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Existence of solutions for a bi-nonlocal fractional p-Kirchhoff type problem

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ABSTRACT

In this paper, we are concerned with the existence of nonnegative solutions for a p-Kirchhoff type problem driven by a non-local integro-differential operator with homogeneous Dirichlet boundary data. As a particular case, we study the following problem

$$M\left(x, [u]_{s,p}^p\right)(-\Delta)_p^s u = f\left(x, u, [u]_{s,p}^p\right) \ \text{ in } \Omega,$$
$$u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega, \quad [u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx\, dy,$$

where $(-\Delta)_p^s$ is a fractional p-Laplace operator, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $M : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function satisfying the Ambrosetti–Rabinowitz type condition. The existence of nonnegative solutions is obtained by using the Mountain Pass Theorem and an iterative scheme. The main feature of this paper lies in the fact that the Kirchhoff function M depends on $x \in \Omega$ and the nonlinearity f depends on the energy of solutions.

1. Introduction and main results

In recent years, a great deal of attention has been paid to the study of problems involving fractional and nonlocal operators, both in the pure mathematical research and in the concrete real-world applications, such as, optimization, finance, continuum mechanics, phase transition phenomena, population dynamics, and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [1,2] and references therein. Especially, the fractional Laplacian operators of the form $(-\Delta)^s$ can be viewed as the infinitesimal generators of stable Lévy processes, see for instance [3]. Some interesting topics concerning the fractional Laplacian, such as, the nonlinear fractional Schrödinger equation (see [2,4,5]), the fractional porous medium equation (see [6,7]) and so on, have attracted considerable attention. There is no doubt that the literature on fractional and nonlocal operators is quite large, here we would like to mention a few, see for example [8,1,9]. For the basic properties of fractional Sobolev spaces, we refer the reader to [10] and references therein.

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In this paper we are interested in the existence of solutions for the following problem

$$\begin{cases} M\left(x, [u]^p_{s,p,K}\right) \mathcal{L}_K u = f\left(x, u, [u]^p_{s,p,K}\right) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $N > ps$ with $s \in (0, 1)$, $[u]^p_{s,p,K} = \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x - y) dx\, dy$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $M : \Omega \times \mathbb{R}^+_0 \to \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \times \mathbb{R}^+_0 \to \mathbb{R}$ is a Carathéodory function and $\mathcal{L}^p_K$ is a nonlocal operator defined as

$$\mathcal{L}^p_K \varphi(x) = \lim_{\varepsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y)) K(x - y)\, dy, \quad x \in \mathbb{R}^N,$$

along any $\varphi \in C^\infty_0(\mathbb{R}^N)$, where $1 < p < \infty$, $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\varepsilon > 0$ at the center $x \in \mathbb{R}^N$ and $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function with the following property

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N), \quad \text{where } \gamma(x) = \min\{|x|^p, 1\}; \\ \text{there exists } K_0 > 0 \text{ such that } K(x) \geq K_0\, |x|^{-(N+ps)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1.2}$$

A typical example for K is given by singular kernel $K(x) = |x|^{-(N+ps)}$. In this case, problem (1.1) becomes

$$\begin{cases} M\left(x, [u]^p_{s,p}\right) (-\Delta)^s_p u = f\left(x, u, [u]^p_{s,p}\right) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $[u]^p_{s,p} = \iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p\, |x - y|^{-(N+ps)}\, dx\, dy$, $(-\Delta)^s_p$ is the fractional p-Laplace operator, for example, see [11–13] for more details. When $p = 2$, $M = 1$ and f depend only on x and u, problem (1.3) reduces to the fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

A distinguished characterization of the fractional operator $(-\Delta)^s$ in (1.4) is the nonlocality, in the sense that this operator takes care of the behavior of the solution in the whole space. This is in contrast with the usual elliptic partial differential equations, which are governed by local differential operators like the Laplace operator. Of course, there are the other explanations for this feature, see for example [14,15]. The functional space that takes into account this boundary condition was introduced in [16]. In [17], the authors get the existence of nontrivial weak solutions of problem (1.4) by using the Mountain Pass Theorem. See also [15,16] for the related discussions.

Recently, *Fiscella* and *Valdinoci* in [18] first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff model. Under some suitable conditions, the authors obtained the existence of nontrivial solutions by using the Mountain Pass Theorem and a truncation argument on M. In this paper, the conditions imposed on the Kirchhoff function $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ are that M is an increasing and continuous function and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for any $t \geq 0$, see also [19] and references therein. However, the increasing condition rules out the non-monotone case, for example,

$$M(t) = (1 + t)^k + (1 + t)^{-1} \quad \text{with } 0 < k < 1 \text{ for all } t \geq 0. \tag{1.5}$$

For this purpose, *Xiang* et al. in [13] studied the existence of solutions to a class of fractional p-Kirchhoff equations, where the Kirchhoff function M is positive and continuous and satisfies the condition: there exists $\theta > 0$ such that $\theta \mathscr{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\mathscr{M}(t) = \int_0^t M(\tau) d\tau$. Obviously, the Kirchhoff function M satisfies (1.5). See also [20–22,11,12,23–25] for some recent results in this direction.

In the present paper, motivated by the above papers, we study the existence of weak solutions for a Kirchhoff type problem (1.1) involving nonlocal fractional operator. It is worth noticing that there are a few of authors dedicated to studying the Kirchhoff problems in which kirchhoff function M depends on $x \in \Omega$. Especially, *Corrêa* and *Figueiredo* in [26] first considered a problem whose equation is of the form

$$\begin{cases} -M\left(x, \displaystyle\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

They proved the existence of positive solutions by using an iterative device introduced in *De Figueiredo* et al. [27]. Note that for M nonhomogeneous we lose the variational structure and hence variational techniques cannot be used, at least in a direct way. Very recently, *Chung* [28] studied the existence of positive solutions for a nonlocal problem with dependence on the gradient in the Laplacian setting, in which the Kirchhoff function M depends on $x \in \Omega$. To our best knowledge, there is no result exploring the Kirchhoff problems, in which the Kirchhoff function may depend on $x \in \Omega$ and the nonlinearity may depend on the energy of solutions, in the setting of fractional Laplacian. It is worthy pointing out that problem like (1.6) arises in various situations. For example, u could describe the density of a population (bacteria, for instance) subject to spreading. In addition, the diffusion coefficient M is supposed to depend on the entire population in the domain Ω, instead of the local density.

