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On a $p(\cdot)$-biharmonic problem with no-flux boundary condition

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ABSTRACT

The study of fourth order partial differential equations has flourished in the last years, however, a $p(\cdot)$-biharmonic problem with no-flux boundary condition has never been considered before, not even for constant p. This is an important step further, since surfaces that are impermeable to some contaminants are appearing quite often in nature, hence the significance of such boundary condition. By relying on several variational arguments, we obtain the existence and the multiplicity of weak solutions to our problem. We point out that, although we use a mountain pass type theorem in order to establish the multiplicity result, we do not impose an Ambrosetti–Rabinowitz type condition, nor a symmetry condition, on our nonlinearity f.

1. Introduction

Fourth order PDEs have various applications, to micro-electro-mechanical systems, phase field models of multiphase systems, thin film theory, thin plate theory, surface diffusion on solids, interface dynamics, flow in Hele–Shaw cells, see for example [1–3]. Therefore many authors focused on the study of such problems with constant exponents, like Molica Bisci and Repovš [4], Candito and Molica Bisci [5], or Liu and Squassina [6] etc. At the same time, many applications are generated by the elliptic problems with variable exponents, which have a large range of applications, due to electrorheological fluids [7–15], thermorheological fluids [16], elastic materials [17,18], image restoration [19], mathematical biology [20], dielectric breakdown and electrical resistivity [21], polycrystal plasticity [22] and sandpile growth [23]. At the interplay of these two research directions, a natural interest goes to the $p(\cdot)$-biharmonic problems. This trend is quite fresh, starting probably in 2009, with the papers [24,25], where the authors considered problems with the Navier boundary condition

$$u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

The line of investigation was continued by several authors, see [26–33]. Notice that all these studies focus on problems with the Navier boundary condition (1) and only one of them, [29], also considers the Neumann type boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 \quad \text{on } \partial\Omega.$$

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But if we think at the applicability to real-life situations, when the surfaces are impermeable to some contaminants, we are drawn to the no-flux boundary problems. Hence, inspired by the previous studies [34,35], where second order problems with no-flux boundary conditions are treated in the framework of the variable exponent spaces, we propose the following problem.

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{for} \quad x \in \Omega, \\ u \equiv \text{constant}, \qquad \Delta u = 0 & \text{for} \quad x \in \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u)\, dS = 0, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with sufficiently smooth boundary, $\lambda > 0$, and the exponent p is log-Hölder continuous, that is, for each $i \in \{1, \ldots, N\}$ there exists $\bar{c} > 0$ such that

$$|p(x) - p(y)| \leq \frac{\bar{c}}{-\log|x-y|} \quad \text{for all } x,\ y \in \Omega,\ \ 0 < |x-y| \leq \frac{1}{2},$$

and

$$1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty \quad \text{for all } x \in \Omega.$$

For simplicity, we denote

$$h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x) \quad \text{and} \quad h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x).$$

We will work under the following hypotheses:

(H1) $a \in L^\infty(\Omega)$ and there exists $a_0 > 0$ such that $a(x) \geq a_0$ for all $x \in \Omega$;
(H2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $t_0 > 0$ and a ball B with $\overline{B} \subset \Omega$ such that

$$\int_B F(x, t_0)\, dx > 0,$$

where F represents the antiderivative of f, that is, $F(x,t) = \int_0^t f(x,s)\, ds$;
(H3) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p(x)-1}} = 0$ uniformly with respect to $x \in \Omega$;
(H4) $\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p(x)-1}} = 0$ uniformly with respect to $x \in \Omega$.

Note that all the necessary details regarding the definition and the properties of the variable exponent spaces involved in the investigation of our problem will be provided in the next section. It is worth mentioning though, that, since the class of problems represented by (2) was not introduced before, not even for the constant case, we will need to introduce a new space on which it is more appropriate to search for weak solutions to (2). Depending on the values taken by λ, we establish an existence and a multiplicity result. For the existence result we rely on a classical theorem from the field of calculus of variations, sometimes referred to as a Weierstrass-type theorem. For the second solution, we make use of a mountain pass type theorem, without imposing the usual Ambrosetti–Rabinowitz growth condition, that is, there exist $\theta > p^+$ and $l > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t \quad \text{for all } |t| > l \text{ and a.e. } x \in \Omega.$$

The celebrated mountain pass theorem of Ambrosetti and Rabinowitz has provided lots of applications during the years and represents the key ingredient to the weak solvability for numerous problems. However, for the multiplicity of solutions, all the adaptations of the mountain pass theorem are relying on additional symmetry conditions on the nonlinearity f:

$$f(x,-t) = -f(x,t) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{3}$$

with the help of which we can get the existence of an unbounded sequence of weak solutions. This was the case for the fourth order PDEs with variable exponent treated by [26,28,24,25,30]. Other multiplicity results, which do not impose condition (3) on the nonlinearity, were provided due to various three critical points theorems of Ricceri type, see [27,29]. Our problem is the first variable exponent problem of fourth order for which the multiplicity of solutions is obtained by applying a different strategy. For second order problems with variable exponents for which the same strategy is applied we refer to [34–36].

