Further on Set-Valued Equilibrium Problems and Applications to Browder Variational Inclusions

Boualem Alleche & Vicențiu D. Rădulescu

Journal of Optimization Theory and Applications

ISSN 0022-3239
Volume 175
Number 1

J Optim Theory Appl (2017) 175:39-58
DOI 10.1007/s10957-017-1169-1

Springer

J Optim Theory Appl (2017) 175:39–58
DOI 10.1007/s10957-017-1169-1

Further on Set-Valued Equilibrium Problems and Applications to Browder Variational Inclusions

Boualem Alleche[1] · **Vicenţiu D. Rădulescu**[2,3]

Received: 27 April 2017 / Accepted: 6 September 2017 / Published online: 12 September 2017

Abstract In this paper, we introduce some concepts of convexity and semicontinuity for real set-valued mappings similar to those of real single-valued mappings. Then, we obtain different results on the existence of solutions of set-valued equilibrium problems generalizing in a common way several old ones for both single-valued and set-valued equilibrium problems. Applications to Browder variational inclusions, with weakened conditions on the involved set-valued operator, are given.

Keywords Equilibrium problem · Set-valued mapping · Convexity · Semicontinuity · Variational inclusion

Mathematics Subject Classification 26A15 · 26B25 · 26E25 · 47J20 · 49J35

1 Introduction

Browder variational inclusions appear in the literature as a generalization of Browder–Hartman–Stampacchia variational inequalities. These inequality problems are presented as a weak type of multivalued variational inequalities, since they involve

✉ Vicenţiu D. Rădulescu
vicentiu.radulescu@imar.ro; vicentiu.radulescu@math.cnrs.fr

Boualem Alleche
alleche.boualem@univ-medea.dz; alleche.boualem@gmail.com

1 Laboratoire LPTEA, Faculté des Sciences, Université de Médéa, 26000 Medea, Algeria

2 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

3 Department of Mathematics, University of Craiova, Street A.I. Cuza No 13, 200585 Craiova, Romania

set-valued mappings in their definition. Browder variational inclusions have many applications, including applications to the surjectivity of set-valued mappings and to nonlinear elliptic boundary value problems. In recent studies, Browder variational inclusions have been reformulated by means of set-valued equilibrium problems and different results have been carried out; see [1–5].

Although set-valued equilibrium problems have already been considered in the literature, the authors have focused on the applications to Browder variational inclusions, or to other areas such as fixed point theory and economic equilibrium theory. It should be mentioned here that the results obtained in these papers on set-valued equilibrium problems are very general and need to be improved. When these results are applied to single-valued equilibrium problems, their assumptions become simple conditions of continuity and convexity. On the other hand, single-valued equilibrium problems have known in last decades several important and deep advancements.

Set-valued equilibrium problems have been recently investigated in [2,4] under mild conditions of continuity and also under the notion of self-segment-dense subset first introduced in [6]. We focus in this paper only on continuity and convexity. We are aware of the rich development in last years of the field of equilibrium problems, which has taken different directions and involved several tools such as vector equilibrium problems. It is worthwhile noticing that equilibrium problems have many applications in different areas of mathematics, including optimization problems, fixed point theory and Nash equilibrium problems.

In this paper, we continue developing our approach and deal with equilibrium problems involving (extended) real single-valued and set-valued bifunctions only. Motivated by the importance of Browder variational inclusions, we have been attracted by the gap between the level of the existing results on set-valued equilibrium problems and that of single-valued equilibrium problems. Then, we have decided to deeply investigate the convexity and the semicontinuity of (extended) real set-valued mappings in details. In Sect. 2, we present different concepts of convexity and semicontinuity of (extended) real set-valued mappings and obtain some results and characterizations. In Sect. 3, we obtain three main results on the existence of solutions of strong and weak set-valued equilibrium problems generalizing those for both set-valued and single-valued equilibrium problems. Section 4 contains applications to Browder variational inclusions in the realm of real normed vector spaces. Results on the existence of solutions of Browder variational inclusions involving set-valued operators, with bounded in norm values and satisfying a condition related to the existence of a maximum rather than the weak* compactness, are presented. Results involving demicontinuous set-valued operators are also given.