In this spirit, we suppose that $M : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

(M_1) There exist $m_0, m_1 > 0$ such that $m_0 \le M(x,t) \le m_1$ for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}_0^+$;
(M_2) There exist constants $R_1 > 0$ and $L_1 = L_1(R_1) > 0$ such that

$$|M(x, t_1^p) - M(x, t_2^p)| \le L_1 |t_1 - t_2|^d, \quad \forall x \in \overline{\Omega} \text{ and } t_1, t_2 \in [0, R_1],$$

where

$$d = \begin{cases} p-1 & \text{if } p \ge 2, \\ 1 & \text{if } \max\{1, 2N/(N+2s)\} < p < 2. \end{cases}$$

As $p = 2$, a typical example for M is given by

$$M(x,t) = \begin{cases} a(x) + b(x)t^m & \text{if } t \in [0,1], \\ a(x) + b(x) & \text{if } t \in [1, \infty), \end{cases}$$

with $m > 0$, $a, b \in C(\overline{\Omega})$, $\inf_{x\in\Omega} a(x) > 0$ and $b(x) \ge 0$ for all $x \in \Omega$.

Also, we assume that $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function satisfying:

(f_1) $f(x,\xi,\eta) \ge 0$ for all $\xi \ge 0$, $\eta \in \mathbb{R}_0^+$ and $x \in \overline{\Omega}$; $f(x,\xi,\eta) = 0$ for all $\xi < 0$, $\eta \in \mathbb{R}_0^+$ and $x \in \overline{\Omega}$.
(f_2) $\lim_{\xi\to 0} \frac{f(x,\xi,\eta)}{|\xi|^{p-1}} = 0$ uniformly for all $\eta \in \mathbb{R}_0^+$ and $x \in \overline{\Omega}$.
(f_3) There exists $q \in (p, Np/(N-sp))$ such that

$$\lim_{\xi\to+\infty} \frac{f(x,\xi,\eta)}{|\xi|^{q-1}} = 0 \quad \text{uniformly for all } \eta \in \mathbb{R}_0^+ \text{ and } x \in \overline{\Omega}.$$

(f_4) There exists $\mu > p$ such that for all $\xi, \eta \in \mathbb{R}_0^+$ and a.e. $x \in \overline{\Omega}$,

$$0 < \mu F(x,\xi,\eta) := \mu \int_0^\xi f(x,\tau,\eta)d\tau \le \xi f(x,\xi,\eta).$$

(f_5) There exist positive constants A_1, A_2 such that

$$F(x,\xi,\eta) \ge A_1 \xi^\mu - A_2 \quad \text{for all } \xi,\ \eta \in \mathbb{R}_0^+ \text{ and } x \in \overline{\Omega}.$$

A simple example of f is given by $f(x,\xi,\eta) = g(\eta)\xi_+^{l-1}$ for all $(x,\xi,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}_0^+$, where $p < l < q$, $\xi_+ := \max\{\xi, 0\}$, and $g \in L^\infty(\Omega)$ such that $0 < c \le g(\eta)$ for some constant c.

In order to perform the Mountain Pass Theorem, here we take advantage of the well-known Ambrosetti–Rabinowitz type condition (f_4). Because of the dependence of the function M on x, we have to add the condition (f_5) in order to justify the geometrical conditions of the Mountain Pass Theorem. Moreover, we have to "freeze" the term containing the energy of solutions in the Kirchhoff function M and also in the nonlinearity f, then an iterative scheme where any "approximated" problem has a nontrivial nonnegative Mountain Pass solution is applied to obtain the desired solution after the verification of boundedness. Indeed, the idea is borrowed from [27], we also refer to the subsequent literature, for example, [29–32] for its applications to semilinear and quasilinear elliptic problems in the Laplacian setting. More precisely, we first consider the following problem:

$$\begin{cases} \mathcal{L}_K u = \dfrac{f(x,u,\|\omega\|_{W_0}^p)}{M(x,\|\omega\|_{W_0}^p)} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.7}$$

for each $\omega \in W_0$. Obviously, problem (1.7) is variational and can be treated by variational methods. Here we say that a function $u \in W_0$ is called to be a (weak) solution of problem (1.7) if

$$\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy = \int_\Omega \frac{f(x,u(x),\|\omega\|_{W_0}^p)\varphi(x)}{M\big(x,\|\omega\|_{W_0}^p\big)}dx, \tag{1.8}$$

for any $\varphi \in W_0$, where the space W_0 will be introduced in Section 2. Now we are ready to give the preliminary result of our paper:

Theorem 1.1. *Let $\omega \in W_0$ and $K : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}^+$ be a function satisfying (1.2). Assume that the hypotheses (M_1)–(M_2) and (f_1)–(f_5) are satisfied. Then there exist positive constants K_1, K_2, K_3, K_4 independent of ω such that problem (1.7) has a solution u_ω satisfying $K_1 \le \|u_\omega\|_{W_0} \le K_2$, $\|u_\omega\|_{L^q(\Omega)} \le K_3$ and $\|u_\omega\|_\infty \le K_4$, which is positive a.e. in Ω and $u_\omega = 0$ a.e. in $\mathbb{R}^N\setminus\Omega$.*

Remark 1.1. For each $\omega \in W_0$, by (f_2), (f_3) and $\|u_\omega\|_\infty \le K_4$ there exists a constant $C^* > 0$ independent of ω such that

$$\left(\int_\Omega \left|f(x, u_\omega, \|\omega\|^p_{W_0})\right|^{p/(p-1)} dx\right)^{(p-1)/p} \le C^*. \tag{1.9}$$

The main result will be established by an iterative method which depends on the solvability of problem (1.7). Especially, the main result involves the positive number C_p, which appears in the following well-known vector inequalities: there exists $C_p > 0$ such that

$$C_p\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta) \ge \begin{cases} |\xi-\eta|^p & \text{if } p \ge 2,\\ |\xi-\eta|^2(|\xi|^p + |\eta|^p)^{(p-2)/p} & \text{if } 1 < p < 2,\end{cases} \tag{1.10}$$

for all $\xi, \eta \in \mathbb{R}^N$. To get the main result, we also need the following technical assumption:

(f_6) There exist constants $R_2 > 0$, $L_2 = L_2(R_1, R_2) > 0$ and $L_3 = L_3(R_1, R_2) > 0$ such that

$$|f(x, \xi_1, \eta^p) - f(x, \xi_2, \eta^p)| \le L_2|\xi_1 - \xi_2|^d,$$

for all $x \in \overline{\Omega}$, $\eta \in [0, R_1]$, $\xi_1 \in [0, R_2]$, $\xi_2 \in [0, R_2]$, and

$$|f(x, \xi, \eta_1^p) - f(x, \xi, \eta_2^p)| \le L_3|\eta_1 - \eta_2|^d,$$

for all $x \in \overline{\Omega}$, $\eta_1 \in [0, R_1]$, $\eta_2 \in [0, R_1]$, $\xi \in [0, R_2]$, where d comes from assumption (M_2).

