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INFINITELY MANY SOLUTIONS FOR THE DIRICHLET PROBLEM ON THE SIERPINSKI GASKET

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We study the nonlinear elliptic equation $\Delta u(x) + a(x)u(x) = g(x)f(u(x))$ on the Sierpinski gasket and with zero Dirichlet boundary condition. By extending a method introduced by Faraci and Kristály in the framework of Sobolev spaces to the case of function spaces on fractal domains, we establish the existence of infinitely many weak solutions.

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1. Introduction

Many physical problems on fractal domains lead to nonlinear models involving reaction-diffusion equations, problems on elastic fractal media or fluid flow through fractal regions, etc. The prevalence of fractal-like objects in nature has led both mathematicians and physicists to study various processes on fractals.

In recent years there has been an increasing interest in studying nonlinear partial differential equations on fractals, also motivated and stimulated by the considerable amount of literature devoted to the definition of a Laplace-type operator for functions on fractal domains.

We cannot expect the solutions of partial differential equations on fractal domains to behave like the solutions of their Euclidean analogues. For example, Barlow and Kigami [1] proved that many fractals have Laplacian eigenfunctions vanishing identically on large open sets, whereas the eigenfunctions of the Laplace operator are analytic in $\mathbb{R}^n$. Among the recent contributions to the theory of nonlinear elliptic equations on fractals we refer to Bockelman and Strichartz [2], Falconer [3], Falconer and Hu [5], Hu [7], Hua and Zhenya [8], Strichartz [16]. The main tools used in these papers to prove the existence of at least one nontrivial solution or of multiple solutions for nonlinear elliptic equations with zero Dirichlet boundary conditions are certain minimax results (mountain pass theorems, saddle-point theorems), respectively, minimization procedures.

In the present paper, we propose a different method to study these types of equations. To apply this method we have to impose on the nonlinear term of the elliptic equation other conditions such as those used in the papers mentioned before. Instead of requiring that the nonlinear term should satisfy certain symmetry properties, this term has to have an oscillating behavior. This method has been used successfully to prove, in the framework of standard Sobolev spaces, the existence of infinitely many solutions, respectively, for Dirichlet problems on bounded domains (Saint Raymond [14]), for one-dimensional scalar field equations and systems (Faraci and Kristály [6]), or for homogeneous Neumann problems (Kristály and Motreanu [11]). The aim of the present paper is to show that the methods used in Faraci and Kristály [6] can be successfully adapted to prove the existence of infinitely many (weak) solutions for nonlinear elliptic equations with zero Dirichlet boundary conditions on the Sierpinski gasket.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$, by $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidian norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$.

2. The Sierpinski Gasket

In its initial representation that goes back to the pioneering papers of the Polish mathematician Waclaw Sierpinski (1882–1969), the *Sierpinski gasket* is the connected subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of 4^{-1} the area, removing the corresponding open triangle from each of the three constituent triangles, and continuing this way. The gasket can also be obtained as the closure of the set of vertices arising in this construction. Over the years, the Sierpinski gasket showed both to be extraordinarily useful in representing roughness in nature and man's works. We refer to Strichartz [15] for an elementary introduction to this subject and to Strichartz [17] for important applications to differential equations on fractals.

We now rigorously describe the construction of the Sierpinski gasket in a general setting. Let $N \geq 2$ be a natural number and let $p_1, \ldots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \ldots, N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i.$$

Obviously, every S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathcal{S} := \{S_1, \dots, S_N\}$ and denote by $F : \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$F(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known (see, for example, [4, Theorem 9.1]) that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor of the family* $\mathcal{S}$, such that $F(V) = V$ (that is, V is a fixed point of the map F). The set V is called the *Sierpinski gasket* (SG for short) in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows: Put $V_0 := \{p_1, \dots, p_N\}, V_m := F(V_{m-1})$, for $m \geq 1$, and $V_* := \cup_{m\geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i = \overline{1, N}$, we have $V_0 \subseteq V_1$. Hence $F(V_*) = V_*$. Taking into account that the maps S_i, $i = \overline{1, N}$, are homeomorphisms, we conclude that $\overline{V_*}$ is a fixed point of F. On the other hand, denoting by C the convex hull of the set $\{p_1, \dots, p_N\}$, we observe that $S_i(C) \subseteq C$ for $i = \overline{1, N}$. Thus $V_m \subseteq C$ for every $m \in \mathbb{N}$, and so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is non-empty and compact, and hence $V = \overline{V_*}$. In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. The set V_0 is called the *intrinsic boundary* of the SG.

