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TURING PATTERNS IN GENERAL REACTION-DIFFUSION SYSTEMS OF BRUSSELATOR TYPE

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We study the reaction-diffusion system

$$\begin{cases} u_t - d_1\Delta u = a - (b+1)u + f(u)v & \text{in } \Omega\times(0,T),\\ v_t - d_2\Delta v = bu - f(u)v & \text{in } \Omega\times(0,T),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{on } \Omega,\\ \dfrac{\partial u}{\partial\nu}(x,t) = \dfrac{\partial u}{\partial\nu}(x,t) = 0 & \text{on } \partial\Omega\times(0,T). \end{cases}$$

Here Ω is a smooth and bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $a, b, d_1, d_2 > 0$ and $f \in C^1[0,\infty)$ is a non-decreasing function. The case $f(u) = u^2$ corresponds to the standard Brusselator model for autocatalytic oscillating chemical reactions. Our analysis points out the crucial role played by the nonlinearity f in the existence of Turing patterns. More precisely, we show that if f has a sublinear growth then no Turing patterns occur, while if f has a superlinear growth then existence of such patterns is strongly related to the inter-dependence between the parameters a, b and the diffusion coefficients d_1, d_2.

Keywords: Turing patterns; reaction-diffusion system; Brusselator model; stability; steady-state solutions.

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1. Introduction

Many physical, chemical, biological, environmental and even sociological processes are driven by reaction-diffusion systems. These are multi-component models involving two different mechanisms: on one hand, there is diffusion, a random particle movement, and on the other hand, there are chemical, biological or sociological reactions representing instantaneous interactions, which depend on the state variables themselves and possibly also explicitly on the particles' position.

In the early fifties, Turing [24], a British mathematician, proposed a model that accounts for pattern formation in morphogenesis. Turing [24] suggested that under certain conditions, chemicals can react and diffuse in such a way to produce steady-state heterogeneous spatial patterns of chemical or morphogen concentrations. He showed that a system of two reacting and diffusing chemicals could give rise to spatial patterns from initial near-homogeneity. The idea behind Turing's model is the existence of a low-range diffusing activator and a wide-range diffusing inhibitor. The activator production is inhibited by the presence of inhibitors and enhanced by the presence of the activator. In contrast, the inhibitor is not self-enhancing, that is, its production is not linked to the presence of other inhibitors, but to the presence of activators. Turing systems show a very rich behavior from the pattern formation point of view, varying from spots to stripes and from lamellar to chaotic structures.

Lately, many Turing-type models described by coupled systems of reaction-diffusion equations have been used for generating patterns in both organic and inorganic systems.

In this paper, we shall be concerned with Turing patterns in a general Brusselator model for autocatalytic oscillating chemical reactions. An autocatalytic reaction is one in which a species acts to increase the rate of its producing reaction. In many autocatalytic systems complex dynamics are seen, including multiple steady-states and periodic orbits. The study of oscillating reactions has only been the subject of interest for the last fifty years, starting with the Belousov–Zhabotinsky chemical reactions.

There is now a large number of real and hypothetical systems that provide insight into the complex behavior of autocatalytic systems. Among them we mention Brusselator model [19], Gray–Scott model [8], Lengyel–Epstein model [12], Oregonator model [6], Schnakenberg model [21], Sel'klov model [22].

In this paper we shall consider the system

$$\begin{cases} u_t - d_1 \Delta u = a - (b+1)u + f(u)v & \text{in } \Omega \times (0,T), \\ v_t - d_2 \Delta v = bu - f(u)v & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{on } \Omega, \\ \dfrac{\partial u}{\partial \nu}(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth and bounded domain, a, b, d_1, d_2 are positive constants and $f \in C^1(0,\infty) \cap C[0,\infty)$ is non-negative and non-decreasing function such that $f > 0$ in $(0,\infty)$. The initial data u_0, v_0 are non-negative continuous functions in $\overline{\Omega}$. The case $f(s) = s^2$ in system (1) corresponds to the Brusselator model introduced by Prigogine and Lefever [19] in 1968. It consists on the following four intermediate reaction steps

$$A \to X, \quad B + X \to Y + D, \quad 2X + Y \to 3X, \quad X \to E.$$

The global reaction is $A + B \to D + E$ and corresponds to the transformation of input products A and B into output products D and E. The unknowns u, v in system (1) represent the concentrations of two intermediary reactants having the diffusion rates $d_1, d_2 > 0$ while a, $b > 0$ are fixed concentrations. The Brusselator system has been extensively investigated in the last decades from both analytical and numerical point of view (see, for instance, [1, 4, 5, 7, 10, 11, 16, 18, 25, 27]).

The analysis in this paper reveals the fact that the dynamics of the evolution system (1) and its associated steady-state is strongly related to the behavior of the nonlinearity f. Throughout this paper, we shall assume that f satisfies one of the following hypotheses:

either f is sublinear, that is,

(f1) the mapping $(0, \infty) \ni s \to \frac{f(s)}{s}$ is non-increasing;
of f has a superlinear character, namely,
(f2) the mapping $(0, \infty) \ni s \to \frac{f(s)}{s}$ is non-decreasing.

A particular attention will be paid to the steady-states to (1), that is, solutions of

$$\begin{cases} -d_1 \Delta u = a - (b+1)u + f(u)v & \text{in } \Omega, \\ -d_2 \Delta v = bu - f(u)v & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases} \tag{2}$$

It is easily seen that there exists a unique uniform steady-state of (2), namely

$$(u, v) = \left(a, \frac{ab}{f(a)} \right). \tag{3}$$

In this paper, we shall investigate the asymptotic stability of the above constant solution. In particular, we shall see that if f has a sublinear growth, then the constant solution (u, v) defined by (3) is uniformly asymptotically stable. Moreover, in the sublinear case on f we prove that (3) is the unique solution of system (2), so there are no Turing patterns in this case. In turn, if f satisfies (f2), the analysis of (2) is more involved. The existence of Turing patterns (and implicitly of non-constant solutions to (2)) is strongly dependent on the diffusion coefficients d_1, d_2 and on the parameters a, b. The most important issue in the study of steady-state solutions are the *a priori* estimates. Using a similar approach to that in [7], we are able to find precise upper and lower bounds for solutions to (2) in terms of a, b, d_1, d_2 for any dimension $N \geq 1$. This allows us to extend the study of the standard Brusselator system started in [4, 7, 18]. As a consequence, we are able to provide existence results in terms of a, b, d_1 and d_2 in case where f has a superlinear growth.