Now we are in a position to state the main result as follows:

Theorem 1.2. *Let $K : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}^+$ be a function satisfying (1.2). Suppose that M satisfies (M_1)–(M_2) and f satisfies (f_1)–(f_6). If one of the following conditions holds:*

$$0 < \frac{C_p m_0 2^{2/p} K_2^{2-p}\widetilde{C}}{m_0 - L_2 C_p K_2^{2-p}(2C_*)^{2/p}} < 1, \quad \text{if } 1 < p < 2;\ 0 < \frac{C_p m_0 \widetilde{C}}{m_0 - L_2 C_p C_*} < 1, \quad \text{if } p \ge 2,$$

where $C_* > 0$ is the number given in Lemma 2.1, $\widetilde{C} = \left(L_3|\Omega|^{(p-1)/p} m_0 C_*^{1/p} + L_1 C^* C_*^{1/p}\right)/m_0^2$, then problem (1.1) has a solution $u \in W_0 \bigcap L^\infty(\Omega)$, which is positive a.e. in Ω and $u = 0$ a.e. in $\mathbb{R}^N\setminus\Omega$.

This paper is organized as follows. In Section 2, we give some related definitions and results in fractional Sobolev space W_0. In Section 3, using the Mountain Pass Theorem and an iterative scheme, we give the proof of the main result.

2. Preliminaries

In this section, we first recall some basic results, which will be used in the next section. Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \dfrac{Np}{N-sp} & \text{if } sp < N,\\ \infty & \text{if } sp \ge N.\end{cases}$$

In the following, we denote $Q = \mathbb{R}^{2N}\setminus\mathcal{O}$, where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{C}(\Omega) = \mathbb{R}^N\setminus\Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\iint_Q |u(x) - u(y)|^p K(x-y)\,dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\iint_Q |u(x) - u(y)|^p K(x-y)\,dxdy\right)^{1/p}.$$

It is easy to prove that $\|\cdot\|_W$ is a norm on W. We shall work in the closed linear subspace

$$W_0 = \left\{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\},$$

endowed with the norm

$$\|u\|_{W_0} := [u]_{s,p,K} = \left(\iint_Q |u(x)-u(y)|^p K(x-y)dxdy\right)^{1/p}.$$

Then $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex Banach space, see [13]. Moreover, $C_0^\infty(\Omega)$ is dense in W_0, see [33] for more details.

Lemma 2.1 (*See [13, Lemma 2.3]*). *Let $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfy assumption (1.2). Then there exists a positive constant $C_0 = C_0(N, p, s)$ such that for any $v \in W_0$ and $1 \le q \le p_s^*$*

$$\begin{aligned}\|v\|^p_{L^q(\Omega)} &\le C_0 \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\le \frac{C_0}{K_0}\iint_Q |v(x)-v(y)|^p K(x-y)dxdy\\ &:= C_* \iint_Q |v(x)-v(y)|^p K(x-y)dxdy.\end{aligned}$$

Lemma 2.2 (*See [13, Lemma 2.5]*). *Let $K : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}^+$ satisfy (1.2) and let v_j be a bounded sequence in W_0. Then, there exists $v \in L^\nu(\mathbb{R}^N)$ with $v = 0$ a.e. in $\mathbb{R}^N\setminus\Omega$ such that up to a subsequence,*

$$v_j \to v \text{ strongly in } L^\nu(\Omega) \quad \text{as } j \to \infty,$$

for any $\nu \in [1, p_s^*)$.

The following strong maximum principle will be used to obtain the positivity of solutions in the proof of our results:

Lemma 2.3 (*See [14, Proposition 2.2]*). *If $u \in W_0\setminus\{0\}$ is such that $u(x) \ge 0$ a.e. in Ω and*

$$\langle \mathcal{L}(u), \varphi\rangle := \iint_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy \ge 0$$

for each $\varphi \in W_0$, $\varphi(x) \ge 0$ a.e. in Ω, then $u(x) > 0$ a.e. in Ω.

3. Proof of Theorems 1.1 and 1.2

As usual, a weak solution of problem (1.7) is obtained as a critical point of the associated functional $I_\omega : W_0 \to \mathbb{R}$ given by

$$I_\omega(u) = \frac{1}{p}\iint_Q |u(x)-u(y)|^p K(x-y)dxdy - \int_\Omega \frac{F(x,u,\|\omega\|^p_{W_0})}{M(x,\|\omega\|^p_{W_0})}dx, \tag{3.1}$$

where $F(x, u, \|\omega\|^p_{W_0}) = \int_0^u f(x,\xi,\|\omega\|^p_{W_0})d\xi$ and $u \in W_0$. Obviously, the energy functional $I : W_0 \to \mathbb{R}$ associated with problem (1.7) is well defined. A similar discussion as in [13] gives that I_ω is of class C^1 on W_0 and its derivative is given by

$$\langle I'_\omega(u), v\rangle = \iint_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy - \int_\Omega \frac{f(x,u,\|\omega\|^p_{W_0})}{M(x,\|\omega\|^p_{W_0})}dx, \quad \forall v \in W_0.$$

The proof of Theorem 1.1 is divided into several lemmas. We show that the functional I_ω has the geometry of the mountain pass theorem, that it satisfies the Palais–Smale condition and finally that the obtained solutions have the uniform bounds.

Lemma 3.1. *Fix $\omega \in W_0$. Let $K : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}^+$ be a function satisfying (1.2) and suppose that M satisfies (M_1) and (M_2) and f satisfies (f_1)–(f_3). Then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$I_\omega(u) \ge \alpha > 0,$$

for any $u \in W_0$ with $\|u\|_{W_0} = \rho$.

