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Nonsmooth dynamical systems: From the existence of solutions to optimal and feedback control ☆

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A B S T R A C T

In this paper, we investigate a nonlinear and nonsmooth dynamics system (NNDS, for short) involving two multi-valued maps which are a convex subdifferential operator and a generalized subdifferential operator in the sense of Clarke, respectively. Under general assumptions, by using

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a surjectivity theorem for multi-valued mappings combined with the theory of nonsmooth analysis and arguments on pseudomonotone operators, the existence of a solution to (NNDS) is proved. Then, an optimal control problem governed by (NNDS) is introduced, and a solvability result for the optimal control problem is established. Moreover, we study a nonlinear feedback control system driven by (NNDS) and an u.s.c. multi-valued feedback map, and employ the Kakutani-Ky Fan fixed point theorem to obtain an existence theorem of solutions for the feedback control problem. Finally, we deliver a convergence result in the sense of Kuratowski describing the changes in the set of solutions for the feedback control problem as the initial data x_0 is perturbed in Hilbert space H.

1. Introduction

In numerous complicated physical processes and engineering applications, mathematical models often lead to inequalities instead of the more commonly seen equations. Variational-hemivariational inequalities present a particular class of inequalities, in which both convex and nonconvex functional are involved. More precisely speaking, the study of variational-hemivariational inequalities requires arguments of Convex Analysis, including properties of the subdifferential of a convex function, and arguments of Nonsmooth Analysis, including properties of the generalized subdifferential in the sense of Clarke, defined for locally Lipschitz functions which are nonconvex in general. The literature on variational-hemivariational inequalities has been significantly enlarged in recently years, mainly because of their multiple relevant applications to various fields. Some representative references include: Han-Migórski-Sofonea [12] obtained the existence, uniqueness and the continuous dependence of the solution for a class of variational-hemivariational inequalities of elliptic type, and applied the linear finite element method to the inequality for deriving an optimal order error estimate and a convergence result; by using the notion of the stable φ-quasimonotonicity and KKM principle, Tang-Huang [47] derived the necessary and sufficient condition to the existence of solutions for an elliptic variational-hemivariational inequality in a Banach space; Bartosz-Sofonea [3] considered a new class of first order evolutionary variational-hemivariational inequalities and employed Rothe method to prove the existence and uniqueness of the inequality, then they used the abstract results to study a quasi-static frictionless problem for Kelvin–Voigt viscoelastic materials; Migórski-Khan-Zeng [32] utilized Kluge fixed point theorem and the Minty approach to an inverse problem of parameter identification in a nonlinear quasi-variational-hemivariational inequality posed in a Banach space, then the authors explored an identification inverse problem in a complicated mixed elliptic boundary value problem with p-Laplace operator and an implicit obstacle. We also refer the reader to [2,13,17,19–21,27–29,32,33,39–41,46,48] and the references therein for a more detailed discussion of this topic.

Given an evolution triple of spaces $V \hookrightarrow H \hookrightarrow V^*$ such that the embedding from V into H is compact, a reflexive Banach space Z, $0 < T < \infty$, a nonlinear operator $A\colon [0,T] \times V \to V^*$, a proper convex and lower semicontinuous function $\varphi\colon Z \to \overline{\mathbb{R}}$, a linear operator $\beta\colon V \to Z$, a function $J\colon [0,T] \times H \to \mathbb{R}$, we formulate a nonlinear and nonsmooth evolution inclusion problem with two multi-valued terms which are a convex subdifferential operator and a generalized subdifferential operator in the sense of Clarke, respectively, as follows:

Problem 1.1. Given a function $f \in \mathcal{H} := L^2(0,T;H)$, find a function $x\colon [0,T] \to V$ such that

$$\begin{cases} x'(t) + A(t,x(t)) + \beta^*\partial_C\varphi(\beta x(t)) + \partial J(t,x(t)) \ni f(t) \text{ for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $\partial_C\varphi\colon Z \to 2^{Z^*}$ and $\partial J(t,\cdot)\colon H \to 2^H$ are the convex subdifferential operator of φ, and the generalized Clarke subdifferential operator of $J(t,\cdot)$, respectively.

It should be mentioning that operator A involved in Problem 1.1 can describe exactly and effectively various natural phenomena and constitutive laws arising in engineering applications, for example, nonlinear elasticity operators, Navier-Stokes operators, nonlinear diffusion operators and so forth, see [24,27,31,35,36,49,52–55]. Under the assumptions that $x \mapsto A(t,x)$ is strongly monotone and Lipschitz, $w \mapsto J(t,w)$ is relaxed monotone and φ is continuous, Han-Migórski-Sofonea [14] proved that Problem 1.1 has a unique solution. However, in many real-life problems, the nonlinear operator could be $x \mapsto A(t,x)$ not strongly monotone and Lipschitz. For example, let Ω be a bounded domain in $\mathbb{R}^N$, $V = H_0^1(\Omega)$, and $A\colon [0,T] \times V \to V^*$ be defined by

$$\begin{aligned} \langle A(t,u),v\rangle = \int_\Omega & k_1(z,t)(\nabla u(z),\nabla v(z))_{\mathbb{R}^N} - k_2(t)(u(z)^{\frac{1}{2}}v(z) + \log|u(z)|v(z)) \\ & - l(z,u(z),\nabla u(z))v(z)\,dz \end{aligned} \tag{1.2}$$

for all $u,v \in V$ and a.e. $t \in [0,T]$. It is not difficult to see that $x \mapsto A(t,x)$ not strongly monotone and Lipschitz in general. In this moment, the theoretical results obtained in [14] can not able to solve those problems involving the nonlinear operator A defined in (1.2). Therefore, a natural question arises that may we establish the generalized theoretical result to fill this gap. Based on this motivation, in Section 3, without the strong monotonicity of $x \mapsto A(t,x)$ and relaxed monotonicity of $w \mapsto J(t,w)$, we are devoted to establish an existence theorem for Problem 1.1 in the general functional framework.

The optimal control of hemivariational/variational-hemivariational inequalities as a new research branch of the optimal control theory has been widely studied in various perspectives. In the following, we provide a brief review of some of the related developments. Li-Liu [23] considered an optimal control problem governed by a differential hemivariational inequality on Banach spaces, and applied an extension of Filippov's theorem to establish the sensitivity properties of the optimal control problem. Peng-Kunisch [45] dealt with the optimality system of an optimal control problem involving a nonlinear elliptic inclusion and a nonsmooth cost functional where the existence of optimal pairs was proved and necessary optimality conditions of first order were derived by applying an adapted penalty method and nonsmooth analysis techniques. More recently, an optimal control problem described a generalized nonlinear quasi-variational-hemivariational inequality involving a multi-valued map was investigated by Zeng-Migórski-Khan [51], the authors examined a solvability result for the optimal control problem by employing the geometric version of the Hahn-Banach theorem together with Tychonov fixed point principle and Weierstrass type theorem. Although the theory and computational techniques for optimal control of hemivariational/variational-hemivariational inequalities have been studied for quite some time now (see e.g. [25,26,37,38]), it seems that there are still many unanswered questions and many interesting ideas are still in the making.

Denote by $\Gamma(f)$ the solution set of Problem 1.1 corresponding to $f \in \mathcal{H}$. To the best of the authors' knowledge, if $\Gamma(f)$ is nonempty and is not a singleton set, it is still an open problem whether the following optimal control problem is solvable:

Problem 1.2. Find $(x^*, f^*) \in \mathcal{W} \times \mathcal{H}$ such that $x^* \in \Gamma(f^*)$ and

$$Q(x^*, f^*) \leq Q(x, f) \text{ for all } x \in \Gamma(f) \text{ and } f \in \mathcal{H}, \tag{1.3}$$

where the cost function $Q\colon \mathcal{W} \times \mathcal{H} \to \mathbb{R}$ is defined by

$$Q(x, f) := \frac{\alpha_1}{2}\|f - g\|_{\mathcal{H}}^2 + \frac{\alpha_2}{2}\|x(T) - z_0\|_H^2 + \frac{\alpha_3}{2}\|x - y_0\|_{\mathcal{V}}^2, \tag{1.4}$$

and $\alpha_1, \alpha_2, \alpha_3 > 0$ are regularization parameters.

Here g, z_0 and y_0 are given target profiles, which will be specialized in Section 4. To deliver a positive answer for the above issue, the second contribution of the paper is to explore, in Section 4, a sufficient theorem for the existence of an optimal control to Problem 1.2.

Moreover, there are many real-life problems which could be modeled by different feedback control systems, for example, trajectory planning of a robot manipulator, guidance of a tactical missile toward a moving target, regulation of room temperature, and control of string vibrations. The third aim of the paper is to deal with the following feedback control system:

Problem 1.3. Find functions $x\colon [0,T] \to V$ and $u\colon [0,T] \to Y$ such that

$$\begin{cases} x'(t) + A(t,x(t)) + \beta^*\partial_C\varphi(\beta x(t)) + \partial J(t,x(t)) \ni B(t,x(t))u(t) \text{ for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1.5}$$

and

$$u(t) \in U(t,x(t)) \text{ for a.e. } t \in [0,T]. \tag{1.6}$$

To highlight the general form of our problem, we list the following particular cases of Problem 1.3.

- If $\varphi \equiv 0$, then Problem 1.3 becomes the one, which has been studied by Bin-Liu [5]. Under the assumptions that B is uniformly bounded and U is continuous in the sense of Hausdorff metric, [5] gave the existence of a feedback control pair.
- Papageorgiou-Rădulescu-Repovš [44] considered Problem 1.3, under the functional framework $V = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $A \equiv 0$, $J \equiv 0$, B is Lipscthiz continuous with respect to the second variable, U is uniformly bounded and continuous in the sense of Hausdorff metric.
- If V is a Hilbert space, $A(t,x) = A(x)$ and $B(t,x) = M$ for all $x \in H$ and a.e. $t \in [0,T]$ with $M \in L(Y,H)$, $\varphi \equiv 0$, then Problem 1.3 reduces to the one, which was investigated by Huang-Liu-Zeng [16].

In contrast to all the aforementioned papers, in this paper, we do not assume that U is continuous in the sense of Hausdorff metric and uniformly bounded, B is Lipschitz and uniformly bounded. Additionally, the method applied in this paper is different from that used in [5,44] and [16]. More precisely, our approach is based on the Kakutani–Ky Fan fixed point theorem. Moreover, we also deliver a new convergence result in the sense of Kuratowski describing the changes in the set of solutions for the feedback control problem as the initial data x_0 is perturbed in Hilbert space H.

2. Notation and preliminary material

In this section we briefly review some basic notation and preliminary results which will be used in the rest of the paper. For more details on the material presented below we refer to the monographs [7–9,15,18,30,50].

Let X be a Hausdorff topological space and $D \subset X$. We denote by 2^X the set of all nonempty subsets of X. Also, we denote by "$\to$" and "$\xrightarrow{w}$" the strong and the weak convergences, respectively, in various spaces which will be specified. Let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple of spaces (or a Gelfand triple of spaces, i.e., V is a separable and reflexive Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous and V is dense in H) and Y, Z be reflexive Banach spaces. In the sequel, we denote by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)_H$ the duality pairing between V^* and V, and the inner product of H, respectively. Given $0 < T < \infty$, in what follows, we introduce the following Bochner-Lebesgue spaces

$$\mathcal{V} = L^2(0,T;V),\ \mathcal{H} = L^2(0,T;H),\ \mathcal{V}^* = L^2(0,T;V^*),\ \mathcal{Y} = L^2(0,T;Y),$$
$$\mathcal{Y}^* = L^2(0,T;Y^*),\ \mathcal{Z} = L^2(0,T;Z),\ \mathcal{Z}^* = L^2(0,T;Z^*),\ \mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\},$$

where the time derivative $v' = \frac{\partial v}{\partial t}$ is understood in the sense of vector-valued distributions. Since spaces $\mathcal{V}$ and $\mathcal{V}^*$ are reflexive Banach spaces, then it is not difficult to see that $\mathcal{W}$ endowed with the graph norm

$$\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*} \text{ for all } v \in \mathcal{W},$$

is a separable and reflexive Banach space and the embeddings $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ are continuous. Besides, the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous too, where $C(0,T;H)$ is the space of continuous functions on $[0,T]$ with values in H. In what follows, we denote by $\langle\langle \cdot, \cdot \rangle\rangle$ and $((\cdot, \cdot))$ the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$, and the inner product of $\mathcal{H}$, respectively.

We next recall a number of basic definitions.

Definition 2.1. Let (X, τ) be a Hausdorff topological space and $\{A_n\} \subset 2^X$. The τ-Kuratowski lower limit of the sequence $\{A_n\}$ is the set given by

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{n\to\infty} x_n,\ x_n \in A_n \text{ for all } \ n \geq 1 \right\},$$

and the τ-Kuratowski upper limit of the sequence $\{A_n\}$ is the set given by

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

We say that the sequence $\{A_n\}$ τ-converges in the sense of Kuratowski if

$$\tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n.$$

In this case the set $A = \tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n$, is called τ-Kuratowski limit of the sequence $\{A_n\}$.

Definition 2.2. Let $(V, \|\cdot\|_V)$ be a reflexive Banach space, let $L\colon D(L) \subset V \to V^*$ be a linear maximal monotone operator, and let $\mathcal{B}\colon V \to 2^{V^*}$ be a multivalued operator. We said that $\mathcal{B}$ is pseudomonotone with respect to $D(L)$ (or L-pseudomonotone), if the following conditions hold:

(i) $\mathcal{B}$ has nonempty, bounded, closed and convex values;
(ii) $\mathcal{B}$ is upper semicontinuous from each finite-dimensional subspace of V to V^* endowed with the weak topology;
(ii) if $\{u_n\} \subset D(L)$ with $u_n \xrightarrow{w} u$ in V, $Lu_n \xrightarrow{w} Lu$ in V^* and $u_n^* \in \mathcal{B}u_n$ is such that

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0,$$

then, $u^* \in \mathcal{B}u$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Let $(V, \|\cdot\|_V)$ be a Banach. A function $\varphi\colon V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called proper, convex, and lower semicontinuous, if it fulfills the conditions

$$\begin{aligned}
&\mathrm{D}(\varphi) := \{u \in V \mid \varphi(u) < +\infty\} \ne \emptyset,\\
&\varphi(\lambda u + (1-\lambda)v) \le \lambda\varphi(u) + (1-\lambda)\varphi(v) \quad \text{for all } \lambda \in [0,1] \quad \text{and } u, v \in V,\\
&\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n) \quad \text{for all sequences } \{u_n\} \subset V \text{ with } u_n \to u,
\end{aligned}$$

respectively. We also denote by $\partial_C\varphi\colon D(\varphi) \to 2^{V^*}$ the subdifferential operator of φ defined by

$$\partial_C\varphi(u) := \{\eta \in V^* \mid \langle \eta, v - u\rangle \le \varphi(v) - \varphi(u) \text{ for all } v \in V\} \text{ for all } u \in D(\varphi).$$

A function $J\colon V \to \mathbb{R}$ is called locally Lipschitz continuous at $u \in V$, if there exist a neighborhood $N(u)$ of u and a constant $L_u > 0$ such that

$$|J(w) - J(v)| \le L_u \|w - v\|_V \quad \text{for all } \; w, v \in N(u).$$

The generalized directional derivative in the sense of Clarke of J at a point $u \in V$ in the direction $v \in V$, denoted by $J^0(u; v)$, is defined by

$$J^0(u; v) = \limsup_{\lambda\to 0^+,\, z\to u} \frac{J(z + \lambda v) - J(z)}{\lambda}.$$

Then, the generalized gradient of J at $u \in V$ is defined by

$$\partial J(u) = \{\, \xi \in V^* \mid J^0(u; v) \ge \langle \xi, v\rangle \text{ for all } \; v \in V \,\}.$$

The generalized gradient and generalized directional derivative of a locally Lipschitz function enjoy several properties that we recall below, following [30, Proposition 3.23].