2. Some preliminaries

We introduce some notation that will clarify what follows. Thus, when we refer to a Banach space X, we denote by $X^\star$ its dual and by $\langle \cdot, \cdot \rangle$ the duality pairing between $X^\star$ and X. By $|\cdot|$ we denote the absolute value of a number, or the Euclidean norm when it is defined on $\mathbb{R}^N$ ($N \geq 2$), respectively the Lebesgue measure, when it is applied to a set.

We recall the definitions of the variable exponent Lebesgue and Sobolev spaces and some of their basic properties, but much more details can be found in the comprehensive works [37–39]. As stated from the beginning, everywhere below we consider p to be log-Hölder continuous with $1 < p^- \le p^+ < \infty$.

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \ : \ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

This space is equipped with the Luxemburg norm,

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0 : \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\},$$

and it is a separable and reflexive Banach space, see [40, Theorem 2.5, corollary 2.7]. Also, we have the following continuous embedding result.

Theorem 1 *([40, Theorem 2.8]). If $0 < |\Omega| < \infty$ and $p_1, p_2 \in C(\overline{\Omega}; \mathbb{R})$, $1 \le p_i^- \le p_i^+ < \infty$ $(i = 1, 2)$, are such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.*

The $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space is represented by $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx,$$

and we have some useful properties connecting this application to the Luxemburg norm, see for example [41, Theorem 1.3, Theorem 1.4]. If $u \in L^{p(\cdot)}(\Omega)$, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1\ (= 1;\ > 1) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1); \tag{4}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \ \Rightarrow\ \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}; \tag{5}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \ \Rightarrow\ \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}; \tag{6}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} \to 0\ (\to \infty) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) \to 0\ (\to \infty). \tag{7}$$

If, in addition, $(u_n)_n \subset L^{p(\cdot)}(\Omega)$, then

$$\begin{aligned} &\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \ \Leftrightarrow\ \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0 \\ &\Leftrightarrow\ (u_n)_n \text{ converges to } u \text{ in measure and } \lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u). \end{aligned} \tag{8}$$

Moreover, we benefit from a Hölder type inequality:

$$\left| \int_\Omega u(x) v(x)\, dx \right| \le 2\, \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{9}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ (see [40, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega)$ the dual of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$, see [40, Corollary 2.7].

Passing to the definition of the Sobolev space with variable exponent, $W^{k,p(\cdot)}(\Omega)$, we set

$$W^{k,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega),\ |\alpha| \le k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$ endowed with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)},$$

is a separable and reflexive Banach space too, see [40, Theorem 3.1].

The log-Hölder continuity of the exponent p plays a decisive role in the following density results.

Theorem 2 *(see [42, Theorem 3.7] and [38, Section 6.5.3]). Assume that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded domain with Lipschitz boundary and p is log-Hölder continuous with $1 < p^- \le p^+ < \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(\cdot)}(\Omega)$.*

Notice that the functions from $C^{0,\mu}(\Omega)$ are log-Hölder continuous. Also, it is important to mention that although the log-Hölder continuity of the exponent is a sufficient condition for the above density result, it is not always necessary, see [38,43].

Moreover, the following embedding theorem takes place.

Theorem 3 (*See [41, Theorem 2.3] and [38, Section 6]*). *Let us consider $q \in C(\overline{\Omega};\mathbb{R})$ such that $1 < q^- \le q^+ < \infty$ and $q(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, where*

$$p_k^*(x) = \begin{cases} \dfrac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \ge N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \ge 1$.

Then there is a continuous embedding

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

If we replace $\le$ with $<$ the embedding is compact.

Let us denote by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. In fact, we are interested in the properties of the spaces $W^{2,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and $W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$. Due to the log-Hölder continuity of the exponent p, the space $W_0^{1,p(\cdot)}(\Omega)$ coincides with

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega) :\ u = 0 \text{ on } \partial\Omega\right\},$$

and it can be endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

due to the following Poincaré type inequality (see [44, Proposition 2.3]):

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{10}$$

where C is a positive constant. The space $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}\right)$ is a separable and reflexive Banach space (see [44, Proposition 2.1]).