The family $\mathcal{S}$ of similarities satisfies the open set condition (see [4, p. 129]) with the interior $\operatorname{int} C$ of C. (Note that $\operatorname{int} C \neq \emptyset$ since the points $p_1, \dots, p_N$ are affine independent.) Thus, by [4, Theorem 9.3], the Hausdorff dimension d of V satisfies the equality

$$\sum_{i=1}^{N} \left(\frac{1}{2}\right)^d = 1.$$

Hence $d = \frac{\ln N}{\ln 2}$, and $0 < \mathcal{H}^d(V) < \infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V, and so $\mu(V) = 1$. The following property of μ will be important for the proof of the main result

$$\mu(B) > 0, \quad \text{for every non-empty open subset } B \text{ of } V. \tag{2.1}$$

In other words, the support of μ coincides with V. To prove (2.1), let B be a non-empty open subset of V and fix an arbitrary element $x \in B$. Then (see [9, 3.1(iii)]) the equality $F(V) = V$ yields the existence of a function $\phi : \mathbb{N}^* \to \{1, \dots, N\}$ such that x is the unique element in the intersection of the members of the following sequence of sets

$$V \supseteq V_{i_1} \supseteq V_{i_1 i_2} \supseteq \cdots \supseteq V_{i_1 i_2 \cdots i_n} \supseteq \cdots,$$

where $V_{i_1 \cdots i_n} := (S_{\phi(1)} \circ \cdots \circ S_{\phi(n)})(V)$ for every $n \in \mathbb{N}^*$. Assuming that

$$V_{i_1 \cdots i_n} \backslash B \neq \emptyset, \quad \text{for every } n \in \mathbb{N}^*,$$

there exists an element $x_n \in V_{i_1 \cdots i_n} \backslash B$ for every $n \in \mathbb{N}^*$. Since

$$|x_n - x| \leq \operatorname{diam} V_{i_1 \cdots i_n} = \left(\frac{1}{2}\right)^n \operatorname{diam} V, \quad \text{for all } n \in \mathbb{N}^*,$$

the sequence (x_n) converges to x. Thus there is an index n_0 with $x_n \in B$ for all $n \geq n_0$, that is, a contradiction. We conclude that there is $n \in \mathbb{N}^*$ such that

$$V_{i_1 \cdots i_n} \subseteq B.$$

It follows that $\mu(V_{i_1 \cdots i_n}) \leq \mu(B)$. On the other hand, by the scaling property of the Hausdorff measure (see [4, 2.1]), we have that

$$\mu(V_{i_1 \cdots i_n}) = \left(\frac{1}{2}\right)^{nd} \cdot \mu(V) > 0,$$

and so $\mu(B) > 0$.

3. The Space $H_0^1(V)$

We retain the notations from the previous section and briefly recall from Falconer and Hu [5] the following notions (see also Hu [7] and Kozlov [10] for the case $N = 3$). Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) | u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_s$. For a function $u : V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y| = 2^{-m}}} (u(x) - u(y))^2. \tag{3.1}$$

We have $W_m(u) \leq W_{m+1}(u)$ for every natural m, and so we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{3.2}$$

Define now

$$H_0^1(V) := \{u \in C_0(V) \,|\, W(u) < \infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V, \mu)$ (equipped with the usual $\|\cdot\|_2$ norm). We now endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: For $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u, v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y \in V_m \\ |x-y| = 2^{-m}}} (u(x) - u(y))(v(x) - v(y)).$$

Put

$$\mathcal{W}(u, v) = \lim_{m \to \infty} \mathcal{W}_m(u, v).$$

Then $\mathcal{W}(u,v) \in \mathbb{R}$ and $H_0^1(V)$, equipped with the inner product $\mathcal{W}$ (which obviously induces the norm $\|\cdot\|$), becomes a real Hilbert space. Moreover,

$$\|u\|_s \leq (2N+3)\|u\|, \quad \text{for every } u \in H_0^1(V), \tag{3.3}$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_s) \tag{3.4}$$

is compact.

