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A NONLINEAR EIGENVALUE PROBLEM WITH $p(x)$-GROWTH AND GENERALIZED ROBIN BOUNDARY VALUE CONDITION

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Abstract. We are concerned with the study of the following nonlinear eigenvalue problem with Robin boundary condition

$$\begin{cases} -\mathrm{div}\,(a(x,\nabla u)) = \lambda b(x,u) & \text{in } \Omega \\ \dfrac{\partial A}{\partial n} + \beta(x)c(x,u) = 0 & \text{on } \partial\Omega. \end{cases}$$

The abstract setting involves Sobolev spaces with variable exponent. The main result of the present paper establishes a sufficient condition for the existence of an unbounded sequence of eigenvalues. Our arguments strongly rely on the Lusternik-Schnirelmann principle. Finally, we focus to the following particular case, which is a $p(x)$-Laplacian problem with several variable exponents:

$$\begin{cases} -\mathrm{div}\,(a_0(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda b_0(x)|u|^{q(x)-2}u & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial n} + \beta(x)|u|^{r(x)-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Combining variational arguments, we establish several properties of the eigenvalues family of this nonhomogeneous Robin problem.

1. Introduction.

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. Assume that the functions $a(x,\xi) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $b(x,y) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $c(x,y) : \overline{\partial\Omega} \times \mathbb{R} \to \mathbb{R}$ are the derivatives with respect to the second variable of the mappings A, B, C respectively. Let $\frac{\partial A}{\partial n}$ denote the outer normal derivative of A on $\partial\Omega$ and assume that $\beta : \mathbb{R} \to [0,+\infty)$ is a given function such that $L^\infty(\partial\Omega)$.

In this work, we are concerned with the following eigenvalue problem with Robin boundary condition space as follows

$$(R) \qquad \begin{cases} -\mathrm{div}\,(a(x,\nabla u)) = \lambda b(x,u) & \text{in } \Omega \\ \dfrac{\partial A}{\partial n} + \beta(x)c(x,u) = 0 & \text{on } \partial\Omega. \end{cases}$$

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The study of nonlinear eigenvalue problems is of central interest in nonlinear analysis, starting with the pioneering papers of F. Browder [4, 5, 6, 7]. In the present paper, by considering the corresponding energy functional and seeking the weak solutions, we establish the existence of eigenfunctions corresponding to eigenvalues in some appropriate Banach space.

The Sobolev space $W^{1,p}(\Omega)$, where p is *constant*, is suitable for studying of many problems in physics and mechanics. In some cases the standard approach based on the theory of classical L^p and $W^{1,p}$ Lebesgue and Sobolev spaces is not adequate in the framework of material with non-homogeneities. For instance, electro-rheological fluids (sometimes referred to as "smart fluids") or phenomena in image processing are described in a correct manner by mathematical models in which the exponent p is allowed to vary. This leads us to the study of variable exponents Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. These relevant applications motivate the growing interest of mathematicians for the study of nonlinear problems with $p(x)$-growth conditions; we refer, e.g., to [1, 2, 12, 20, 22, 25, 28] and the references therein.

We deal with boundary data which is a combination of Dirichlet and Neumann conditions. This is called *Robin boundary condition* (or third type boundary condition), named after the French mathematician Victor Gustave Robin. This boundary condition is commonly used in solving Sturm-Liouville problems, which appear in many contexts in science and engineering. In addition, the Robin boundary condition is a general form of the insulating boundary condition for convectiondiffusion equations. For instance, in many physical problems such as heat conduction or chemical reaction, the flow across the boundary surface is proportional to the difference between the surrounding density and the density just inside Ω. Consider $\partial u/\partial \nu$ as the outward normal derivative of u on $\partial\Omega$ and denote the surrounding density by $\rho_0(x,t)$. According to [17, pp. 5-6],

$$\frac{\partial u}{\partial \nu} = \beta(\rho_0 - u),$$

where β is a proportionality constant, which can vary from point to point on $\partial\Omega$. Since ρ_0 is known, the boundary condition can be written as

$$\frac{\partial u}{\partial \nu} + \beta u = \rho(x,t) \quad \text{for } x \in \partial\Omega,\ t > 0,$$

where $\rho = \beta\rho_0$. We also point out that the most often used boundary condition in optical imaging is the Robin boundary condition, which is also referred as the *partial current boundary condition*; see [3, 23].

The paper is organized as follows. We first present some of the main properties of variable exponent Lebesgue and Sobolev spaces as well as the abstract Lusternik-Schnirelmann (L-S) principle [15]. Then we establish the existence of infinitely many eigenpairs in relationship with the L-S principle. Section 4 includes an illustration of the main result in the particular case of $p(x)$-Laplace operators. The final section of this paper is devoted to study of the infimum of the eigenvalue set and some related properties.

2. **Preliminaries.**

Let $p \in C(\overline{\Omega})$ and

$$1 \leq p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

The variable exponent Lebesgue space is

$$L^{p(x)}(\Omega) = \{u: \ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < \infty\},$$

which is a Banach space with the norm

$$|u|_{L^{p(x)}(\Omega)} = \inf\left\{\sigma > 0 : \ \int_\Omega |\frac{u}{\sigma}|^{p(x)} dx \leq 1\right\}.$$

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega); \ |\nabla u| \in L^{p(x)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Throughout this paper, we use the symbols X, E and $\mathbb{E}$ instead of $W^{1,p(x)}(\Omega)$, $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ respectively.