2 Notations and Preliminary Results

In all the paper, $\mathbb{R} =]-\infty, +\infty[$ denotes the set of real numbers and $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$. We also make use of the following notation: $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\mathbb{R}_-^* = -\mathbb{R}_+^*$.

In the sequel, $\overline{\mathbb{R}}$ will be endowed with the topology extended from the usual topology of $\mathbb{R}$ and with the usual operations involving $+\infty$ and $-\infty$. For a subset A of a Hausdorff topological space X, we denote by cl A, the closure of A.

By a set-valued mapping $F : X \rightrightarrows Y$, we mean a mapping F from a set X to the collection of nonempty subsets of a set Y. In the present paper, a mapping $f : X \to Y$ and the set-valued mapping $F : X \rightrightarrows Y$ defined by $F(x) := \{f(x)\}$ for every $x \in X$, will be identified and both will be called a single-valued mapping. That is, a single-valued mapping is a "classical" mapping or a set-valued mapping with singleton values. By a real set-valued mapping, we mean a set-valued mapping with values in $\mathbb{R}$. A real single-valued mapping is a single-valued mapping with values in $\mathbb{R}$. When $\overline{\mathbb{R}}$ is used instead of $\mathbb{R}$, we talk about extended real single-valued or extended real set-valued mappings.

In the sequel, for a real normed vector space X, we denote by X^*, the dual space of X, and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X.

Let C be a nonempty subset of a real topological Hausdorff vector space in the general settings, and $\Phi : C \times C \rightrightarrows \mathbb{R}$ a set-valued mapping called *a set-valued bifunction*. The *strong set-valued equilibrium problem* is a problem of the form

$$\text{find } x_0 \in C \text{ such that } \Phi(x_0, y) \subset \mathbb{R}_+ \quad \forall y \in C. \tag{Ssvep}$$

The *weak set-valued equilibrium problem* is a problem of the form

$$\text{find } x_0 \in C \text{ such that } \Phi(x_0, y) \cap \mathbb{R}_+ \neq \emptyset \quad \forall y \in C. \tag{Wsvep}$$

In the special case where Φ is a single-valued mapping, the strong and the weak set-valued equilibrium problems are the same and coincide with what is often called, an equilibrium problem in the sense of Blum, Muu and Oettli or inequality of Ky Fan-type due to their contribution to the field. Examples of equilibrium problems abound in the literature since they encompass different kinds of problems such as problems of optimization, hierarchical minimization problems, variational inequality problems and complementarity problems.

In the case where Φ is a set-valued mapping, set-valued equilibrium problems also encompass different other problems such as multivalued variational inequalities and Browder variational inequalities. They also have applications to different problems such as fixed point theory and economic equilibrium theory.

Example 2.1 Let C be a nonempty, closed and convex subset of a real normed vector space X. Endowed with the weak topology, X is a real topological Hausdorff vector space. Let $F : C \rightrightarrows X^*$ be a set-valued operator. The problem

$$\text{find } x_0 \in C \text{ such that } \left\{\langle x_0^*, y - x_0\rangle : x_0^* \in F(x_0)\right\} \subset \mathbb{R}_+ \quad \forall y \in C,$$

is an example of a strong set-valued equilibrium problem in the real topological Hausdorff vector space X. The problem

$$\text{find } x_0 \in C \text{ such that } \left\{\langle x_0^*, y - x_0\rangle : x_0^* \in F(x_0)\right\} \cap \mathbb{R}_+ \neq \emptyset \quad \forall y \in C,$$

is an example of a weak set-valued equilibrium problem in the real topological Hausdorff vector space X.

In practice, examples are often taken in the settings of real normed vector spaces which are special cases of real topological Hausdorff vector spaces. The following example is an application of the strong set-valued equilibrium problems to fixed point theory.