Obviously, the choice of the norms has a major influence on the development of the argumentation. Generally, we know that if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, then $(X\cap Y, \|\cdot\|_{X\cap Y})$ is a Banach space too, where $\|u\|_{X\cap Y} = \|u\|_X + \|u\|_Y$. In our case, we have,

$$\|u\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{W^{2,p(\cdot)}(\Omega)} + \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \sum_{|\alpha|=2}\|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Furthermore, $\left(W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)}\right)$ is a separable and reflexive Banach space. In addition, we know that $\|\cdot\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)}$ and $\|\Delta(\cdot)\|_{L^{p(\cdot)}(\Omega)}$ are equivalent norms on $W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$, see [45, Theorem 4.4]. However, taking into account the particularity of problem (2), which represents the subject of our investigation, the following representation of the norm might be best:

$$\|u\|_a = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\} \tag{11}$$

for all $u \in W^{2,p(\cdot)}(\Omega)$ or $W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$. The previously defined norm represents a norm on both $W^{2,p(\cdot)}(\Omega)$ or $W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$ and it is equivalent to the usual norm defined here, see [29, Remark 2.1]. Moreover, the modular inequalities that were appropriate for the norm of the Lebesgue space, can be extended to this situation, by proceeding similarly to [41, Theorems 1.2–1.3]. More precisely, for any a taken as in (H1), we consider $\Lambda : W^{2,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_\Omega \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right] dx. \tag{12}$$

Let us fix $u \in W^{2,p(\cdot)}(\Omega)\setminus\{0\}$. It is trivial to see that $\Lambda(\mu\, u)$ is even, and, for $\mu \in [0,\infty)$, $\Lambda(\mu\, u)$ increases strictly. Also, let $\mu_n \to \mu$. Since $(\mu_n)_n$ is bounded, $1 < p^- \le p^+ < \infty$ and a satisfies (H1), by Lebesgue's Dominated Convergence Theorem we deduce that $\Lambda(\mu_n u) \to \Lambda(\mu\, u)$, hence $\Lambda(\mu\, u)$ is continuous.

Based on these properties of Λ, we have the following consequence.

Corollary 1. *Let $u \in W^{2,p(\cdot)}(\Omega)\setminus\{0\}$. Then $\|u\|_a = |\kappa|$ if and only if $\Lambda\left(\frac{u}{\kappa}\right) = 1$.*

Proof. Without loss of generality, we can suppose that $\kappa > 0$ because $\Lambda(\mu u)$ is even.

To show the direct implication, we consider that $\|u\|_a = \kappa$. Note that $\Lambda(0\cdot u) = 0$ and that, whenever $\mu \to \infty$, $\Lambda(\mu u) \to \infty$. Therefore the continuity of $\Lambda(\mu u)$ ensures the existence of a $\mu_0 \in (0, \infty)$ with the property that

$$\Lambda\left(\frac{u}{\mu_0}\right) = 1. \tag{13}$$

By (11),

$$\kappa = \inf\left\{\mu > 0 : \Lambda\left(\frac{u}{\mu}\right) \leq 1\right\}. \tag{14}$$

Since, by (13), $\mu_0 \in \left\{\mu > 0 : \Lambda\left(\frac{u}{\mu}\right) \leq 1\right\}$, relation (14) gives us

$$\kappa \leq \mu_0. \tag{15}$$

At the same time,

$$\mu \geq \mu_0 \quad \text{for all } \mu \in (0, \infty) \text{ such that } \Lambda\left(\frac{u}{\mu}\right) \leq 1$$

because $\Lambda\left(\frac{u}{\mu_0}\right) = 1$ and $\Lambda(\mu u)$ increases strictly for $\mu \in [0, \infty)$. The previous inequality indicates that μ_0 represents a lower bound for the set $\left\{\mu > 0 : \Lambda\left(\frac{u}{\mu}\right) \leq 1\right\}$, thus, by (14),

$$\kappa \geq \mu_0. \tag{16}$$

Putting together (13), (15) and (16), we have obtained that $\|u\|_a = \kappa$ implies $\Lambda\left(\frac{u}{\kappa}\right) = 1$.

For the reciprocal implication, let us assume that $\Lambda\left(\frac{u}{\kappa}\right) = 1$. By proceeding as above, we first notice that $\kappa \in \left\{\mu > 0 : \Lambda\left(\frac{u}{\mu}\right) \leq 1\right\}$, hence, by (11), $\kappa \geq \|u\|_a$. Then, using again the monotonicity of $\Lambda(\mu u)$ for $\mu > 0$, we deduce that μ_0 represents a lower bound for the set $\left\{\mu > 0 : \Lambda\left(\frac{u}{\mu}\right) \leq 1\right\}$, so $\kappa \leq \|u\|_a$ and the conclusion follows. $\square$

Now we are able to prove the modular-type inequalities that we previously mentioned.