We refer to [13, 18, 19] for basic information about the variable exponent spaces but we recall in the following some of their relevant properties:

(i) The space $(E, |.|_E)$ is a separable, uniform convex Banach space and its conjugate space is $E' := \mathbf{L}^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. Moreover, for all $u \in E$ and $v \in E'$, we have

$$\left|\int_\Omega uv dx\right| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_E |u|_{E'},$$

which is the Hölder inequality for variable exponents.

(ii) If $p_1, p_2 \in C(\overline{\Omega})$ and $1 < p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then the embedding $\mathbf{L}^{p_2(x)}(\Omega) \hookrightarrow \mathbf{L}^{p_1(x)}(\Omega)$ is continuous.

(iii) For all $u \in E$ we have

$$\min(|u|_E^{p^+}, |u|_E^{p^-}) \leq \int_\Omega |u|^{p(x)} dx \leq \max(|u|_E^{p^+}, |u|_E^{p^-}).$$

(iv) X is a separable, reflexive Banach space.

(v) There is a compact and continuous embedding $X \hookrightarrow\hookrightarrow \mathbf{L}^{q(.)}(\Omega)$, where $q \in \mathbf{C}(\overline{\Omega})$, $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

(vi) There is a constant $C > 0$, such that

$$|u|_E \leq C|\nabla u|_E \quad \text{for all } u \in E.$$

Thus we can use $|\nabla u|_E$ as an equivalent norm of u that is $\|u\|_X$.

(vii) If Ω has a smooth boundary $\partial\Omega$ and $p \in C(\overline{\Omega})$ then there exists a compact embedding $X \hookrightarrow\hookrightarrow \mathbf{L}^{q(.)}(\partial\Omega)$, where $q \in C(\partial\Omega)$, $1 \leq q(x) < p^{\partial}(x)$ for any $x \in \partial\Omega$ and

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}; & p(x) < N \\ \infty; & p(x) \geq N. \end{cases}$$

As pointed out in [19, pp. 8-9], the function spaces with variable exponent have some striking properties, such as:

(i) If $1 < p^- \leq p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the mean continuity property. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \leq C\,|f|_{p(x)}\,\|g\|_{L^1}$$

holds if and only if p is constant.

2.1. **Lusternik-Schnirelmann principle.** We recall a version of the L-S principle, which was discussed in Browder [7] and Zeidler [26, 27]. Let X be a real reflexive Banach space and F, G be two functionals on X. Consider the following eigenvalue problem:

$$F'(u) = \mu G'(u); \qquad u \in S_G,\ \mu \in \mathbb{R}, \tag{1}$$

where $S_{G,\alpha} = \{u \in X; G(u) = \alpha\}$ is the α-level set of G for $\alpha > 0$.

Then u is a solution of (1), if and only if u is critical point of F with respect to $S_{G,\alpha}$.

We assume that the following hypotheses are fulfilled:

(LS1) F, G are even functionals and $F, G \in C^1(X, \mathbb{R})$ with $F(0) = G(0) = 0$.

(LS2) F' is strongly continuous, that is, $u_n \rightharpoonup u$ in X implies $F'(u_n) \to F'(u)$. Moreover, $\langle F'(u), u\rangle = 0$, $u \in \overline{\operatorname{co} S_{G,\alpha}}$ implies $F(u) = 0$, where $\overline{\operatorname{co} S_{G,\alpha}}$ is the closed convex hull of $S_{G,\alpha}$.

(LS3) G' is continuous, bounded and satisfies (S_0) condition, that is,

$$u_n \rightharpoonup u, \quad G'(u_n) \rightharpoonup v, \quad \langle G'(u_n), u_n\rangle \to \langle v, u\rangle \quad \text{imply} \quad u_n \to u.$$

(LS4) The level set $S_{G,\alpha}$ is bounded and $u \neq 0$ implies

$$\langle G'(u), u\rangle > 0, \qquad \lim_{t \to +\infty} G(tu) = +\infty, \qquad \inf_{u \in S_{G,\alpha}} \langle G'(u), u\rangle > 0.$$

For $n \in \mathbb{N}$, let $\mathbf{A}_n$ denote the class of all compact, symmetric subsets K of $S_{G,\alpha}$ such that $F(u) > 0$ on K and

$$\gamma(K) := \inf\{k \in \mathbb{N}; \exists h : K \to \mathbb{R}^k \backslash \{0\} \text{ such that } h \text{ is continuous and odd}\} \geq n.$$

Let

$$a_{n,\alpha} := \begin{cases} \sup_{\mathbf{H}\in\mathbf{A}_n} \inf_{u\in\mathbf{H}} F(u) & \text{if } \mathbf{A}_n \neq \emptyset \\ 0; & \text{if } \mathbf{A}_n = \emptyset. \end{cases}$$

We now present L-S principle and we refer to Zeidler [26] for more details.

Theorem 2.1. *Under assumptions (LS1)-(LS4), the following assertions hold:*

(i) If $a_{n,\alpha} > 0$, then problem (1) possesses a pair $(\pm u_{n,\alpha}, \mu_{n,\alpha})$ of eigenpairs such that $F(u_{n,\alpha}) = a_{n,\alpha}$ and $\mu_{n,\alpha} = \frac{<F'(u_{n,\alpha}),u_{n,\alpha}>}{<G'(u_{n,\alpha}),u_{n,\alpha}>}$.

(ii) For all α we have $\infty > a_{1,\alpha} \geq a_{2,\alpha} \geq \ldots \geq 0$ and $a_{n,\alpha} \to 0$ as $n \to \infty$.

Definition 2.2. The functional T is called conjugational function on a Banach space E provided that for any $u \in E$ we have $T(u) \in E'$, where E' is the conjugate (dual) space of E.