We now state a useful property of the space $H_0^1(V)$ which shows, together with the facts that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space and that $H_0^1(V)$ is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$. (In fact, it is the analogue in the case of $H_0^1(V)$ of a property stated in Marcus and Mizel [13] for Sobolev spaces.)

Lemma 3.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \geq 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L \cdot \|u\|$.*

Proof. It is clear that $h \circ u \in C_0(V)$. For every $m \in \mathbb{N}$ we have, by (3.1) and the Lipschitz property of h, that

$$W_m(h \circ u) \leq L^2 \cdot W_m(u).$$

Hence $W(h \circ u) \leq L^2 \cdot W(u)$, according to (3.2). Thus $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L \cdot \|u\|$. $\square$

4. The Dirichlet Problem on the Sierpinski Gasket

We keep the notations from the previous sections. We also recall from Falconer and Hu [5] (respectively, from Hu [7] and Kozlov [10] in the case $N = 3$) that one can define in a standard way a linear, bijective and self-adjoint operator $\Delta : D \to L^2(V,\mu)$, where D is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), \|\cdot\|)$), such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \quad \text{for every } (u,v) \in D \times H_0^1(V).$$

The operator Δ is called the *weak Laplacian* on V.

Given $a : V \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : V \to \mathbb{R}$, with appropriate properties, we can formulate now the following *Dirichlet problem on the SG*: Find functions $u \in H_0^1(V)$ such that

$$(\mathrm{P}) \begin{cases} \Delta u(x) + a(x)u(x) = g(x)f(u(x)), & \forall\, x \in V \setminus V_0. \\ u|_{V_0} = 0. \end{cases}$$

A function $u \in H_0^1(V)$ is called a *weak solution of* (P) if

$$\mathcal{W}(u,v) - \int_V a(x)u(x)v(x)d\mu + \int_V g(x)f(u(x))v(x)d\mu = 0, \quad \forall\, v \in H_0^1(V).$$

The aim of the paper is to prove the following result concerning the existence of multiple weak solutions of problem (P):

Theorem 4.1. *Assume that the following conditions hold*:

(C1) $a \in L^1(V,\mu)$ *and* $a \le 0$ *a.e. in* V.

(C2) $f:\mathbb{R} \to \mathbb{R}$ *is continuous such that*

- (1) *There exist two sequences* (a_k) *and* (b_k) *in* $]0,\infty[$ *with* $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ *and such that* $f(s) \le 0$ *for every* $s \in [a_k, b_k]$.
- (2) *Either* $\sup\{s < 0 \mid f(s) > 0\} = 0$, *or there is a* $\delta > 0$ *with* $f|_{[-\delta,0]} = 0$.

(C3) $F:\mathbb{R} \to \mathbb{R}$, *defined by* $F(s) = \int_0^s f(t)dt$, *is such that*

- (3) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2}$,
- (4) $\limsup_{s\to 0^+} \frac{F(s)}{s^2} = \infty$.

(C4) $g \in C(V)$ *with* $g \le 0$ *and such that the restriction of* g *to every open subset of* V *is not identically* 0.

Then, there is a sequence (u_k) *of pairwise distinct weak solutions of problem* (P) *such that* $\lim_{k\to\infty} \|u_k\| = 0$. *In particular,* $\lim_{k\to\infty} \|u_k\|_s = 0$.

Remark 4.2. The conditions (1) and (2) of Theorem 4.1 on the nonlinear term $f:\mathbb{R} \to \mathbb{R}$ of problem (P) show that this function has an oscillating behavior at 0. In Faraci and Kristály [6] there is given the following example for a function satisfying the conditions (C2) and (C3) of the theorem: Let $0 < \alpha < 1 < \beta$ and $f:\mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $f(t) = |t|^\alpha \max\{0, \sin |t|^{-1}\} + |t|^\beta \min\{0, \sin |t|^{-1}\}$ for $t \ne 0$.

5. Preparatory Results

Let $a, g \in L^1(V,\mu)$ and $f:\mathbb{R} \to \mathbb{R}$ be continuous. Define $F:\mathbb{R} \to \mathbb{R}$ by $F(s) = \int_0^s f(t)dt$. The map $I: H_0^1(V) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_V a(x)u^2(x)d\mu + \int_V g(x)F(u(x))d\mu, \quad \forall\, u \in H_0^1(V), \tag{5.1}$$

will turn out to be under suitable assumptions the energy functional attached to problem (P) (see Proposition 5.3 below). To see this we first recall a few basic notions.