Proof. By (f_2) and (f_3), for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for any $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$, we have

$$|f(x,\xi,\|\omega\|^p_{W_0})| \le p\varepsilon|\xi|^{p-1} + qC(\varepsilon)|\xi|^{q-1}. \tag{3.2}$$

It follows from (3.2) that

$$|F(x,\xi,\|\omega\|^p_{W_0})| \le \varepsilon|\xi|^p + C(\varepsilon)|\xi|^q. \tag{3.3}$$

Let $u \in W_0$. By (3.3), (M_1) and Lemma 2.1, we obtain

$$
\begin{aligned}
I_\omega(u) &= \frac{1}{p}\iint_Q |u(x)-u(y)|^p K(x-y)dxdy - \int_\Omega \frac{F(x,u,\|\omega\|_{W_0}^p)}{M(x,\|\omega\|_{W_0}^p)}dx \\
&\geq \frac{1}{p}\iint_Q |u(x)-u(y)|^p K(x-y)dxdy - \frac{1}{m_0}\int_\Omega \varepsilon|u(x)|^p + C(\varepsilon)|u(x)|^q dx \\
&\geq \frac{1}{p}\|u\|_{W_0}^p - \frac{C_0\varepsilon}{m_0}\|u\|_{W_0}^p - \frac{C(\varepsilon)}{m_0}\left(\frac{C_0}{k_0}\right)^{q/p}\|u\|_{W_0}^q. \qquad (3.4)
\end{aligned}
$$

Choosing $\varepsilon = m_0/(2pC_0)$, we have by (3.4) that

$$
I_\omega(u) \geq \frac{1}{2p}\|u\|_{W_0}^p - C\|u\|_{W_0}^q \geq \|u\|_{W_0}^p\left(\frac{1}{2p} - C\|u\|_{W_0}^{q-p}\right),
$$

where C is a constant only depending on N, s, p, m_0, k_0. Now, let $\|u\|_{W_0} = \rho > 0$. Since $q > p$, we can choose ρ sufficiently small such that $m_0/(2p) - C\rho^{q-p} > 0$, so that

$$
I_\omega(u) \geq \rho^p\left(\frac{1}{2p} - C\rho^{q-p}\right) =: \alpha > 0.
$$

Thus, the proof is complete. □

Lemma 3.2. *Fix $\omega \in W_0$. Let $K : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}^+$ be a function satisfying (1.2) and suppose that M satisfies (M_1) and (M_2) and f satisfies (f_1)–(f_3). Then there exists $e \in C_0^\infty(\Omega)$ such that $\|e\|_{W_0} \geq \rho$ and $I_\omega(e) < \alpha$, where ρ and α are given in Lemma 3.1.*

Proof. From (f_5) and (M_2), we have

$$
\begin{aligned}
I_\omega(tu_0) &= \frac{1}{p}\|tu_0\|_{W_0}^p - \int_\Omega F(x, tu_0(x), \|\omega\|_{W_0}^p)dx \\
&\leq \frac{1}{p}t^p - \frac{A_1 t^\mu}{m_1}\int_\Omega |u_0(x)|^\mu dx - \frac{A_2}{m_1}|\Omega|,
\end{aligned}
$$

where $u_0 \in W_0$ satisfying $\|u_0\|_{W_0} = 1$, $|\Omega|$ is the Lebesgue measure of Ω in $\mathbb{R}^N$. Since $\mu > p$ by assumption (f_4), passing to the limit as $t \to \infty$, we obtain that $I(tu_0) \to -\infty$. Thus, the assertion follows by taking $e = Tu_0$ with T sufficiently large. □

Definition 3.1. We say that I satisfies (PS) condition in W_0, if for any sequence $\{u_n\}_n \subset W_0$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence of $\{u_n\}_n$.

Lemma 3.3. *Fix $\omega \in W_0$. Let $K : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}^+$ be a function satisfying (1.2) and suppose that M satisfies (M_1) and (M_2) and f satisfies (f_1)–(f_7). Then the functional I_ω satisfies (PS) condition.*

Proof. For any sequence $\{u_n\}_n \subset W_0$ such that $I_\omega(u_n)$ is bounded and $I'_\omega(u_n) \to 0$ as $n \to \infty$, there exists $C > 0$ such that $|\langle I'_\omega(u_n), u_n\rangle| \leq C\|u_n\|_{W_0}$ and $|I_\omega(u_n)| \leq C$. Thus, by (M_1), (M_2) and (f_2), we get

$$
\begin{aligned}
C + C\|u_n\|_{W_0} &\geq I_\omega(u_n) - \frac{1}{\mu}\langle I'_\omega(u_n), u_n\rangle \\
&= \frac{1}{p}\|u_n\|_{W_0}^p - \frac{1}{\mu}\|u_n\|_{W_0}^p - \frac{1}{\mu}\int_\Omega (\mu F(x,u_n,\|\omega\|_{W_0}^p) - f(x,u_n,\|\omega\|_{W_0}^p)u_n)dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{W_0}^p,
\end{aligned}
$$

where C denotes various positive constants. Hence, $\{u_n\}_n$ is bounded in W_0.

For simplicity, we first introduce a notation. Let $\varphi \in W_0$ be fixed and denote by B_φ the linear functional on W_0 defined by

$$
B_\varphi(v) = \iint_Q |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(v(x)-v(y))K(x-y)dxdy
$$

for all $v \in W_0$. Clearly, by the Hölder inequality, B_φ is also continuous, being

$$
|B_\varphi(v)| \leq \|\varphi\|_{W_0}^{p-1}\|v\|_{W_0} \quad \text{for all } v \in W_0.
$$

Since W_0 is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}_n$ such that $u_n \rightharpoonup u_\omega$ weakly in W_0 as $n \to \infty$. Then $\langle I'_\omega(u_n), u_n - u_\omega\rangle \to 0$, that is,

$$\begin{aligned}\langle I'_\omega(u_n), u_n - u_\omega\rangle &= B_{u_n}(u_n - u_\omega) - \int_\Omega \frac{f(x, u_n, \|\omega\|^p_{W_0})(u_n - u_\omega)}{M(x, \|\omega\|^p_{W_0})}dx \\ &\longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