Example 2.2 Let C be a nonempty, closed and convex subset of a real normed vector space X and $F : C \rightrightarrows X$ is a set-valued operator. Solving the strong set-valued equilibrium problem

$$\text{find } x_0 \in C \text{ such that } \operatorname{dist}(y, F(x_0)) - \operatorname{dist}(x_0, F(x_0)) + [0, +\infty[\subset \mathbb{R}_+ \quad \forall y \in C,$$

provides us with a tool to obtain a version of *Kakutani fixed point theorem* on the existence of fixed points of F; see [2, Theorem 4.5].

The following example is an application of the weak set-valued equilibrium problems to economic equilibrium theory.

Example 2.3 Consider the simplex

$$M^n := \left\{ x := (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}$$

and a set-valued mapping $C : M^n \rightrightarrows \mathbb{R}^n$. According to Debreu–Gale–Nikaïdo theorem, under some conditions and if the Walras law holds which states that for every (x, y) in the graph of C we have $\langle y, x \rangle \geq 0$, then there exists $\tilde{x} \in M^n$ such that $C(\tilde{x}) \cap \mathbb{R}^n_+ \neq \emptyset$. For every $x \in M^n$ and $y \in \mathbb{R}^n$, we set

$$\sigma(C(x), y) := \sup_{z \in C(x)} \langle z, y \rangle.$$

Solving the weak set-valued equilibrium problem

$$\text{find } x_0 \in M^n \text{ such that }]-\infty, \sigma(C(x_0), y)] \cap \mathbb{R}_+ \neq \emptyset \quad \forall y \in C,$$

provides us with a tool to obtain a version of Debreu–Gale–Nikaïdo-type theorem on the existence of $\tilde{x} \in M^n$ such that $C(\tilde{x}) \cap \mathbb{R}^n_+ \neq \emptyset$. This result is a *Debreu–Gale–Nikaïdo-type theorem*, which extends the famous classical result in economic equilibrium theory by weakening the conditions on the collective Walras law. It has been obtained in [4, Theorem 5.1] under the weakened condition of assuming that the Walras law holds only on a self-segment-dense subset D of M^n.

2.1 Concepts of Convexity

The notions of convexity and concavity of set-valued mappings have been considered in the literature as a generalization of convexity and concavity of real single-valued mappings. However, these notions are not limited to real set-valued mappings and so, they are not really adapted and very general. Applied to real single-valued mappings, they are in fact too stronger than the convexity and concavity and produce a sort of "linearity on line segments".

Here, we develop weaker notions of convexity and concavity for real set-valued mappings which generalize both those for set-valued mappings and those for real single-valued mappings. For more details on the rich field about convexity and related notions of real single-valued mappings, we refer to [7].

Let C be a nonempty and convex subset of a real topological Hausdorff vector space. Recall that a real set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is said to be *convex* on C, if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\sum_{i=1}^{n} \lambda_i F(x_i) \subset F\left(\sum_{i=1}^{n} \lambda_i x_i\right),$$

where the sum denotes here the usual Minkowski sum of sets. It follows that a set-valued mapping is convex on C if and only if its graph is convex. We recall that a real single-valued mapping is convex if and only if its epigraph is convex.

The real set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is said to be *concave* on C, if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$F\left(\sum_{i=1}^{n} \lambda_i x_i\right) \subset \sum_{i=1}^{n} \lambda_i F(x_i).$$

We first introduce the notion of convexly quasi-convexity for real set-valued mappings which generalizes both the convexity of set-valued mappings and the quasi-convexity of real single-valued mappings.

A set-valued mapping $F : C \rightrightarrows \mathbb{R}$ will be said *convexly quasi-convex* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$, then for every $\{z_1, \ldots, z_n\}$ with $z_i \in F(x_i)$ for every $i = 1, \ldots, n$, there exists $z \in F\left(\sum_{i=1}^{n} \lambda_i x_i\right)$ such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

For $\lambda \in \mathbb{R}$, we set $[F \leq \lambda] := \{x \in C : F(x) \cap]-\infty, \lambda] \neq \emptyset\}$.