Definition 5.1. Let E be a real Banach space and $T: E \to \mathbb{R}$ a functional. We say that T is *Fréchet differentiable at* $u \in E$ if there exists a continuous linear map $dT(u): E \to \mathbb{R}$, called the *Fréchet differential of* T *at* u, such that

$$\lim_{v\to 0} \frac{|T(u+v) - T(u) - dT(u)(v)|}{\|v\|} = 0.$$

The functional T is *Fréchet differentiable on* E if T is Fréchet differentiable at every point $u \in E$. A point $u \in E$ is a *critical point of* T if T is Fréchet differentiable at u and if $dT(u) = 0$.

Remark 5.2. Note that if the functional $T: E \to \mathbb{R}$ has in $u \in E$ a local extremum and if T is Fréchet differentiable at u, then u is a critical point of T.

Proposition 5.3. *Suppose that $a \in L^1(V, \mu)$, $f: \mathbb{R} \to \mathbb{R}$ is continuous and $g \in C(V)$. Then the functional $I: H_0^1(V) \to \mathbb{R}$ defined by (5.1) is Fréchet differentiable on $H_0^1(V)$. Moreover, for every $u, v \in H_0^1(V)$ the following equality holds*

$$dI(u)(v) = \mathcal{W}(u,v) - \int_V a(x)u(x)v(x)d\mu + \int_V g(x)f(u(x))v(x)d\mu.$$

In particular, $u \in H_0^1(V)$ is a weak solution of problem (P) *if and only if u is a critical point of I.*

Proof. See [5, Proposition 2.19]. □

Remark 5.4. If $a \in C(V)$, $f: \mathbb{R} \to \mathbb{R}$ is continuous and $g \in C(V)$, then, using the regularity result in [5, Lemma 2.16], it follows that every weak solution of problem (P) is also a strong solution (as defined in [5]).

We next do some preparation in order to state in Proposition 5.9 an important property of the map I.

Definition 5.5. Let X be a topological Hausdorff space. A map $h: X \to \mathbb{R}$ is called *sequentially lower semicontinuous at $x \in X$* if for every sequence (x_n) in X converging to x the inequality

$$h(x) \le \liminf_{n\to\infty} h(x_n)$$

holds. The map $h: X \to \mathbb{R}$ is *sequentially lower semicontinuous on X* if h is sequentially lower semicontinuous at every point $x \in X$.

Lemma 5.6. *Let $a, g \in L^1(V, \mu)$ and let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Then the functional $I: H_0^1(V) \to \mathbb{R}$ defined by (5.1) is weakly sequentially lower semicontinuous on $H_0^1(V)$.*

Proof. We first prove that the maps $u \in H_0^1(V) \mapsto \int_V a(x)u^2(x)d\mu \in \mathbb{R}$ and $u \in H_0^1(V) \mapsto \int_V g(x)F(u(x))d\mu \in \mathbb{R}$ are both weakly sequentially continuous on $H_0^1(V)$. To this end let $u \in H_0^1(V)$ and consider a sequence (u_n) weakly converging to u in $H_0^1(V)$. Since the embedding (3.4) is compact, the sequence (u_n) converges to u in $C_0(V)$, and so does (u_n^2) to u^2. The relations

$$\begin{aligned}\left|\int_V a(x)u_n^2(x)d\mu - \int_V a(x)u^2(x)d\mu\right| &\le \int_V |a(x)|\cdot|u_n^2(x) - u^2(x)|d\mu \\ &\le \|u_n^2 - u^2\|_s \cdot \int_V |a(x)|d\mu\end{aligned}$$

show that the map $u \in H_0^1(V) \mapsto \int_V a(x)u^2(x)d\mu \in \mathbb{R}$ is weakly sequentially continuous at u, and hence on $H_0^1(V)$, because u was arbitrarily chosen.