The outline of the paper is as follows. Section 2 is devoted to time-dependent solutions of (1). The main ingredient in proving the existence of such solutions if the invariant region method in the spirit of [23] (see also [26]) combined with *a priori* estimates. Then, uniform stability of the constant solution (3) is investigated. In Sec. 3, we discuss the existence and non-existence of non-constant solutions to the steady-state system (2). Here, we point out the role played by each parameter a, b, d_1 and $d_2 > 0$.

2. The Evolution System (1)

2.1. *Existence of global solutions*

In this part, we establish the existence of global solutions to (1). Our first result result concerns the case where f is sublinear.

Theorem 2.1. *Assume that f satisfies* (f1) *and* $\lim_{s\to\infty} f(s)/s = 0$. *Then, for any $a, b, d_1, d_2 > 0$ and any non-negative continuous functions u_0, v_0, the system* (1) *has at least one global solution.*

Proof. The proof relies on the invariant region method (see, e.g., [23, 26]). To this aim, we rewrite the system (1) is the vectorial form

$$\mathbf{w}_t = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Delta \mathbf{w} + F(\mathbf{w}) \quad \text{in } \Omega \times (0, \infty), \tag{4}$$

where $\mathbf{w} = (u, v)^T$ and

$$F(\mathbf{w}) = \begin{pmatrix} a - (b+1)u + f(u)v \\ bu - f(u)v \end{pmatrix}.$$

We claim that the rectangle $\Sigma := [0, c_1] \times [0, c_2]$ is an invariant region for (4) provided $c_1, c_2 > 0$ are large enough. In view of $(f1)$ we can choose $c_1 > \max\{2a, \|u_0\|_{L^\infty}\}$ such that

$$\frac{(b+1/2)c_1}{f(c_1)} > \|v_0\|_{L^\infty}$$

and define

$$c_2 := \frac{(b+1/2)c_1}{f(c_1)}.$$

We also write Σ as

$$\Sigma = \{\mathbf{w} = (u, v)^T \in C(\overline{\Omega}) \cap C(\overline{\Omega}) : G_i(\mathbf{w}) \le 0, 1 \le i \le 4\},$$

where

$$G_1(\mathbf{w}) = -u, \quad G_2(\mathbf{w}) = u - c_1, \quad G_3(\mathbf{w}) = -v, \quad G_4(\mathbf{w}) = v - c_2.$$

It is obviously that the initial data (u_0, v_0) belongs to the interior of Σ. If $\mathbf{w} = (u,v)^T \in \partial\Sigma$, by the definition of c_1 and c_2 we have

$$\begin{aligned}
\nabla G_1 \cdot F_{|_{u=0}} &= -a - f(0)v < 0,\\
\nabla G_2 \cdot F_{|_{u=c_1}} &= a - (b+1)c_1 + f(c_1)v \le a - (b+1)c_1 + f(c_1)c_2 = a - \frac{c_1}{2} \le 0,\\
\nabla G_3 \cdot F_{|_{v=0}} &= -bu \le 0,\\
\nabla G_4 \cdot F_{|_{v=c_2}} &= bu - f(u)v = u\left(b - \frac{f(u)}{u}\right) \le u\left(b - \frac{f(c_1)}{c_1}c_2\right) < 0.
\end{aligned}$$

By [23, Theorem 14.13] it follows that Σ is invariant for (4). Thus, there exists a global solution of (4). □

Next, we turn our attention to the case where f is superlinear. For the standard Brusselator model, that is, $f(u) = u^2$, the existence of global solution to (1) was obtained by Rothe [20]. Here, the existence of global solution to (1) is derived for more general nonlinearities f under the restriction $d_1 = d_2$ and the initial data u_0 is strictly positive in $\overline{\Omega}$.

Theorem 2.2. *Assume that $d_1 = d_2 > 0$, the initial data u_0, v_0 are continuous function in $\overline{\Omega}$ such that $u_0 > 0$, $v_0 \ge 0$ in $\overline{\Omega}$ and the nonlinearity f satisfies* (f2) *and* $\lim_{s\to 0} f(s)/s = 0$. *Then, for any $a, b > 0$, the system* (1) *has a global solution.*

Proof. With the change of variable we can assume $d_1 = d_2 = 1$. For $\varepsilon > 0$ small enough we consider the related problem

$$\begin{cases}
u_t - \Delta u = a - (b+1)u + f(u)v & \text{in } \Omega \times (0,\infty),\\
v_t - \Delta v = bu - f(u)v & \text{in } \Omega \times (0,\infty),\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x) + \varepsilon & \text{on } \Omega,\\
\dfrac{\partial u}{\partial \nu}(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty),
\end{cases} \tag{5}$$

By standard parabolic arguments, there exists a classical solution $(u^\varepsilon, u^\varepsilon)$ of (5) in a maximal interval $(0, T^\varepsilon_{\max})$. We claim that $T^\varepsilon_{\max} = \infty$. First, by (5) we have that U^ε satisfies

$$u^\varepsilon_t - \Delta u^\varepsilon + (b+1)u^\varepsilon \ge a > 0 \quad \text{in } \Omega \times (0, T^\varepsilon_{\max}).$$

Since $u_0 > 0$ in $\overline{\Omega}$, there exists a constant $k > 0$ independent of ε such that

$$u^\varepsilon \ge k \quad \text{in } \Omega \times (0, T^\varepsilon_{\max}). \tag{6}$$