For example, if $u \in L^p(\Omega)$ then $T(u) = |u|^{p-1} \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$, so T is a conjugational function on $L^p(\Omega)$.

Let us consider the following assumptions:

(1) $a(x,\xi) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $b(x,y) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $c(x,y) : \overline{\partial\Omega} \times \mathbb{R} \to \mathbb{R}$ are the continuous derivatives with respect to their second variable of the mappings A, B, C respectively; namely, there exist $A(x,\xi) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, $B(x,y) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $C(x,y) : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ such that $a(x,\xi) = \nabla_\xi A(x,\xi)$, $b(x,y) = \frac{\partial B}{\partial y}$ and $c(x,y) = \frac{\partial C}{\partial y}$.

(2) A, B, C are even with respect to their second component and $A(x,0) = B(x,0) = C(x,0) = 0$. Moreover, A and C are coercive with respect to second component, that is, $\lim_{t\to\infty} A(x,t\xi) = \lim_{t\to\infty} C(x,ty) = \infty$ for any $\xi \in \mathbb{R}^N$ and $y \in \mathbb{R}$.

(3) $a(x,\xi)$ is conjugational function with respect to ξ on E and $b(x,y), c(x,y)$ are continuous conjugational on the Banach spaces E and $\mathbb{E}$ respectively.

(4) There exists a positive constant γ such that $a(x,\xi) \cdot \xi \geq \gamma A(x,\xi) \geq 0$, $b(x,y)y \geq \gamma B(x,y) \geq 0$ and $c(x,y)y \geq \gamma C(x,y) \geq 0$; for any $x, y \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

(5) There exists a function $r = r(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ where for any fixed $x \in \Omega$ we have $p(x) = O(r(x,t))$, (that is, $\lim_{t\longrightarrow 0} \frac{r(x,t)}{t^{p(x)}} \neq 0$) and

$$A(x, \frac{\xi+\psi}{2}) \leq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\psi) - r(x, |\xi - \psi|),$$

for all $x \in \overline{\Omega}$ and $\xi, \psi \in \mathbb{R}^N$.

(6) $\beta : \partial\Omega \longrightarrow [0, +\infty)$ belongs to $L^\infty(\partial\Omega)$.

3. **Main existence result.** A pair $(u,\lambda) \in X\backslash\{0\} \times \mathbb{R}$ is called eigenpair of problem (R) if for all $v \in X$,

$$\int_\Omega a(x,\nabla u)\nabla v dx + \int_{\partial\Omega} \beta(x)c(x,u)v ds = \lambda \int_\Omega b(x,u)v dx. \tag{2}$$

In this section, by using the Lusternik-Schnirelmann principle, we find an unbounded sequence of eigenpairs for problem (R), as it is stated the following result.

Theorem 3.1. *Consider problem* (R) *and assume that hypotheses* (1)-(6) *are fulfilled. Then for any* $\alpha > 0$ *there exists a nondecreasing sequence of nonnegative eigenvalue* $\{\lambda_{n,\alpha}\}$ *of* (R) *such that*

$$\lambda_{n,\alpha} = \frac{1}{\mu_{n,\alpha}} \to \infty \quad \text{as } n \to \infty.$$

Moreover, every $\mu_{n,\alpha}$ *is an eigenvalue of the corresponding equation* $F'(u) = \mu G'(u)$, *where*

$$F(u) = \int_\Omega B(x,u)dx, \quad G(u) = \int_\Omega A(x,\nabla u)dx + \int_{\partial\Omega} \beta(x)C(x,u)ds. \tag{3}$$

Lemma 3.2. *Let* F *as defined in (3). Then* F' *satisfies hypotheses* $(LS1)$ *and* $(LS2)$.

Proof. Let $u_n \rightharpoonup u$, hence by the compact embedding $X \hookrightarrow\hookrightarrow E$ we obtain $u_n \to u$ in E. Thus, by the variable exponent Hölder inequality combined with hypothesis (3) we have

$$\begin{aligned}|\langle F'(u_n) - F'(u), v\rangle| &\le \int_\Omega |(b(x,u_n) - b(x,u))v|dx \\ &\le \kappa \|b(x,u_n) - b(x,u)\|_{E'} \|v\|_E \to 0.\end{aligned}$$

Moreover by condition (4), we conclude the entire assertions. □

The proof of the following lemma is inspired by the proof of Lemma 3.11 from [16].

Lemma 3.3. *Let* G *be defined in (3). Then hypothesis* $(LS3)$ *is satisfied.*

Proof. Using the variable exponent Hölder inequality combined with the compact embedding $X \hookrightarrow\hookrightarrow \mathbb{E}$ and hypothesis (3) we have

$$\begin{aligned}\|G'\|_{X*} &= \sup\{|\langle G'(u), v\rangle|; \|v\| \le 1\} \\ &\le \sup|\int_\Omega a(x,\nabla u)\nabla v dx + \int_{\partial\Omega} \beta(x)c(x,u)v ds| \\ &\le \kappa\|a(x,\nabla u)\|_{E'}\|v\|_E + \beta^+\|c(x,u)\|_{\mathbb{E}'}\|v\|_{\mathbb{E}} \\ &\le \kappa\|a(x,\nabla u)\|_{E'}\|v\|_E + \kappa\beta^+\|c(x,u)\|_{\mathbb{E}'}\|v\| < \infty.\end{aligned}$$

It follows that G' is bounded.

