying Ekeland's variational principle to the functional $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that $T_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon$ and for all $u \neq u_\epsilon$, $T_\lambda(u_\epsilon) < T_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{\overline{\Phi}}$. Since $T_\lambda(u_\epsilon) \leqslant \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon \leqslant \inf_{B_\rho(0)} T_\lambda + \epsilon < \inf_{\partial B_\rho(0)} T_\lambda$, we deduce that $u_\epsilon \in B_\rho(0)$. Define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = T_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{\overline{\Phi}}$. Then u_ϵ is a minimum point of I_λ and thus $t^{-1}[I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)] \geqslant 0$ for small $t > 0$ and any $v \in B_1(0)$. Letting $t \to 0$ it follows that $\langle T'_\lambda(u_\epsilon), v\rangle + \epsilon \cdot \|v\|_{\overline{\Phi}} > 0$, hence $\|T'_\lambda(u_\epsilon)\| \leqslant \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that $T_\lambda(w_n) \to \underline{c}$ and $T'_\lambda(w_n) \to 0$. Moreover, $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Actually, with similar arguments as those used in the end of the proof of Theorem 1.1 a) we can show that $\{w_n\}$ converges strongly to w in E. So, $T_\lambda(w) = \underline{c} < 0$ and $T'_\lambda(w) = 0$. We conclude that w is a nontrivial weak solution of problem (2). The proof of Theorem 1.1 b) is complete.

Finally, we show that Theorem 1.1 c) holds true. In order to do that we first point out that by Theorem 1.1 b) it follows that there exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ problem (2) has a nontrivial weak solution. In order to show that there exists $\lambda^{\star\star} > 0$ such that for any $\lambda > \lambda^{\star\star}$ problem (2) has a nontrivial weak solution we prove that T_λ possesses a nontrivial global minimum point in E. Indeed, it is not difficult to show that T_λ is weakly lower semicontinuous and coercive on E. So, by Theorem 1.2 in [11], there exists a global minimizer $u_\lambda \in E$ of T_λ and, thus, a weak solution of problem (2). We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $v_0 \in C_0^\infty(\Omega) \subset E$ such that $v_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \leqslant v_0(x) \leqslant t_0$ in $\Omega \setminus \Omega_1$. We have

$$T_\lambda(v_0) = \int_\Omega \left\{ \sum_{i=1}^N \Phi_i\big(|\partial_i v_0|\big) - \frac{\lambda}{q(x)} |v_0|^{q(x)} \right\} \mathrm{d}x \leqslant L - \frac{\lambda}{q^+} \int_{\Omega_1} |v_0|^{q(x)} \, \mathrm{d}x \leqslant L - \frac{\lambda}{q^+} t_0^{q^-} |\Omega_1|,$$

where L is a positive constant. Thus, there exists $\lambda^{\star\star} > 0$ such that $T_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^{\star\star}, \infty)$. It follows that $T_\lambda(u_\lambda) < 0$ for any $\lambda \geqslant \lambda^{\star\star}$ and thus u_λ is a nontrivial weak solution of problem (2) for λ large enough. The proof of Theorem 1.1 c) is complete. □

We refer to [5] for complete proofs and additional results.

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