Moreover, by Lemma 2.2, up to a subsequence,

$$u_n \to u \text{ strongly in } L^q(\Omega) \text{ and a.e. in } \Omega.$$

Using (f_3) and the Hölder inequality, we obtain

$$\begin{aligned}\left|\int_\Omega \frac{f(x, u_n, \|\omega\|^p_{W_0})(u_n - u_\omega)}{M(x, \|\omega\|^p_{W_0})}dx\right| &\le \frac{C}{m_0}\int_\Omega (1 + |u_n|^{q-1})|u_n - u_\omega|dx \\ &\le \frac{C}{m_0}\left(|\Omega|^{(q-1)/q} + \|u_n\|^{q-1}_{L^q(\Omega)}\right)\|u_n - u_\omega\|_{L^q(\Omega)} \\ &\longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned} \tag{3.6}$$

where $C > 0$ is a constant. Inserting (3.6) into (3.5), we get

$$\lim_{n\to\infty} B_{u_n}(u_n - u_\omega) = 0.$$

Furthermore, by the weak convergence of $\{u_n\}_n$ in W_0 we get

$$B_{u_n}(u_n - u_\omega) - B_{u_\omega}(u_n - u_\omega) \to 0 \quad \text{as } n \to \infty.$$

Using the well-known vector inequalities (1.10), we obtain for $p > 2$

$$\|u_n - u_\omega\|^p_{W_0} \le C_p\,[B_{u_n}(u_n - u_\omega) - B_{u_\omega}(u_n - u_\omega)] = o(1), \tag{3.7}$$

and for $1 < p < 2$

$$\begin{aligned}\|u_n - u_\omega\|^p_{W_0} &\le C_p^{p/2}\,[B_{u_n}(u_n - u_\omega) - B_{u_\omega}(u_n - u_\omega)]^{p/2}\left(\|u_n\|^p_{W_0} + \|u_\omega\|^p_{W_0}\right)^{(2-p)/2} \\ &\le C_p^{p/2}\,[B_{u_n}(u_n - u) - B_{u_\omega}(u_n - u)]^{p/2}\left(\|u_n\|^{p(2-p)/2}_{W_0} + \|u_\omega\|^{p(2-p)/2}_{W_0}\right) \\ &\le C[B_{u_n}(u_n - u_\omega) - B_{u_\omega}(u_n - u_\omega)]^{p/2} = o(1), \end{aligned} \tag{3.8}$$

where $C > 0$ is a constant. Combining (3.7) with (3.8), we get that $u_n \to u$ strongly in W_0 as $n \to \infty$. Therefore, I satisfies (PS) condition. □

Since Lemmas 3.1–3.2 hold, the Mountain Pass Theorem [34, Theorem 6.1] gives that problem (3.1) has a nonnegative solution u_w satisfying

$$I'_\omega(u_\omega) = 0,\ c_\omega = I_\omega(u_\omega) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\omega(\gamma(t)) > I(0) = 0, \tag{3.9}$$

where $\Gamma = \{\gamma \in C([0, 1], W_0) : \gamma(0) = 0, \gamma(1) = e\}$ and e from Lemma 3.2. Thus, $u_\omega \not\equiv 0$. Moreover, $u_\omega \ge 0$ a.e. in $\mathbb{R}^N$. Indeed,

$$\langle I'_\omega(u_\omega), u^-_\omega\rangle = B_{u_\omega}(u^-_\omega) - \int_\Omega \frac{f(x, u_\omega, \|\omega\|^p_{W_0})u^-_\omega}{M(x, \|\omega\|^p_{W_0})}dx = 0,$$

where $u^-_\omega = \max\{-u_\omega, 0\} \in W_0$. Combining this with (f_1), we have

$$\iint_Q |u_\omega(x) - u_\omega(y)|^{p-2}(u_\omega(x) - u_\omega(y))(u^-_\omega(x) - u^-_\omega(y))K(x - y)dxdy = 0.$$

From which together with the following fact:

$$(u_\omega(x) - u_\omega(y))(u^-_\omega(x) - u^-_\omega(y)) \le -\left|u^-_\omega(x) - u^-_\omega(y)\right|^2, \quad \text{a.e. } x, y \in \mathbb{R}^N,$$

we get that $(u_\omega)^-(x) = (u_\omega)^-(y)$ for a.e. $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Since $(u_\omega)^-(x) = 0$ a.e. in $\mathbb{R}^N\setminus\Omega$, we obtain that $(u_\omega)^-(x) = 0$ a.e. in $\mathbb{R}^N$, that is, $u_\omega \ge 0$ a.e. in $\mathbb{R}^N$. Notice that (f_1) implies

$$\langle \mathcal{L}_K(u_\omega), v\rangle = \int_\Omega \frac{f(x, u_\omega, \|\omega\|^p_{W_0})}{M(x, \|\omega\|^p_{W_0})}v dx \ge 0,$$

for each $v \in W_0$, $v(x) \ge 0$ a.e. in Ω. Then Lemma 2.3 means the positivity of u_ω a.e. in Ω with $u = 0$ in $\mathbb{R}^N\setminus\Omega$.

Lemma 3.4. *Let $\omega \in W_0$. There exists $K_1 > 0$ independent of ω such that $\|u_\omega\|_{W_0} \geq K_1$.*

Proof. Using u_ω as a test function in (1.8), we obtain by (f_2) and (f_3) that

$$\begin{aligned}\iint_Q |u_\omega(x)-u_\omega(y)|^p K(x-y)dxdy &= \int_\Omega \frac{f(x,u_\omega,\|\omega\|_{W_0}^p)}{M(x,\|\omega\|_{W_0}^p)}dx\\ &\leq \frac{\varepsilon}{m_0}\int_\Omega |u_\omega|^p dx + \frac{C_\varepsilon}{m_0}\int_\Omega |u_\varepsilon|^q dx\\ &\leq \frac{\varepsilon C_*}{m_0}\|u_\omega\|_{W_0}^p + \frac{C_\varepsilon}{m_0}C_*^{q/p}\|u_\omega\|_{W_0}^q,\end{aligned}$$

for any $\varepsilon > 0$. Then

$$\left(1-\frac{\varepsilon C_*}{m_0}\right)\|u_\omega\|_{W_0}^p \leq \frac{C_\varepsilon}{m_0}C_*^{q/p}\|u_\omega\|_{W_0}^q.$$

From this and $q > p$, by taking $\varepsilon > 0$ small enough, we get

$$\|u\|_{W_0} \geq \left[\frac{m_0}{C_\varepsilon}C_*^{q/p}\left(1-\frac{\varepsilon C_0}{m_0 K_0}\right)\right]^{1/(q-p)} := K_1 > 0.$$

So the assertion follows. □

Lemma 3.5. *Let $\omega \in W_0$. Then there exists $K_2 > 0$ independent of ω such that $\|u_\omega\|_{W_0} \leq K_2$.*