Proposition 2.3. *Assume that $J\colon V \to \mathbb{R}$ is a locally Lipschitz function. Then:*

(i) *for every* $u \in V$*, the function* $V \ni v \mapsto J^0(u; v) \in \mathbb{R}$ *is positively homogeneous and subadditive, i.e.,*

$$J^0(u; \lambda v) = \lambda J^0(u; v) \quad \text{and} \quad J^0(u; v_1 + v_2) \leq J^0(u; v_1) + J^0(u, ; v_2)$$

for all $u, v, v_1, v_2 \in V$, $\lambda \geq 0$*;*

(ii) *for every* $v \in V$*, it holds* $J^0(u; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial J(u)\}$*;*

(iii) *for every* $u \in V$*, the gradient* $\partial J(u)$ *is a nonempty, convex, weakly** *compact subset of* V^* *which is bounded by the Lipschitz constant* $L_u > 0$ *of* J *near* u*;*

(iv) *the function* $V \times V \ni (u, v) \mapsto J^0(u; v) \in \mathbb{R}$ *is upper semicontinuous.*

(v) *the multifunction* $V \ni u \mapsto \partial J(u) \subset V^*$ *is upper semicontinuous (u.s.c.) from* V *into the space* V^* *endowed with the weak** *topology.*

Furthermore, we consider the important concept of strongly-quasi boundedness for set-valued operators (see for instance [10, Definition 2.14]).

Definition 2.4. Let $(V, \|\cdot\|_V)$ be a reflexive Banach space with its dual V^* and $A\colon D(A) \subset V \to 2^{V^*}$ be a multivalued mapping. A is called to be strongly-quasi bounded, if for each $M > 0$, there exists $K_M > 0$ satisfying if $u \in D(A)$ and $u^* \in Au$ are such that

$$\langle u^*, u\rangle \leq M \quad \text{and} \quad \|u\|_V \leq M,$$

then $\|u^*\|_{V^*} \leq K_M$.

Obviously, it is not easy to verify that a multivalued operator is strongly-quasi bounded by using the definition. However, Browder-Hess in [6, Proposition 14] provided the following criterion to validate the strongly-quasi boundedness for multivalued operators.

Proposition 2.5. *Let* E *be a reflexive Banach space with its dual* E^**. If* $A\colon D(A) \subset E \to 2^{E^*}$ *is a monotone operator such that* $0 \in intD(A)$*, then* A *is strongly-quasi bounded.*

Moreover, we recall the following surjectivity result for the sum of operators in Banach spaces (its detailed proof can be found in [10, Theorem 3.1]), which will play a significant role in the proof of the existence of a solution for the parabolic variational-hemivariational inequality, Problem 1.1.

Theorem 2.6. *Let* $(E, \|\cdot\|_E)$ *be a reflexive, strictly convex Banach space,* $\mathcal{L}\colon D(\mathcal{L}) \subset E \to E^*$ *be a linear, densely defined and maximal monotone operator,* $\mathcal{A}\colon E \to 2^{E^*}$ *be a bounded and* $\mathcal{L}$*-pseudomonotone operator such that*

$$\langle \mathcal{A}u, u\rangle_{E^* \times E} \geq r(\|u\|_E)\|u\|_E \quad \text{for all} \quad u \in E,$$

where $r\colon \mathbb{R}_+ \to \mathbb{R}$ *is a function satisfying* $r(s) \to +\infty$ *as* $s \to +\infty$*. If the multivalued mapping* $\mathcal{B}\colon D(\mathcal{B}) \subset E \to 2^{E^*}$ *is a maximal monotone operator which is strongly-quasi bounded and* $0 \in \mathcal{B}(0)$*, then* $\mathcal{L} + \mathcal{A} + \mathcal{B}$ *is surjective, namely,* $R(\mathcal{L} + \mathcal{A} + \mathcal{B}) = E^*$*.*

We end the section by recalling the Kakutani–Ky Fan theorem for a reflexive Banach space, see e.g. [43, Theorem 2.6.7], which will be applied to prove the solvability of the feedback control problem, Problem 1.3.

Theorem 2.7. *Let* E *be a reflexive Banach space and* $D \subseteq E$ *be a nonempty, bounded, closed and convex set. Let* $\Lambda\colon D \to 2^D$ *be a set-valued map with nonempty, closed and convex values such that its graph is sequentially closed in* $E_w \times E_w$ *topology. Then* Λ *has a fixed point.*

3. Parabolic variational-hemivariational inequalities

The section is devoted to the study of a class of abstract variational-hemivariational inequalities of parabolic type in the framework of an evolution triple of spaces (or a Gelfand triple of spaces), $V \hookrightarrow H \hookrightarrow V^*$. Under the mild conditions, the existence of a solution for the inequality problem, Problem 1.1, is established in which our method is based on a surjectivity theorem for set-valued mappings that we use for the sum of a maximal monotone and strongly-quasi bounded operator, a linear densely defined and maximal monotone operator $\mathcal{L}$, and a bounded pseudomonotone operator with respect to $\mathcal{L}$. Furthermore, we apply monotone arguments to reveal the uniqueness of solution for the solution to Problem 1.1.

The concept of the solutions of Problem 1.1 is understood as follows.

Definition 3.1. We say that $x \in \mathcal{W}$ is a solution to Problem 1.1, if there exist two functions $\xi\colon [0,T] \to H$ and $\eta\colon [0,T] \to V^*$ such that

$$\begin{cases} x'(t) + A(t, x(t)) + \xi(t) + \eta(t) = f(t) \text{ for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{3.1}$$

and

$$\eta(t) \in \beta^* \partial_C \varphi(\beta x(t)) \quad \text{and} \quad \xi(t) \in \partial J(t, x(t)) \quad \text{for a.e. } t \in [0,T]. \tag{3.2}$$

In order to obtain the solvability of Problem 1.1, we impose the following assumptions on the data of Problem 1.1.

<u>$H(A)$</u>: $A\colon [0,T] \times V \to V^*$ is such that

(i) for each $x \in V$, the function $t \mapsto A(t,x)$ is measurable on $[0,T]$;
(ii) $x \mapsto A(t,x)$ is pseudomonotone for a.e. $t \in [0,T]$;

(iii) there exist $a_A \in L^2(0,T)_+$ and $b_A > 0$ such that

$$\|A(t,x)\|_{V^*} \le a_A(t) + b_A\|x\|_V$$

for all $x \in V$ and a.e. $t \in [0,T]$;

(iv) there exist $m_A > 0$, $c_A \in L^2(0,T)_+$ and $d_A \in L^1(0,T)_+$ satisfying

$$\langle A(t,x), x\rangle \ge m_A\|x\|_V^2 - c_A(t)\|x\|_V - d_A(t)$$

for a.e. $t \in [0,T]$ and $x \in V$;

(v) there exists $m_A > 0$ such that

$$\langle A(t,x) - A(t,y), x - y\rangle \ge m_A\|x - y\|_V^2$$

for a.e. $t \in [0,T]$ and $x, y \in V$.

Remark 3.2. Let $V = H_0^1(\Omega)$, $k_1 \in L^\infty(\Omega \times (0,T))_+$ be such that $k_1(z,t) \ge m_{k_1} > 0$ for a.e. $(z,t) \in \Omega \times [0,T]$, $k_2 \in L^2(0,T)$, and $l\colon \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function such that: (i) there exist $\alpha \in L^2(\Omega)$ and $a_1, a_2 \ge 0$ satisfying $|f(z,s,\xi)| \le a_1\|\xi\|_{\mathbb{R}^N} + a_2|s| + \alpha(z)$ for a.e. $z \in \Omega$, for all $(s,\xi) \in \mathbb{R}^{N+1}$; (ii) there exists $w \in L^1(\Omega)_+$ and $b_1, b_2 \ge 0$ with $b_2\hat{c}^2 < m_{k_1}$ satisfying $f(z,s,\xi)s \le b_1\|\xi\|_{\mathbb{R}^N} + b_2|s|^2 + w(z)$ for a.e. $z \in \Omega$ for all $(s,\xi) \in \mathbb{R}^{N+1}$, where $\hat{c}$ is the smallest constant such that $\|v\|_{L^2(\Omega)} \le \hat{c}\|\nabla v\|_{L^2(\Omega;\mathbb{R}^N)}$ for all $v \in H_0^1(\Omega)$. Then, arguing as in the proof of [11, Theorem 3.2], it is not difficult to prove that the function $A\colon [0,T] \times V \to V^*$ defined in (1.2) is not strongly monotone and Lipschitz, but it satisfies conditions $H(A)$(i)–(iv).

<u>$H(J)$</u>: $J\colon [0,T] \times H \to \mathbb{R}$ is such that

(i) for all $x \in H$, the function $t \mapsto J(t,x)$ is measurable on $[0,T]$;

(ii) $x \mapsto J(t,x)$ is locally Lipschitz on H for a.e. $t \in [0,T]$;

(iii) there are a function $a_J \in L^2(0,T)_+$ and a constant $b_J > 0$ such that

$$\|\partial J(t,x)\|_H \le a_J(t) + b_J\|x\|_H$$

for all $x \in H$ and a.e. $t \in [0,T]$;

(iv) there exist $\theta \in [1,2]$, $c_J \in L^2(0,T)_+$ and $d_J \in L^1(0,T)_+$ such that

$$J^0(t,x;-x) \le m_J\|x\|_H^\theta + c_J(t)\|x\|_H + d_J(t)$$

for all $x \in H$ and a.e. $t \in [0,T]$, where $m_J > 0$ is such that

$$\begin{cases} m_J > 0 & \text{if } \theta \in [1,2), \\ m_J\|\gamma\|^2 < m_A & \text{if } \theta = 2, \end{cases}$$

where $m_A > 0$ is given in $H(A)$(iv) and γ is the embedding operator from V into H;

(v) there exists a constant $m_J > 0$ with $m_J\|\gamma\|^2 < m_A$ such that

$$J^0(t, x; y - x) + J^0(t, y; x - y) \leq m_J \|x - y\|_H^2$$

for all $x, y \in H$ and a.e. $t \in [0, T]$, where $m_A > 0$ is given in $H(A)$(v).

$\underline{H(0)}$: $x_0 \in V$ and $\beta\colon V \to Z$ is linear and bounded.

$\underline{H(\varphi)}$: $\varphi\colon Z \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function such that $0 \in \partial_C \varphi(\beta x_0)$ (or $\beta x_0 \in \mathrm{int} D(\varphi)$).

Let us consider the mappings $\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{N}\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $\Phi\colon \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$\mathcal{A}(x)(t) = A(t, x(t) + x_0) \text{ for a.e. } t \in [0, T], \tag{3.3}$$

$$\mathcal{N}(y) := \{\xi \in \mathcal{H} \mid \xi(t) \in \partial J(t, y(t) + x_0) \text{ for a.e. } t \in [0, T]\}, \tag{3.4}$$

$$\Phi(x) = \int_0^T \varphi(\beta(x(t) + x_0))\, dt \tag{3.5}$$

for all $(x, y) \in \mathcal{V} \times \mathcal{H}$.

Lemma 3.3. *Assume that $H(\varphi)$ holds. Then the function Φ defined in (3.5) is proper, convex and lower semicontinuous.*

Proof. It is obvious from $H(\varphi)$ that $\Phi(0) = \int_0^T \varphi(\beta x_0)\, dt = T\varphi(\beta x_0) < +\infty$. Besides, the convexity of φ guarantees that Φ is convex as well. Let $\{x_n\} \subset \mathcal{V}$ be a sequence with $x_n \to x$ in $\mathcal{V}$ as $n \to \infty$ for some $x \in \mathcal{V}$. So, we may assume that $x_n(t) \to x(t)$ in V as $n \to \infty$ for a.e. $t \in [0, T]$. It follows from [1, Proposition 1.10] that there are constants $c_\varphi, d_\varphi \geq 0$ such that

$$\varphi(z) \geq -c_\varphi \|z\|_Z - d_\varphi \text{ for all } z \in Z. \tag{3.6}$$

Applying Fatou lemma, see e.g. [30, Theorem 1.64], it yields

$$\liminf_{n\to\infty} \Phi(x_n) = \liminf_{n\to\infty} \int_0^T \varphi(\beta(x_n(t) + x_0))\, dt \geq \int_0^T \liminf_{n\to\infty} \varphi(\beta(x_n(t) + x_0))\, dt$$

$$\geq \int_0^T \varphi(\beta(x(t) + x_0))\, dt = \Phi(x),$$

where we have used the lower semicontinuity of φ. This means that Φ is lower semicontinuous. □

Also, we define $\mathcal{L}\colon D(\mathcal{L}) \subset \mathcal{V} \to \mathcal{V}^*$ by

$$\mathcal{L}x = \frac{\partial x}{\partial t} \text{ for all } x \in D(\mathcal{L}), \tag{3.7}$$

where $D(\mathcal{L})$ is the effective domain of $\mathcal{L}$ given by

$$D(\mathcal{L}) = \mathcal{W}_0 := \{x \in \mathcal{W} \mid x(0) = 0\}.$$

We now introduce an auxiliary problem as follows.

Problem 3.4. Find a function $x \in \mathcal{W}$ such that

$$\mathcal{L}x + \mathcal{A}(x) + \mathcal{N}(x) + \partial_C \Phi(x) \ni f \text{ in } \mathcal{V}^*. \tag{3.8}$$

Next, we provide the following lemma, which will be used to prove the solvability of Problem 1.1.