Next, we suppose that $u_n \rightharpoonup u$, $G'(u_n) \rightharpoonup v$ and $\langle G'(u_n), u_n\rangle \to \langle v, u\rangle$ for some $v \in X^*$ and $u \in X$ and we prove that $u_n \to u$. Since X is uniformly convex Banach space, a weak convergence and norm convergence imply strong convergence in the norm topology. Therefore to prove that $u_n \to u$ in X, we only need to show that $\|u_n\| \to \|u\|$. By Sobolev compact embedding we have $u_n \to u$ in E and so

$$\langle G'(u_n) - G'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

We have

$$\begin{aligned}\langle G'(u_n) - G'(u), u_n - u\rangle = &\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u))(\nabla u_n - \nabla u)dx \\ &+ \int_{\partial\Omega} \beta(x)(c(x,u_n) - c(x,u))(u_n - u)ds.\end{aligned}$$

By hypothesis (3) the second term in the right-hand side of the above relation tends to zero and hence

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u))(\nabla u_n - \nabla u)dx \to 0. \tag{4}$$

Fix $\varepsilon > 0$. By hypothesis (5) we deduce that $u \mapsto \int_\Omega A(x, \nabla u)dx$ is a convex functional. Therefore

$$\begin{aligned}\int_\Omega A(x,\nabla v)dx &\geq \int_\Omega A(x,\nabla u)dx + \int_\Omega a(x,\nabla u)(\nabla v - \nabla u)dx \\ &\geq \int_\Omega A(x,\nabla u)dx - \int_\Omega |a(x,\nabla u)||(\nabla v - \nabla u)|dx \\ &\geq \int_\Omega A(x,\nabla u)dx - \kappa \|a(x,\nabla u)\|_{E'} \|\|\nabla v - \nabla u\|_E \\ &\geq \int_\Omega A(x,\nabla u)dx - \kappa' \|u - v\| \geq \int_\Omega A(x,\nabla u)dx - \varepsilon,\end{aligned} \tag{5}$$

for all $v \in X$ with $\|u - v\| < \frac{\varepsilon}{\kappa'}$. The positive constants κ, κ' are derived from Hölder's inequality and the boundedness of $a(x, \nabla u)$ in E'. We conclude that $u \mapsto \int_\Omega A(x, \nabla u)dx$ is lower semi-continuous and so weakly lower semi-continuous. Thus by definition, if $u_n \rightharpoonup u$ in X we have

$$\int_\Omega A(x,\nabla u)dx \leq \liminf \int_\Omega A(x,\nabla u_n)dx.$$

On the other hand, from (4) and (5) we conclude that

$$\int_\Omega A(x,\nabla u)dx = \lim \int_\Omega A(x,\nabla u_n)dx.$$

Taking into account that $\frac{u_n+u}{2} \rightharpoonup u$ in X and using the previous argument we obtain

$$\int_\Omega A(x,\nabla u)dx \leq \liminf \int_\Omega A(x,\nabla(\frac{u_n+u}{2}))dx. \tag{6}$$

If u_n were not converge strongly in X, there exist $\varepsilon > 0$ and a subsequence of u_n, still denoted by u_n, such that $\|u_n - u\| > \varepsilon$. Next, by assumption (5) we deduce that

$$\begin{aligned}&\frac{1}{2}\int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{2} A(x,\nabla u_n)dx \\ &\geq \int_\Omega A(x, \frac{\nabla u_n + \nabla u}{2})dx + \int_\Omega r(x, |\nabla u_n - \nabla u|)dx \\ &\geq \int_\Omega A(x, \frac{\nabla u_n + \nabla u}{2})dx + \kappa_0 \int_\Omega (\nabla u_n - \nabla u)^{p(x)}dx \\ &\geq \int_\Omega A(x, \tfrac{\nabla u_n + \nabla u}{2})dx + \kappa_0 \varepsilon^{p_0},\end{aligned}$$

where p_0 is equal to p^- if $\|u_n - u\| > 1$ and is p^+ in otherwise. Taking $n \to \infty$ we deduce that

$$\int_\Omega A(x,\nabla u)dx - \kappa_0\varepsilon^{p_0} \geq \limsup \int_\Omega A(x, \frac{\nabla u_n + \nabla u}{2})dx,$$

which contradicts (6). It follows that u_n converges strongly to u in X, so hypothesis $(LS3)$ is fulfilled. □

Proof of Theorem 3.1. In the view of conditions (3) and (4) we see that the functionals F, G are well defined on X and that they belong to $C^1(X, \mathbb{R})$ with the derivatives given by

$$\langle F'(u), v\rangle = \int_\Omega b(x,u)v dx,$$

$$\langle G'(u), v\rangle = \int_\Omega a(x,\nabla u)\nabla v dx + \int_{\partial\Omega} \beta(x)c(x,u)v ds,$$

for all $v \in X$.

Then $S_{G,\alpha}$ is bounded and for $u \neq 0$, $\langle G'(u), u\rangle > 0$, $\lim_{t\to+\infty} G(tu) = +\infty$ and $\inf_{u\in S_{G,\alpha}} \langle G'(u), u\rangle > 0$.

The eigenvalue problem $F'(u) = \mu G'(u)$ is equivalent to

$$\int_\Omega b(x,u)vdx = \mu(\int_\Omega a(x,\nabla u)\nabla v dx + \int_{\partial\Omega} \beta(x)c(x,u)vds)$$

for any $v \in X$. Equivalently, we have for all $v \in X$

$$\int_\Omega a(x,\nabla u)\nabla v dx + \int_{\partial\Omega} \beta(x)c(x,u)vds = \frac{1}{\mu}\int_\Omega b(x,u)vdx.$$

Finally, by Lemmas 3.2 and 3.3 combined with Theorem 2.1, we conclude the proof. $\square$

4. **A special case in the $p(x)$-Laplacian class.** One of the most important eigenvalue problems are those concerned with the p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

with different boundary conditions. We now consider the more general differential $p(x)$-Laplace operator defined as

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

Nonlinear eigenvalue problems for $p(x)$-Laplacian operator similar to eigenvalue problems for p-Laplacian with Dirichlet, Neumann and Steklov boundary condition have been investigated previously, see, e.g., [8, 9, 10, 19].