Since $\lim_{s\to 0} f(s)/s = 0$, we can choose $k > 0$ small enough such that

$$v_0 + 1 \le \frac{bk}{f(k)} \quad \text{in } \overline{\Omega}. \tag{7}$$

The function v satisfies

$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon = bu^\varepsilon - f(u^\varepsilon)v^\varepsilon & \text{in } \Omega \times (0, T_{\max}^\varepsilon), \\ v^\varepsilon(x,0) = v_0(x) + \varepsilon & \text{on } \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \nu}(x,t) = 0 & \text{on } \partial\Omega \times (0, T_{\max}^\varepsilon). \end{cases} \tag{8}$$

Using (6) and that fact that f satisfies (f2), we can easily deduce that the interval

$$\Sigma := [0, bk/f(k)]$$

is an invariant region for Eq. (8). This means that

$$v(x,t) \le \frac{bk}{f(k)} = \text{const.} \quad \text{in } \overline{\Omega} \times (0, T_{\max}^\varepsilon). \tag{9}$$

Adding the first two equations in (5), we have

$$(u^\varepsilon + u^\varepsilon)_t - \Delta(u^\varepsilon + u^\varepsilon) + \frac{1}{d_1}(u^\varepsilon + u^\varepsilon) \le a + \frac{bk}{d_1 f(k)} \quad \text{in } \Omega \times (0, T_{\max}^\varepsilon).$$

By maximum principle we deduce that $u^\varepsilon + v^\varepsilon \le M$ in $\overline{\Omega} \times (0, T_{\max}^\varepsilon)$, for some constant $M > 0$ independent of ε. Therefore, for $\varepsilon > 0$ small enough, u^ε, u^ε satisfy

$$\varepsilon \le u^\varepsilon, v^\varepsilon \le M \quad \text{in } \overline{\Omega} \times (0, T_{\max}^\varepsilon).$$

This yields $T_{\max}^\varepsilon = \infty$, so u^ε and v^ε exist globally. Also by standard parabolic arguments and up to a subsequence, u^ε and v^ε converge to some functions u and v which are global solutions to (1). This finishes the proof of Theorem 2.2. □

2.2. *Stability of the uniform steady-state*

The linearization of (4) at $\mathbf{w}_0 = (a, ab/f(a))^T$ is

$$\mathbf{w}_t = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Delta \mathbf{w} + \nabla F(\mathbf{w}_0) \cdot \mathbf{w} + \mathcal{O}(\|\mathbf{w} - \mathbf{w}_0\|^2). \tag{10}$$

Denote by

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$$

the eigenvalues of $-\Delta$ with homogeneous Neumann boundary condition. For any $k \ge 0$ we also denote by $e(\mu_k)$ the multiplicity of μ_k. Consider

$$\mathbf{X} = \left\{ \mathbf{w} = (u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} \tag{11}$$

and decompose

$$\mathbf{X} = \bigoplus_{k \ge 0} \mathbf{X}_k, \tag{12}$$

where $\mathbf{X}_k$ denotes the eigenspace corresponding to μ_k, $k \ge 0$.

Theorem 2.3. *Assume that*

$$f(a) > \frac{baf'(a)}{f(a)} - b - 1 \tag{13}$$

and the first eigenvalue μ_1 *of the Dirichlet operator subject to homogeneous Neumann condition satisfies*

$$\mu_1 > \frac{1}{d_1}\left(\frac{baf'(a)}{f(a)} - b - 1\right) - \frac{f(a)}{d_2}. \tag{14}$$

Then the steady-state $\mathbf{w}_0$ *is uniformly asymptotically stable.*

Proof. Define $\mathcal{L} : \mathbf{X} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ by

$$\mathcal{L} = \begin{pmatrix} d_1\Delta + \dfrac{baf'(a)}{f(a)} - b - 1 & f(a) \\ b - \dfrac{baf'(a)}{f(a)} & d_2\Delta - f(a) \end{pmatrix}$$

Then $\mathbf{X}_k$ is invariant for $\mathcal{L}$ and ξ_k is an eigenvalue of $\mathcal{L}$ on $\mathbf{X}_k$ if and only if ξ is an eigenvalue of the matrix

$$A_k = \begin{pmatrix} -d_1\mu_k + \dfrac{baf'(a)}{f(a)} - b - 1 & f(a) \\ b - \dfrac{baf'(a)}{f(a)} & -d_2\mu_k - f(a) \end{pmatrix}.$$

The determinant and trace of A_k are

$$\begin{aligned} \det(A_k) &= \mu_k\left[d_1 d_2 \mu_k + d_1 f(a) - d_2\left(\frac{baf'(a)}{f(a)} - b - 1\right)\right] + f(a), \\ \mathrm{Tr}(A_k) &= \frac{baf'(a)}{f(a)} - b - 1 - f(a) - (d_1 + d_2)\mu_k. \end{aligned} \tag{15}$$

Remark that for any $k \geq 0$ we have

$$\det(A_k) > 0 > \mathrm{Tr}(A_k).$$

Denote by ξ_k^+ and ξ_k^- the two eigenvalues of A_k, $k \geq 0$.

If ξ_k^+, ξ_k^- are complex numbers, then by (14) we have

$$\begin{aligned} \mathrm{Re}(\xi_k^+) = \mathrm{Re}(\xi_k^-) &= \frac{1}{2}\mathrm{Tr}(A_k) \\ &\leq \frac{1}{2}\left(\frac{baf'(a)}{f(a)} - b - 1 - f(a)\right) < 0. \end{aligned}$$

If ξ_k^+, ξ_k^- are real numbers, then by (14) we have

$$\begin{aligned}
\xi_k^- \le \xi_k^+ &= \frac{\mathrm{Tr}(A_k) + \sqrt{\mathrm{Tr}^2(A_k) - 4\det(A_k)}}{2} \\
&= \frac{2\det(A_k)}{\mathrm{Tr}(A_k) - \sqrt{\mathrm{Tr}^2(A_k) - 4\det(A_k)}} \\
&\le \frac{\det(A_k)}{\mathrm{Tr}(A_k)} \\
&< 0.
\end{aligned}$$

Since $\mu_k \to \infty$ as $k \to \infty$, from the above estimate we deduce $\xi_k^+ \to -\infty$ as $k \to \infty$.