Lemma 3.5. *Assume that $H(A)$(i)–(iv), $H(J)$(i)–(iv) and $H(0)$ hold. Then the set-valued mapping $\mathcal{A} + \mathcal{N}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is bounded and pseudomonotone with respect to $D(\mathcal{L})$ ($\mathcal{L}$-pseudomonotone, for short) such that*

$$\langle\langle \mathcal{A}(x) + \mathcal{N}(x), x\rangle\rangle \geq r(\|x\|_{\mathcal{V}})\|x\|_{\mathcal{V}} \textit{ for all } x \in \mathcal{V}, \tag{3.9}$$

where $r\colon \mathbb{R}_+ \to \mathbb{R}$ is such that $r(s) \to +\infty$ as $s \to \infty$.

Proof. We first show that $\mathcal{N}$ is well-defined. For any $x \in \mathcal{H}$ fixed, hypotheses $H(J)$(i), (ii) and [30, Proposition 3.44] imply that $t \mapsto \partial J(t, x(t))$ is measurable on $[0, T]$. Yankov-von Neumann-Aumann selection theorem (see [15, Theorem 2.14, p. 158] or [18, Theorem 1.3.1]) indicates that there is a measurable selection $\xi\colon [0, T] \to H$ of $t \mapsto \partial J(t, x(t))$, i.e., $\xi(t) \in \partial J(t, x(t))$ for a.e. $t \in [0, T]$. But, the estimate (see hypothesis $H(J)$(iii))

$$\|\xi(t)\|_H \leq a_J(t) + b_J\|x(t)\|_H \text{ for a.e. } t \in [0, T],$$

entails that $\xi \in \mathcal{H}$. Hence, $\mathcal{N}$ is well-defined.

Let $x \in \mathcal{V}$ be fixed. For any $\xi \in \mathcal{N}(x)$, we use conditions $H(A)$(iii) and $H(J)$(iii) to obtain

$$\begin{aligned}&\|\mathcal{A}(x) + \xi\|_{\mathcal{V}^*}\\ &\leq \|\mathcal{A}(x)\|_{\mathcal{V}^*} + \|\xi\|_{\mathcal{V}^*} \leq \left(\int_0^T \|A(t, x(t) + x_0)\|_{V^*}^2\, dt\right)^{\frac{1}{2}} + \left(\int_0^T \|\xi(t)\|_{V^*}^2\, dt\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \left(\int_0^T \|A(t,x(t)+x_0)\|_{V^*}^2\,dt\right)^{\frac{1}{2}} + \|\gamma^*\| \left(\int_0^T \|\xi(t)\|_H^2\,dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{2}\left(\int_0^T a_A(t)^2 + b_A^2\|x(t)+x_0\|_V^2\,dt\right)^{\frac{1}{2}} + \sqrt{2}\|\gamma^*\| \left(\int_0^T a_J(t)^2 + b_J^2\|x(t)+x_0\|_H^2\,dt\right)^{\frac{1}{2}}.$$

This means that for each $x \in \mathcal{V}$ the set $\mathcal{A}(x)+\mathcal{N}(x)$ is bounded in $\mathcal{V}^*$. Additionally, the convexity and closedness of $\partial J(t,x(t))$ points out that $\mathcal{A}(x)+\mathcal{N}(x)$ is convex and closed in $\mathcal{V}^*$ for each $x \in \mathcal{V}$.

For any $x \in \mathcal{V}$, hypotheses $H(A)$(iii) and (iv) deduce

$$\langle\langle \mathcal{A}(x), x\rangle\rangle = \int_0^T \langle A(t,x(t)+x_0), x(t)+x_0\rangle\,dt - \int_0^T \langle A(t,x(t)+x_0), x_0\rangle\,dt \tag{3.10}$$

$$\geq \int_0^T m_A\|x(t)+x_0\|_V^2 - c_A(t)\|x(t)+x_0\|_V - d_A(t)\,dt$$

$$- \int_0^T (a_A(t) + b_A\|x(t)+x_0\|_V)\|x_0\|_V\,dt.$$

We apply conditions $H(J)$(iii) and (iv) to find

$$((\xi, x)) = -\int_0^T (\xi(t), -x(t)-x_0)_H\,dt - \int_0^T (\xi(t), x_0)_H\,dt$$

$$\geq -\int_0^T J^0(t, x(t)+x_0; -x(t)-x_0)\,dt - \int_0^T \|\xi(t)\|_H\|x_0\|_H\,dt \tag{3.11}$$

$$\geq -m_J \int_0^T \|x(t)+x_0\|_H^\theta - c_J(t)\|x(t)+x_0\|_H - d_J(t)\,dt$$

$$- \int_0^T (a_J(t) + b_J\|x(t)+x_0\|_H)\|x_0\|_H\,dt.$$

From (3.10) and (3.11), we have

$$\langle\langle \mathcal{A}(x)+\mathcal{N}(x), x\rangle\rangle$$

$$\geq \int_0^T m_A\|x(t)+x_0\|_V^2 - c_A(t)\|x(t)+x_0\|_V - d_A(t)\,dt$$

$$
-\int_0^T (a_A(t)+b_A\|x(t)+x_0\|_V)\|x_0\|_V\,dt - m_J\int_0^T \|x(t)+x_0\|_H^\theta - c_J(t)\|x(t)+x_0\|_H
$$
$$
-d_J(t)\,dt - \int_0^T (a_J(t)+b_J\|x(t)+x_0\|_H)\|x_0\|_H\,dt.
$$

Let $\varepsilon = \frac{m_A}{2}$. If $\theta \in [1,2)$, using Young's inequality and Hölder's inequality yields

$$
\begin{aligned}
&\langle\langle \mathcal{A}(x)+\mathcal{N}(x),x\rangle\rangle\\
\geq\ & m_A\|x\|_{\mathcal{V}}^2 - 2m_A\sqrt{T}\|x_0\|_V\|x\|_{\mathcal{V}} - m_AT\|x_0\|_V^2 - \|c_A\|_{L^2(0,T)}\|x\|_{\mathcal{V}}\\
&-\|c_A\|_{L^2(0,T)}\|x_0\|_V^2\sqrt{T} - \|d_A\|_{L^1(0,T)} - \|x_0\|_V\|a_A\|_{L^1(0,T)} - b_A\sqrt{T}\|x_0\|_V\|x\|_{\mathcal{V}}\\
&-b_AT\|x_0\|_V^2 - \varepsilon\|x\|_{\mathcal{V}}^2 - 2\varepsilon\sqrt{T}\|x_0\|\|x\|_{\mathcal{V}}^2 - \varepsilon T\|x_0\|_V^2 - k(\varepsilon) - \|c_J\|_{L^2(0,T)}\|\gamma\|\|x\|_{\mathcal{V}}\\
&-\|c_J\|_{L^2(0,T)}\|x_0\|_H\sqrt{T} - \|d_J\|_{L^1(0,T)} - \|x_0\|_H\|a_J\|_{L^1(0,T)}\\
&-\|x_0\|_Hb_J\|\gamma\|\sqrt{T}\|x\|_{\mathcal{V}} - b_J\|x_0\|_H^2T,
\end{aligned}
\tag{3.12}
$$

with some $k(\varepsilon)>0$, where we have used the element inequality $\|x(t)+x_0\|_V^2 \geq \|x(t)\|_V^2 - 2\|x_0\|_V\|x(t)\|_V - \|x_0\|_V^2$. When $\theta = 2$, we have

$$
\begin{aligned}
&\langle\langle \mathcal{A}(x)+\mathcal{N}(x),x\rangle\rangle\\
\geq\ & (m_A - m_J\|\gamma\|^2)\|x\|_{\mathcal{V}}^2 - 2m_A\sqrt{T}\|x_0\|_V\|x\|_{\mathcal{V}} - m_AT\|x_0\|_V^2 - \|c_A\|_{L^2(0,T)}\|x\|_{\mathcal{V}}\\
&-\|c_A\|_{L^2(0,T)}\|x_0\|_V^2\sqrt{T} - \|d_A\|_{L^1(0,T)} - \|x_0\|_V\|a_A\|_{L^1(0,T)} - b_A\sqrt{T}\|x_0\|_V\|x\|_{\mathcal{V}}\\
&-b_AT\|x_0\|_V^2 - 2m_J\|\gamma\|\sqrt{T}\|x_0\|_H\|x\|_{\mathcal{V}} - m_J\|x_0\|_H^2T - \|c_J\|_{L^2(0,T)}\|\gamma\|\|x\|_{\mathcal{V}}\\
&-\|c_J\|_{L^2(0,T)}\|x_0\|_H\sqrt{T} - \|d_J\|_{L^1(0,T)} - \|x_0\|_H\|a_J\|_{L^1(0,T)}\\
&-\|x_0\|_Hb_J\|\gamma\|\sqrt{T}\|x\|_{\mathcal{V}} - b_J\|x_0\|_H^2T.
\end{aligned}
\tag{3.13}
$$

By virtue of (3.12) and (3.13), it yields

$$
\langle\langle \mathcal{A}(x)+\mathcal{N}(x),x\rangle\rangle \geq r(\|x\|_{\mathcal{V}})\|x\|_{\mathcal{V}},
$$

where $r\colon \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$
r(s) := \begin{cases} \dfrac{m_A}{2}s - L_1 - \dfrac{L_2}{s}, \\ (m_A - m_J\|\gamma\|^2)s - L_3 - \dfrac{L_4}{s}, \end{cases}
$$

and L_1, L_2, L_3, L_4 are given by

$$L_1 := 2m_A\sqrt{T}\|x_0\|_V + \|c_A\|_{L^2(0,T)} + b_A\sqrt{T}\|x_0\|_V + 2\varepsilon\sqrt{T}\|x_0\| + \|c_J\|_{L^2(0,T)}\|\gamma\| \\ +\|x_0\|_H b_J\|\gamma\|\sqrt{T},$$

$$L_2 := m_A T\|x_0\|_V^2 + \|c_A\|_{L^2(0,T)}\|x_0\|_V^2\sqrt{T} + \|d_A\|_{L^1(0,T)} + \|x_0\|_V\|a_A\|_{L^1(0,T)} \\ +b_A T\|x_0\|_V^2 + \varepsilon T\|x_0\|_V^2 + k(\varepsilon) + \|c_J\|_{L^2(0,T)}\|x_0\|_H\sqrt{T} + \|d_J\|_{L^1(0,T)} \\ +\|x_0\|_H\|a_J\|_{L^1(0,T)} + b_J\|x_0\|_H^2 T,$$

$$L_3 := 2m_A\sqrt{T}\|x_0\|_V + \|c_A\|_{L^2(0,T)} + b_A\sqrt{T}\|x_0\|_V + 2m_J\|\gamma\|\sqrt{T}\|x_0\|_H \\ +\|c_J\|_{L^2(0,T)}\|\gamma\| + \|x_0\|_H b_J\|\gamma\|\sqrt{T},$$

$$L_4 := m_A T\|x_0\|_V^2 + \|c_A\|_{L^2(0,T)}\|x_0\|_V^2\sqrt{T} + \|d_A\|_{L^1(0,T)} + \|x_0\|_V\|a_A\|_{L^1(0,T)} \\ +b_A T\|x_0\|_V^2 + m_J\|x_0\|_H^2 T + \|c_J\|_{L^2(0,T)}\|x_0\|_H\sqrt{T} + \|d_J\|_{L^1(0,T)} \\ +\|x_0\|_H\|a_J\|_{L^1(0,T)} + b_J\|x_0\|_H^2 T.$$

Therefore, (3.9) is verified.

Moreover, we show that $\mathcal{A}+\mathcal{N}$ is $\mathcal{L}$-pseudomonotone. It is obvious that for each $x \in \mathcal{V}$, $\mathcal{A}(x) + \mathcal{N}(x)$ is nonempty, bounded, closed and convex in $\mathcal{V}^*$. We now assert that $\mathcal{A}$ is demicontinuous. Let $\{x_n\} \subset \mathcal{V}$ be a sequence satisfying $x_n \to x$ in $\mathcal{V}$ as $n \to \infty$, for some $x \in \mathcal{V}$. Then, without loss of generality, we may assume that $x_n(t) \to x(t)$ in V as $n \to \infty$ for a.e. $t \in [0,T]$. Recall that $z \mapsto A(t,z)$ is pseudomonotone, so, using [30, Theorem 3.69] and hypothesis $H(A)$(iii) conclude that $z \mapsto A(t,z)$ demicontinuous and $\{\mathcal{A}(x_n)\}$ is bounded in $\mathcal{V}^*$. For any $y \in \mathcal{V}$ fixed, employing Lebesgue dominated convergence theorem finds

$$\lim_{n\to\infty} \langle\langle \mathcal{A}(x_n), y\rangle\rangle = \lim_{n\to\infty} \int_0^T \langle A(t, x_n(t) + x_0), y(t)\rangle \, dt$$

$$= \int_0^T \lim_{n\to\infty} \langle A(t, x_n(t) + x_0), y(t)\rangle \, dt = \int_0^T \langle A(t, x(t) + x_0), y(t)\rangle \, dt = \langle\langle \mathcal{A}(x), y\rangle\rangle.$$

The arbitrariness of $y \in \mathcal{V}$ indicates that $\mathcal{A}$ is demicontinuous. Nevertheless, we are going to prove that $\mathcal{N}$ is strongly-weakly u.s.c., i.e., $\mathcal{N}$ is u.s.c. from $\mathcal{H}$ with the norm topology into $\mathcal{H}$ endowed with its weak topology. So, we have to verify that for each weakly closed subset D in $\mathcal{H}$, $\mathcal{N}^-(D)$ is closed in $\mathcal{H}$. Let $\{x_n\} \subset \mathcal{N}^-(D)$ be such that $x_n \to x$ in $\mathcal{H}$ as $n \to \infty$ for some $x \in \mathcal{H}$. Then, it has

$$x_n(t) \to x(t) \text{ in } H \text{ as } n \to \infty \text{ for a.e. } t \in [0,T]. \tag{3.14}$$

Also, we are able to find a sequence $\{\xi_n\} \subset \mathcal{H}$ with $\xi_n \in \mathcal{N}(x_n) \cap D$. But, condition $H(J)$(iii) points out that $\{\xi_n\}$ is bounded in $\mathcal{H}$. Passing to a subsequence if necessary, we may suppose that

$$\xi_n \xrightarrow{w} \xi \text{ in } \mathcal{H} \text{ as } n \to \infty, \tag{3.15}$$

for some $\xi \in \mathcal{H}$. The convergences (3.14)–(3.15) and the convergence theorem of Aubin and Cellina (see for example, [30, Theorem 3.13]) yield that $\xi(t) \in \partial J(t, x(t)+x_0)$ for a.e. $t \in [0,T]$, namely, $\xi \in \mathcal{N}(x) \cap D$ thanks to the weak closedness of D. Hence, $x \in \mathcal{N}^-(D)$. We conclude that $\mathcal{N}$ is strongly-weakly u.s.c. Therefore, $\mathcal{A}+\mathcal{N}$ is strongly-weakly u.s.c. as well, see [18, Theorem 1.2.14].