Since (u_n) converges to u in $C_0(V)$, there is a positive constant ρ such that $\|u\|_s \leq \rho$ and $\|u_n\|_s \leq \rho$ for all $n \in \mathbb{N}$. Use the notation $c := \max_{t\in[-\rho,\rho]} |f(t)|$. Then, for every $n \in \mathbb{N}$,

$$|F(u_n(x)) - F(u(x))| = \left| \int_{u(x)}^{u_n(x)} f(t)dt \right| \leq c \cdot \|u_n - u\|_s.$$

Hence

$$\left| \int_V g(x)F(u_n(x))d\mu - \int_V g(x)F(u(x))d\mu \right| \leq c \cdot \|u_n - u\|_s \cdot \int_V |g(x)|d\mu.$$

We conclude, as above, that the map $u \in H_0^1(V) \mapsto \int_V g(x)F(u(x))d\mu \in \mathbb{R}$ is weakly sequentially continuous on $H_0^1(V)$.

The map $u \in H_0^1(V) \mapsto \|u\|^2 \in \mathbb{R}$ is continuous in the norm topology on $H_0^1(V)$ and convex. Thus, it is weakly sequentially lower semicontinuous on $H_0^1(V)$. We conclude that I is also weakly sequentially lower semicontinuous on $H_0^1(V)$. □

Lemma 5.7. *Let X be a non-empty topological Hausdorff space and let $h : X \to \mathbb{R}$ be a map satisfying the following properties*:

(i) *h is sequentially lower semicontinuous on X,*
(ii) *h is bounded from below on X,*
(iii) *every sequence (x_n) in X such that the sequence $(h(x_n))$ is bounded has a convergent subsequence.*

Then h attains its infimum on X.

Proof. Let (x_n) be a sequence in X such that $\lim h(x_n) = \inf h(X)$. According to (ii), the sequence $(h(x_n))$ is bounded, and so, by (iii), there is a convergent subsequence (x_{n_k}) of (x_n). Let $x \in X$ be the limit point of this subsequence. Using (i), we get

$$h(x) \leq \liminf_{k\to\infty} h(x_{n_k}) = \inf h(X),$$

and hence $h(x) = \inf h(X)$. □

Remark 5.8. Let $a, g \in L^1(V, \mu)$ be so that $a \leq 0$ and $g \leq 0$ a.e. in V, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Consider $u \in H_0^1(V)$ and $d, b \in \mathbb{R}$ such that $d \leq u(x) \leq b$ for every $x \in V$. According to the fact that $g \leq 0$ a.e. in V, we then have

$$\int_V g(x)F(u(x))d\mu \geq \max_{s\in[d,b]} F(s) \cdot \int_V g(x)d\mu. \tag{5.2}$$

For later use we state the following relations concerning the functional $I : H_0^1(V) \to \mathbb{R}$ defined by (5.1): The inequalities (5.2) and $a \leq 0$ a.e. in V imply that

$$I(u) \geq \max_{s\in[d,b]} F(s) \cdot \int_V g(x)d\mu \tag{5.3}$$

and

$$\frac{1}{2}\|u\|^2 \leq I(u) - \max_{s\in[d,b]} F(s) \cdot \int_V g(x)d\mu. \tag{5.4}$$

We also note that for every $x \in V$

$$F(u(x)) \leq |F(u(x))| = \left| \int_0^{u(x)} f(t)dt \right| \leq \max_{t\in[d,b]} |f(t)| \cdot \|u\|_s.$$

As above, we then conclude that

$$I(u) \geq \max_{t\in[d,b]} |f(t)| \cdot \|u\|_s \cdot \int_V g(x)d\mu \tag{5.5}$$

and

$$\frac{1}{2}\|u\|^2 \leq I(u) - \max_{t\in[d,b]} |f(t)| \cdot \|u\|_s \cdot \int_V g(x)d\mu. \tag{5.6}$$

Proposition 5.9. *Let $a, g \in L^1(V,\mu)$ be so that $a \leq 0$ and $g \leq 0$ a.e. in V, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Consider $d, b \in \mathbb{R}$ so that $d < 0 < b$ and put*

$$M := \{u \in H_0^1(V) \,|\, d \leq u(x) \leq b, \,\forall\, x \in V\}.$$

Then the functional $I : H_0^1(V) \to \mathbb{R}$ defined by (5.1) is bounded from below on M and attains its infimum on M.