Proof. Taking a special pass $\overline{\gamma}(t) = te$ and using the definition of c_ω in (3.9), we obtain

$$c_\omega = I_\omega(u_\omega) \leq \max_{0\leq t\leq 1} I_\omega(\overline{\gamma}(t)) \leq \max_{t\geq 0} I_\omega(te), \tag{3.10}$$

where e chosen in Lemma 3.2. By (f_5), we have

$$I_\omega(te) \leq \frac{t^p}{p} - \frac{A_1}{m_1}t^\mu \int_\Omega |e|^\mu dx + \frac{A_2|\Omega|}{m_1} := h(t),$$

whose maximum is achieved at some $\widehat{t_0} > 0$ and the value $h(\widehat{t_0})$ can be taken as $C > 0$. Clearly it is independent of ω. Using (f_4), we have

$$C \geq c_\omega = I_\omega(u_\omega) - \frac{1}{\mu}\langle I'_\omega(u_\omega), u_\omega\rangle \geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_\omega\|_{W_0}^p. \tag{3.11}$$

Then (3.10) together with (3.11) implies that there exists $K_2 := (\mu p C/(\mu - p))^{1/p} > 0$ independent of ω such that $\|u_\omega\|_{W_0} \leq K_2$. □

By Lemmas 3.5 and 2.2, there exists $K_3 > 0$ independent of ω such that $\|u\|_q \leq K_3$. Without loss of generality, we assume that $K_3 \geq 1$.

Lemma 3.6. *There exists $K_4 > 0$ independent of ω such that for every weak solution $u_\omega \in W_0$ of problem (1.7), there holds*

$$\|u_\omega\|_\infty \leq K_4.$$

To prove Lemma 3.6, we need the following lemma (see [35, Lemma 4.1]):

Lemma 3.7. *Let $\{Y_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive numbers satisfying the recursive inequalities*

$$Y_{n+1} \leq Cb^n Y_n^{1+\alpha},$$

where $b > 1$ and $C, \alpha > 0$ are given numbers. If

$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2},$$

then $\{Y_n\}$ converges to zero as $n \to \infty$.

Proof of Lemma 3.6. Fix a nonnegative solution $u_\omega \in W_0$ of (1.7) and set $v = K_3^{-1}u_\omega$. Then v is a non-negative solution of the following problem

$$\begin{cases} \mathcal{L}_K v = \dfrac{K_3^{1-p} f(x, K_3 v, \|\omega\|_{W_0}^p)}{M(x,\|\omega\|_{W_0}^p)} & \text{in } \Omega,\\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.12}$$

For all $n \in \mathbb{N}$, we set $v_n = (v - l + l2^{-n})^+$, where $l \geq 1$ is a constant to be determined later. So $v_n \in W_0$, $\|v_n\|_q \leq 1$, $v_0 = v^+ = v$ and for all $n \in \mathbb{N}$ we have $0 \leq v_{n+1}(x) \leq v_n(x)$ and $v_n(x) \to (v(x) - l)^+$ a.e. in Ω as $n \to \infty$. Moreover, the following inclusion holds (up to a Lebesgue null set)

$$\{x \in \Omega : v_{n+1}(x) > 0\} \subset \left\{x \in \Omega : 0 < v(x) < (2^{n+1} - 1)v_n(x)\right\} \bigcap \left\{x \in \Omega : v_n(x) > 2^{-n-1}l\right\}. \tag{3.13}$$

For all $n \in \mathbb{N}$ we set $R_n = \|v_n\|_q^q$, then $R_0 = \|v\|_q^q$ and $\{R_n\}_n$ is a non-increasing sequence on $[0, 1]$. Next we prove that $R_n \to 0$ as $n \to \infty$. By the Hölder inequality, the fractional Sobolev inequality (see [10, Theorem 6.5]), (3.13) and Chebyshev inequality (see [36, p. 52]), we have for all $n \in \mathbb{N}$

$$\begin{aligned} R_{n+1} &\leq |\{x \in \Omega : v_{n+1} > 0\}|^{1-q/p_s^*} \|v_{n+1}\|_{p_s^*}^q \\ &\leq C \left|\{x \in \Omega : v_n > 2^{-(n+1)}l\}\right|^{1-q/p_s^*} \|v_{n+1}\|_{W_0}^q \\ &\leq C \left(2^{n+1}/l\right)^{q-q^2/p_s^*} R_n^{1-q/p_s^*} \|v_{n+1}\|_{W_0}^q. \end{aligned} \tag{3.14}$$

It follows from (f_2) and (f_3) that there exists a constant $C > 0$ such that for a.e. $x \in \Omega$ and all $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}_0^+$

$$f(x, \xi, \eta) \leq C(|\xi|^{p-1} + |\xi|^{q-1}). \tag{3.15}$$

Then using the following inequality

$$|\xi^+ - \eta^+|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(\xi^+ - \eta^+) \quad \text{for all } \xi, \eta \in \mathbb{R},$$

and testing (3.12) with v_{n+1}, together with (3.13), (M_1) and the Hölder inequality, we have

$$\begin{aligned} \|v_{n+1}\|_{W_0}^p \leq B_v(v_{n+1}) &= \int_\Omega \frac{1}{M(x, \|\omega\|_{W_0}^p)} f(x, K_3 v, \|\omega\|_{W_0}^p) v_{n+1}(x) dx \\ &\leq C \int_{\{x \in \Omega : v_{n+1} > 0\}} \left(|v(x)|^{p-1} + |v(x)|^{q-1}\right) v_{n+1}(x) dx \\ &\leq C \int_{\{x \in \Omega : v_{n+1} > 0\}} \left[(2^{n+1} - 1)^{p-1} v_n^p + (2^{n+1} - 1)^{q-1} v_n^q\right] dx \\ &\leq C 2^{(n+1)(q-1)} (R_n^{p/q} + R_n) \\ &\leq C 2^{(n+1)(q-1)} R_n^{p/q}, \end{aligned}$$

where C denotes various positive constants independent of n, ω. Hence we deduce from (3.14) that

$$R_{n+1} \leq C l^{q^2/p_s^* - q} 2^{(q - q^2/p_s^* + q^2/p - q/p)(n+1)} R_n^{2-q/p_s^*} := \overline{C} l^{q^2/p_s^* - q} 2^{\lambda n} R_n^{1+\beta}, \tag{3.16}$$

where $\overline{C} = 2^\lambda C > 0$ is a constant independent of n and ω, $\lambda := q - q^2/p_s^* + q^2/p - q/p > 0$ and $0 < \beta := 1 - q/p_s^* < 1$. Now we choose $l \geq 1$ such that

$$R_0 = \int_\Omega v(x)^q dx \leq (\overline{C} l^{q^2/p_s^* - q})^{-1/\beta} 2^{-\lambda/\beta^2}.$$