Let sequences $\{x_n\} \subset \mathcal{V}$ and $\{x_n^*\} \subset \mathcal{V}^*$ be such that

$$\begin{cases} x_n \xrightarrow{w} x \text{ in } \mathcal{V}, \\ \mathcal{L}x_n \xrightarrow{w} \mathcal{L}x \text{ in } \mathcal{V}^*, \\ x_n^* \in \mathcal{A}(x_n) + \mathcal{N}(x_n), \text{ i.e., } x_n^* = \mathcal{A}(x_n) + \xi_n \text{ with } \xi_n \in \mathcal{N}(x_n), \\ x_n^* \xrightarrow{w} x^* \text{ in } \mathcal{V}^*, \\ \limsup_{n\to\infty} \langle\langle x_n^*, x_n - x \rangle\rangle \le 0. \end{cases} \tag{3.16}$$

This means that $\{x_n\}$ is bounded in $\mathcal{W}$ and $\{\xi_n\}$ is bounded in $\mathcal{H}$ (see $H(J)$(iii)). So, we may assume that the following convergences and (3.15) hold

$$x_n \xrightarrow{w} x \text{ in } \mathcal{W} \text{ and } x_n \to x \text{ in } \mathcal{H}, \text{ as } n \to \infty. \tag{3.17}$$

Keeping in mind that for each $x \in \mathcal{H}$ the set $\mathcal{N}(x)$ is bounded, closed and convex in $\mathcal{H}$, so, $\mathcal{N}(x)$ is weakly closed as well. The latter together with the strong-weak upper semicontinuity of $\mathcal{N}$ and [18, Theorem 1.1.4] concludes that $\mathcal{N}$ is closed from $\mathcal{H}$ into $w-\mathcal{H}$ (i.e., the graph of $\mathcal{N}$ is closed in $\mathcal{H} \times w-\mathcal{H}$). Taking into account (3.16)–(3.17) and the closedness of $\mathcal{N}$, we have $\xi \in \mathcal{N}(x)$. Then, it has $\mathcal{A}(x_n) = x_n^* - \xi_n \xrightarrow{w} x^* - \xi$ in $\mathcal{V}^*$ as $n \to \infty$. But, (3.16) deduces

$$\begin{aligned} 0 &\ge \limsup_{n\to\infty} \langle\langle x_n^*, x_n - x\rangle\rangle \ge \limsup_{n\to\infty} \langle\langle \mathcal{A}(x_n), x_n - x\rangle\rangle + \liminf_{n\to\infty} ((\xi_n, x_n - x)) \\ &= \limsup_{n\to\infty} \langle\langle \mathcal{A}(x_n), x_n - x\rangle\rangle, \end{aligned} \tag{3.18}$$

where the last equation is obtained by using (3.17). However, hypotheses $H(A)$ allow us to apply [42, Proposition 1] to reveal that $\mathcal{A}$ is pseudomonotone. We use (3.16), (3.18) and [30, Proposition 3.66] to get

$$\mathcal{A}(x_n) \xrightarrow{w} \mathcal{A}(x) \text{ and } \lim_{n\to\infty} \langle\langle \mathcal{A}(x_n), x_n\rangle\rangle = \langle\langle \mathcal{A}(x), x\rangle\rangle.$$

From the analysis above, we confess that $\mathcal{A}(x) = x^* - \xi$, i.e., $\mathcal{A}(x) + \xi \in \mathcal{A}(x) + \mathcal{N}(x)$ and $\lim_{n\to\infty} \langle\langle x_n^*, x_n\rangle\rangle = \langle\langle \mathcal{A}(x) + \xi, x\rangle\rangle = \langle\langle x^*, x\rangle\rangle$. Consequently, $\mathcal{A} + \mathcal{N} \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is pseudomonotone with respect to $D(\mathcal{L})$. □

Theorem 3.6. *Assume that* $H(A)$(i)–(iv), $H(J)$(i)–(iv), $H(0)$ *and* $H(\varphi)$ *hold. Then for each* $f \in \mathcal{H}$, *Problem 1.1 admits a solution.*

Proof. We will utilize Theorem 2.6 to prove the desired conclusion. It follows from [9, Section 8.5] that $\mathcal{L}$ defined in (3.7) is closed, linear, densely defined, and maximal monotone. Besides, Lemma 3.3 and [9, Theorem 6.3.19, p. 48] infer that $\partial_C\Phi$ is a maximal monotone operator with $0 \in \partial_C\Phi(0)$, due to $H(\varphi)$. On the other side, we apply [4, Proposition 2.7] to obtain that $\mathrm{int}D(\Phi) \subset D(\partial_C\Phi)$, that is, $\mathrm{int}D(\Phi) \subset \mathrm{int}D(\partial_C\Phi)$. This combined with $H(\varphi)$ and Proposition 2.5 implies that $\partial_C\Phi$ is strongly-quasi bounded.

To conclude, all conditions of Theorem 2.6 are verified. Using this theorem, we admit that there exists a function $\overline{x} \in \mathcal{W}$ such that

$$\mathcal{L}\overline{x} + \mathcal{A}(\overline{x}) + \mathcal{N}(\overline{x}) + \partial_C\Phi(\overline{x}) \ni f \text{ in } \mathcal{V}^*.$$

Set $x = \overline{x} + x_0$, then, x solves the above problem

$$\begin{cases} x'(t) + A(t, x(t)) + \xi(t) + \eta(t) = f(t) \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

with $\xi \in \mathcal{N}(x - x_0)$ and $\eta \in \partial_C\Phi(x - x_0)$. Whereas, by virtue of [30, Proposition 3.46] and $H(\varphi)$, it gives

$$\partial_C\Phi(x - x_0) = S^2_{\varphi(x(\cdot))} := \{\eta \in \mathcal{V}^* \mid \eta(t) \in \beta^*\partial_C\varphi(\beta(x(t))) \text{ for a.e. } t \in [0, T]\}.$$

Then, we have $\xi(t) \in \partial J(t, x(t))$ and $\eta(t) \in \beta^*\partial_C\varphi(\beta(x(t)))$ for a.e. $t \in [0, T]$. Consequently, $x \in \mathcal{W}$ solves Problem 1.1. □

We end the section to deliver the unique solvability of Problem 1.1.

Theorem 3.7. *Assume that $H(A)$(i)–(iii), (v), $H(J)$(i)–(iii), (v), $H(0)$ and $H(\varphi)$ are satisfied. Then for each $f \in \mathcal{H}$, Problem 1.1 has a unique solution.*

Proof. We will use Theorem 3.6 to prove the existence of solutions to Problem 1.1. Therefore, we have to verify the validity of $H(A)$(iv) and $H(J)$(iv). Indeed, $H(A)$(iii) and (v) imply

$$\langle A(t, x), x\rangle = \langle A(t, x) - A(t, 0), x\rangle + \langle A(t, 0), x\rangle \geq m_A\|x\|_V^2 - \|A(t, 0)\|_{V^*}\|x\|_V$$
$$\geq m_A\|x\|_V^2 - a_A(t)\|x\|_V$$

for all $x \in V$ and a.e. $t \in [0, T]$. This means that $H(A)$(iv) is satisfied with $c_A = a_A$ and $d_A \equiv 0$. For any $y \in H$ and a.e. $t \in [0, T]$, we have

$$J^0(t, y; -y) = J^0(t, y; -y) + J^0(t, 0; y) - J^0(t, 0; y) \leq m_J\|y\|_H^2 + a_J(t)\|y\|_H.$$

So, $H(J)$(iv) is available with $\theta = 2$, $c_J = a_J$ and $d_J \equiv 0$. Employing Theorem 3.6, we conclude that Problem 1.1 is solvable.

For the uniqueness, let $x, y \in \mathcal{W}$ be two solutions to Problem 1.1. We are able to find functions $\xi_x(t) \in \partial J(t, x(t))$, $\xi_y(t) \in \partial J(t, y(t))$, $\eta_x(t) \in \beta^* \partial_C \varphi(\beta x(t))$ and $\eta_y(t) \in \beta^* \partial_C \varphi(\beta y(t))$ for a.e. $t \in [0, T]$. A simple calculating gives

$$\langle x'(t) - y'(t), x(t) - y(t)\rangle + \langle A(t, y(t)) - A(t, x(t)), y(t) - x(t)\rangle$$

$$+(\xi_x(t) - \xi_y(t), y(t) - x(t))_H = \langle \eta_y(t) - \eta_x(t), x(t) - y(t)\rangle \leq 0$$

for a.e. $t \in [0, T]$, where the last inequality is obtained by using the monotonicity of $\partial_C \varphi$. Integrating the inequality above over $[0, t]$ and using hypotheses $H(A)(\mathrm{v})$ and $H(J)(\mathrm{v})$, we have

$$\frac{1}{2}\|x(t) - y(t)\|_H^2 + (m_A - m_J\|\gamma\|^2) \int_0^t \|x(s) - y(s)\|_V^2 \, ds \leq 0$$

for all $t \in [0, T]$. This concludes that Problem 1.1 has a unique solution. □

4. Optimal control

In this section, we are interesting in the investigation of a nonlinear optimal control problem, Problem 1.2, governed by the nonlinear and nonsmooth dynamics system, Problem 1.1. Our main goal is to establish a sufficient theorem for examining the existence of an optimal control of Problem 1.2.

In the sequel, we denote by $\Gamma\colon \mathcal{H} \to 2^{\mathcal{W}}$ by the solution mapping of Problem 1.1, i.e.,

$$\Gamma(f) := \{x \in \mathcal{W} \mid x \text{ is a solution to Problem 1.1 corresponding to } f\}. \tag{4.1}$$

To obtain the existence of an optimal control to Problem 1.2, we make the following assumptions.

<u>$H(\varphi)'$</u>: $\varphi\colon Z \to \mathbb{R}$ is a convex and lower semicontinuous function such that there exists $b_\varphi > 0$ satisfying

$$\|\partial_C \varphi(z)\|_{Z^*} \leq b_\varphi(1 + \|z\|_Z)$$

for all $z \in Z$.

<u>$H(\beta)'$</u>: $\beta\colon V \to Z$ is a bounded, linear and compact operator such that its Nemytskii operator $\overline{\beta}\colon \mathcal{W} \subset \mathcal{V} \to \mathcal{Z}$ is compact, where $\mathcal{Z} := L^2(0, T; Z)$.

<u>$H(1)$</u>: $g \in \mathcal{H}$, $z_0 \in H$ and $y_0 \in \mathcal{V}$.

Remark 4.1. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a smooth boundary $\partial\Omega$ such that $\partial\Omega$ is divided into two measurable and disjoint parts Γ_1 and Γ_2 with

$meas(\Gamma_1) > 0$. Also let $V = \{v \in H^1(\Omega;\mathbb{R}^N) \mid v = 0 \text{ on } \Gamma_1\}$, $Z = L^2(\Gamma_2;\mathbb{R}^N)$, and β be the trace operator from V to Z. From the trace theorem, it is not difficult to prove that β is bounded, linear and compact, and the function $\varphi\colon Z \to \mathbb{R}$ defined by $\varphi(v) = \int_{\Gamma_2} \|v(z)\|_{\mathbb{R}^N}\, d\Gamma$ for all $v \in Z$ satisfies $H(\varphi)'$. Moreover, [34, Theorem 2.18] reveals that the Nemytskii operator $\overline{\beta}\colon \mathcal{W} \subset \mathcal{V} \to \mathcal{Z}$ corresponding to β is compact.

We provide the following lemma to reveal the critical properties of solution map Γ for Problem 1.1, which will play a significant role for the proof of the main results of the paper.

Lemma 4.2. *Suppose that $H(A)$(i)–(iv), $H(J)$(i)–(iv), $H(0)$, $H(\beta)'$ and $H(\varphi)'$ are fulfilled. Then, the statements hold*

(i) Γ *maps bounded sets of $\mathcal{H}$ into bounded sets of $\mathcal{W}$;*
(ii) Γ *is weakly-weakly u.s.c. from $\mathcal{H}$ into $\mathcal{W}$ (i.e., Γ is u.s.c. from $w-\mathcal{H}$ into $w-\mathcal{W}$), and weakly-strongly u.s.c. from $\mathcal{H}$ into $C(0,T;H)$ (i.e., Γ is u.s.c. from $w-\mathcal{H}$ into $C(0,T;H)$);*
(iii) *for each bounded and closed subset $I \subset \mathcal{H}$, the set $\Gamma(I)$ is compact in $C(0,T;H)$, namely, Γ is compact from $\mathcal{H}$ into $C(0,T;H)$.*