Proposition 1. *For $u,\ u_n \in W^{2,p(\cdot)}(\Omega)$ we have*

$$\|u\|_a < (=; > 1) \Leftrightarrow \Lambda(u) < (=; > 1), \tag{17}$$

$$\|u\|_a \leq 1 \Rightarrow \|u\|_a^{p^+} \leq \Lambda(u) \leq \|u\|_a^{p^-}, \tag{18}$$

$$\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \Lambda(u) \leq \|u\|_a^{p^+}, \tag{19}$$

$$\|u_n\|_a \to 0\ (\to \infty) \quad \Leftrightarrow \Lambda(u_n) \to 0\ (\to \infty). \tag{20}$$

Proof. For $\|u\|_a = 0$, $\Lambda(u) = 0$ and there is nothing to prove, thus we focus on the situation when $\|u\|_a \neq 0$. Let us denote $\|u\|_a = \kappa$. By Corollary 1 we have that $\Lambda\left(\frac{u}{\kappa}\right) = 1$.

If $\kappa = 1$, we immediately get that $\Lambda(u) = 1$. Using again Corollary 1 we easily notice that the vice-versa holds too: if $\Lambda(u) = 1$, then $\|u\|_a = 1$.

If $\kappa < 1$, the definition (12) enables us to write

$$\frac{1}{\kappa^{p^-}}\Lambda(u) \leq \Lambda\left(\frac{u}{\kappa}\right) \leq \frac{1}{\kappa^{p^+}}\Lambda(u).$$

Since $\Lambda\left(\frac{u}{\kappa}\right) = 1$, we arrive at

$$\kappa^{p^+} \leq \Lambda(u) \leq \kappa^{p^-} < 1.$$

Similarly, if $\kappa > 1$,

$$1 < \kappa^{p^-} \leq \Lambda(u) \leq \kappa^{p^+},$$

so the direct implication of (17) is proved, together with relations (18) and (19). Actually, the reciprocal implication of (17) is also true. Indeed, let us assume for example that $\Lambda(u) < 1$. Then it is clear that $\|u\|_a < 1$, otherwise, if $\|u\|_a \geq 1$, then, from what we have proved above, we get $\Lambda(u) \geq 1$, which contradicts our initial assumption. The case when $\Lambda(u) > 1$ is similar.

Passing to the proof of (20), if $\|u_n\|_a \to 0$, then (18) implies

$$0 \le \Lambda(u_n) \le \|u_n\|_a^{p^-} \to 0,$$

while if $\|u_n\|_a \to \infty$, then (19) implies

$$\Lambda(u_n) \ge \|u_n\|_a^{p^-} \to \infty.$$

Reciprocal, if $\Lambda(u_n) \to 0$, we use (17) and (18) to arrive at $0 \le \|u_n\|_a \le (\Lambda(u_n))^{1/p^+} \to 0$, while if $\Lambda(u_n) \to \infty$, we use (17) and (19) to arrive at $\|u_n\|_a \ge (\Lambda(u_n))^{1/p^+} \to \infty$. $\square$

Since we are getting closer to our goal, that is, the discussion of problem (2), it is time to introduce the space where we will search for weak solutions to our problem and to establish some of its main properties.

3. Weak solvability of the problem

When treating a problem with no-flux boundary condition, we need to choose a variable exponent space that is more appropriate for our study than the ones presented in the previous section. Therefore we introduce the following subspace of $W^{2,p(\cdot)}(\Omega)$.

$$V = \left\{u \in W^{2,p(\cdot)}(\Omega) : \, u\big|_{\partial\Omega} \equiv \text{constant}\right\}.$$

Notice that V can be viewed also as

$$V = \left\{u + c : \, u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega), \; c \in \mathbb{R}\right\} \tag{21}$$

and we can prove the following result.

Theorem 4. $\left(V, \|\cdot\|_{W^{2,p(\cdot)}(\Omega)}\right)$ *is a separable and reflexive Banach space.*

Proof. Our goal is to prove that V is a closed subspace of the separable and reflexive Banach space $\left(W^{2,p(\cdot)}(\Omega), \|\cdot\|_{W^{2,p(\cdot)}(\Omega)}\right)$. Let $(v_n)_n \subset V$ be such that it converges to $v \in W^{2,p(\cdot)}(\Omega)$. In order to prove our claim it is sufficient to show that $v \in V$.