In this section, we consider the following class of degenerate $p(x)$-Laplacian problems with Robin boundary condition

$$(P1) \qquad \begin{cases} -\operatorname{div}(a_0(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda b_0(x)|u|^{q(x)-2}u & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial n} + \beta(x)|u|^{r(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q : \Omega \to [1,\infty)$ belong to $C(\overline{\Omega})$, such that $1 \le p^- \le p^+ < \infty$ and $1 \le q(x) < p^*(x)$ for all $x \in \Omega$. We assume that $r : \partial\Omega \to [1,\infty)$ belongs to $C(\partial\Omega)$ such that $1 \le r(x) < p^\partial(x)$ for all $x \in \partial\Omega$. The potentials $a_0, b_0 : \Omega \to [0,\infty)$ and $\beta : \partial\Omega \to [0,\infty)$ belong to $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ respectively.

It is obvious that by setting $a(x,\xi) := a_0(x)|\xi|^{p(x)-2}\xi$, $b(x,y) := b_0(x)|y|^{q(x)-2}y$ and $c(x,y) := |y|^{r(x)-2}y$ in problem (R), conditions (1)-(5) are satisfied.

We check in what follows hypothesis $(LS3)$. For this purpose, we first recall the following Simon inequalities [24]

$$\begin{cases} (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \ge \dfrac{1}{2^p}|\xi-\eta|^p & \text{if } p \ge 2 \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta)(|\xi|+|\eta|)^{2-p} \ge (p-1)|\xi-\eta|^p & \text{if } 1 < p < 2. \end{cases} \tag{7}$$

We refer to [11] for some applications of Simon's inequalities.

Next, applying Simon's inequalities to

$$I_n(x) := (a(x,\nabla u_n) - a(x,\nabla u))(\nabla u_n - \nabla u),$$

we obtain

$$\begin{aligned} I_n(x) \ge & \left(\frac{a_0(x)}{2^{p(x)}}|\nabla u_n - \nabla u|^{p(x)}\right)\chi_{\mathbf{\Omega}^+}(x) \\ & + a_0(x)(p(x)-1)\frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}}\chi_{\mathbf{\Omega}^-}(x), \end{aligned} \tag{8}$$

where $\Omega^+ = \{x \in \Omega;\ p(x) \geq 2\}$ and $\Omega^- = \Omega \backslash \Omega^+$. Consequently, for some positive constant κ we have

$$\int_{\Omega^+} |\nabla u_n - \nabla u|^{p(x)} dx \leq \kappa \int_\Omega I_n(x) dx. \tag{9}$$

From the last term in (8), we get

$$\int_{\Omega^-} |\nabla u_n - \nabla u|^{p(x)} dx \leq \kappa \int_{\Omega^-} I_n(x)^{\frac{p(x)}{2}} (|\nabla u_n| + |\nabla u|)^{(2-p(x))\frac{p(x)}{2}} dx =: \kappa J.$$

Since $\lim_{n\to\infty} I_n = 0$, we can assume that $0 \leq \int_\Omega I_n(x)dx < 1$. By Young's inequality, we obtain

$$J = \int_{\Omega^-} I_n(x)^{\frac{p(x)}{2}} (\int_\Omega I_n(y)dy)^{-\frac{p(x)}{2}} (|\nabla u_n| + |\nabla u|)^{(2-p(x))\frac{p(x)}{2}} (\int_\Omega I_n(y)dy)^{\frac{p(x)}{2}} dx$$

$$\leq (\int_\Omega I_n(y)dy)^{\frac{1}{2}} \int_{\Omega^-} \frac{p(x)}{2} (I_n(x)(\int_\Omega I_n(y)dy)^{-1}) + \frac{2-p(x)}{2} (|\nabla u_n| + |\nabla u|)^{p(x)} dx$$

$$\leq (\int_\Omega I_n(y)dy)^{\frac{1}{2}} (1 + \int_\Omega (|\nabla u_n| + |\nabla u|)^{p(x)} dx).$$

Hence by inserting (9), we obtain

$$\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \leq \kappa \left(\int_\Omega I_n(y)dy\right)^{1/2} \left(1 + \int_\Omega (|\nabla u_n| + |\nabla u|)^{p(x)} dx\right).$$

Thus $\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \to 0$ as $n \to 0$, and so $u_n \to u$ in X.

By the above argument we have the following result.

Theorem 4.1. *Consider problem (P_1) with the above mentioned assumptions on p, q, r, a_0, b_0, β. Then for any $\alpha > 0$ there exists a nondecreasing sequence of non-negative eigenvalue for (P_1), named $\{\lambda_{n,\alpha}\}$ such that $\lambda_{n,\alpha} \to \infty$ as $n \to \infty$.*

5. The set of eigenvalues.

Set

$$\Lambda := \{\mu \in \mathbb{R}; \mu \text{ is an eigenvalue of } F'(u) = \mu G'(u)\}.$$

Then

$$\Lambda = \bigcup_{n=1}^{\infty} \bigcup_{\alpha>0} \mu_{n,\alpha};$$

where for any $\alpha > 0$, $\{\mu_{n,\alpha}\}_{n=1}^\infty$ is the eigenvalue sequence of $F'(u) = \mu G'(u)$ in α-level set of G.