Proof. (i) Let O be a bounded subset of $\mathcal{H}$. For any $x \in \Gamma(O)$, we can find functions $f \in O$, $\xi \in \mathcal{H}$ and $\eta \in \mathcal{V}^*$ such that (3.1) and (3.2) hold. We use the integration by parts formula and hypothesis $H(A)$(iv) to get

$$\int_0^t \langle x'(s), x(s)\rangle\, ds = \frac{1}{2}(\|x(t)\|_H^2 - \|x(0)\|_H^2) \tag{4.2}$$

and

$$\int_0^t \langle A(s,x(s)), x(s)\rangle\, ds \geq m_A \int_0^t \|x(s)\|_V^2 - c_A(s)\|x(s)\|_V - d_A(s)\, ds \tag{4.3}$$

for all $t \in [0,T]$. Applying condition $H(J)$(iv), it gives

$$\begin{aligned}\int_0^t (\xi(s), -x(s))_H\, ds &\leq \int_0^t J^0(s,x(s);-x(s))\, ds \\ &\leq \int_0^t m_J\|x(s)\|_H^\theta + c_J(s)\|x(s)\|_H + d_J(s)\, ds \end{aligned} \tag{4.4}$$

for all $t \in [0,T]$. It follows from $H(\varphi)'$ that

$$\int_0^t \langle \eta(s), -x(s)\rangle\, ds \le \int_0^t \varphi(\beta 0) - \varphi(\beta x(s))\, ds \le T(\varphi(\beta 0) + d_\varphi) + \int_0^t c_\varphi \|\beta x(s)\|_Z\, ds \tag{4.5}$$

for all $t \in [0, T]$, where the constants c_φ and d_φ are given in (3.6). Besides, it has

$$\int_0^t (f(s), x(s))_H\, ds \le \|f\|_{\mathcal{H}} \|x\|_{L^2(0,t;H)} \tag{4.6}$$

for all $t \in [0, T]$. Multiplying (3.1) by $x(t)$ and integrating the resulting equality over $[0, t]$, we can use the estimates (4.2)–(4.6) to obtain

$$\begin{aligned}
&\frac{1}{2}(\|x(t)\|_H^2 - \|x(0)\|_H^2) + m_A \int_0^t \|x(s)\|_V^2 - c_A(s)\|x(s)\|_V - d_A(s)\, ds \\
\le &\int_0^t m_J \|x(s)\|_H^\theta + c_J(s)\|x(s)\|_H + d_J(s)\, ds + T(\varphi(\beta 0) + d_\varphi) + \int_0^t c_\varphi \|\beta x(s)\|_Z\, ds \\
&+ \|f\|_{\mathcal{H}} \|x\|_{L^2(0,t;H)}
\end{aligned}$$

for all $t \in [0, T]$. Furthermore, when $\theta \in [1, 2)$, we utilize Young's inequality and Hölder's inequality to get

$$\begin{aligned}
&\frac{1}{2}(\|x(t)\|_H^2 - \|x(0)\|_H^2) + (m_A - 2\varepsilon) \int_0^t \|x(s)\|_V^2\, ds \\
\le &\frac{\|c_A\|_{L^2(0,T)} + \|c_J\|_{L^2(0,T)}}{4\varepsilon} + \|d_A\|_{L^1(0,T)} + 3\varepsilon \int_0^t \|x(s)\|_H^2\, ds + c(\varepsilon) \\
&+ \|d_J\|_{L^1(0,T)} + T(\varphi(\beta 0) + d_\varphi) + \frac{\|\beta\|^2 c_\varphi^2 T}{4\varepsilon} + \frac{\|f\|_{\mathcal{H}}^2}{4\varepsilon}
\end{aligned}$$

for all $t \in [0, T]$, with some $c(\varepsilon) > 0$; if $\theta = 2$, we also obtain

$$\begin{aligned}
&\frac{1}{2}(\|x(t)\|_H^2 - \|x(0)\|_H^2) + (m_A - 2\varepsilon) \int_0^t \|x(s)\|_V^2\, ds \\
\le &\frac{\|c_A\|_{L^2(0,T)} + \|c_J\|_{L^2(0,T)}}{4\varepsilon} + \|d_A\|_{L^1(0,T)} + \int_0^t (2\varepsilon + m_J)\|x(s)\|_H^2\, ds \\
&+ \|d_J\|_{L^1(0,T)} + T(\varphi(\beta 0) + d_\varphi) + \frac{\|\beta\|^2 c_\varphi^2 T + \|f\|_{\mathcal{V}^*}^2}{4\varepsilon}
\end{aligned}$$

for all $t \in [0, T]$. We are now in a position to invoke Gronwall's inequality to find that $\Gamma(O)$ is bounded both in $\mathcal{V}$ and $C(0, T; H)$.

Moreover, it follows from (3.1) that

$$\|x'(t)\|_{V^*} \leq \|A(t, x(t))\|_{V^*} + \|\xi(t)\|_{V^*} + \|\eta(t)\|_{V^*} + \|f(t)\|_{V^*}$$

for a.e. $t \in [0, T]$. This combined with hypotheses $H(A)$(iii), $H(J)$(iii) and $H(\varphi)'$ deduces

$$\begin{aligned} \|x'(t)\|_{V^*} \leq a_A(t) + b_A\|x(t)\|_V + \|\gamma^*\|(a_J(t) + b_J\|x(t)\|_H) \\ + \|\beta^*\|b_\varphi(1 + \|\beta\|\|x(t)\|_V) + \|\gamma^*\|\|f(t)\|_H \end{aligned} \tag{4.7}$$

for a.e. $t \in [0, T]$. Recall that $\Gamma(O)$ is bounded both in $\mathcal{V}$ and $C(0, T; H)$, we conclude that $\Gamma(O)$ is bounded in $\mathcal{W}$ too.

(ii) We first prove that Γ is u.s.c. from $w - \mathcal{H}$ into $w - \mathcal{W}$. It is sufficient to show that for each weakly closed set D in $\mathcal{W}$, the set $\Gamma^-(D)$ is weakly closed in $\mathcal{H}$. Let $\{f_n\} \subset \Gamma^-(D)$ be a sequence such that

$$f_n \xrightarrow{w} f \text{ in } \mathcal{H} \text{ as } n \to \infty, \tag{4.8}$$

for some $f \in \mathcal{H}$. Then, we are able to find a sequence $\{x_n\} \subset \mathcal{W}$ satisfying $x_n \in \Gamma(f_n) \cap D$ for each $n \in \mathbb{N}$, namely, for each $n \in \mathbb{N}$ we have

$$\begin{cases} x_n'(t) + A(t, x_n(t)) + \xi_n(t) + \eta_n(t) = f_n(t) \text{ for a.e. } t \in [0, T], \\ x_n(0) = x_0, \end{cases} \tag{4.9}$$

where $\xi_n \in \mathcal{H}$ and $\eta_n \in \mathcal{V}^*$ are such that

$$\eta_n(t) \in \beta^* \partial_C \varphi(\beta x_n(t)) \quad \text{and} \quad \xi_n(t) \in \partial J(t, x_n(t)) \quad \text{for a.e. } t \in [0, T]. \tag{4.10}$$

By virtue of assertion (i), we conclude that the sequence $\{x_n\}$ is bounded in $\mathcal{W}$. Then, passing to a relabeled subsequence, we can assume that

$$x_n \xrightarrow{w} x \text{ in } \mathcal{W} \text{ and } x_n \to x \text{ in } \mathcal{H} \text{ as } n \to \infty, \tag{4.11}$$

with some $x \in \mathcal{W}$, due to the compactness of the embedding from $\mathcal{W}$ into $\mathcal{H}$. So, we may say that

$$x_n(t) \to x(t) \text{ in } H \text{ as } n \to \infty \tag{4.12}$$

for a.e. $t \in [0, T]$. Using $H(J)$(iii), we can see that $\{\xi_n\}$ is bounded in $\mathcal{H}$. Without loss of generality, it has $\xi_n \xrightarrow{w} \xi$ in $\mathcal{H}$ for some $\xi \in \mathcal{H}$ as $n \to \infty$. This together with the convergence (4.12) and the convergence theorem of Aubin and Cellina (see for example, [30, Theorem 3.13]) gives $\xi(t) \in \partial J(t, x(t))$ for a.e. $t \in [0, T]$.

We multiply (4.9) by $x_n(t) - x(t)$ and integrate the resulting equality over $[0, t]$ to obtain

$$\int_0^t \langle x_n'(s), x_n(s) - x(s)\rangle + \langle A(s, x_n(s)), x_n(s) - x(s)\rangle \, ds$$
$$= \int_0^t (\xi_n(s), x(s) - x_n(s))_H + \langle \eta_n(s), x(s) - x_n(s)\rangle + (f_n(s), x_n(s) - x(s))_H \, ds$$

for all $t \in [0, T]$. Note that

$$\int_0^t \langle x_n'(s), x_n(s) - x(s)\rangle \, ds = \frac{1}{2}(\|x(t) - x_n(t)\|_H^2 - \|x(0) - x_n(0)\|_H^2) + \int_0^t \langle x'(s), x_n(s) - x(s)\rangle \, ds$$

$$\int_0^t \langle \eta_n(s), x(s) - x_n(s)\rangle \, ds \le \int_0^t \varphi(\beta x(s)) - \varphi(\beta x_n(s)) \, ds$$

for all $t \in [0, T]$, we have

$$\frac{1}{2}\|x(t) - x_n(t)\|_H^2 + \int_0^t \langle A(s, x_n(s)), x_n(s) - x(s)\rangle \, ds + \int_0^t \langle x'(s), x_n(s) - x(s)\rangle \, ds$$
$$\le \int_0^t (\xi_n(s), x(s) - x_n(s))_H + (f_n(s), x_n(s) - x(s))_H \, ds + \int_0^t \varphi(\beta x(s)) - \varphi(\beta x_n(s)) \, ds \tag{4.13}$$

for all $t \in [0, T]$. Recall that (4.12) and the inequality

$$\int_0^t (\xi_n(s), x(s) - x_n(s))_H \, ds \le \int_0^t J^0(t, x_n(s); x(s) - x_n(s)) \, ds,$$

we apply Fatou lemma and Proposition 2.3(iv) to find

$$\limsup_{n\to\infty} \int_0^t (\xi_n(s), x(s) - x_n(s))_H \, ds \le \limsup_{n\to\infty} \int_0^t J^0(t, x_n(s); x(s) - x_n(s)) \, ds$$

$$\leq \int_0^t \limsup_{n\to\infty} J^0(t, x_n(s); x(s) - x_n(s))\, ds \leq 0 \tag{4.14}$$

for all $t \in [0, T]$. The convergences (4.8) and (4.12) imply

$$\lim_{n\to\infty} \int_0^t (f_n(s), x_n(s) - x(s))_H\, ds = 0 \text{ and } \lim_{n\to\infty} \int_0^t \langle x'(s), x_n(s) - x(s)\rangle\, ds = 0 \tag{4.15}$$

for all $t \in [0, T]$. Nevertheless, arguing as in the proof of Lemma 3.3 indicates that the function $y \mapsto \int_0^t \varphi(\beta y(s))\, ds$ is convex and lower semicontinuous on $L^2(0, t; V)$, so, it is weakly lower semicontinuous on $L^2(0, t; V)$ too. Hence,

$$\limsup_{n\to\infty} \int_0^t \varphi(\beta x(s)) - \varphi(\beta x_n(s))\, ds \leq \int_0^t \varphi(\beta x(s))\, ds - \liminf_{n\to\infty} \int_0^t \varphi(\beta x_n(s))\, ds \leq 0 \tag{4.16}$$

for all $t \in [0, T]$. Passing to the upper limit as $n \to \infty$ into inequality (4.13) and using (4.14)–(4.16), it yields

$$\limsup_{n\to\infty} \int_0^t \langle A(s, x_n(s)), x_n(s) - x(s)\rangle\, ds \leq 0.$$

But, hypothesis $H(A)$, [42, Proposition 1] and convergence (4.11) turn out that

$$\begin{cases} A(\cdot, x_n(\cdot)) \xrightarrow{w} A(\cdot, x(\cdot)) \text{ in } L^2(0, t; V^*) \text{ and} \\ \int_0^t \langle A(s, x_n(s)), x_n(s)\rangle\, ds \to \int_0^t \langle A(s, x(s)), x(s)\rangle\, ds \end{cases} \text{as } n \to \infty \tag{4.17}$$

for all $t \in [0, T]$. Remembering that $\eta_n(t) \in \beta^* \partial_C \varphi(\beta x_n(t))$ for a.e. $t \in [0, T]$, we can find $\zeta_n \colon [0, T] \to Z^*$ such that $\eta_n(t) = \beta^* \zeta_n(t)$ for a.e. $t \in [0, T]$. Condition $H(\varphi)'$ ensures that $\{\zeta_n\}$ is bounded in $\mathcal{Z}^*$, so, passing to a subsequence if necessary, we may suppose that $\zeta_n \xrightarrow{w} \zeta$ in $\mathcal{Z}^*$ as $n \to \infty$ for some $\zeta \in \mathcal{Z}^*$, hence, $\eta_n \xrightarrow{w} \overline{\beta}^* \zeta := \eta$ in $\mathcal{V}^*$ as $n \to \infty$, where $\overline{\beta}^* : \mathcal{Z}^* \to \mathcal{V}^*$ is the dual operator of $\overline{\beta}$. For any $y \in \mathcal{V}$, we have

$$\langle \zeta_n, \overline{\beta}(y - x_n)\rangle_{\mathcal{Z}} = \int_0^T \langle \zeta_n(t), \beta(y(t) - x_n(t))\rangle_Z\, dt = \int_0^T \langle \eta_n(t), y(t) - x_n(t)\rangle\, dt$$

$$= \langle\langle \eta_n, y - x_n \rangle\rangle \leq \int_0^T \varphi(\beta y(t)) - \varphi(\beta x_n(t))\, dt,$$

where we have used the hypothesis $H(\beta)'$. Passing to the upper limit as $n \to \infty$ into the inequality above, it gives

$$\begin{aligned} &\langle\langle \eta, y - x \rangle\rangle = \langle\langle \overline{\beta}^* \zeta, y - x \rangle\rangle = \langle \zeta, \overline{\beta}(y - x)\rangle_{\mathcal{Z}} = \lim_{n\to\infty} \langle \zeta_n, \overline{\beta}(y - x_n)\rangle_{\mathcal{Z}} \\ &\leq \int_0^T \varphi(\beta y(t))\, dt - \liminf_{n\to\infty} \int_0^T \varphi(\beta x_n(t))\, dt \leq \int_0^T \varphi(\beta y(t)) - \varphi(\beta x(t))\, dt. \end{aligned}$$

Combining the above estimates and [30, Proposition 3.46], we obtain $\eta(t) \in \beta^* \partial_C \varphi(\beta x(t))$ for a.e. $t \in [0, T]$. Under the analysis above, we can see that

$$x_n' + A(\cdot, x_n(\cdot)) + \xi_n + \eta_n - f_n \xrightarrow{w} x' + A(\cdot, x(\cdot)) + \xi + \eta - f \text{ in } \mathcal{V}^* \text{ as } n \to \infty,$$

with $\eta(t) \in \beta^* \partial_C \varphi(\beta x(t))$ and $\xi(t) \in \partial J(t, x(t))$ for a.e. $t \in [0, T]$. Then, from the fact $x_n(0) \to x_0$ as $n \to \infty$, we can see that x solves Problem 1.1 associated with f, namely, $x \in \Gamma(f)$. Hence, $x \in \Gamma(f) \cap D$, due to the weak closedness of D. This concludes that Γ is u.s.c. from $w - \mathcal{H}$ into $w - \mathcal{W}$.