Taking into account (21), we are aware of the fact that there exist $(u_n)_n \subset W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ and $(c_n)_n \subset \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$v_n = u_n + c_n.$$

The equivalence of the norms $\|\cdot\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)}$ and $\|\Delta(\cdot)\|_{L^{p(\cdot)}(\Omega)}$ on $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ enables us to write

$$\|u_n - u_m\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)} \le c\|\Delta(u_n - u_m)\|_{L^{p(\cdot)}(\Omega)} \le c\sum_{i=1}^{N}\left\|\frac{\partial^2}{\partial x_i^2}(u_n + c_n - u_m - c_m)\right\|_{L^{p(\cdot)}(\Omega)}$$

where c represents a generic positive constant that may vary along the calculus, as it is the case for the remaining of our paper. Consequently,

$$\|u_n - u_m\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)} \le c\|v_n - v_m\|_{W^{2,p(\cdot)}(\Omega)}. \tag{22}$$

But $(v_n)_n$ is converging to v in $\left(W^{2,p(\cdot)}(\Omega), \|\cdot\|_{W^{2,p(\cdot)}(\Omega)}\right)$, hence it is a Cauchy sequence, and (22) implies that $(u_n)_n$ is a Cauchy sequence in the Banach space $\left(W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega), \; \|\cdot\|_{W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)}\right)$. It follows immediately that $(u_n)_n$ is converging to a function $\overline{u} \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$.

On the other hand, we have the continuous embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$, so

$$\|c_n - c_m\|_{L^1(\Omega)} \le c\|c_n - c_m\|_{L^{p(\cdot)}(\Omega)} \le c\|v_n - v_m\|_{L^{p(\cdot)}(\Omega)} + c\|u_m - u_n\|_{L^{p(\cdot)}(\Omega)}. \tag{23}$$

Since both $(v_n)_n$ and $(u_n)_n$ are Cauchy sequences in $W^{2,p(\cdot)}(\Omega)$, respectively in $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$, by the definition of the corresponding norms and by the boundedness of Ω, we infer that $(c_n)_n$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Therefore $(c_n)_n$ is converging to a $\overline{c} \in \mathbb{R}$ and we have obtained that $v = \overline{u} + \overline{c} \in V$, which completes our proof. $\square$

Now that we have established some basic properties of the space V, we are ready to introduce the definition of a weak solution to our problem. To this purpose, we consider a smooth function u that verifies (2) and, by applying Green's formula, we get

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v \, dx + \int_{\partial\Omega} \frac{\partial}{\partial\nu}\left(|\Delta u|^{p(x)-2}\Delta u\right) v \, dx - \int_{\partial\Omega} |\Delta u|^{p(x)-2}\Delta u \frac{\partial v}{\partial\nu}\, dx + \int_\Omega a(x)|u|^{p(x)-2}uv \, dx$$
$$= \lambda \int_\Omega f(x,u)v \, dx \quad \text{for all } v \in C^\infty(\Omega).$$

Taking into consideration the fact that V is a closed subspace of $\left(W^{2,p(\cdot)}(\Omega), \|\cdot\|_{W^{2,p(\cdot)}(\Omega)}\right)$ together with the density result Theorem 2 and the boundary conditions, we arrive at the following formulation.

Definition 1. We say that $u \in V$ is a weak solution of the boundary value problem (2) if and only if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\, dx + \int_\Omega a(x)|u|^{p(x)-2}uv\, dx - \lambda \int_\Omega f(x,u)v\, dx = 0 \quad \text{for all } v \in V.$$

To be able to find a weak solution to (2), we rely on the critical point theory, thus to problem (2) we associate the functional

$$I : V \to \mathbb{R}, \qquad I = I_1 - \lambda I_2,$$

where

$$I_1(u) = \int_\Omega \frac{1}{p(x)}\left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right] dx \quad \text{and} \quad I_2(u) = \int_\Omega F(x,u)\, dx. \tag{24}$$

By proceeding similarly to [29, Proposition 2.5], one can establish the following.

Proposition 2. *Let $I_1 : V \to \mathbb{R}$ be the above defined functional.*

(i) *I_1 is of class C^1, with the Gâteaux derivative defined by*

$$\langle I_1'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\, dx + \int_\Omega a(x)|u|^{p(x)-2}uv\, dx.$$

(ii) *I_1 is (sequentially) weakly lower semicontinuous, that is, for any $u \in V$ and any subsequence $(u_n)_n \subset V$ such that $u_n \rightharpoonup u$ weakly in V, there holds*

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n).$$

(iii) *$I_1' : V \to V^*$ is of type (S+), that is, $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} I_1'(u_n)(u_n - u) \le 0$ imply that $u_n \to u$.*

Thus, due to the properties fulfilled by f, we can easily deduce that I is of class C^1, with Gâteaux derivative defined by

$$\langle I'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\, dx + \int_\Omega a(x)|u|^{p(x)-2}uv\, dx - \lambda \int_\Omega f(x,u)v\, dx$$

so any critical point of I is a weak solution to (2). Therefore, in what follows we focus on studying the existence and the multiplicity of the nontrivial critical points of I.