In all eigenvalue problems, the infimum of the eigenvalue set Λ is a central topic in nonlinear spectral theory. In the process of finding eigenvalues by (L-S) principle, if all eigenvalues $\mu_{n,\alpha}$ are independent of α, we have

$$0 = \inf \Lambda < \max \Lambda = \mu_{1,\alpha} =: \mu_1. \tag{10}$$

However, in the general case we do not know if $\Lambda^* := \sup \Lambda < \infty$. In fact,

$$\Lambda^* = \sup_{\alpha>0} \mu_{1,\alpha}. \tag{11}$$

Now consider the eigenvalue set of problem (R), which is denoted by Λ_R. We observe that for $\lambda \in \Lambda_R$ then there exists $u = u_\lambda \in X\backslash\{0\}$ such that

$$\lambda = \frac{\int_\Omega a(x,\nabla u)\nabla u dx + \int_{\partial\Omega} \beta(x)c(x,u)u ds}{\int_\Omega b(x,u)u dx}\,. \tag{12}$$

Moreover, there exist $n \in \mathbb{N}$ and $\alpha > 0$ such that $\lambda = \lambda_{n,\alpha}$, $u = u_\lambda \in S_{G,\alpha}$, that is,

$$\int_\Omega A(x,\nabla u)dx + \int_{\partial\Omega} \beta(x)C(x,u)ds = \alpha\,.$$

Thus, by taking into account assumption (4) we have

$$\lambda = \lambda_{n,\alpha} \geq \gamma \frac{\alpha}{\int_\Omega b(x,u)u dx}\,. \tag{13}$$

Since the last term is dependent on α then we cannot conclude $\inf \Lambda_R$ is equal to zero or not. On the other hand, it is obvious that $\sup \Lambda_R = \infty$, by the first equality in (10).

It is known that in the case $p = q = r \equiv$constant we have $\inf \Lambda_{P_1} > 0$, see [14].

In the following we present some sufficient conditions for problem (P_1) in order to obtain that $\inf \Lambda_{P_1}$ is either zero or positive. Let us to refer [8, 9], where related arguments are provided. We start with the following preliminary result.

Lemma 5.1. *Consider problem (P_1) and the set*

$$\Pi := \left\{ \frac{\int_\Omega a_0(x)|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{r(x)}ds}{\int_\Omega b_0(x)|u|^{q(x)}dx};\ u \in X\backslash\{0\} \right\}.$$

Then $\inf \Lambda_{P_1} > 0$ if and only if $\inf \Pi > 0$.

Proof. The necessity is obvious, since by (12) we have $\Lambda_{P_1} \subseteq \Pi$ and so $\inf \Pi \leq \inf \Lambda_{P_1}$.

Recall that in problem (P_1) we have

$$\langle F'(u), v\rangle = \int_\Omega b_0(x)|u|^{q(x)-2}uv dx$$

and

$$\langle G'(u), v\rangle = \int_\Omega a_0(x)|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{r(x)-2}uv ds,$$

where $F(u) = \int_\Omega \frac{b_0(x)}{p(x)}|u|^{q(x)}dx$ and $G(u) = \int_\Omega \frac{a_0(x)}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)}|u|^{p(x)}ds$. Let $\overline{\Pi} := \{\frac{G(u)}{F(u)}; u \in X\backslash\{0\}\}$. Note that if $\inf \Pi = 0$ then $\inf \overline{\Pi} = 0$, since

$$\frac{q^-}{\max\{p^+, r^+\}} \inf \Pi \leq \inf \overline{\Pi} \leq \frac{q^+}{\min\{p^-, r^-\}} \inf \Pi.$$

So for any $\varepsilon > 0$ there exists $u_\varepsilon \in X\backslash\{0\}$ such that $\frac{G(u_\varepsilon)}{F(u_\varepsilon)} < \varepsilon$. Let $G(u_\varepsilon) = \alpha$. By the definition of $a_{1,\alpha}$ we have $\frac{\alpha}{a_{1,\alpha}} \leq \frac{G(u_\varepsilon)}{F(u_\varepsilon)} < \varepsilon$ and so $\lambda_{1,\alpha} \leq \frac{\max\{p^+,r^+\}}{q^-}\varepsilon$. Moreover from (11), we know that $\inf \Lambda_{P_1} = \inf_{\alpha>0} \lambda_{1,\alpha}$. Since ε is arbitrary we conclude that $\inf \Lambda_{P_1} = 0$. $\square$

Proposition 1. *Consider problem (P_1) and suppose that there exists an open subset U of Ω such that p has either a local minimum or a local maximum at $x_0 \in U$ and $p(x_0) > q(x_0)$. Moreover, for some $\varepsilon_r, \varepsilon_q > 0$, we assume that $r^-_{B(\partial U,\varepsilon_r)\cap\partial\Omega} > q^+_{B(x_0,\varepsilon_q)}$. Then $\inf \Lambda_{P_1} = 0$.*

Proof. Assume that p has a local minimum at $x_0 \in U$, the other case being similar. Since $p(x_0) > q(x_0)$, there exists $\varepsilon_0 > 0$ such that $p(x_0) - q(x_0) > \frac{\varepsilon_0}{2}$. Without loss of generality, suppose that $\overline{U} \subset \Omega$ and $p(x) > p(x_0)$ for all $x \in \partial U$.