Hence, in both the above cases we can find $\delta > 0$ such that the spectrum of $\mathcal{L}$ lies in the region $\{z \in \mathbb{C} : \mathrm{Re}(z) < -\delta\}$. By [9, Theorem 5.1.1], we obtain that $\mathbf{w}_0$ is asymptotically uniformly stable for (4). This ends the proof. □

If f satisfies (f1) then $\frac{baf'(a)}{f(a)} - b - 1 < 0$ so that both conditions (13) and (14) are satisfied. In this case we obtain.

Corollary 2.4. *If f satisfies* (f1) *then $\mathbf{w}_0$ is uniformly asymptotically stable.*

2.3. *Diffusion-driven instability*

In this part, we point out that under certain conditions on the parameters a and b, the uniform steady-state (u_0, v_0) defined by (3) can be linearly stable in the absence of diffusion but unstable in the presence of diffusion. This is the well-known phenomenon of *diffusion-driven instability* emphasized by Turing in his pioneering work [24].

Let us consider the spatially homogeneous system corresponding to (1):

$$\begin{cases} \dfrac{du}{dt} = a - (b+1)u + f(u)v\,, & t > 0, \\ \dfrac{dv}{dt} = bu - f(u)v\,, & t > 0, \end{cases} \tag{16}$$

We have the following result.

Theorem 2.5. *Assume that*

$$f(a) > \frac{baf'(a)}{f(a)} - b - 1 > 0. \tag{17}$$

Then, there exist $d^*, D^* > 0$ such that for all

$$0 < d_1 < d^* \quad \textit{and} \quad d_2 > D^*,$$

the steady-state $\mathbf{w}_0 = (a, ba/f(a))^T$ is uniformly asymptotically stable for the system (16) *and instable for the system* (1), *that is, Turing instabilities occur.*

Remark that (17) does not hold if f satisfies (f1).

Proof. Using the same approach as in Theorem 2.3 we have that $\mathbf{w}_0$ is uniformly asymptotically stable for (16) provided (17) holds. Also by (17) we can choose $D^* > 0$ large enough such that

$$\mu_1 D^* \left(\frac{baf'(a)}{f(a)} - b - 1 \right) > f(a).$$

Using (15), for all $d_2 > D^*$ we have

$$\lim_{d_1 \searrow 0} \det(A_1) \leq f(a) - \mu_1 D^* \left(\frac{baf'(a)}{f(a)} - b - 1 \right) < 0.$$

Therefore we can find $d^* > 0$ such that

$$\det(A_1) < 0 \quad \text{for all } 0 < d_1 < d^*, \;\; d_2 > D^*.$$

This implies that A_1, and so the operator $\mathcal{L}$, has at least one positive eigenvalue. By [9, Corollary 5.1.1] it follows that $\mathbf{w}_0$ is uniformly asymptotically instable. This finishes the proof. □

3. Steady-State Solutions

In this section, we shall be concerned with steady-state solutions to (1). Basic to our subsequent analysis is the following result which is due to Lou and Ni (see [13, Proposition 2.2] or [14, Lemma 2.1]).

Lemma 3.1. *Let $g \in C^1(\overline{\Omega} \times \mathbb{R})$.*

(1) *If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta w + g(x, w) \geq 0 \quad \textit{in } \Omega, \qquad \frac{\partial w}{\partial n} \leq 0 \quad \textit{on } \partial\Omega,$$

and $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.

(2) *If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta w + g(x, w) \leq 0 \quad \textit{in } \Omega, \qquad \frac{\partial w}{\partial n} \geq 0 \quad \textit{on } \partial\Omega,$$

and $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Using the above result, we first derive that if f satisfies (f1) then (2) has no non-constant solutions. More precisely we have.

Theorem 3.2. *Assume that f satisfies* (f1). *Then, $(u, v) = (a, \frac{ab}{f(a)})$ is the unique solution of system* (2).

Proof. Let (u, v) be a classical solution of (2). Let also x_1 (respectively, x_2) be a maximum point of u (respectively, v) and x_3 (respectively, x_4) be a minimum point of u (respectively, v) in $\overline{\Omega}$. Using Lemma 3.1(i) in the first equation of (2) we have

$$(b + 1)u(x_1) \leq a + f(u(x_1))v(x_1). \tag{18}$$

Now, Lemma 3.1(i) applied to the second equation in (2) yields

$$bu(x_2) \geq f(u(x_2))v(x_2),$$

that is, $v(x_2) \leq b\frac{u(x_2)}{f(u(x_2))}$. By virtue of (f1) we next derive

$$v(x_1) \leq v(x_2) \leq b\frac{u(x_2)}{f(u(x_2))} \leq b\frac{u(x_1)}{f(u(x_1))}. \tag{19}$$

Therefore (18) and (19) imply $(b+1)u(x_1) \leq a + bu(x_1)$, that is,

$$u \leq u(x_1) \leq a \quad \text{in } \Omega. \tag{20}$$

On the other hand, Lemma 3.1(ii) applied to the second equation of (2) leads us to $v(x_4) \geq b\frac{u(x_4)}{f(u(x_4))}$. Again by (f1) it follows that

$$v(x_3) \geq v(x_4) \geq b\frac{u(x_4)}{f(u(x_4))} \geq b\frac{u(x_3)}{f(u(x_3))}. \tag{21}$$

Next, Lemma 3.1(ii) applied to the first equation in (2) yields

$$(b+1)u(x_3) \geq a + f(u(x_3))v(x_3) \geq a + bu(x_3),$$

which implies

$$u \geq u(x_3) \geq a \quad \text{in } \Omega. \tag{22}$$

Now (20) and (22) produce $u \equiv a$ in Ω and by (2) we also have $v \equiv ab/f(a)$. This ends the proof. $\square$

When f satisfies (f2) the analysis of the steady-state system (2) is more delicate. In some cases, depending of the parameters a, b, d_1, d_2 we obtain the existence of non-constant solutions to (1). We start this study with the following crucial result that provides *a priori* estimates for solutions to (2).