4. The existence result

We base our first result on the classical theorem of calculus of variations:

Theorem 5. *see [46, Section 2, Theorem 1.2] Assume that X is a reflexive Banach space of norm $\|\cdot\|_X$ and the functional $\Phi : X \to \mathbb{R}$ is*

(i) *coercive on X, that is, $\Phi(u) \to \infty$ as $\|u\|_X \to \infty$;*
(ii) *(sequentially) weakly lower semicontinuous on X.*

Then Φ is bounded from below on X and attains its infimum in X.

By applying this result to the functional I, we prove the existence of a nontrivial weak solution.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded domain with smooth boundary and p be log-Hölder continuous with $1 < p^- \le p^+ < \infty$ for all $x \in \Omega$. Assume hypotheses (H1)–(H3) take place. Then there exists a constant $\lambda_0 > 0$ such that problem (2) has at least one nontrivial weak solution in V for every $\lambda > \lambda_0$.*

Proof. Let us first deal with the coercivity of I. Hypothesis (H3) implies that, for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $|t| > \delta_\varepsilon$ and all $x \in \Omega$ we have

$$|f(x,t)| \le \varepsilon |t|^{p(x)-1}.$$

For the moment, we arbitrarily fix $\varepsilon > 0$. Then the continuity of f in its second argument indicates that for all $t \in \mathbb{R}$ and all $x \in \Omega$ there exists $c_0 > 0$ such that

$$|f(x,t)| \le c_0 + \varepsilon |t|^{p(x)-1}. \tag{25}$$

Taking into account the definition of I (see (24)) and (25), we arrive at

$$I(u) \geq \frac{1}{p^+} \int_\Omega \left[|\Delta u|^{p(x)} + (a(x) - \lambda\varepsilon)|u|^{p(x)} \right] dx - \lambda\, c_0 \|u\|_{L^1(\Omega)}.$$

Let us choose ε such that $\varepsilon < a_0/\lambda$ because in this way

$$\widetilde{a} = a - \lambda\varepsilon$$

verifies (H1). We know that V is endowed with the norm $\|\cdot\|_{W^{2,p(\cdot)}(\Omega)}$ which is equivalent to the norm $\|\cdot\|_{\widetilde{a}}$ introduced by (11). Then for any $u \in V$ with $\|u\|_{\widetilde{a}} \geq 1$, inequality (19) leads to

$$I(u) \geq \frac{1}{p^+} \|u\|_{\widetilde{a}}^{p^-} - \lambda\, c_0 \|u\|_{L^1(\Omega)}. \tag{26}$$

At the same time, we have that $1 < p_2^\star(x)$ for all $x \in \overline{\Omega}$, therefore by Theorem 3 and by (26) we deduce that there exists $c > 0$ such that

$$I(u) \geq \frac{1}{p^+} \|u\|_{\widetilde{a}}^{p^-} - \lambda\, c \|u\|_{\widetilde{a}},$$

hence I is coercive.

Moving further, to the weakly lower semicontinuity of I, we already know that I_1 is weakly lower semicontinuous, by Proposition 2. To investigate if this property holds for I_2 too, we assume $u_n \rightharpoonup u$ in V. But V is a closed subspace of $W^{2,p(\cdot)}(\Omega)$ thus the compact embedding produced by Theorem 3 gives us

$$u_n \to u \quad \text{in } L^{p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^1(\Omega). \tag{27}$$

Using the mean value theorem, there exists v which takes values strictly between the values of u and u_n such that

$$|I_2(u_n) - I_2(u)| \leq \int_\Omega |F(x, u_n) - F(x, u)|\, dx \leq \int_\Omega |u_n - u| \sup_{x\in\Omega} |f(x, v(x))|\, dx,$$

hence, by (25) and (27) the functional I_2 is weakly continuous, so I is weakly continuous also. Consequently, we obtain the weakly lower semicontinuity of I.