So there exists $\varepsilon_p > 0$ such that

$$p(x) - \varepsilon_0 > p(x_0) \quad \text{for all } x \in B(\partial U, \varepsilon_p). \tag{14}$$

Also, there exists $\varepsilon'_q > 0$ such that

$$|q(x) - q(x_0)| < \frac{\varepsilon_0}{2} \quad \text{for all} \;\; x \in B(x_0, \varepsilon'_q). \tag{15}$$

Define $u_0 \in C^\infty(\Omega)$ with $|\nabla u_0| \le C_0$ and $0 \le u_0(x) \le 1$ for any $x \in \Omega$ where $u_0(x) = 0$ for $x \notin U \cup B(\partial U, \varepsilon)$ and $u_0(x) = 1$ on $U \backslash B(\partial U, \varepsilon)$; where $\varepsilon = \min(\varepsilon_p, \varepsilon_r)$. Obviously, for any $\alpha > 0$ there exists $t > 0$ such that $tu_0 \in S_{G,\alpha}$, i.e., $G(tu_0) = \alpha$; and $t \to 0$ as $\alpha \to 0$.

Using (12), we have

$$\begin{aligned}
\inf \Lambda_{P_1} \le \lambda_{1,\alpha} \le \kappa \frac{F(tu_0)}{G(tu_0)} &\le \kappa \frac{\int_\Omega a_0(x) |\nabla t u_0|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |tu_0|^{r(x)} ds}{\int_\Omega b_0(x) |tu_0|^{q(x)} dx} \\
&= \kappa \frac{\int_{B(\partial U,\varepsilon)\cap\Omega} a_0(x) t^{p(x)} |\nabla u_0|^{p(x)} dx + \int_{\partial\Omega \cap B(\partial U,\varepsilon)} \beta(x) t^{r(x)} |u_0|^{r(x)} ds}{\int_{(U \cup B(\partial U,\varepsilon))\cap\Omega} b_0(x) t^{q(x)} |u_0|^{q(x)} dx} \\
&\le \kappa \frac{t^{p(\xi_1)} \int_{B(\partial U,\varepsilon)\cap\Omega} a_0(x) |\nabla u_0|^{p(x)} dx + t^{r(\xi_2)} \int_{\partial\Omega\cap B(\partial U,\varepsilon)} \beta(x) |u_0|^{r(x)} ds}{t^{q(\xi_3)} \int_{B(x_0,\varepsilon_1)} b_0(x) |u_0|^{q(x)} dx} \\
&\le \kappa \frac{C_1 a_0^+ (C_0 t)^{p(\xi_1)} + t^{r(\xi_2)} \int_{\partial\Omega\cap B(\partial U,\varepsilon_1)} \beta(x) |u_0|^{r(x)} ds}{C_3 b_0^- t^{q(\xi_3)}},
\end{aligned}$$

for some positive constants κ, $\xi_1 \in B(\partial U, \varepsilon) \cap \Omega$, $\xi_2 \in \partial\Omega \cap B(\partial U, \varepsilon)$, $\xi_3 \in B(x_0, \varepsilon_1)$, $C_1 = |B(\partial U, \varepsilon) \cap \Omega|$, $C_3 = |B(x_0, \varepsilon_1)|$ and $\varepsilon_1 = \min(\varepsilon_q, \varepsilon'_q)$. Using (14) and (15) we obtain $p(\xi_1) - q(\xi_3) > \varepsilon_0$. Moreover $r(\xi_2) - q(\xi_3) > 0$, Thus we get

$$\lambda_{1,\alpha} \le C t^{p(\xi_1) - q(\xi_3)} + C' t^{r(\xi_2) - q(\xi_3)} \to 0$$

as $t \to 0$, which this complete the proof. $\square$

Theorem 5.2. *Consider problem (P_1) and suppose that for an open subset U of Ω and a compact subset V of U with positive measure we have $q^+(V) < p^-(U \backslash V)$. Then $\lambda_{1,\alpha} \to 0$ as $\alpha \to 0$ and consequently* $\inf \Lambda_{p_1} = 0$.

Proof. For any $\delta > 0$ define $U_\delta := \{x \in U; \mathrm{dist}(x, \partial U) < \delta\}$. So there exists $\delta > 0$ such that $V \cap U_\delta = \emptyset$ and $q^+(V) < p^-(U_\delta)$. Let $u_0 \in X$ such that $u_0(x) = 0$ for $x \in \Omega \backslash U$ and $u_0(x) = 1$ for $x \in U \backslash U_\delta$. Moreover, for all $\alpha > 0$ there exists $t = t_\alpha$ such that $tu_0 \in S_{G,\alpha}$ and $t \to 0$ as $\alpha \to 0$. So we have

$$\begin{aligned}
\lambda_{1,\alpha} \le& \frac{\int_\Omega a_0(x) |\nabla t u_0|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |tu_0|^{r(x)} ds}{\int_\Omega b_0(x) |tu_0|^{q(x)} dx} \\
\le& \frac{t^{p^-(U_\delta)} \int_{U_\delta} a_0(x) |\nabla u_0|^{p(x)} dx}{t^{q^+(V)} \int_V b_0(x) |u_0|^{q(x)} dx} \to 0;
\end{aligned}$$

as $t \to 0$. So the proof is complete. $\square$

By a similar method we can derive the following theorem.

Theorem 5.3. *Consider problem (P_1) and suppose that for an open subset U of Ω and a compact subset V of U with positive measure we have $q^-(V) > p^+(U\backslash V)$. Then $\lambda_{1,\alpha} \to 0$ as $\alpha \to \infty$ and consequently $\inf \Lambda_{P_1} = 0$.*

Denote $(\widetilde{P_1})$ for problem (P_1) when $q(\cdot) := p(\cdot)$.