Theorem 3.3. *Assume that f satisfies* (f2)*. Then, any solution (u, v) of* (2) *satisfies*

$$\frac{a}{b+1} \leq u \leq a + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)f\left(\frac{a}{b+1}\right)} \quad \textit{in } \Omega, \tag{23}$$

and

$$\frac{ab}{(b+1)f\left(a + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)f\left(\frac{a}{b+1}\right)}\right)} \leq v \leq \frac{ab}{(b+1)f\left(\frac{a}{b+1}\right)} \quad \textit{in } \Omega. \tag{24}$$

Proof. Consider first a minimum point $x_0 \in \overline{\Omega}$ of u. By Lemma 3.1(ii) it follows

$$a - (b+1)u(x_0) + f(u(x_0))v(x_0) \leq 0$$

which implies $u(x_0) \geq a/(b+1)$. Hence

$$u \geq \frac{a}{b+1} \quad \text{in } \Omega. \tag{25}$$

At maximum point of v we have $bu - f(u)v \geq 0$, that is, $v \leq bu/f(u)$. By virtue of (f2) and (25) we deduce

$$v \leq \frac{ab}{(b+1)f\left(\dfrac{a}{b+1}\right)} \quad \text{in } \Omega. \tag{26}$$

Let $w = d_1 u + d_2 v$. Adding the first two relations in (2) we have

$$-\Delta w = a - u \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Let now $x_1 \in \overline{\Omega}$ be a maximum point of w. According to Lemma 3.1(i) we have $a - u(x_1) \geq 0$, that is, $u(x_1) \leq a$. By virtue of (26), for all $x \in \overline{\Omega}$ we have

$$d_1 u(x) \leq w(x) \leq w(x_1) \leq d_1 a + d_2 \cdot \frac{ab}{(b+1)f\left(\dfrac{a}{b+1}\right)} \quad \text{in } \Omega.$$

This yields

$$u \leq a + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)f\left(\dfrac{a}{b+1}\right)} \quad \text{in } \Omega. \tag{27}$$

We have proved that u satisfies (23). Again by Lemma 3.1(ii), at minimum points of v we have $bu - f(u)v \leq 0$, which yields $v \geq bu/f(u)$. Combining this inequality with (27) we obtain the first estimate in (24). This concludes our proof. □

From the estimates (23) and (24) in Theorem 3.3 we derive the following:

Proposition 3.4. *Assume that f satisfies* (f2) *and let $a, b, D_1, D_2 > 0$ be fixed. Then, there exist two positive constants $C_1, C_2 > 0$ depending on a, b, D_1, D_2 such that for all*

$$d_1 \geq D_1, \quad 0 < d_2 \leq D_2,$$

any solution (u, v) of (2) *satisfies*

$$C_1 < u, v < C_2 \quad \text{in } \overline{\Omega}.$$

Furthermore, by standard elliptic arguments and Theorem 3.3 we now obtain:

Proposition 3.5. *Assume that f satisfies* (f2) *and let $a, b, D_1, D_2 > 0$ be fixed. Then, for any positive integer $k \geq 1$ there exists a constant*

$$C = C(a, b, D_1, D_2, k, N, \Omega) > 0$$

such that for all

$$d_1 \geq D_1, \quad 0 < d_2 \leq D_2,$$

any solution (u, v) of (2) satisfies

$$\|u\|_{C^k(\overline{\Omega})} + \|v\|_{C^k(\overline{\Omega})} \le C.$$

In particular, any solution of (2) belongs to $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

Theorem 3.6. (i) *Let $a, b, d_2 > 0$ be fixed. There exists $D = D(a, b, d_2) > 0$ such that system (2) has no non-constant solutions for all $d_1 > D$.*

(ii) *Let $a, d_1, d_2 > 0$ be fixed. There exists $B = B(a, d_1, d_2) > 0$ such that system (2) has no non-constant solutions for all $0 < b < B$.*

Proof. (i) Remark first that if (u, v) is a solution of (2), then, integrating the two equations in (2) over Ω and adding them up we have

$$\int_\Omega u(x)dx = a|\Omega|. \tag{28}$$

Lemma 3.7. *Let $a, b, d_2 > 0$ be fixed and let $\{\delta_n\} \subset (0, \infty)$ be such that $\delta_n \to \infty$ as $n \to \infty$. If (u_n, v_n) is a solution of (2) with $d_1 = \delta_n$ then*

$$(u_n, v_n) \to \left(a, \frac{ab}{f(a)}\right) \quad \text{in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \quad \text{as } n \to \infty. \tag{29}$$

Proof. By Proposition 3.5, the sequence $\{(u_n, v_n)\}$ is bounded in $C^3(\overline{\Omega}) \times C^3(\overline{\Omega})$. Hence, passing to a subsequence if necessary, $\{(u_n, v_n)\}$ converges in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ to some $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. We divide by δ_n in the corresponding equation to u_n and then we pass to the limit with $n \to \infty$. We obtain that (u, v) satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ -d_2 \Delta v = bu - f(u)v & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Also, u_n and u satisfy (28). Now, the first equation in (30) together with $\partial u/\partial \nu = 0$ on $\partial\Omega$ implies that u is constant. Combining this fact with (28) it follows that $u \equiv a$. Thus, from (30), v satisfies

$$-d_2 \Delta v = ab - f(a)v \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Multiplying the above equality with $ab - f(a)v$ and then integrating over Ω we obtain

$$0 \le \frac{d_2}{f(a)} \int_\Omega |\nabla(ab - f(a)v)|^2 dx = -\int_\Omega (ab - f(a)v)^2 dx \le 0.$$