By Lemma 3.7, we obtain that $R_n \to 0$ as $n \to \infty$, so that $v(x) \leq l$ a.e. in Ω. Hence, we have $u_\omega \in L^\infty(\Omega)$ and $\|u_\omega\|_\infty \leq K_3 l$. It follows from the choice of l and Lemma 3.5 that there exists $K_4 > 0$ independent of ω such that $\|u_\omega\|_\infty \leq K_4$. This completes the proof. □

Proof of Theorem 1.2. By Theorem 1.1, we can construct a sequence $\{u_n\}_n$ of nonnegative solutions as

$$\begin{cases} \mathcal{L}_K u_n = \dfrac{f(x, u_n, \|u_{n-1}\|_{W_0}^p)}{M(x, \|u_{n-1}\|_{W_0}^p)} & \text{in } \Omega \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathcal{E}_n$}$$

obtained by the Mountain Pass Theorem, starting with an arbitrary $u_0 \in W_0 \bigcap L^\infty(\Omega)$. By Lemmas 3.5 and 3.6, we obtain that $\|u_n\|_{W_0} \leq K_2$ and $\|u_n\|_\infty \leq K_4$. Using $(\mathcal{E}_n)$ and $(\mathcal{E}_{n+1})$, we get

$$B_{u_n}(u_{n+1}(x) - u_n(x)) = \int_\Omega \frac{f(x, u_n(x), \|u_{n-1}\|_{W_0}^p)(u_{n+1} - u_n)}{M\left(x, \|u_{n-1}\|_{W_0}^p\right)} dx \tag{3.17}$$

and

$$B_{u_{n+1}}(u_{n+1}(x) - u_n(x)) = \int_\Omega \frac{f(x, u_{n+1}(x), \|u_n\|_{W_0}^p)(u_{n+1} - u_n)}{M\left(x, \|u_n\|_{W_0}^p\right)} dx. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}
&B_{u_{n+1}}(u_{n+1}(x)-u_n(x)) - B_{u_n}(u_{n+1}(x)-u_n(x)) \\
&= \int_\Omega \frac{f(x,u_{n+1},\|u_n\|_{W_0}^p)(u_{n+1}-u_n)}{M(x,\|u_n\|_{W_0}^p)} - \frac{f(x,u_n,\|u_{n-1}\|_{W_0}^p)(u_{n+1}-u_n)}{M(x,\|u_{n-1}\|_{W_0}^p)}dx \\
&= \frac{1}{M(x,\|u_n\|_{W_0}^p)}\int_\Omega f(x,u_{n+1},\|u_n\|_{W_0}^p) - f(x,u_n,\|u_{n-1}\|_{W_0}^p)(u_{n+1}-u_n)dx \\
&\quad + \int_\Omega f(x,u_n,\|u_{n-1}\|_{W_0}^p)\left(\frac{1}{M(x,\|u_n\|_{W_0}^p)} - \frac{1}{M(x,\|u_{n-1}\|_{W_0}^p)}\right)(u_{n+1}-u_n)dx. \qquad (3.19)
\end{aligned}$$

We first consider the case $p \geq 2$. Applying the Hölder inequality, the fractional Sobolev embedding, (M_1) and (f_7), we obtain

$$\begin{aligned}
&\frac{1}{M(x,\|u_n\|_{W_0}^p)}\left|\int_\Omega f(x,u_{n+1},\|u_n\|_{W_0}^p) - f(x,u_n,\|u_{n-1}\|_{W_0}^p)(u_{n+1}-u_n)dx\right| \\
&\leq \frac{1}{m_0}\int_\Omega \left|f(x,u_{n+1},\|u_n\|_{W_0}^p) - f(x,u_n,\|u_{n-1}\|_{W_0}^p)\right| |u_{n+1}-u_n|dx \\
&\leq \frac{1}{m_0}\int_\Omega \left|f(x,u_{n+1},\|u_n\|_{W_0}^p) - f(x,u_n,\|u_n\|_{W_0}^p)\right| |u_{n+1}-u_n|dx \\
&\quad + \frac{1}{m_0}\int_\Omega \left|f(x,u_n,\|u_n\|_{W_0}^p) - f(x,u_n,\|u_{n-1}\|_{W_0}^p)\right| |u_{n+1}-u_n|dx \\
&\leq \frac{L_2}{m_0}\int_\Omega |u_{n+1}-u_n|^p dx + \frac{L_3}{m_0}\int_\Omega \|u_n-u_{n-1}\|_{W_0}^{p-1}|u_{n+1}-u_n|dx \\
&\leq \frac{L_2 C_*}{m_0}\|u_{n+1}-u_n\|_{W_0}^p + \frac{L_3|\Omega|^{(p-1)/p}C_*^{1/p}}{m_0}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|u_{n+1}-u_n\|_{W_0}, \qquad (3.20)
\end{aligned}$$

where $L_2 = L_2(K_2, K_4) > 0$ and $L_3 = L_3(K_2, K_4) > 0$. Furthermore, we deduce from (M_2), (f_7) and (1.9) that

$$\begin{aligned}
&\left|\int_\Omega f(x,u_n,\|u_{n-1}\|_{W_0}^p)\left(\frac{1}{M(x,\|u_n\|_{W_0}^p)} - \frac{1}{M(x,\|u_{n-1}\|_{W_0}^p)}\right)(u_{n+1}-u_n)dx\right| \\
&\leq \frac{1}{m_0^2}\int_\Omega \left|f(x,u_n,\|u_{n-1}\|_{W_0}^p)\right| \left|M(x,\|u_n\|_{W_0}^p) - M(x,\|u_{n-1}\|_{W_0}^p)\right| |u_{n+1}-u_n|dx \\
&\leq \frac{L_1}{m_0^2}\|u_n-u_{n-1}\|_{W_0}^{p-1}\int_\Omega \left|f(x,u_n,\|u_{n-1}\|_{W_0}^p)\right| |u_{n+1}-u_n|dx \\
&\leq \frac{L_1}{m_0^2}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|f(x,u_n,\|u_{n-1}\|_{W_0}^p)\|_{p/(p-1)}\|u_{n+1}-u_n\|_p \\
&\leq \frac{L_1 C^* C_*^{1/p}}{m_0^2}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|u_{n+1}-u_n\|_{W_0}, \qquad (3.21)
\end{aligned}$$