Proof. Obviously the set M is non-empty (it contains the constant 0 function) and convex. Since the inclusion (3.4) is continuous, M is closed in the norm topology on $H_0^1(V)$. It follows that M is also closed in the weak topology on $H_0^1(V)$. Consider M to be endowed with the relative weak topology. We will show that the restriction $I|_M$ of I to M satisfies the hypotheses (i)–(iii) of Lemma 5.7. Condition (i) results from Lemma 5.6, and (ii) is a consequence of (5.3). Hence I is bounded from below on M. To verify (iii), let (u_n) be a sequence in M such that the sequence $(I(u_n))$ is bounded. Inequality (5.4) then yields that (u_n) is (norm) bounded. By the reflexivity of the Hilbert space $H_0^1(V)$ and the closedness of M in the weak topology we conclude that the sequence (u_n) has a weakly convergent subsequence in M. Lemma 5.7 implies now that I attains its infimum on M. □

6. Proof of Theorem 4.1

Throughout this section we assume that the conditions C(1)–C(4) in the hypotheses of Theorem 4.1 are satisfied.

Case 1. Suppose first that the equality $\sup\{s < 0 \mid f(s) > 0\} = 0$ in condition (2) of (C2) holds. Then there exists a strictly increasing sequence (c_k) of negative reals such that $\lim c_k = 0$ and $f(c_k) > 0$ for every natural k. By continuity of f there exists another sequence (d_k) such that $d_k < c_k < d_{k+1}$ and $f(t) > 0$ for every $t \in [d_k, c_k]$ and every natural k.

Case 2. If we have in (2) that there is a $\delta > 0$ with $f|_{[-\delta,0]} = 0$, then choose a strictly increasing sequence (c_k) of negative reals greater than $-\delta$ such that $\lim c_k = 0$. Construct the sequence (d_k) such that $d_k < c_k < d_{k+1}$ and $f(t) = 0$ for every $t \in [d_k, c_k]$ and every natural k.

In both cases, since $F(s) = \int_0^s f(t)dt$ for every $s \in \mathbb{R}$, it follows that

$$F(s) \leq F(c_k), \quad \text{for every } s \in [d_k, c_k]. \tag{6.1}$$

Using condition (1) of (C2), we have that

$$F(s) \leq F(a_k), \quad \text{for every } s \in [a_k, b_k]. \tag{6.2}$$

For every $k \in \mathbb{N}$ set now

$$M_k := \{u \in H_0^1(V) \,|\, d_k \leq u(x) \leq b_k, \ \forall\, x \in V\}.$$

The proof of Theorem 4.1 includes the following main steps contained in the next lemmas:

(i) we show that the map $I : H_0^1(V) \to \mathbb{R}$ defined by (5.1) has at least one critical point in each of the sets M_k;

(ii) we will show that there are infinitely many pairwise distinct such critical points;

(iii) by Proposition 5.3 we know that each of these critical points is a weak solution of problem (P).

Lemma 6.1. *For every $k \in \mathbb{N}$ there is an element $u_k \in M_k$ such that the following conditions hold*:

(i) $I(u_k) = \inf I(M_k)$,

(ii) $c_k \leq u_k(x) \leq a_k$, *for every $x \in V$.*

Proof. Fix $k \in \mathbb{N}$. According to Proposition 5.9, there is an element $\tilde{u}_k \in M_k$ such that $I(\tilde{u}_k) = \inf I(M_k)$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} c_k, & t < c_k \\ x, & t \in [c_k, a_k] \\ a_k, & t > a_k. \end{cases}$$

Note that $h(0) = 0$ and that h is a Lipschitz map with Lipschitz constant $L = 1$. According to Lemma 3.1 the map $u_k := h \circ \tilde{u}_k$ belongs to $H_0^1(V)$ and

$$\|u_k\| \leq \|\tilde{u}_k\|. \tag{6.3}$$

Moreover, u_k belongs to M_k and obviously satisfies condition (ii) to be proved. We next show that (i) also holds. For this set

$$V_1 := \{x \in V \,|\, \tilde{u}_k(x) < c_k\}, \quad V_2 := \{x \in V \,|\, \tilde{u}_k(x) > a_k\}.$$

Then

$$u_k(x) = \begin{cases} c_k, & x \in V_1 \\ \tilde{u}_k(x), & x \in V \backslash (V_1 \cup V_2) \\ a_k, & x \in V_2. \end{cases}$$

It follows that

$$(u_k(x))^2 \le (\tilde{u}_k(x))^2, \quad \text{for every } x \in V. \tag{6.4}$$

