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Partial Differential Equations

Infinitely many solutions for a class of nonlinear elliptic problems on fractals

Infinité de solutions pour une classe de problèmes elliptiques non linéaires sur des fractales

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A R T I C L E   I N F O

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A B S T R A C T

We study the nonlinear problem $\Delta u + a(x)u = \lambda g(x) f(u)$ in $V \setminus V_0$, $u = 0$ on V_0, where V is the Sierpiński gasket, V_0 is its intrinsic boundary, Δ denotes the weak Laplace operator, λ is a positive parameter, and f has an oscillatory behaviour either near the origin or at infinity. In both cases, we establish the existence of infinitely many solutions, which either converge to zero or have larger and larger energies.

R É S U M É

On étudie le problème non linéaire $\Delta u + a(x)u = \lambda g(x) f(u)$ dans $V \setminus V_0$, $u = 0$ sur V_0, où V est le joint de culasse de Sierpiński, V_0 est sa frontière intrinsèque, Δ dénote l'opérateur de Laplace au sens faible, λ est un paramètre positif et f a un comportement oscillatoire autour de l'origine ou à l'infini. Dans les deux cas on établit l'existence d'une infinité de solutions, qui ou bien convergent vers à zéro, ou bien ont des énergies de plus en plus grandes.

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Soit V le joint de culasse de Sierpiński (ainsi appelé par Mandelbrot [7]) et soit V_0 sa frontière intrinsèque. Le but de cette Note est d'étudier le problème de Dirichlet

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda g(x) f\big(u(x)\big) & \text{si } x \in V \setminus V_0, \\ u = 0 & \text{sur } V_0, \end{cases} \tag{P}$$

où λ est un paramètre positif et les potentiels a et g satisfont les conditions suivantes :

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(h_1) $a \in L^1(V,\mu)$ et $a \leqslant 0$ presque partout dans V, où μ est la mesure normalisée $\log N/\log 2$-dimensionnelle de Hausdorff sur V ;
(h_2) $g \in C(V)$, $g \leqslant 0$ et la restriction de g sur tout sous-ensemble ouvert de V n'est pas identiquement nulle.

Dans le cas où le terme non linéaire f a un comportement oscillatoire autour de l'origine, on montre que si $\lambda > 0$ est suffisament petit, alors le problème (P) admet une infinité de solutions. Plus précisément, on a la propriété suivante de multiplicité :

Théorème 0.1. *Soit $f : \mathbb{R} \to [0,+\infty)$ une fonction continue. On suppose que*

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty \quad et \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty,$$

où $F(t) := \int_0^t f(s)\,\mathrm{d}s$.

Alors, pour tout

$$\lambda \in \left(0, -\frac{1}{2(2N+3)^2(\int_V g(x)\,\mathrm{d}\mu)\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}}\right),$$

il existe une suite $\{v_n\}$ de solutions faibles du problème (P) qui converge vers zéro dans l'espace $H_0^1(V)$.

La preuve du Théorème 0.1 repose sur un résultat de multiplicité établit par Bonanno et Molica Bisci (voir [1, Theorem 2.1]), qui étend le principe variationnel de Ricceri [8]. Avec des arguments similaires on peut traiter le cas suivant, qui correspond à un comportement oscillatoire de la nonlinéarité à l'infini :

Théorème 0.2. *Soit $f : \mathbb{R} \to [0,+\infty)$ une fonction continue. On suppose que*

$$\liminf_{\xi\to +\infty} \frac{F(\xi)}{\xi^2} < +\infty \quad et \quad \limsup_{\xi\to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Alors, pour tout

$$\lambda \in \left(0, -\frac{1}{2(2N+3)^2(\int_V g(x)\,\mathrm{d}\mu)\liminf_{\xi\to +\infty}\frac{F(\xi)}{\xi^2}}\right),$$

il existe une suite $\{v_n\}$ de solutions faibles du problème (P) qui est non bornée dans l'espace $H_0^1(V)$.

1. Introduction

In this Note we are concerned with a class of nonlinear Dirichlet problems on the Sierpiński gasket V in $\mathbb{R}^{N-1}$, see Sierpiński [9]. We refer to Falconer [4] for the construction of the rigorous construction of this fractal set. Let V_0 denote the intrinsic boundary of V. By Theorem 9.3 in Falconer [4], the Hausdorff (fractal) dimension d of V satisfies $d = \log N/\log 2$ and $0 < \mathcal{H}^d(V) < \infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$.

Let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V. The following property of μ will be useful in the sequel:

$$\mu(B) > 0, \qquad \text{for every nonempty open subset } B \text{ of } V. \tag{2}$$

In other words, the support of μ coincides with V; see, for instance, Breckner, Rădulescu and Varga [3].