where $L_1 = L_1(K_2) > 0$. Inserting (3.20) and (3.21) into (3.19), we get

$$\begin{aligned}
&B_{u_{n+1}}(u_{n+1}(x)-u_n(x)) - B_{u_n}(u_{n+1}(x)-u_n(x)) \\
&\leq \frac{L_2 C_*}{m_0}\|u_{n+1}-u_n\|_{W_0}^p + \frac{L_3|\Omega|^{(p-1)/p}C_*^{1/p}}{m_0}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|u_{n+1}-u_n\|_{W_0} \\
&\quad + \frac{L_1 C^* C_*^{1/p}}{m_0^2}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|u_{n+1}-u_n\|_{W_0}. \qquad (3.22)
\end{aligned}$$

Since $p \geq 2$, we deduce from (1.10) that

$$\frac{1}{C_p}\|u_{n+1}-u_n\|_{W_0}^p \leq \frac{L_2 C_*}{m_0}\|u_{n+1}-u_n\|_{W_0}^p + \widetilde{C}\|u_n-u_{n-1}\|_{W_0}^{p-1}\|u_{n+1}-u_n\|_{W_0}, \qquad (3.23)$$

where

$$\widetilde{C} = \frac{L_3|\Omega|^{(p-1)/p}m_0 C_*^{1/p} + L_1 C^* C_*^{1/p}}{m_0^2}.$$

It follows from (3.23) that

$$\|u_{n+1}-u_n\|_{W_0} \le \kappa \|u_n-u_{n-1}\|_{W_0}, \tag{3.24}$$

where $\kappa = \left(C_p m_0 \widetilde{C} / \left(m_0 - L_2 C_* C_p\right)\right)^{1/(p-1)}$. Since $0 < \kappa < 1$, we conclude from (3.24) that the sequence $\{u_n\}_n$ converges strongly in W_0 to some $u \in W_0$, which is the solution of problem (1.1). By u_n is positive a.e. in Ω and $\|u_n\|_{W_0} \ge K_1$ and $\|u_n\|_\infty \le K_3$ for all $n \in \mathbb{N}$, we get that $u > 0$ a.e. in Ω and $u \in L^\infty(\Omega)$.

It remains to consider the case $1 < p < 2$. By (1.10), we have

$$\begin{aligned}\|u_{n+1}-u_n\|_{W_0}^p &\le C_p^{p/2}\left[B_{u_{n+1}}(u_{n+1}-u_n) - B_{u_n}(u_{n+1}-u_n)\right]^{p/2}\left(\|u_{n+1}\|_{W_0}^p + \|u_n\|_{W_0}^p\right)^{(2-p)/2}\\ &\le C_p^{p/2}\left[B_{u_{n+1}}(u_{n+1}-u_n) - B_{u_n}(u_{n+1}-u_n)\right]^{p/2}\left(\|u_{n+1}\|_{W_0}^{p(2-p)/2} + \|u_n\|_{W_0}^{p(2-p)/2}\right)\\ &\le 2C_p^{p/2}K_2^{p(2-p)/2}\left[B_{u_{n+1}}(u_{n+1}-u_n) - B_{u_n}(u_{n+1}-u_n)\right]^{p/2}. \end{aligned}\tag{3.25}$$

A similar discussion as (3.20) and (3.21) gives that

$$\begin{aligned}&B_{u_{n+1}}(u_{n+1}(x)-u_n(x)) - B_{u_n}(u_{n+1}(x)-u_n(x))\\ &\quad\le \frac{L_2 C_*^{2/p}}{m_0}\|u_{n+1}-u_n\|_{W_0}^2 + \frac{L_3|\Omega|^{(p-1)/p}C_*^{1/p}}{m_0}\|u_n-u_{n-1}\|_{W_0}\|u_{n+1}-u_n\|_{W_0}\\ &\qquad + \frac{L_1 C^* C_*^{1/p}}{m_0^2}\|u_n-u_{n-1}\|_{W_0}\|u_{n+1}-u_n\|_{W_0}. \end{aligned}\tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\|u_{n+1}-u_n\|_{W_0}^2 \le C_p 2^{2/p} K_2^{2-p}\left(\frac{L_2 C_*^{2/p}}{m_0}\|u_{n+1}-u_n\|_{W_0}^2 + \widetilde{C}\|u_n-u_{n-1}\|_{W_0}\|u_{n+1}-u_n\|_{W_0}\right),$$

which implies that

$$\|u_{n+1}-u_n\|_{W_0} \le C_p 2^{2/p} K_2^{2-p}\left(\frac{L_2 C_*^{2/p}}{m_0}\|u_{n+1}-u_n\|_{W_0} + \widetilde{C}\|u_n-u_{n-1}\|_{W_0}\right). \tag{3.27}$$

Therefore, we get

$$\|u_{n+1}-u_n\|_{W_0} \le \widehat{\kappa}\|u_n-u_{n-1}\|_{W_0},$$

where

$$\widehat{\kappa} = \frac{C_p 2^{2/p} K_2^{2-p}\widetilde{C}}{1 - \left(C_p K_2^{2-p}(2C_*)^{2/p} L_2\right)/m_0}.$$

Since $0 < \widehat{\kappa} < 1$, we obtain that $\{u_n\}_n$ is a Cauchy sequence in W_0. Hence the sequence $\{u_n\}_n$ converges strongly in W_0 to some $u \in W_0$, which is a nonnegative solution of problem (1.1). □

Remark 3.1. (a) It is easy to see from the proof of Theorem 1.1 that the nonnegativity of solutions in Ω to problem (1.7) can be obtained immediately if (f_1) is replaced with weaker assumption (f_1'): $f(x,\xi,\eta) = 0$ for all $\xi < 0$, $\eta \in [0,\infty)$ and a.e. $x \in \Omega$.

(b) Evidently, Theorems 1.1–1.2 still hold if $f(x,\xi,\eta)$ is independent of η. Furthermore, we can remove condition (f_5) since (f_4) implies (f_5) if $f(x,\xi) \in C(\Omega \times \mathbb{R})$.

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