Next, we are going to illustrate that Γ is u.s.c. from $w - \mathcal{H}$ into $C(0, T; H)$. So, we have to verify that $\Gamma^-(E)$ is weakly closed in $\mathcal{H}$, for each closed set E of $C(0, T; H)$. Let $\{f_n\}$ be a sequence such that (4.8) is available. Then, there is a sequence $\{x_n\} \subset \mathcal{W}$ such that (4.9) and (4.10) hold. As before we did, it is not difficult to prove that (4.11) hold with $x \in \Gamma(f)$. Our aim is to show that $x \in E$. From (3.1) and (4.9), we have

$$\begin{aligned} &\langle x'(t) - x_n'(t), x(t) - x_n(t)\rangle + \langle A(t, x(t)) - A(t, x_n(t)), x(t) - x_n(t)\rangle \\ = &(\xi_n(t) - \xi(t), x(t) - x_n(t))_H + \langle \eta_n(t) - \eta(t), x(t) - x_n(t)\rangle \\ &+ (f_n(t) - f(t), x(t) - x_n(t))_H \end{aligned}$$

for a.e. $t \in [0, T]$. Keeping in mind that $z \mapsto \partial_C \varphi(z)$ is monotone, it has

$$\langle \eta_n(t) - \eta(t), x(t) - x_n(t)\rangle = \langle \zeta_n(t) - \zeta(t), \beta(x(t) - x_n(t))\rangle_Z \leq 0$$

for a.e. $t \in [0, T]$. Then, we obtain

$$\frac{1}{2}\|x(t) - x_n(t)\|_H^2 + \int_0^t \langle A(s, x(s)) - A(s, x_n(s)), x(s) - x_n(s)\rangle\, ds$$

$$\leq \int_0^t (\xi_n(s) - \xi(s), x(s) - x_n(s))_H + (f_n(s) - f(s), x(s) - x_n(s))_H\, ds$$

for all $t \in [0, T]$. Letting $\lim_{n\to\infty}$ into the inequality above, we utilize (4.8), (4.11), (4.14), (4.15) and (4.17) to get

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2}\|x(t) - x_n(t)\|_H^2 \\
&\le \lim_{n\to\infty} \int_0^t \langle A(s, x(s)) - A(s, x_n(s)), x_n(s) - x(s)\rangle \, ds \\
&\quad + \lim_{n\to\infty} \int_0^t (\xi_n(s) - \xi(s), x(s) - x_n(s))_H + (f_n(s) - f(s), x(s) - x_n(s))_H \, ds \\
&\le 0
\end{aligned}
$$

for all $t \in [0, T]$. This means that $x_n \to x$ in $C(0, T; H)$. Recall that E is closed in $C(0, T; H)$ and $\{x_n\} \subset E$, we have $x \in E$, namely, $x \in \Gamma^-(E)$. Consequently, Γ is u.s.c. from $w - \mathcal{H}$ to $C(0, T; H)$.

(iii) Let I be a bounded closed set of $\mathcal{H}$. Let sequence $\{x_n\} \subset \Gamma(I)$ be arbitrary. Because I is bounded in $\mathcal{H}$, from assertion (i), we can see that $\{x_n\}$ is bounded in $\mathcal{W}$. Passing to a relabeled subsequence if necessary, we may assume that (4.11) is available. In fact, for each $n \in \mathbb{N}$, there exists $f_n \in I$ such that $x_n \in \Gamma(f_n)$. Using the boundedness and closedness of I again, it can say that (4.8) holds with some $f \in I$. Applying the same arguments as in the proof of assertion (ii), we can see that there exists a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, such that $x_n \to x$ in $C(0, T; H)$ as $n \to \infty$ with $x \in \Gamma(f) \subset \Gamma(I)$. Therefore, we conclude that the set $\Gamma(I)$ is compact in $C(0, T; H)$. □

Remark 4.3. Furthermore, from the proof of Lemma 4.2, it is not difficult to prove that Γ is also closed from $w - \mathcal{H}$ into $w - \mathcal{W}$ (i.e., the graph of Γ is closed in $w - \mathcal{H} \times w - \mathcal{W}$), and for each $f \in \mathcal{H}$, $\Gamma(f)$ is bounded and weakly closed in $\mathcal{W}$, and compact in $C(0, T; H)$.

Particularly, if Problem 1.1 has a unique solution, i.e., Γ is a single-valued operator, then we have the following corollary.

Corollary 4.4. *Suppose that* $H(A)$(i)–(iii), (v), $H(J)$(i)–(iii), (v), $H(0)$, $H(\beta)'$ *and* $H(\varphi)'$ *are fulfilled. Then, the statements hold*

(i) $\Gamma\colon \mathcal{H} \to \mathcal{W}$ *is a bounded mapping;*
(ii) Γ *is continuous from* $w - \mathcal{H}$ *into* $w - \mathcal{W}$*;*
(iii) $\Gamma\colon \mathcal{H} \to C(0, T; H)$ *is compact (therefore,* Γ *is continuous from* $w - \mathcal{H}$ *into* $C(0, T; H)$*).*

We now give the existence theorem for Problem 1.2 as follows.

Theorem 4.5. *Assume that $H(A)$(i)–(iv), $H(J)$(i)–(iv), $H(0)$, $H(\beta)'$, $H(\varphi)'$ and $H(1)$ are fulfilled. Then, the set of solutions to Problem 1.2 is nonempty and weakly compact in $\mathcal{W} \times \mathcal{H}$.*

Proof. By virtue of definition of Q (see (1.4)), we can see that $Q(x, f) \geq 0$ for all $x \in \mathcal{W}$ and $f \in \mathcal{H}$. Let $\{(x_n, f_n)\} \subset \mathcal{W} \times \mathcal{H}$ with $x_n \in \Gamma(f_n)$ for each $n \in \mathbb{N}$, be a minimizing sequence of Problem 1.2, i.e.,

$$\lim_{n\to\infty} Q(x_n, f_n) = \inf_{f\in\mathcal{H}, x\in\Gamma(f)} Q(x, f) := \rho \geq 0. \tag{4.18}$$

But, the estimate

$$Q(x_n, f_n) \geq \frac{\alpha_1}{2}\|f_n - g\|^2_{\mathcal{H}},$$

indicates that $\{f_n\}$ is bounded in $\mathcal{H}$. Passing to a subsequence if necessary, we may assume that

$$f_n \xrightarrow{w} f^* \text{ in } \mathcal{H} \text{ as } n \to \infty, \tag{4.19}$$

for some $f^* \in \mathcal{H}$.

Lemma 4.2(i) points out that $\{x_n\}$ is bounded in $\mathcal{W}$. So, passing to a relabeled subsequence, we can suppose that

$$x_n \xrightarrow{w} x^* \text{ in } \mathcal{W} \text{ and } \mathcal{V} \text{ as } n \to \infty, \tag{4.20}$$

for some $x^* \in \mathcal{W}$. Since the embedding from $\mathcal{W}$ into $C(0, T; H)$ is continuous, so, we have

$$x_n(T) \xrightarrow{w} x^*(T) \text{ in } H \text{ as } n \to \infty. \tag{4.21}$$

The convergences (4.19) and (4.20), and the closedness of Γ entail that $x^* \in \Gamma(f^*)$ (see Remark 4.3).

We use convergences (4.19)–(4.21) to yield

$$\begin{aligned}
&\liminf_{n\to\infty} Q(x_n, f_n)\\
&\geq \liminf_{n\to\infty} \frac{\alpha_1}{2}\|f_n - g\|^2_{\mathcal{H}} + \liminf_{n\to\infty} \frac{\alpha_2}{2}\|x_n(T) - z_0\|^2_H + \liminf_{n\to\infty} \frac{\alpha_3}{2}\|x_n - y_0\|^2_{\mathcal{V}}\\
&\geq \frac{\alpha_1}{2}\|f^* - g\|^2_{\mathcal{H}} + \frac{\alpha_2}{2}\|x^*(T) - z_0\|^2_H + \frac{\alpha_3}{2}\|x^* - y_0\|^2_{\mathcal{V}}\\
&= Q(x^*, f^*).
\end{aligned}$$

This combined with the fact $x^* \in \Gamma(f^*)$ and (4.18) deduces

$$\inf_{f\in\mathcal{H},x\in\Gamma(f)} Q(x,f)\le Q(x^*,f^*)\le \liminf_{n\to\infty} Q(x_n,f_n)=\inf_{f\in\mathcal{H},x\in\Gamma(f)} Q(x,f).$$

This implies that $(x^*,f^*)\in\mathcal{W}\times\mathcal{H}$ with $x^*\in\Gamma(f^*)$ is an optimal control of Problem 1.2.

Finally, we show that the set of solutions to Problem 1.2 is weakly compact in $\mathcal{W}\times\mathcal{H}$. Let $\{(x_n,f_n)\}\subset\mathcal{W}\times\mathcal{H}$ with $x_n\in\Gamma(f_n)$ is an any solution sequence of Problem 1.2. Therefore, $\{f_n\}$ and $\{x_n\}$ are bounded in $\mathcal{H}$ and $\mathcal{W}$, respectively. This allows us to assume that (4.19)–(4.21) hold with $x^*\in\Gamma(f^*)$. An easy computing finds

$$\begin{aligned}
&\inf_{f\in\mathcal{H},x\in\Gamma(f)} Q(x,f)=\liminf_{n\to\infty} Q(x_n,f_n)\\
&\ge \liminf_{n\to\infty}\frac{\alpha_1}{2}\|f_n-g\|_{\mathcal{H}}^2+\liminf_{n\to\infty}\frac{\alpha_2}{2}\|x_n(T)-z_0\|_H^2+\liminf_{n\to\infty}\frac{\alpha_3}{2}\|x_n-y_0\|_{\mathcal{V}}^2\\
&\ge \frac{\alpha_1}{2}\|f^*-g\|_{\mathcal{H}}^2+\frac{\alpha_2}{2}\|x^*(T)-z_0\|_H^2+\frac{\alpha_3}{2}\|x^*-y_0\|_{\mathcal{V}}^2\\
&=Q(x^*,f^*)\ge \inf_{f\in\mathcal{H},x\in\Gamma(f)} Q(x,f).
\end{aligned}$$

This turns out that (x^*,f^*) with $x^*\in\Gamma(f^*)$ is a solution of Problem 1.2. Consequently, the set of solutions to Problem 1.2 is weakly compact in $\mathcal{W}\times\mathcal{H}$. □

5. Feedback control

The main goal of the section is to explore the existence of a feedback control pair to the nonlinear and nonsmooth feedback dynamic system, Problem 1.3, in which our method is based on Kakutani-Ky Fan fixed point theorem for set-valued mappings and the theory of nonsmooth analysis. Then, the compactness of the solution set of Problem 1.3 is proved. Finally, we obtain a convergence result in the sense of Kuratowski which describes the changes in the set of solutions for Problem 1.3 as the initial data x_0 is perturbed in Hilbert space H.

To this end, we make the following assumptions.

$\underline{H(B)}$: $B\colon [0,T]\times H\to L(Y,H)$ is such that

(i) $t\mapsto B(t,x)u$ is measurable on $[0,T]$ for any $(x,u)\in H\times Y$;
(ii) $x\mapsto B^*(t,x)u$ is continuous for all $u\in H$ and a.e. $t\in[0,T]$, where $B^*(t,x)\in L(H^*,Y^*)$ is the dual operator of $B(t,x)$;
(iii) for any $x\in H$ and a.e. $t\in T$, there exist $c_B\in L^2(0,T)_+$ and $d_B>0$ such that

$$\|B(t,x)\|_{L(Y,H)}\le c_B(t)+d_B\|x\|_H$$

for all $x\in H$ and a.e. $t\in[0,T]$;
(iv) there exists $c_B\in L^2(0,T)_+$ with $c_Bc_U\in L^2(0,T)_+$ such that

$$\|B(t,x)\|_{L(Y,H)}\le c_B(t)$$

for all $x \in H$ and a.e. $t \in [0, T]$, where c_U is given in $H(U)$(iv) (see below).

$\underline{H(U)}$: $U\colon [0, T] \times H \to 2^Y$ is such that

(i) for each $x \in H$, the function $t \mapsto U(t, x)$ is measurable on $[0, T]$;
(ii) U has nonempty, bounded, closed and convex values;
(iii) for a.e. $t \in [0, T]$, the function $x \mapsto U(t, x)$ is upper semicontinuous;
(iv) there exist $c_U \in L^2(0, T)_+$ and $d_U > 0$ such that

$$\|U(t, x)\|_Y \le c_U(t) + d_U \|x\|_H$$

for all $x \in H$ and a.e. $t \in [0, T]$;
(v) there exists a function $c_U \in L^2(0, T)_+$ with $c_B c_U \in L^2(0, T)_+$ such that

$$\|U(t, x)\|_Y \le c_U(t)$$

for all $x \in H$ and a.e. $t \in [0, T]$, where c_B is given in $H(B)$(iii).

Definition 5.1. We say that a pair of functions $(x, u) \in \mathcal{W} \times \mathcal{Y}$ is a solution (or a feedback control pair) to Problem 1.3, if there exist $\xi \in \mathcal{H}$ and $\eta \in \mathcal{V}^*$ such that

$$\begin{cases} x'(t) + A(t, x(t)) + \xi(t) + \eta(t) = B(t, x(t))u(t) \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{5.1}$$

and

$$\xi(t) \in \partial J(t, x(t)),\ \eta(t) \in \beta^* \partial_C \varphi(\beta x(t)),\ \text{ and } \ u(t) \in U(t, x(t)) \text{ for a.e. } t \in [0, T]. \tag{5.2}$$

In what follows, we denote by $\mathcal{S}(x_0)$ the solution set of Problem 1.3 corresponding to the initial data $x_0 \in V$.

Lemma 5.2. *Suppose that $H(A)$(i)–(iv), $H(J)$(i)–(iv), $H(0)$, $H(\varphi)'$, $H(B)$(i)–(ii) and $H(U)$(i)–(iii) are fulfilled. If $H(B)$(iii) and $H(U)$(v), or $H(B)$(iv) and $H(U)$(iv), hold, then there exists a constant $M > 0$ such that for each $(x, u) \in \mathcal{S}(x_0)$ we have*

$$\|x\|_{\mathcal{C}} \le M, \quad \|x\|_{\mathcal{W}} \le M \quad \text{and} \quad \|u\|_{\mathcal{Y}} \le M, \tag{5.3}$$

where $\|x\|_{\mathcal{C}} := \inf_{t \in [0,T]} \|x(t)\|_H$.

Proof. Assume that $\mathcal{S}(x_0) \neq \emptyset$, let $(x, u) \in \mathcal{S}(x_0)$ be an any solution of Problem 1.3. Then, we can find $\xi \in \mathcal{H}$ and $\eta \in \mathcal{V}^*$ such that (5.1) and (5.2) hold. By virtue of hypotheses $H(B)$(iii) and $H(U)$(v), we have

$$\int_0^t (B(s,x(s))u(s),x(s))_H\,ds \le \int_0^t \|B(s,x(s))u(s)\|_H\|x(s)\|_H\,ds$$

$$\le \int_0^t (c_B(s)+d_B\|x(s)\|_H)c_U(s)\|x(s)\|_H\,ds \tag{5.4}$$

for all $t\in[0,T]$. Likewise, from $H(B)$(iv) and $H(U)$(iv), it has

$$\int_0^t (B(s,x(s))u(s),x(s))_H\,ds \le \int_0^t c_B(s)(c_U(s)+d_U\|x(s)\|_H)\|x(s)\|_H\,ds \tag{5.5}$$

for all $t\in[0,T]$.