Hence $v \equiv \frac{ab}{f(a)}$ and the proof follows. □

We first introduce the function spaces

$$H^2_{\mathrm{n}}(\Omega) = \left\{ w \in W^{2,2}(\Omega) : \frac{\partial w}{\partial \nu} = 0 \right\}, \quad L^2_0(\Omega) = \left\{ w \in L^2(\Omega) : \int_\Omega w = 0 \right\}.$$

Thus, letting $w = u - a$, by (28) and the standard elliptic regularity, system (2) is equivalent to

$$\begin{cases} -\Delta w = \delta(a - (b+1)(w+a) + f(w+a)v) & \text{in } \Omega, \\ -d_2\Delta v = b(w+a) - f(w+a)v & \text{in } \Omega, \\ w \in H^2_{\rm n}(\Omega) \cap L^2_0(\Omega), \quad v \in H^2_{\rm n}(\Omega), \end{cases} \tag{31}$$

where $\delta = 1/d_1$. Define

$$\mathcal{F} : \mathbb{R} \times (H^2_{\rm n}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\rm n}(\Omega) \to L^2_0(\Omega) \times L^2(\Omega),$$

by

$$\mathcal{F}(\delta, w, v) = \begin{pmatrix} \Delta w + \delta\mathcal{P}(a - (b+1)(w+a) + f(w+a)v) \\ d_2\Delta v + b(w+a) - f(w+a)v \end{pmatrix},$$

where $\mathcal{P} : L^2(\Omega) \to L^2_0(\Omega)$ is the projection operator from $L^2(\Omega)$ onto $L^2_0(\Omega)$, namely,

$$\mathcal{P}(z) = z - \frac{1}{|\Omega|}\int_\Omega z(x)dx, \quad \text{for all } z \in L^2(\Omega).$$

Now (31) is equivalent to

$$\mathcal{F}(\delta, w, v) = \mathbf{0}. \tag{32}$$

Indeed, if $\mathcal{F}(\delta, w, v) = \mathbf{0}$, then

$$d_2\Delta v + b(w+a) - f(w+a)v = 0 \quad \text{in } \Omega, \quad v \in H^2_{\rm n}(\Omega).$$

It is easy to see that the above relations imply $b(w+a) - f(w+a)v \in L^2_0(\Omega)$. Since $w \in L^2_0(\Omega)$, this yields

$$a - (b+1)(w+a) + f(w+a)v \in L^2_0(\Omega),$$

so that

$$\mathcal{P}(a - (b+1)(w+a) + f(w+a)v) = a - (b+1)(w+a) + f(w+a)v.$$

Therefore (31) is satisfied.

With the same method as in the proof of Lemma 3.7 we have that the equation $\mathcal{F}(0, w, v) = \mathbf{0}$ has the unique solution $(w, v) = (0, ab/f(a))$. Next it is easy to see that

$$D_{(w,v)}\mathcal{F}(0, 0, ab/f(a)) : (H^2_{\rm n}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\rm n}(\Omega) \to L^2_0(\Omega) \times L^2(\Omega),$$

is given by

$$D_{(w,v)}\mathcal{F}(0, 0, ab/f(a)) = \begin{pmatrix} \Delta & 0 \\ b\dfrac{f(a) - af'(a)}{f(a)} & d_2\Delta - f(a) \end{pmatrix}.$$

Thus $D_{(w,v)}\mathcal{F}(0, 0, ab/f(a))$ is invertible and we are in the frame of the Implicit Function Theorem. It follows that there exists $\delta_0, r > 0$ such that $(0, 0, ab/f(a))$ is the unique solution of

$$\mathcal{F}(\delta, w, v) = \mathbf{0} \quad \text{in } [0, \delta_0] \times B_r\left(0, \frac{ab}{f(a)}\right),$$

where $B_r(0, \frac{ab}{f(a)})$ denotes the open ball in $(H^2_{\rm n}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\rm n}(\Omega)$ centered at $(0, ab/f(a))$ and having the radius $r > 0$.

Let now $\{\delta_n\}$ be a sequence of positive real numbers such that $\delta_n \to \infty$ as $n \to \infty$ and let (u_n, v_n) be an arbitrary solution of (2) for a, b, d_2 fixed and $d_1 = \delta_n$. Letting $w_n = u_n - a$, it follows that

$$\mathcal{F}\left(\frac{1}{\delta_n}, w_n, v_n\right) = \mathbf{0}.$$

According to Lemma 3.7 we have

$$(w_n, v_n) \to \left(0, \frac{ab}{f(a)}\right) \quad \text{in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \quad \text{as } n \to \infty.$$

This means that for $n \geq 1$ large enough there holds $(1/\delta_n, w_n, v_n) \in (0, \delta_0) \times B_r(0, \frac{ab}{f(a)})$ which yields $(w_n, v_n) = (0, \frac{ab}{f(a)})$. Hence, for $d_1 = 1/\delta_n$ small enough, system (2) has only the constant solution $(a, \frac{ab}{f(a)})$. The proof of (ii) is similar. $\square$

3.1. *Existence results*

Let $\mathbf{X}$ be the space defined in (11) and let

$$\mathbf{X}^+ = \{(u, v) \in X \,:\, u, v > 0 \text{ in } C(\overline{\Omega})\}.$$

We write the system (2) in the form

$$-\Delta \mathbf{w} = \mathcal{G}(\mathbf{w}), \quad \mathbf{w} \in \mathbf{X}^+, \tag{33}$$

where

$$\mathcal{G}(\mathbf{w}) = \begin{pmatrix} \dfrac{1}{d_1}(a - (b+1)u + f(u)v) \\ \dfrac{1}{d_2}(bu - f(u)v) \end{pmatrix}.$$