Now we are in position to apply Theorem 5 and to find $u_1 \in V$ in which I attains its infimum, hence u_1 represents a weak solution to problem (2). Furthermore, for all $\lambda > 0$,

$$I(u_1) \leq I(u) \quad \text{for all } u \in V. \tag{28}$$

Given the ball B provided by hypothesis (H2), we can take $\varepsilon > 0$ sufficiently small such that

$$\overline{B_\varepsilon} := \overline{\{x \in \Omega \,|\, \operatorname{dist}(x, B) \leq \varepsilon\}} \subset \Omega.$$

Furthermore, we can construct the following C_c^1 function:

$$u_\varepsilon(x) := \begin{cases} t_0, & \text{when } x \in B, \\ 0, & \text{when } x \in \Omega \setminus B_\varepsilon. \end{cases}$$

Then

$$I(u_\varepsilon) \leq I_1(u_\varepsilon) - \lambda \int_B F(x, t_0)\, dx - \lambda \int_{B_\varepsilon \setminus B} F(x, u_\varepsilon)\, dx.$$

By the definition of F we are able to fix ε_0 sufficiently small such that there exists a positive constant α_0 with the property that

$$I(u_{\varepsilon_0}) \leq I_1(u_{\varepsilon_0}) - \lambda\alpha_0 \int_B F(x, t_0)\, dx.$$

Now, by taking

$$\lambda_0 := \frac{I_1(u_{\varepsilon_0})}{\alpha_0 \int_B F(x, t_0)\, dx} > 0 \tag{29}$$

we deduce that $I(u_{\varepsilon_0}) < 0$ for all $\lambda > \lambda_0$. By choosing $u = u_{\varepsilon_0}$ in (28) we obtain that u_1 is nontrivial for all $\lambda > \lambda_0$ because $I(0) = 0$, and we have completed our proof. $\square$

5. The multiplicity result

For the multiplicity result of this paper we rely on a variant of the celebrated mountain pass theorem (see for example [47–49]) of Ambrosetti and Rabinowitz.

Theorem 7. *Let $(X, \|\cdot\|_X)$ be a Banach space. Assume that $\Phi \in C^1(X;\mathbb{R})$ satisfies the Palais–Smale condition, that is, any sequence $(u_n)_n \subset X$ such that $(\Phi(u_n))_n$ is bounded and $\Phi'(u_n) \to 0$ in $X^\star$ as $n \to \infty$, contains a convergent subsequence. Also, assume that Φ has a mountain pass geometry, that is,*

(i) *there exist two constants $\tau > 0$ and $\rho \in \mathbb{R}$ such that $\Phi(u) \geq \rho$ if $\|u\|_X = \tau$;*
(ii) *$\Phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > \tau$ and $\Phi(e) < \rho$.*

Then Φ has a critical point $u_0 \in X \setminus \{0, e\}$ with critical value

$$\Phi(u_0) = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} \Phi(u) \geq \rho > 0,$$

where $\mathcal{P}$ denotes the class of the paths $\gamma \in C([0, 1]; X)$ joining 0 to e.

Now we are able to prove the following.

Theorem 8. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary and p be log-Hölder continuous with $1 < p^- \leq p^+ < \infty$ for all $x \in \Omega$. Assume hypotheses* (H1)–(H4) *take place. Then there exists $\lambda_0 > 0$ such that problem (2) has at least two nontrivial weak solutions in V for every $\lambda > \lambda_0$.*

Proof. By Theorem 6 we have already established that problem (2) has at least one nontrivial weak solution $u_1 \in V$ for every $\lambda > \lambda_0$, where λ_0 is the one defined by (29). To deduce the existence of a second nontrivial weak solution for problem (2), we will show that I satisfies the hypotheses of Theorem 7. We begin with the Palais–Smale condition. Let us consider a sequence $(u_n)_n \subset V$ with the property that there exists $M > 0$ such that

$$|I(u_n)| \leq M, \quad \text{and,} \quad \text{when } n \to \infty, \qquad I'(u_n) \to 0 \quad \text{in } V^*. \tag{30}$$

We recall that in the proof of Theorem 6 we have established the coercivity of I, so by (30) we infer that $(u_n)_n$ is bounded. Moreover, V is a reflexive Banach space and a closed subspace of $W^{2,p(\cdot)}(\Omega)$, thus there exists $u_0 \in V \subset W^{2,p(\cdot)}(\Omega)$ such that, passing eventually to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in } W^{1,\vec{p}(\cdot)}(\Omega). \tag{31}$$

Applying again Theorem 3 we deduce that

$$u_n \to u_0 \quad \text{in } L^1(\Omega), \qquad u_n \to u_0 \quad \text{in } L^{p^-}(\Omega) \quad \text{and} \quad u_n \to u_0 \quad \text{in } L^{p(\cdot)}(\Omega). \tag{32}$$