Furthermore, if $x \in V_1$ then $\tilde{u}_k(x) \in [d_k, c_k[$, and hence $F(\tilde{u}_k(x)) \le F(c_k) = F(u_k(x))$, by (6.1). Analogously, if $x \in V_2$, then (6.2) yields $F(\tilde{u}_k(x)) \le F(a_k) = F(u_k(x))$. Thus

$$F(\tilde{u}_k(x)) \le F(u_k(x)), \quad \text{for every } x \in V. \tag{6.5}$$

The inequalities (6.3)–(6.5) imply, together with the fact that $a \le 0$ and $g \le 0$ a.e. in V, that

$$I(\tilde{u}_k) - I(u_k) = \frac{1}{2}\|\tilde{u}_k\|^2 - \frac{1}{2}\|u_k\|^2 - \frac{1}{2}\int_V a(x)(\tilde{u}_k^2(x) - u_k^2(x))d\mu$$
$$+ \int_V g(x)(F(\tilde{u}_k(x)) - F(u_k(x)))d\mu \ge 0.$$

Thus $I(\tilde{u}_k) \ge I(u_k)$. Since $I(\tilde{u}_k) = \inf I(M_k)$ and since $u_k \in M_k$, we conclude that $I(u_k) = \inf I(M_k)$, and thus (i) is also fulfilled. $\square$

Lemma 6.2. *For every $k \in \mathbb{N}$ let $u_k \in M_k$ be a function satisfying the conditions* (i) *and* (ii) *of Lemma 6.1. The functional I has then in u_k a local minimum* (*with respect to the norm topology on $H_0^1(V)$*), *for every $k \in \mathbb{N}$. In particular*, (u_k) *is a sequence of weak solutions of problem* (P).

Proof. Fix $k \in \mathbb{N}$. Suppose to the contrary that I has not in u_k a local minimum. This implies the existence of a sequence (w_n) in $H_0^1(V)$ converging to u_k in the norm topology such that

$$I(w_n) < I(u_k), \quad \text{for every } n \in \mathbb{N}.$$

In particular, $w_n \notin M_k$, for all $n \in \mathbb{N}$. Choose a real number ε such that

$$0 < \varepsilon < \frac{1}{2}\min\{b_k - a_k, c_k - d_k\}.$$

In view of (3.3), the sequence (w_n) converges to u_k in the supremum norm topology on $C(V)$. Hence there is an index $m \in \mathbb{N}$ such that

$$\|w_m - u_k\|_s \le \varepsilon.$$

For every $x \in V$ we then have according to (ii) of Lemma 6.1

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \le \varepsilon + u_k(x) \le \frac{b_k - a_k}{2} + a_k < b_k$$

and

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \geq -\varepsilon + u_k(x) \geq \frac{d_k - c_k}{2} + c_k > d_k.$$

Thus $w_m \in M_k$, that is a contradiction. We conclude that I has in u_k a local minimum. The last assertion of the lemma follows now from Remark 5.2 and Proposition 5.3. □

Lemma 6.3. *For every $k \in \mathbb{N}$ put $\gamma_k := \inf I(M_k)$. Then $\gamma_k < 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \gamma_k = 0$.*

Proof. Lemma 3.1 implies that $|u| \in H_0^1(V)$ whenever $u \in H_0^1(V)$. Thus we can pick a function $u \in H_0^1(V)$ such that $u(x) \geq 0$ for every $x \in V$ and such that there is an element $x_0 \in V$ with $u(x_0) > 1$. It follows that $D := \{x \in V \mid u(x) > 1\}$ is a non-empty open subset of V. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = \min\{t, 1\}$, for every $t \in \mathbb{R}$. Then $h(0) = 0$ and h is a Lipschitz map with Lipschitz constant $L = 1$. Lemma 3.1 yields that $v := h \circ u \in H_0^1(V)$. Moreover, $v(x) = 1$ for every $x \in D$, and $0 \leq v(x) \leq 1$ for every $x \in V$.