Suppose that $H(B)$(iii) and $H(U)$(v) are fulfilled. Multiplying (5.1) by $x(t)$ and integrating the resulting equality over $[0,t]$, we use the estimates (4.2)–(4.6) and (5.4) to obtain

$$\frac{1}{2}(\|x(t)\|_H^2-\|x(0)\|_H^2)+m_A\int_0^t \|x(s)\|_V^2-c_A(s)\|x(s)\|_V-d_A(s)\,ds$$

$$\le \int_0^t m_J\|x(s)\|_H^\theta+c_J(s)\|x(s)\|_H+d_J(s)\,ds+T(\varphi(\beta 0)+d_\varphi)$$

$$+\int_0^t c_\varphi\|\beta x(s)\|_Z\,ds+\int_0^t (c_B(s)+d_B\|x(s)\|_H)c_U(s)\|x(s)\|_H\,ds$$

for all $t\in[0,T]$. Furthermore, when $\theta\in[1,2)$, we utilize Young's inequality and Hölder's inequality to get

$$\frac{1}{2}(\|x(t)\|_H^2-\|x(0)\|_H^2)+(m_A-2\varepsilon)\int_0^t \|x(s)\|_V^2\,ds$$

$$\le \frac{\|c_A\|_{L^2(0,T)}+\|c_J\|_{L^2(0,T)}}{4\varepsilon}+\|d_A\|_{L^1(0,T)}+\int_0^t (3\varepsilon+d_Bc_U(s))\|x(s)\|_H^2\,ds+c_1(\varepsilon)$$

$$+\|d_J\|_{L^1(0,T)}+T(\varphi(\beta 0)+d_\varphi)+\frac{\|\beta\|^2c_\varphi^2T+\|c_Bc_U\|_{L^2(0,T)}^2}{4\varepsilon}$$

for all $t\in[0,T]$, with some $c_1(\varepsilon)>0$; if $\theta=2$, we also obtain

$$\frac{1}{2}(\|x(t)\|_H^2-\|x(0)\|_H^2)+(m_A-2\varepsilon)\int_0^t \|x(s)\|_V^2\,ds$$

$$\leq \frac{\|c_A\|_{L^2(0,T)} + \|c_J\|_{L^2(0,T)}}{4\varepsilon} + \|d_A\|_{L^1(0,T)} + \int_0^t (2\varepsilon + d_B c_U(s) + m_J)\|x(s)\|_H^2\, ds$$

$$+\|d_J\|_{L^1(0,T)} + T(\varphi(\beta 0) + d_\varphi) + \frac{\|\beta\|^2 c_\varphi^2 T + \|c_B c_U\|_{L^2(0,T)}^2}{4\varepsilon}$$

for all $t \in [0,T]$. We are now in a position to invoke Gronwall's inequality to find a constant $M_0 > 0$ such that

$$\|x(t)\|_H \leq M_0 \text{ for all } t \in [0,T] \text{ and } \|x\|_{\mathcal{V}} \leq M_0. \tag{5.6}$$

However, while $H(B)$(iv) and $H(U)$(iv) are satisfied, we could apply the similar arguments to verify the estimates (5.6).

Furthermore, it follows from (5.1) that

$$\|x'(t)\|_{V^*} \leq \|A(t,x(t))\|_{V^*} + \|\xi(t)\|_{V^*} + \|\eta(t)\|_{V^*} + \|B(t,x(t))u(t)\|_{V^*}$$

for a.e. $t \in [0,T]$. If $H(B)$(iii) and $H(U)$(v) hold, then we use hypotheses $H(A)$(iii), $H(J)$(iii) and $H(\varphi)'$ to find

$$\|x'(t)\|_{V^*} \leq a_A(t) + b_A\|x(t)\|_V + \|\gamma^*\|(a_J(t) + b_J\|x(t)\|_H)$$
$$+\|\beta^*\|b_\varphi(1 + \|\beta\|\|x(t)\|_V) + \|\gamma^*\|(c_B(t) + d_B\|x(t)\|_H)c_U(t) \tag{5.7}$$

for a.e. $t \in [0,T]$. But, when $H(B)$(iv) and $H(U)$(iv) hold, we can calculate to get the similar estimate as (5.7). Taking into account (5.6) and (5.7), we can see that there exists $M_1 > 0$ such that

$$\|x'\|_{\mathcal{V}} \leq M_1. \tag{5.8}$$

However, (5.6), and $H(U)$(iv) or $H(U)$(v) ensure that

$$\|u\|_{\mathcal{Y}} \leq M_2, \tag{5.9}$$

with some $M_2 > 0$. So, from (5.6), (5.8) and (5.9), we can take $M = \max\{M_2, M_0 + M_1\}$ to conclude that the estimates (5.3) are valid. □

We, first, give the following existence theorem to Problem 1.3.

Theorem 5.3. *Assume that* $H(A)$(i)–(iii), (v), $H(J)$(i)–(iii), (v), $H(0)$, $H(\varphi)'$, $H(B)$(i)–(ii) *and* $H(U)$(i)–(iii) *are fulfilled. If* $H(B)$(iii) *and* $H(U)$(v), *or* $H(B)$(iv) *and* $H(U)$(iv), *hold, then for each* $x_0 \in \mathcal{V}$ *Problem 1.3 has at least one solution, i.e.,* $\mathcal{S}(x_0) \neq \emptyset$.

Proof. Let us introduce the set-valued mapping $S_U : \mathcal{W} \to 2^{\mathcal{Y}}$ defined by

$$S_U(x) := \{u \in \mathcal{Y} \mid u(t) \in U(t, x(t)) \text{ for a.e. } t \in [0,T]\} \text{ for all } x \in \mathcal{W}. \tag{5.10}$$

Recall that $t \mapsto U(t,y)$ is measurable on $[0,T]$ for all $y \in H$ and $y \mapsto U(t,y)$ is u.s.c. (see hypothesis $H(U)$(iii)) for a.e. $t \in [0,T]$, so, for each $x \in \mathcal{W}$, the function $t \mapsto U(t,x(t))$ is measurable on $[0,T]$ too. Therefore, applying Yankov-von Neumann-Aumann selection theorem (see [15, Theorem 2.14, p. 158] or [18, Theorem 1.3.1]), for each $x \in \mathcal{W}$, $U(\cdot, x(\cdot))$ has a measurable selector $u : [0,T] \to Y$ with $u(t) \in U(t,x(t))$ for a.e. $t \in [0,T]$. From condition $H(U)$(iv) or $H(U)$(v), we have

$$\|u(t)\|_Y \le a_U(t) + c_U \|x(t)\|_H \text{ or } \|u(t)\|_Y \le a_U(t) \text{ for a.e. } t \in [0,T],$$

thus, $u \in \mathcal{Y}$. This means that $S_U : \mathcal{W} \to 2^{\mathcal{Y}}$ is well-defined. In addition, hypothesis $H(U)$(ii) ensures that S_U has nonempty, closed and convex values.

We affirm that S_U is weakly-weakly u.s.c., i.e., S_U is u.s.c. from $w - \mathcal{W}$ to $w - \mathcal{Y}$. It is enough to show that for any weakly closed set D in $\mathcal{Y}$ the set $S_U^-(D)$ is weakly closed in $\mathcal{W}$. Let $\{x_n\} \subset S_U^-(D)$ be such that

$$x_n \xrightarrow{w} x \text{ in } \mathcal{W} \text{ as } n \to \infty, \tag{5.11}$$

for some $x \in \mathcal{W}$. Then, the compactness of the embedding from $\mathcal{W}$ into $\mathcal{H}$ indicates that

$$x_n \to x \text{ in } \mathcal{H} \text{ as } n \to \infty. \tag{5.12}$$

In the meanwhile, we are able to find a sequence $\{u_n\} \subset \mathcal{Y}$ such that

$$u_n \in S_U(x_n) \cap D \text{ for each } n \in \mathbb{N}, \tag{5.13}$$

i.e., $u_n(t) \in U(t, x_n(t))$ for a.e. $t \in [0,T]$. Condition $H(U)$(iv) or $H(U)$(v) implies that the sequence $\{u_n\}$ is bounded in $\mathcal{Y}$. Without loss of generality, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } \mathcal{Y} \text{ as } n \to \infty, \tag{5.14}$$

for some $u^* \in \mathcal{Y}$. Applying Mazur's Theorem (see e.g. [22, Chapter 2, Corollary 2.8]), for each $l \in \mathbb{N}$, there exists a sequence $\{a_{il}\}_{i \ge 1} \subset \mathbb{R}$ with $a_{il} \ge 0$ and $\sum_{i \ge 1} a_{il} = 1$ such that

$$u_l(\cdot) := \sum_{i \ge 1} a_{il} u_{i+l}(\cdot) \to u^* \text{ in } \mathcal{Y} \text{ as } l \to \infty.$$

Hence, we may assume that

$$u_l(t) \to u^*(t) \text{ in } Y \text{ as } l \to \infty \text{ for a.e. } t \in [0,T]. \tag{5.15}$$

Convergence (5.12) allows us to suppose that

$$x_n(t) \to x(t) \text{ in } H \text{ as } n \to \infty \text{ for a.e. } t \in [0,T]. \tag{5.16}$$

Since $x \mapsto U(t,x)$ is u.s.c., then for any $\varepsilon > 0$ and $k > 0$ large enough, we have

$$u_k(t) \in U(t, x_k(t)) \subset U(t, x(t)) + O_\varepsilon \text{ for a.e. } t \in [0,T],$$

where O_ε is an open ball with radius $\varepsilon > 0$ centered at 0_Y. Indeed, we also have

$$u_l(t) \in U(t, x(t)) + O_\varepsilon \text{ for a.e. } t \in [0,T],$$

due to the convexity of $U(t,x(t)) + O_\varepsilon$. This combined with the convergence (5.15) deduces

$$u^*(t) \in \overline{U(t,x(t)) + O_\varepsilon} \text{ for a.e. } t \in [0,T].$$

Letting $\varepsilon \to 0$, we have

$$u^*(t) \in \overline{U(t,x(t))} \text{ for a.e. } t \in [0,T].$$

Since U has closed values, so, we get $u^*(t) \in \overline{U(t,x(t))} = U(t,x(t))$ for a.e. $t \in [0,T]$. This means that $u \in S_U(x)$. But, the weak closedness of D infers that $u \in D$, that is, $x \in S_U^-(D)$. To summary, S_U is u.s.c. from $w-\mathcal{W}$ to $w-\mathcal{Y}$.

Given $u \in \mathcal{Y}$, let us consider the problem: find $x \in \mathcal{W}$ such that

$$\begin{cases} x'(t) + A(t,x(t)) + \xi(t) + \eta(t) = B(t,x(t))u(t) \text{ for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{5.17}$$

and

$$\xi(t) \in \partial J(t,x(t)) \quad \text{ and } \quad \eta(t) \in \beta^* \partial_C \varphi(\beta x(t)) \text{ for a.e. } t \in [0,T]. \tag{5.18}$$

Arguing as in the proof of Theorems 3.6 and 3.7, we can see that for each $u \in \mathcal{Y}$, problem (5.17)–(5.18) has a unique solution $x \in \mathcal{W}$. Let us introduce a function $F\colon \mathcal{Y} \to \mathcal{W}$ defined by $F(u) = x(u)$, where $x(u) \in \mathcal{W}$ is the unique solution of problem (5.17)–(5.18) corresponding to $u \in \mathcal{Y}$. Applying the same arguments as in the proof of Lemma 4.2(i), we can see that F is a bounded map.

We assert that F is weakly-weakly continuous, namely, F is continuous from $w-\mathcal{Y}$ into $w-\mathcal{W}$. Let $\{u_n\} \subset Y$ be such that $u_n \xrightarrow{w} u$ in $\mathcal{Y}$ as $n \to \infty$ for some $u \in \mathcal{Y}$. Then, for each $n \in \mathbb{N}$, we can find sequences $\{\xi_n\} \subset \mathcal{H}$ and $\{\eta_n\} \subset \mathcal{V}^*$ such that

$$\begin{cases} x_n'(t) + A(t,x_n(t)) + \xi_n(t) + \eta_n(t) = B(t,x_n(t))u_n(t) \text{ for a.e. } t \in [0,T], \\ x_n(0) = x_0, \end{cases}$$

and

$$\xi_n(t) \in \partial J(t, x_n(t)) \quad \text{and} \quad \eta_n(t) \in \beta^* \partial_C \varphi(\beta x_n(t)) \text{ for a.e. } t \in [0, T].$$

Because the sequence $\{x_n\}$ with $x_n = F(u_n)$ is bounded in $\mathcal{W}$. Hence, passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $\mathcal{W}$ as $n \to \infty$ for some $x \in \mathcal{W}$. We will show that $x = F(u)$. A simple calculating gives (see (4.13) for example)

$$\begin{aligned} &\frac{1}{2}\|x(T) - x_n(T)\|_H^2 + \int_0^T \langle A(s, x_n(s)), x_n(s) - x(s)\rangle \, ds \\ &\le \int_0^T (\xi_n(s), x(s) - x_n(s))_H + (B(s, x_n(s))u_n(s), x_n(s) - x(s))_H \, ds \\ &\quad + \int_0^T \varphi(\beta x(s)) - \varphi(\beta x_n(s)) \, ds + \int_0^T \langle x(s), x(s) - x_n(s)\rangle \, ds. \end{aligned} \tag{5.19}$$

For any $y \in \mathcal{H}$, it has

$$\int_0^T (B(t, x_n(t))u_n(t), y(t))_H \, dt = \int_0^T \langle u_n(t), B^*(t, x_n(t))y(t)\rangle_Y \, dt.$$

The continuity of $x \mapsto B^*(t, x)$ and Lebesgue dominated convergence theorem entail