Proposition 2.1 *Let C be a nonempty and convex subset of a real topological Hausdorff vector space. A set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is convexly quasi-convex on C if and only if the set $[F \leq \lambda]$ is convex, for every $\lambda \in \mathbb{R}$.*

Proof Let $\lambda \in \mathbb{R}$. Let $\{x_1, \ldots, x_n\} \subset [F \leq \lambda]$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ be such that $\sum_{i=1}^n \lambda_i = 1$. For every $i = 1, \ldots, n$, choose $z_i \in F(x_i) \cap]-\infty, \lambda]$. Since F is convexly quasi-convex, let $z \in F\left(\sum_{i=1}^n \lambda_i x_i\right)$ be such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

Then $z \leq \lambda$, and therefore, $\sum_{i=1}^n \lambda_i x_i \in [F \leq \lambda]$.

Conversely, let $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ be such that $\sum_{i=1}^n \lambda_i = 1$. Take $\{z_1, \ldots, z_n\}$ with $z_i \in F(x_i)$, for every $i = 1, \ldots, n$. Put $\lambda = \max\{z_i : i = 1, \ldots, n\} \in \mathbb{R}$. We have $x_i \in [F \leq \lambda]$, for every $i = 1, \ldots, n$. By convexity of $[F \leq \lambda]$, it follows that $\sum_{i=1}^n \lambda_i x_i \in [F \leq \lambda]$ which means that there exists $z \in F\left(\sum_{i=1}^n \lambda_i x_i\right)$ such that $z \leq \lambda$. We conclude that $z \leq \max\{z_i : i = 1, \ldots, n\}$. □

Now, we introduce the notion of concavely quasi-convexity for real set-valued mappings which generalizes both the concavity of set-valued mappings and the quasi-convexity of real single-valued mappings.

A set-valued mapping $F : C \rightrightarrows \mathbb{R}$ will be said *concavely quasi-convex* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$, then for every $z \in F\left(\sum_{i=1}^n \lambda_i x_i\right)$, there exist $\{z_1, \ldots, z_n\}$ with $z_i \in F(x_i)$ for every $i = 1, \ldots, n$ such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

For $\lambda \in \mathbb{R}$, we set $[F \subseteq \lambda] := \{x \in C : F(x) \subset]-\infty, \lambda]\}$.

Proposition 2.2 *Let C be a nonempty and convex subset of a real topological Hausdorff vector space. If a set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is concavely quasi-convex on C, then the set $[F \subseteq \lambda]$ is convex, for every $\lambda \in \mathbb{R}$.*

Proof Let $\lambda \in \mathbb{R}$. Let $\{x_1, \ldots, x_n\} \subset [F \subseteq \lambda]$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ be such that $\sum_{i=1}^n \lambda_i = 1$. Take $z \in F\left(\sum_{i=1}^n \lambda_i x_i\right)$. Since F is concavely quasi-convex, let $z_i \in F(x_i)$ for every $i = 1, \ldots, n$ be such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

Since $\max\{z_i : i = 1, \ldots, n\} \leq \lambda$ and z is arbitrary in $F\left(\sum_{i=1}^n \lambda_i x_i\right)$, then $\sum_{i=1}^n \lambda_i x_i \in [F \subseteq \lambda]$. □

Note that, if f is a real single-valued mapping, then $[f \leq \lambda] = [f \subseteq \lambda]$, for every $\lambda \in \mathbb{R}$. We have the following result.

Proposition 2.3 *Let C be a nonempty and convex subset of a real topological Hausdorff vector space. For a real single-valued mapping $f : C \to \mathbb{R}$, the following conditions are equivalent*

1. *f is quasi-convex on C,*
2. *f is convexly quasi-convex on C,*
3. *f is concavely quasi-convex on C.*

Example 2.4 Consider a quasi-convex function $f : \mathbb{R} \to \mathbb{R}$ which is not convex, and let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be the set-valued mapping defined by $F(x) := \{f(x)\}$, for every $x \in \mathbb{R}$. As mentioned before, f is identified to F in our purpose. By Proposition 2.3, F is convexly quasi-convex and concavely quasi-convex single-valued mapping, but it is neither convex nor concave in the sense of set-valued mapping.