It is more convenient to write (33) in the form

$$\mathcal{F}(\mathbf{w}) = \mathbf{0}, \quad \mathbf{w} \in \mathbf{X}^+, \tag{34}$$

where

$$\mathcal{F}(\mathbf{w}) = \mathbf{w} - (\mathbf{I} - \Delta)^{-1}(\mathcal{G}(\mathbf{w}) + \mathbf{w}), \quad \mathbf{w} \in \mathbf{X}^+. \tag{35}$$

Let $\mathbf{w}_0 = (a, ab/f(a))^T$ be the uniform steady state solution of (2). Then

$$\nabla \mathcal{F}(\mathbf{w}_0) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\mathbf{I} + A),$$

where

$$A := \nabla\mathcal{G}(\mathbf{w}_0) = \begin{pmatrix} \frac{1}{d_1}\left(b\frac{af'(a)-f(a)}{f(a)} - 1\right) & \frac{f(a)}{d_1} \\ -\frac{b}{d_2}\frac{af'(a)-f(a)}{f(a)} & -\frac{f(a)}{d_2} \end{pmatrix}.$$

If $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible, by [15, Theorem 2.8.1] the index of $\mathcal{F}$ at $\mathbf{w}_0$ is given by

$$\mathrm{index}(\mathcal{F}, \mathbf{w}_0) = (-1)^\gamma, \tag{36}$$

where γ denotes the number of the negative eigenvalues of $\nabla\mathcal{F}(\mathbf{w}_0)$. On the other hand, using the decomposition (12) we have that $\mathbf{X}_i$ is an invariant space under $\nabla\mathcal{F}(\mathbf{w}_0)$ and $\xi \in \mathbb{R}$ is an eigenvalue of $\nabla\mathcal{F}(\mathbf{w}_0)$ in $\mathbf{X}_i$ if and only if ξ is an eigenvalue of $(\mu_i+1)^{-1}(\mu_i\mathbf{I} - A)$. Therefore, $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible if and only if for any $i \geq 0$ the matrix $(\mu_i\mathbf{I} - A)$ is invertible.

Let us define

$$H(a, b, d_1, d_2, \mu) = \det(\mu\mathbf{I} - A). \tag{37}$$

Then, if $(\mu_i\mathbf{I} - A)$ is invertible for any $i \geq 0$, with the same arguments as in [17] we have

$$\gamma = \sum_{\substack{i\geq 0,\\ H(a,b,d_1,d_2,\mu_i)<0}} e(\mu_i). \tag{38}$$

A straightforward computation yields

$$H(a, b, d_1, d_2, \mu) = \mu^2 - \left(\frac{abf'(a) - (b+1)f(a)}{d_1 f(a)} - \frac{f(a)}{d_2}\right)\mu + \frac{f(a)}{d_1 d_2}.$$

If

$$b\frac{af'(a) - f(a)}{f(a)} > \left(1 + \sqrt{\frac{d_1}{d_2}f(a)}\right)^2, \tag{39}$$

then the equation $H(\mu) = 0$ has two positive solutions $\mu^\pm(a, b, d_1, d_2)$ given by

$$\mu^\pm(a, b, d_1, d_2) = \frac{1}{2}(\theta(a, b, d_1, d_2) \pm \sqrt{\theta(a, b, d_1, d_2)^2 - 4f(a)/(d_1 d_2)}),$$

where

$$\theta(a, b, d_1, d_2) = \frac{abf'(a) - (b+1)f(a)}{d_1 f(a)} - \frac{f(a)}{d_2}.$$

With the same method as in [17] (see also [7, 18]) we have the following result.

Theorem 3.8. *Assume that condition* (39) *holds and there exist* $i > j \geq 0$ *such that*

(i) $\mu_i < \mu^+(a, b, d_1, d_2) < \mu_{i+1}$ *and* $\mu_j < \mu^-(a, b, d_1, d_2) < \mu_{j+1}$;
(ii) $\sum_{k=j+1}^{i} e_k$ *is odd.*

Then (2) *has at least one non-constant solution.*

Proof. The proof uses some topological degree arguments (see [2, 3]). By Theorem 3.6(i) we can fix $D > d_1$ such that

(a) system (2) with diffusion coefficients D and d_2 has no non-constant solutions;
(b) $H(a, b, D, d_2, \mu) > 0$ for all $\mu \geq 0$.

Further, by Proposition 3.4 one can find $C_1, C_2 > 0$ depending on a, b, d_1, d_2 such that for any $d \geq d_1$, any solution (u, v) of (2) with diffusion coefficients d and d_2 satisfies

$$C_1 < u, v < C_2 \quad \text{in } \overline{\Omega}.$$

Set

$$\mathcal{M} = \{(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : C_1 < u, v < C_2 \text{ in } \overline{\Omega}\},$$

and define

$$\Psi : [0, 1] \times \mathcal{M} \to C(\overline{\Omega}) \times C(\overline{\Omega}),$$

by

$$\Psi(t, \mathbf{w}) = (-\Delta + \mathbf{I})^{-1} \begin{pmatrix} u + \left(\dfrac{1-t}{D} + \dfrac{t}{d_1}\right)(a - (b+1)u + f(u)v) \\ v + \dfrac{1}{d_2}(bu - f(u)v) \end{pmatrix}.$$

It is easy to see that solving (2) is equivalent to find a fixed point of $\Psi(1, \cdot)$ in $\mathcal{M}$. Further, from the definition of $\mathcal{M}$ and Proposition 3.4, we have that $\Psi(t, \cdot)$ has no fixed points in $\partial\mathcal{M}$ for all $0 \leq t \leq 1$. Therefore, the Leray–Schauder topological degree $\deg(\mathbf{I} - \Psi(t, \cdot), \mathcal{M}, 0)$ is well defined.