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T (B(t, x_n(t))u_n(t), y(t))_H \, dt \\ &= \lim_{n\to\infty} \int_0^T \langle u_n(t), [B^*(t, x_n(t)) - B^*(t, x(t))]y(t)\rangle_Y \, dt \\ &\quad + \lim_{n\to\infty} \int_0^T \langle u_n(t), B^*(t, x(t))y(t)\rangle_Y \, dt \\ &= \int_0^T \langle u(t), B^*(t, x(t))y(t)\rangle_Y \, dt = \int_0^T (B(t, x(t))u(t), y(t))_H \, dt, \end{aligned} \tag{5.20}$$

where we have used the convergence $x_n(t) \to x(t)$ in H as $n \to \infty$ for a.e. $t \in [0, T]$, owing to the compactness of the embedding from $\mathcal{W}$ into $\mathcal{H}$. So, $B(\cdot, x_n(\cdot))u_n(\cdot) \xrightarrow{w} B(\cdot, x(\cdot))u(\cdot)$ in $\mathcal{H}$ as $n \to \infty$. Passing to the upper limit as $n \to \infty$ into (5.19), we have

$$\limsup_{n\to\infty} \int_0^T \langle A(s, x_n(s)), x_n(s) - x(s)\rangle \, ds \le 0, \tag{5.21}$$

where we have applied the weak lower semicontinuity of $x \mapsto \int_0^T \varphi(\beta x(t))\, dt$ and the boundedness of $\{\xi_n\}$ (see hypothesis $H(J)$(iii)). From (5.21) and the convergence, $x_n \xrightarrow{w} x$ in $\mathcal{W}$ as $n \to \infty$, we can invoke [42, Proposition 1] to obtain

$$A(\cdot, x_n(\cdot)) \xrightarrow{w} A(\cdot, x(\cdot)) \text{ in } \mathcal{V}^* \text{ as } n \to \infty.$$

Using the similar arguments as in the proof of Lemma 4.2(ii), we may say that $\xi_n \xrightarrow{w} \xi$ in $\mathcal{H}$ as $n \to \infty$ with some $\xi \in \mathcal{H}$ and $\xi(t) \in \partial J(t, x(t))$ for a.e. $t \in [0, T]$, and $\eta_n \xrightarrow{w} \eta$ in $\mathcal{V}^*$ as $n \to \infty$ with some $\eta \in \mathcal{V}^*$ such that $\eta(t) \in \beta^* \partial_C \varphi(\beta x(t))$ for a.e. $t \in [0, T]$. Therefore, we have

$$\begin{aligned} & x_n'(\cdot) + A(\cdot, x_n(\cdot)) + \xi_n(\cdot) + \eta_n(\cdot) - B(\cdot, x_n(\cdot))u_n(\cdot) \\ \xrightarrow{w} \; & x'(\cdot) + A(\cdot, x(\cdot)) + \xi(\cdot) + \eta(\cdot) - B(\cdot, x(\cdot))u(\cdot) \text{ in } \mathcal{V}^*, \end{aligned}$$

$x_n(0) \to x_0$ and $\xi(t) \in \partial J(t, x(t))$, $\eta(t) \in \beta^* \partial_C \varphi(\beta x(t))$ for a.e. $t \in [0, T]$. This means that x is the unique solution of problem (5.17) associated with $u \in \mathcal{Y}$, i.e., $x = F(u)$. Therefore, we conclude that the whole sequence $\{x_n\}$ weakly converges to $F(u)$, namely, F is weakly-weakly continuous from $\mathcal{H}$ into $\mathcal{W}$.

Furthermore, we introduce a set-valued mapping $\Lambda\colon \mathcal{W} \times \mathcal{Y} \to 2^{\mathcal{W}\times\mathcal{Y}}$ defined by

$$\Lambda(x, u) = (F(u), S_U(x)) \text{ for all } (x, u) \in \mathcal{W} \times \mathcal{Y}. \tag{5.22}$$

It is obvious that Λ has nonempty closed and convex values in $\mathcal{W}\times\mathcal{Y}$. Let $\mathcal{D}$ be a subset of $\mathcal{W} \times \mathcal{Y}$ defined by

$$\mathcal{D} := \{(x, u) \in \mathcal{W} \times \mathcal{Y} \mid \|x\|_{\mathcal{W}} \le M \text{ and } \|u\|_{\mathcal{Y}} \le M\},$$

where $M > 0$ is given in Lemma 5.2. Following the proof of Lemma 5.2, it is not difficult to prove that Λ maps $\mathcal{D}$ into itself.

However, the weak-weak upper semicontinuity of Λ (i.e., Λ is u.s.c. from $w - \mathcal{W}$ into $w - \mathcal{Y}$) and [18, Theorem 1.1.4] imply that the graph of Λ is sequentially closed in $w - \mathcal{W} \times w - \mathcal{Y}$. Consequently, all conditions of Theorem 2.7 are verified with $D = \mathcal{D}$ and $E = \mathcal{W} \times \mathcal{Y}$. Using this theorem, we infer that there exists $(x^*, u^*) \in \mathcal{W} \times \mathcal{Y}$ such that $(x^*, u^*) \in \Lambda(x^*, u^*)$, i.e., $x^* = F(u^*)$ and $u^* \in S_U(x^*)$. Hence,

$$\begin{cases} {x^*}'(t) + A(t, x^*(t)) + \xi^*(t) + \eta^*(t) = B(t, x^*(t))u^*(t) \text{ for a.e. } t \in [0, T], \\ x^*(0) = x_0, \end{cases}$$

and

$$\xi^*(t) \in \partial J(t, x^*(t)),\ \eta^*(t) \in \beta^*\partial_C\varphi(\beta x^*(t)),\ \text{ and }\ u^*(t) \in U(t, x^*(t)) \text{ for a.e. } t \in [0, T].$$

Therefore, Problem 1.3 admits a feedback control pair, i.e., $\mathcal{S}(x_0) \neq \emptyset$.

It remains us to show that $\mathcal{S}(x_0)$ is weakly compact in $\mathcal{W} \times \mathcal{Y}$. Let $\{(x_n, u_n)\} \subset \mathcal{S}(x_0)$ be a solution sequence of Problem 1.3. Lemma 5.2 concludes that $\{x_n\}$ and $\{u_n\}$ are both bounded in $\mathcal{W}$ and $\mathcal{Y}$, respectively. So, we may assume that (5.11), (5.12) and (5.14) hold. Since the graph of S_U is sequentially weakly-weakly closed and F is weakly-weakly continuous, then we have $x_n = F(u_n) \xrightarrow{w} F(u^*) = x$ and $u^* \in S_U(x)$ with $x(0) = x_0$. This means that $(x, u^*) \in \mathcal{S}(x_0)$. Consequently, $\mathcal{S}(x_0)$ is weakly compact in $\mathcal{W} \times Y$. □

Finally, we give a convergence result in the sense of Kuratowski which describes the changes in the set of solutions for Problem 1.3 as the initial data x_0 is perturbed in Hilbert space H.

Theorem 5.4. *Assume that $H(A)$(i)–(iii), (v), $H(J)$(i)–(iii), (v), $H(0)$, $H(\varphi)'$, $H(B)$(i)–(ii) and $H(U)$(i)–(iii) are fulfilled. If $H(B)$(iii) and $H(U)$(v), or $H(B)$(iv) and $H(U)$(iv), hold, and $\{x_0^n\} \subset V$ is a sequence such that $x_0^n \to x_0$ in H as $n \to \infty$ with $x_0 \in V$, then it is true*

$$\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}(x_0^n) \subset \mathcal{S}(x_0), \tag{5.23}$$

where $w - \limsup_{n\to\infty} \mathcal{S}(x_0^n)$ stands for the Kuratowski upper limit of the sequence $\{\mathcal{S}(x_0^n)\}$ with respect to the weak topology of $\mathcal{W} \times \mathcal{Y}$.

Proof. Let $\{x_0^n\} \subset V$ and $x_0 \in V$ be such that $x_0^n \to x_0$ in H as $n \to \infty$. It follows from Theorem 5.3 that $\mathcal{S}(x_0^n) \neq \emptyset$ for each $n \geq 1$ and $\mathcal{S}(x_0) \neq \emptyset$. Arguing as in the proof of Lemma 5.2, we can see that the set $\bigcup_{n\geq 1} \mathcal{S}(x_0^n)$ is bounded in $\mathcal{W} \times \mathcal{Y}$. Passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } \mathcal{W},\ x_n \to x \text{ in } \mathcal{H}, \text{ and } u_n \xrightarrow{w} u \text{ in } \mathcal{Y} \text{ as } n \to \infty, \tag{5.24}$$

for some $(x, u) \in \mathcal{W} \times \mathcal{Y}$. Therefore, we have $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}(x_0^n)$. In fact, for each $n \geq 1$, we have

$$\begin{cases} x_n'(t) + A(t, x_n(t)) + \xi_n(t) + \eta_n(t) = B(t, x_n(t))u_n(t) \text{ for a.e. } t \in [0, T], \\ x_n(0) = x_0^n, \end{cases} \tag{5.25}$$

with $\xi_n \in \mathcal{H}$ and $\eta_n \in \mathcal{V}^*$ satisfying

$$\xi_n(t) \in \partial J(t, x_n(t)),\ \eta_n(t) \in \beta^*\partial_C\varphi(\beta x_n(t)),\ \text{ and }\ u_n(t) \in U(t, x_n(t)) \text{ for a.e. } t \in [0, T]. \tag{5.26}$$

Since the graph of S_U is sequentially closed in $w-\mathcal{W}\times w-\mathcal{Y}$. Therefore, (5.24) and (5.26) imply that $u\in S_U(x)$, i.e., $u\in\mathcal{Y}$ and $u(t)\in U(t,x(t))$ for a.e. $t\in[0,T]$. Observe that $\{\xi_n\}\subset\mathcal{H}$ and $\{\eta_n\}\subset\mathcal{V}^*$ are both bounded, we may suppose that $\xi_n \xrightarrow{w} \xi$ in $\mathcal{H}$ and $\eta_n \xrightarrow{w} \eta$ in $\mathcal{V}^*$ as $n\to\infty$ for some $(\xi,\eta)\in\mathcal{H}\times\mathcal{V}^*$. As before we did, it is not difficult to verify that $\xi(t)\in\partial J(t,x(t))$ and $\eta(t)\in\beta^*\partial_C\varphi(\beta x(t))$ for a.e. $t\in[0,T]$. Recall that the embedding from $\mathcal{W}$ into $C(0,T;H)$ is continuous, the latter combined with the convergences (5.24) and $x_0^n\to x_0$ in H as $n\to\infty$, reveals that $x_n(0)\to x(0)=x_0$ in H as $n\to\infty$. Additionally, (5.20) indicates that $B(\cdot,x_n(\cdot))u_n(\cdot) \xrightarrow{w} B(\cdot,x(\cdot))u(\cdot)$ in $\mathcal{H}$ as $n\to\infty$.

Multiplying (5.25) by $x_n(t)-x(t)$ and integrating the resulting equality over $[0,T]$, it yields

$$\begin{aligned}
&\int_0^T \langle A(s,x_n(s)),x_n(s)-x(s)\rangle\,ds\\
&\le \int_0^T (\xi_n(s),x(s)-x_n(s))_H+(B(s,x_n(s))u_n(s),x_n(s)-x(s))_H\,ds\\
&\quad+\int_0^T \varphi(\beta x(s))-\varphi(\beta x_n(s))\,ds+\frac{1}{2}\|x(0)-x_n(0)\|_H^2.
\end{aligned}$$

Passing to the upper limit as $n\to\infty$, we have $\limsup_{n\to\infty}\int_0^T\langle A(s,x_n(s)),x_n(s)-x(s)\rangle\,ds\le 0$. This together with [42, Proposition 1] finds that $A(\cdot,x_n(\cdot)) \xrightarrow{w} A(\cdot,x(\cdot))$ in $\mathcal{V}^*$ as $n\to\infty$. Letting $n\to\infty$ in (5.25), we obtain

$$\begin{cases} x'(t)+A(t,x(t))+\xi(t)+\eta(t)=B(t,x(t))u(t) \text{ for a.e. } t\in[0,T],\\ x(0)=x_0.\end{cases}$$

Taking into account $(\xi,\eta,u)\in\mathcal{H}\times\mathcal{V}^*\times\mathcal{Y}$ with $\xi(t)\in\partial J(t,x(t))$, $\eta(t)\in\beta^*\partial_C\varphi(\beta x(t))$ and $u(t)\in U(t,x(t))$ for a.e. $t\in[0,T]$, we can see that $(x,u)\in\mathcal{W}\times\mathcal{Y}$ is a solution to Problem 1.3 associated with the initial condition x_0, i.e., $(x,u)\in\mathcal{S}(x_0)$. This concludes that (5.23) is valid. □

Remark 5.5. In Theorem 5.3, we have proved that Problem 1.3 admits a feedback control pair. But, observer that the framework applied in Theorem 5.3 implies that for each $u\in\mathcal{Y}$ given problem (5.17)–(5.18) has a unique solution $x\in\mathcal{W}$. In this moment, the solution mapping F of problem (5.17)–(5.18) is single-valued. Naturally, an open problem arises whether we can prove the existence of solutions of Problem 1.3 in the situation that F is a set-valued mapping. It should be mentioned that the essential difficulty is that we do not know the convexity of F (see Theorem 3.6), when F is a set-valued mapping.

6. Conclusions

In this paper, under the framework of an evolution triple of spaces, we consider a class of nonlinear and nonsmooth dynamics systems involving two multi-valued terms which are a convex subdifferential operator and a generalized subdifferential operator in the sense of Clarke, respectively. In conclusion, in the paper, we carry out the in-depth research to the nonlinear and nonsmooth dynamics system under consideration from the following three perspectives:

- under quite general assumptions, we establish an existence theorem to nonlinear and nonsmooth dynamics system, Problem 1.1, by employing a surjectivity theorem for set-valued mappings that we use for the sum of a maximal monotone and strongly-quasi bounded operator, a linear densely defined and maximal monotone operator $\mathcal{L}$, and a bounded pseudomonotone operator with respect to $D(\mathcal{L})$;
- an optimal control problem governed by nonlinear and nonsmooth dynamics system, Problem 1.1, is introduced, and the nonemptiness and weak compactness of the set of optimal controls for the optimal control problem are obtained;
- in the convex framework, we investigate a nonlinear feedback control problem described by an upper semicontinuous set-valued mapping and Problem 1.1, and explore the sufficient condition for the existence of solutions of the feedback control problem in which our main tool is the well-known Kakutani-Ky Fan fixed point theorem. In the meantime, a convergence result in the sense of Kuratowski describing the changes in the set of solutions for the feedback control problem as the initial data x_0 is perturbed in Hilbert space H, is delivered.

In fact, problems of this type are encountered in transport optimization, dynamic Nash equilibrium problem of multiple players, dynamic contact problems, fluid mechanics problems with multivalued and nonmonotone boundary conditions, and related fields. In the future we plan to apply the theoretical results established in the current paper to an evolutionary Oseen model for generalized Newtonian fluid with multivalued nonmonotone friction law and leak/slip boundary conditions.

Moreover, we are going to study a new kind of minimizing problems driven by the feedback control system, Problem 1.3, in which the control constraint U and the cost integrand as a function of the control variable are nonconvex (i.e., the minimizing problems will be considered in the nonconvex framework), and to establish the corresponding relaxation-type results for the nonconvex optimal feedback control problems.

Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest. Moreover, the present manuscript has no associated data.

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