Denote by $C_0(V)$ the space of real-valued continuous functions on V and vanishing on V_0. If $u : V \to \mathbb{R}$ and $m \in \mathbb{N}$, we set

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{x,y\in V_m:\ |x-y|=2^{-m}} \big(u(x)-u(y)\big)^2,$$

where V_m is a suitable set associated to the fractal V obtained recursively starting from V_0; see [2].

Then $W_m(u) \leqslant W_{m+1}(u)$ for all m. Denote $W(u) = \lim_{m\to\infty} W_m(u)$. Set

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{x,y\in V_m:\ |x-y|=2^{-m}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)$$

and

$$\mathcal{W}(u,v)=\lim_{m\to\infty}\mathcal{W}_m(u,v).$$

Then $H_0^1(V):=\{u\in C_0(V)\mid W(u)<\infty\}$ is a Hilbert space with respect to the norm induced by the inner product $\mathcal{W}(u,v)$.

As pointed out by Falconer and Hu [5], if $a\in L^1(V)$ and $a\leqslant 0$ in V then the norm $\|u\|_*:=(\mathcal{W}(u,u)-\int_V a(x)u^2\,\mathrm{d}\mu)^{1/2}$, is equivalent with $\sqrt{W(u)}$ in $H_0^1(V)$.

Following Falconer and Hu [5] we can define in a standard way a linear self-adjoint operator $\Delta: Z\to L^2(V,\mu)$, where Z is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$, such that

$$-\mathcal{W}(u,v)=\int_V \Delta u\cdot v\,\mathrm{d}\mu,\quad \text{for every } (u,v)\in Z\times H_0^1(V).$$

The operator Δ is called the weak Laplace operator on V.

2. The nonlinear problem

In this Note we are concerned with the nonlinear problem

$$\begin{cases}\Delta u(x)+a(x)u(x)=\lambda g(x)f\big(u(x)\big) & \text{if } x\in V\setminus V_0,\\ u=0 & \text{on } V_0,\end{cases}\tag{P}$$

where λ is a positive real parameter. We assume that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and that the variable potentials $a,g:V\to\mathbb{R}$ satisfy the following conditions:

(h_1) $a\in L^1(V,\mu)$ and $a\leqslant 0$ almost everywhere in V, where μ denotes the restriction to V of the normalized $\log N/\log 2$-dimensional Hausdorff measure on V;

(h_2) $g\in C(V)$ with $g\leqslant 0$ and such that the restriction of g to every open subset of V is not identically zero.

The first result of this Note establishes the following multiplicity property, provided that the nonlinearity has an oscillatory behaviour near the origin:

Theorem 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function. Assume that*

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}<+\infty\quad \text{and}\quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=+\infty,$$

where $F(t):=\int_0^t f(s)\,\mathrm{d}s$.

Then, for every

$$\lambda\in\left(0,-\frac{1}{2(2N+3)^2(\int_V g(x)\,\mathrm{d}\mu)\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}}\right),$$

there exists a sequence $\{v_n\}$ of pairwise distinct weak solutions of problem (P) *such that* $\lim_{n\to\infty}\|v_n\|=\lim_{n\to\infty}\|v_n\|_\infty=0$.

Proof. We define the functionals $\Phi,\Psi: H_0^1(V)\to\mathbb{R}$ by

$$\Phi(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_V a(x)u^2(x)\,\mathrm{d}\mu\quad\text{and}\quad \Psi(u)=-\int_V g(x)F\big(u(x)\big)\,\mathrm{d}\mu.$$

Fix λ as in the conclusion. Then all critical points of the functional $I_\lambda:=\Phi-\lambda\Psi$ are weak solutions of problem (P). We first observe that $I_\lambda\in C^1(H_0^1(V),\mathbb{R})$. Next, Φ is coercive and, by Lemma 5.6 in Breckner, Rădulescu and Varga [3], the functionals Φ and Ψ are weakly sequentially lower semi-continuous on $H_0^1(V)$. Now, let $\{c_n\}$ be a real sequence such that $\lim_{n\to\infty}c_n=0$ and

$$\lim_{n\to\infty}\frac{F(c_n)}{c_n^2}=\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}.$$

Put $r_n=\frac{c_n^2}{2(2N+3)^2}$ for every $n\in\mathbb{N}$. Due to the compact embedding $(H_0^1(V),\|\cdot\|)\hookrightarrow(C_0(V),\|\cdot\|_\infty)$ (see Fukushima and Shima [6]) we have

$$\big\{v\in H_0^1(V)\bigm|\Phi(v)<r_n\big\}\subseteq\big\{v\in H_0^1(V)\bigm|\|v\|_\infty\leqslant c_n\big\}.$$