Using (35) we have $\mathbf{I} - \Psi(1, \cdot) = \mathcal{F}$. Thus, if (2) has no other solutions except the constant one $\mathbf{w}_0$, then by (36) and (38) we have

$$\deg(\mathbf{I} - \Psi(1, \cdot), \mathcal{M}, 0) = \operatorname{index}(\mathcal{F}, \mathbf{w}_0) = (-1)^{\sum_{k=j+1}^{i} e(\mu_k)} = -1. \tag{40}$$

On the other hand, from the invariance of the Leray–Schauder degree at the homotopy we deduce

$$\deg(\mathbf{I} - \Psi(1, \cdot), \mathcal{M}, 0) = \deg(\mathbf{I} - \Psi(0, \cdot), \mathcal{M}, 0). \tag{41}$$

Remark that by our choice of D, we have that $\mathbf{w}_0$ is the only fixed point of $\Psi(0, \cdot)$. Furthermore by (b) above we have

$$\deg(\mathbf{I} - \Psi(0, \cdot), \mathcal{M}, 0) = \operatorname{index}(\mathbf{I} - \Psi(\cdot, 0), \mathbf{w}_0) = 1. \tag{42}$$

Now, from (40)–(42) we reach a contradiction. Therefore, there exists a non-constant solution of (2). This ends the proof. □

Corollary 3.9. *Let $a, b, d_2 > 0$ be fixed. Assume that*

$$abf'(a) > (b+1)f(a) \tag{43}$$

and all the eigenvalues μ_i have odd multiplicity. Then, there exists a sequence of intervals $\{(k_n, K_n)\}$ with $0 < k_n < K_n < k_{n-1} \to 0$ (as $n \to \infty$) such *that the steady-state system* (2) *has at least one non-constant solution for all* $d_1 \in \bigcup_{n\geq 1}(k_n, K_n)$.

Proof. In view of (43), condition (39) holds for small values of $d_1 > 0$. Also for $a, b, d_2 > 0$ fixed we have

$$\mu^-(a,b,d_1,d_2) \to \frac{f(a)^2}{d_2(abf'(a) - (b+1)f(a))} \quad \text{as } d_1 \to 0.$$
$$\mu^+(a,b,d_1,d_2) \to \infty \quad \text{as } d_1 \to 0.$$

Therefore we can find a sequence of intervals $\{(k_n, K_n)\}_n$ such that

$$\sum_{\substack{i\geq 0,\\ \mu^-(a,b,d_1,d_2)<\mu_i<\mu^+(a,b,d_1,d_2)}} e(\mu_i) \quad \text{is odd} \tag{44}$$

for all $d_1 \in \bigcup_{n\geq 1}(k_n, K_n)$. Therefore, conditions (i) and (ii) in Theorem 3.8 are fulfilled. □

Corollary 3.10. *Let* $a, b, d_1 > 0$ *be fixed. Assume that* (43) *holds and*

$$\sum_{\substack{i\geq 0,\\ 0<\mu_i<\frac{abf'(a)-(b+1)f(a)}{d_1 f(a)}}} e(\mu_i) \quad \textit{is odd.} \tag{45}$$

Then there exists $D > 0$ *such that the steady-state system* (2) *has at least one non-constant solution for any* $d_2 > D$.

Proof. By virtue of (43), for any $d_2 > 0$ large enough condition (39) holds. Also for any a, b, d_1 fixed we have

$$0 < \mu^-(a,b,d_1,d_2) < \mu^+(a,b,d_1,d_2) < \frac{abf'(a) - (b+1)f(a)}{d_1 f(a)}$$

and

$$\mu^-(a,b,d_1,d_2) \to 0, \quad \mu^+(a,b,d_1,d_2) \to \frac{abf'(a) - (b+1)f(a)}{d_1 f(a)} \quad \text{as } d_2 \to \infty.$$

Therefore, for $d_2 > 0$ large, condition (45) implies (i) and (ii) in Theorem 3.8. This concludes the proof. □

Corollary 3.11. *Let* $a, d_1, d_2 > 0$ *be fixed. Assume that* $af'(a) > f(a)$ *and all the eigenvalues* μ_i *have odd multiplicity. Then, there exists a sequence of intervals* $\{(b_n, B_n)\}$ *with* $0 < b_n < B_n < b_{n+1} \to \infty$ (*as* $n \to \infty$) *such that the steady-state system* (2) *has at least one non-constant solution for all* $b \in \cup_{n\geq 1}(b_n, B_n)$.

Proof. We proceed similarly. Since $af'(a) > f(a)$, for large values of b condition (39) is fulfilled. Also for $a, d_1, d_2 > 0$ fixed we have

$$\mu^-(a,b,d_1,d_2) \to 0, \quad \mu^+(a,b,d_1,d_2) \to \infty \quad \text{as } b \to \infty.$$

Hence, we can find a sequence of non-overlapping intervals $\{(b_n, B_n)\}$ such that $b_n \to \infty$ as $n \to \infty$ and (44) holds for all $b \in \bigcup_{n\geq 1}(b_n, B_n)$. $\square$

If $f(s) = s^m$, $m > 1$, then condition (43) is independent of a. We obtain

Corollary 3.12. *Let $f(s) = s^m$, $m > 1$. Assume that $b(m-1) > 1$ and*

$$\sum_{\substack{i\geq 0,\\ 0<\mu_i<(b(m-1)-1)/d_1}} e(\mu_i) \quad \textit{is odd.} \tag{46}$$

Then there exists $A > 0$ such that the steady-state system (2) has at least one non-constant solution for any $0 < a < A$.

Proof. It is easy to see that (39) holds for small values of $a > 0$. As before

$$0 < \mu^-(a, b, d_1, d_2) < \mu^+(a, b, d_1, d_2) < \frac{b(m-1)-1}{d_1}$$

and

$$\mu^-(a, b, d_1, d_2) \to 0\,, \quad \mu^+(a, b, d_1, d_2) \to \frac{b(m-1)-1}{d_1} \quad \text{as } a \to 0.$$

Therefore, for $a > 0$ small, condition (46) implies (i) and (ii) in Theorem 3.8. This ends the proof. $\square$

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