Proposition 2. *(i) If $N = 1$ and p is monotone then $\inf \Lambda_{\widetilde{P_1}} > 0$.*

(ii) If $N > 1$ and there is a vector $l \in \mathbb{R}^N\backslash\{0\}$ such that for any $x \in \Omega$ the function $f(t) = p(x + tl)$ is monotone for $t \in I_x = \{t; x + tl \in \Omega\}$, then $\inf \Lambda_{\widetilde{P_1}} > 0$.

Proof. We have

$$\frac{\int_\Omega a_0(x)|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{r(x)}ds}{\int_\Omega b_0(x)|u|^{q(x)}dx} \geq \frac{\int_\Omega a_0(x)|\nabla u|^{p(x)}dx}{\int_\Omega b_0(x)|u|^{q(x)}dx}.$$

By Theorems 3.2 and 3.3 of [10], we have

$$\inf_{u\in X\backslash\{0\}} \frac{\int_\Omega a_0(x)|\nabla u|^{p(x)}dx}{\int_\Omega b_0(x)|u|^{q(x)}dx} > 0,$$

so we conclude the assertions by Lemma 5.1. □

Definition 5.4. An eigenvalue λ is called principal eigenvalue if there exists a nonnegative eigenfunction corresponding to λ, that is, there exists $u \in X\backslash\{0\}$ such that $u \geq 0$ and (u, λ) is a solution of corresponding problem. Denote the set of principal eigenvalues by Λ^+.

In the case where p, q, r are constant functions in problem (P_1), it is well known that the first eigenvalue is principal, in fact if u is a first eigenfunction, so is $|u|$. When we deal with variable exponent case we consider the set $\Lambda^1 P_1 := \{\lambda_{1,\alpha}; \alpha > 0\}$. Then for any $\lambda \in \Lambda^1 P_1$ there exists $u_{\lambda,\alpha} \in S_{G,\alpha}$ with $F(u_{\lambda,\alpha}) = \sup_{v\in S_{G,\alpha}} F(v)$.

It is obvious that for any $\alpha > 0$, the function $|u_{\lambda,\alpha}|$, where $u_{\lambda,\alpha}$ is corresponding eigenfunction to $\lambda_{1,\alpha} \in \Lambda^1 P_1$, is also eigenfunction corresponding to $\lambda_{1,\alpha}$ and so every element of $\Lambda^1 P_1$ also belongs to $\Lambda^+ P_1$ (the set of principal eigenvalues of problem P_1)

Proposition 3. *Suppose that $\inf \Lambda^+_{\widetilde{P_1}} > 0$ and there exists $\varepsilon > 0$ such that $q(x)+\varepsilon \leq p(x)$ for a.e. $x \in \Omega$. Then $\Lambda^+_{P_1} = (0, \infty)$.*

Proof. We show that for all $\lambda > 0$ problem (P_1) has a nontrivial, nonnegative solution u_λ. Since $\inf \Lambda^+_{\widetilde{P_1}} > 0$, we obtain

$$\vartheta := \inf \frac{\int_\Omega \frac{a_0(x)}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)}|u|^{r(x)}ds}{\int_\Omega \frac{b_0(x)}{p(x)}|u|^{p(x)}dx} > 0\,. \tag{16}$$

Moreover, by our hypotheses we deduce that there exists $R > 0$ large enough such that

$$\frac{\lambda|t|^{q(x)}}{q(x)} < \frac{\vartheta|t|^{p(x)}}{2p(x)} \quad \text{for } |t| > R \ \text{ and } \ \text{a.e. } x \in \Omega. \tag{17}$$

Hence by (16) and (17), the energy functional I_λ corresponding to problem (P_1) is coercive, that is, $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$. Indeed, for all $u \in X$,

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{a_0(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)} |u|^{r(x)} ds - \lambda \int_\Omega \frac{b_0(x)}{q(x)} |u|^{q(x)} dx \\
&= \int_\Omega \frac{a_0(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)} |u|^{r(x)} ds \\
&\quad - \lambda \int_{\Omega_R^+} \frac{b_0(x)}{q(x)} |u|^{q(x)} dx - \lambda \int_{\Omega_R^-} \frac{b_0(x)}{q(x)} |u|^{q(x)} dx \\
&\geq \int_\Omega \frac{a_0(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)} |u|^{r(x)} ds - \frac{\vartheta}{2} \int_\Omega \frac{b_0(x)}{p(x)} |u|^{p(x)} dx - C \\
&\geq \frac{1}{2} \left(\int_\Omega \frac{a_0(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{r(x)} |u|^{r(x)} ds \right) - C \\
&\geq \frac{1}{2} \int_\Omega \frac{a_0(x)}{p(x)} |\nabla u|^{p(x)} dx - C,
\end{aligned}$$

where C is positive constant, $\Omega_R^+ = \Omega \cap \{x; |u(x)| > R\}$ and $\Omega_R^- = \Omega \cap \{x; |u(x)| \leq R\}$. It follows that I_λ has a global minimizer u_0. Moreover, by Theorem 5.2, we have $\inf \Lambda_{P_1} = 0$ and so $\inf \overline{\Pi} = 0$. Thus, there exists $u_\lambda \in X \backslash \{0\}$ such that $\frac{G(u_\lambda)}{F(u_\lambda)} < \lambda$ and hence $I_\lambda(u_\lambda) = G(u_\lambda) - \lambda F(u_\lambda) < 0$. We deduce that $I_\lambda(u_0) < 0$ and thus $u_0 \neq 0$. Since $|u_0|$ is also a global minimizer of I_λ, we get $\lambda \in \Lambda_{P_1}^+$, and so $\Lambda_{P_1}^+ = (0, \infty)$. □

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