Therefore

$$\begin{aligned}\varphi(r_n) &= \inf_{\Phi(u)<r_n} \frac{\sup_{\Phi(v)<r_n} \int_V (-g(x))F(v(x))\,\mathrm{d}\mu + \int_V g(x)F(u(x))\,\mathrm{d}\mu}{r_n - \Phi(u)} \\ &\leqslant \frac{\sup_{\Phi(v)<r_n} \int_V (-g(x))F(v(x))\,\mathrm{d}\mu}{r_n} \leqslant -\left(\int_V g(x)\,\mathrm{d}\mu\right)\frac{\max_{|\xi|\leqslant c_n} F(\xi)}{r_n} \\ &= -\left(\int_V g(x)\,\mathrm{d}\mu\right)\frac{F(c_n)}{r_n} = -2(2N+3)^2\left(\int_V g(x)\,\mathrm{d}\mu\right)\frac{F(c_n)}{c_n^2}.\end{aligned}$$

Thus, since $\liminf_{\xi\to 0^+} F(\xi)/\xi^2 < +\infty$, we deduce that

$$\delta \leqslant \liminf_{n\to\infty} \varphi(r_n) \leqslant -2(2N+3)^2\left(\int_V g(x)\,\mathrm{d}\mu\right)\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} < +\infty.$$

We prove in what follows that 0, which is the unique global minimum of Φ, is not a local minimum of the functional I_λ. Hence, fix a function $u \in H_0^1(V)$ such that there is an element $x_0 \in V$ with $u(x_0) > 1$. It follows that $D := \{x \in V \mid u(x) > 1\}$ is a nonempty open subset of V. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = |\min\{t, 1\}|$ for all $t \in \mathbb{R}$. Since $h(0) = 0$ and h is a Lipschitz function, we have $v := h \circ u \in H_0^1(V)$. Moreover, $v(x) = 1$ for every $x \in D$, and $0 \leqslant v(x) \leqslant 1$ for every $x \in V$. Bearing in mind that $\limsup_{\xi\to 0^+} F(\xi)/\xi^2 = +\infty$, there exists a sequence $\{\xi_n\}$ in $]0,\rho[$ such that $\lim_{n\to\infty} \xi_n = 0$ and $\lim_{n\to\infty} F(\xi_n)/\xi_n^2 = +\infty$. Consider the sequence of functions $\{\xi_n v\} \subset H_0^1(V)$. We have

$$I_\lambda(\xi_n v) = \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v^2(x)\,\mathrm{d}\mu + \lambda F(\xi_n)\int_D g(x)\,\mathrm{d}\mu + \lambda\int_{V\setminus D} g(x)F\big(\xi_n v(x)\big)\,\mathrm{d}\mu.$$

Taking into account that F is positive in $]0,+\infty[$, from hypothesis (h_2), the above equation yields

$$I_\lambda(\xi_n v) \leqslant \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v^2(x)\,\mathrm{d}\mu + \lambda F(\xi_n)\int_D g(x)\,\mathrm{d}\mu, \quad \forall n \in \mathbb{N}.$$

Thus, for every $n \in \mathbb{N}$,

$$\frac{I_\lambda(\xi_n v)}{\xi_n^2} \leqslant \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_V a(x)v^2(x)\,\mathrm{d}\mu + \lambda\frac{F(\xi_n)}{\xi_n^2}\int_D g(x)\,\mathrm{d}\mu.$$

Moreover, by (h_2) and (1), we deduce that $\int_D g(x)\,\mathrm{d}\mu < 0$. Therefore

$$\lim_{n\to\infty}\frac{I_\lambda(\xi_n v)}{\xi_n^2} = -\infty,$$

hence $I_\lambda(\xi_n v) < 0$ for n sufficiently large. Since $I_\lambda(0) = 0$, we conclude that 0 is not a local minimum of I_λ. Moreover, since Φ has 0 as unique global minimum, Theorem 2.1 in Bonanno and Molica Bisci [1] ensures the existence of a sequence $\{v_n\}$ of pairwise distinct critical points of the functional I_λ, such that

$$\lim_{n\to\infty}\left(\|v_n\|^2 - \int_V a(x)v_n^2(x)\,\mathrm{d}\mu\right) = 0.$$

It follows that $\lim_{n\to\infty}\|v_n\| = 0$, which implies $\lim_{n\to\infty}\|v_n\|_\infty = 0$. This completes the proof of Theorem 2.1. □

With similar arguments we can prove the following result in the case where the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ has an oscillatory behaviour at infinity:

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Assume that*

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} < +\infty \quad \textit{and} \quad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for every

$$\lambda \in \left(0, -\frac{1}{2(2N+3)^2(\int_V g(x)\,\mathrm{d}\mu)\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}\right),$$

there exists a sequence of weak solutions of problem (P) *which is unbounded in $H_0^1(V)$.*

We refer to Bonanno, Molica Bisci, and Rădulescu [2] for detailed proofs, examples, and related results.

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