On the other hand, condition (3) of (C3) implies the existence of real numbers $\rho > 0$ and c such that $\frac{F(s)}{s^2} > c$ for every $s \in]0, \rho[$. It follows that

$$F(s) \geq cs^2, \quad \text{for every } s \in [0, \rho[. \tag{6.6}$$

Condition (4) of (C3) yields the existence of a sequence (r_n) in $]0, \rho[$ such that $\lim_{n\to\infty} r_n = 0$ and

$$\lim_{n\to\infty} \frac{F(r_n)}{r_n^2} = \infty. \tag{6.7}$$

We then have for every $n \in \mathbb{N}$

$$\begin{aligned} I(r_n v) = \frac{r_n^2}{2}\|v\|^2 - \frac{r_n^2}{2}\int_V a(x)v^2(x)d\mu + F(r_n)\int_D g(x)d\mu \\ + \int_{V\setminus D} g(x)F(r_n v(x))d\mu. \end{aligned}$$

Using (6.6) and the fact that $g \leq 0$ in V, we get for every $n \in \mathbb{N}$

$$\begin{aligned} I(r_n v) \leq \frac{r_n^2}{2}\|v\|^2 - \frac{r_n^2}{2}\int_V a(x)v^2(x)d\mu + F(r_n)\int_D g(x)d\mu \\ + cr_n^2\int_{V\setminus D} g(x)v^2(x)d\mu. \end{aligned}$$

Thus

$$\frac{I(r_n v)}{r_n^2} \leq \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_V a(x)v^2(x)d\mu + \frac{F(r_n)}{r_n^2}\int_D g(x)d\mu + c\int_{V\setminus D} g(x)v^2(x)d\mu.$$

Condition (C4) and (2.1) imply that $\int_D g(x)d\mu < 0$, and so we get from (6.7) and the above inequality that

$$\lim_{n\to\infty} \frac{I(r_n v)}{r_n^2} = -\infty.$$

Thus there is an index n_0 such that $I(r_n v) < 0$ for every $n \geq n_0$. Fix now $k \in \mathbb{N}$. Since $\lim_{n\to\infty} \|r_n v\|_s = 0$, we get an index $p \geq n_0$ such that $r_p v \in M_k$. Hence $\gamma_k \leq I(r_p v) < 0$.

Let $u_k \in M_k$ be so that $\gamma_k = I(u_k)$. Since $M_k \subseteq M_0$, relation (5.5) yields

$$\gamma_k = I(u_k) \geq \max_{t\in[d_0,b_0]} |f(t)| \cdot \|u_k\|_s \cdot \int_V g(x)d\mu,$$

and hence

$$0 > \gamma_k \geq \max_{t\in[d_0,b_0]} |f(t)| \cdot \max\{b_k, |d_k|\} \cdot \int_V g(x)d\mu.$$

Since $\lim_{k\to\infty} b_k = \lim_{k\to\infty} d_k = 0$, we conclude that $\lim_{k\to\infty} \gamma_k = 0$. □

Proof of Theorem 4.1 concluded. From Lemma 6.2 we know that there is a sequence (u_k) of weak solutions of problem (P) such that $\gamma_k = I(u_k)$, where $\gamma_k = \inf I(M_k)$, for every natural k. Using relation (5.6) and the fact that $\gamma_k \leq 0$, we obtain

$$\frac{1}{2}\|u_k\|^2 \leq - \max_{t\in[d_0,b_0]} |f(t)| \cdot \max\{b_k, |d_k|\} \cdot \int_V g(x)d\mu.$$

Using once again that $\lim_{k\to\infty} b_k = \lim_{k\to\infty} d_k = 0$, we conclude that $\lim_{k\to\infty} \|u_k\| = 0$. Thus also $\lim_{k\to\infty} \|u_k\|_s = 0$, by (3.3).

We know from Lemma 6.3 that $I(u_k) = \gamma_k < 0$, for every natural k, and that $\lim_{k\to\infty} \gamma_k = 0$. Thus we can find a subsequence (u_{k_j}) of the sequence (u_k) consisting of pairwise distinct elements. □

Remark 6.4. (1) If to the hypotheses of Theorem 4.1 one adds the requirement that $a \in C(V)$, then Theorem 4.1 and Remark 5.4 yield the existence of a sequence of pairwise distinct strong solutions of problem (P) converging to 0.

(2) By the same method, one can prove (see also [6]) an analogous result in the case when the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ has an oscillating behavior at ∞. In this case one obtains a sequence (u_k) of pairwise distinct weak solutions of problem (P) such that $\lim_{k\to\infty} \|u_k\| = \infty$.

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