All the above information was obtained starting from the boundedness of $(I(u_n))_n$. By exploiting the second part of relation (30) and the weak convergence from (31) we arrive at

$$\lim_{n\to\infty} \left|\langle I'(u_n), u_n - u_0\rangle\right| = 0.$$

More exactly, we have

$$\begin{aligned} 0 = {} & \lim_{n\to\infty} \int_\Omega |\Delta u_n|^{p(x)-2} \, \Delta u_n \, \Delta(u_n - u_0)\, dx + \lim_{n\to\infty} \int_\Omega a(x)|u_n|^{p(x)-2} u_n(u_n - u_0)\, dx \\ & - \lim_{n\to\infty} \lambda \int_\Omega f(x, u_n)(u_n - u_0)\, dx. \end{aligned} \tag{33}$$

By (H1), (9), (32) and (8), we deduce that

$$\lim_{n\to\infty} \left| \int_\Omega a(x)|u_n|^{p(x)-2} u_n(u_n - u_0)\, dx \right| \leq 2\|a\|_{L^\infty(\Omega)} \lim_{n\to\infty} \left(\| \, |u_n|^{p(x)-1} \|_{L^{p'(\cdot)}(\Omega)} \|u_n - u_0\|_{L^{p(\cdot)}(\Omega)} \right) = 0. \tag{34}$$

On the other hand, by (25), (9), (32) and (8), there exists $c > 0$ such that

$$\lim_{n\to\infty} \left| \int_\Omega f(x, u_n)(u_n - u_0) dx \right| \leq c_0 \lim_{n\to\infty} \|u_n - u_0\|_{L^1(\Omega)} + c \lim_{n\to\infty} \left(\| \, |u_n|^{p^- -1} \|_{L^{(p^-)'}(\Omega)} \|u_n - u_0\|_{L^{p^-}(\Omega)} \right) = 0. \tag{35}$$

Replacing (34) and (35) in (33) we obtain

$$\lim_{n\to\infty} \int_\Omega |\Delta u_n|^{p(x)-2} \, \Delta u_n \, \Delta(u_n - u_0)\, dx = 0,$$

so the weak convergence (31) and Proposition 2 imply that $u_n \to u_0$ in V as $n \to \infty$. With this, we conclude that I verifies the Palais–Smale condition. Let us show now that I has a mountain pass-type geometry too.

We can see immediately that

$$I_1(u) \geq \frac{1}{p^+} \int_\Omega \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right] dx \quad \text{for all } u \in V. \tag{36}$$

Passing to I_2, we make use once again of (H3). For $\varepsilon > 0$ arbitrarily fixed, there exists $\delta_1 \geq 1$ such that for all $|s| > \delta_1$ and all $x \in \Omega$,

$$|f(x, s)| \leq \varepsilon |s|^{r-1}$$

where $p^+ < r < p_2^\star$. At the same time, by (H4), there exists $\delta_2 > 0$ such that for all $|s| < \delta_2$ and all $x \in \Omega$,

$$|f(x, s)| \leq \varepsilon |s|^{p(x)-1}.$$

Putting together the previous two inequalities and the continuity of f in its second argument, for a sufficiently large constant $c > 0$,

$$I_2(u) \leq c\, \|u\|^r_{L^r(\Omega)} + \frac{\varepsilon}{p^-} \int_\Omega |u|^{p(x)}\, dx \quad \text{for all } u \in V. \tag{37}$$

Combining (36) and (37) we get

$$I(u) \leq \left(\frac{1}{p^+} - \frac{\lambda\, \varepsilon}{a_0\, p^-}\right) \int_\Omega \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right] dx - \lambda\, c\, \|u\|^r_{L^r(\Omega)} \quad \text{for all } u \in V,$$

since $a(x) \geq a_0 > 0$ for all $x \in \Omega$.

We arbitrarily take $0 < \tau < 1$. Then, due to the above inequality, for $\|u\|_a = \tau$, relation (18) and Theorem 3 give us

$$I(u) \geq \left(\frac{1}{p^+} - \frac{\lambda\, \varepsilon}{a_0\, p^-}\right) \|u\|_a^{p^+} - \lambda\, c\, \|u\|_a^r. \tag{38}$$

We choose $0 < \varepsilon < a_0\, p^-/(\lambda p^+)$ and, since $p^+ < r$, for $\tau = \|u\|_a < \min\{1, \|u_1\|_a\}$, we can find ρ such that $I(u) \geq \rho > 0 = I(0) > I(u_1)$, where u_1 is the first nontrivial weak solution found by Theorem 6. Therefore I has a mountain pass-type geometry.

Now we can apply Theorem 7 to obtain a second nontrivial weak solution $u_2 \in V \backslash \{0,\ u_1\}$ to problem (2) and our proof is complete. □

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