2.2 Concepts of Continuity

Concerning concepts of continuity of set-valued mappings, lower and upper semicontinuity is the most known among them. However, these concepts applied to single-valued mappings, they produce the continuity, which is too strong in many applications. For more details on the large and rich field about semicontinuity of set-valued mappings with different characterizations, we refer to [8].

Let X and Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping. For a subset B of Y, we define

$$F^{-}(B) := \{x \in X : F(x) \cap B \neq \emptyset\}$$

the lower inverse set of B by F. We also define

$$F^{+}(B) := \{x \in X : F(x) \subset B\}$$

the upper inverse set of B by F.

Recall that the set-valued mapping F is said to be lower semicontinuous at a point $x \in X$ if whenever V is an open subset of Y such that $F(x) \cap V \neq \emptyset$, the lower inverse set $F^{-}(V)$ of V by F is a neighborhood of x.

The set-valued mapping F is said to be upper semicontinuous at a point $x \in X$ if whenever V is an open subset of Y such that $F(x) \subset V$, the upper inverse set $F^{+}(V)$ of V by F is a neighborhood of x.

Following [2], the set-valued mapping F is lower semicontinuous (*resp*. upper semicontinuous) on a subset S of X if it is lower semicontinuous (*resp*. upper semicontinuous) at every point of S. In particular, it is proved that a set-valued mapping $F : X \rightrightarrows Y$ is upper semicontinuous on a subset S of X if and only if for every closed subset B of Y, $F^{-}(B) \cap S = \mathrm{cl}\left(F^{-}(B)\right) \cap S$; see [2, Proposition 2.4.].

As for convexity and concavity, the notions of lower and upper semicontinuity of set-valued mappings as defined above are not limited to extended real set-valued mappings, and therefore, they may be too stronger than the lower and upper semicontinuity of extended real single-valued mappings. Here, we develop weaker notions of lower and upper semicontinuity for extended real set-valued mappings, which generalize those for both set-valued mappings and extended real single-valued mappings.

Let X be a Hausdorff topological space and $F : X \rightrightarrows \overline{\mathbb{R}}$ an extended real set-valued mapping. First, we derive two definitions from lower semicontinuity of set-valued mappings.

We say that F is *l-lower semicontinuous* at $x \in X$ if for every $\lambda \in \mathbb{R}$ such that $F(x) \cap]\lambda, +\infty] \neq \emptyset$, there exists an open neighborhood U of x such that $F\left(x'\right) \cap]\lambda, +\infty] \neq \emptyset$, for every $x' \in U$.

Clearly, the notion of l-lower semicontinuous generalizes both lower semicontinuity of extended real single-valued mappings and lower semicontinuity of set-valued mappings.

We say that F is l-lower semicontinuous on a subset S of X if F is l-lower semicontinuous at every point of S.

Proposition 2.4 *Let X be a Hausdorff topological space, S a subset of X and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Then, F is l-lower semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{+}([-\infty, \lambda]) \cap S = \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right) \cap S.$$

Proof Assume that F is l-lower semicontinuous on S and let $\lambda \in \mathbb{R}$. Let $x \in \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right) \cap S$. If $x \notin F^{+}([-\infty, \lambda])$, then $F(x) \cap]\lambda, +\infty] \neq \emptyset$. Since $x \in S$, then there exists an open neighborhood U of x such that $F\left(x'\right) \cap]\lambda, +\infty[\neq \emptyset$, for every $x' \in U$. It follows that $U \cap F^{+}([-\infty, \lambda]) = \emptyset$, which contradicts the fact that $x \in \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right)$.

Conversely, let $x \in S$ and $\lambda \in \mathbb{R}$ be such that $F(x) \cap]\lambda, +\infty] \neq \emptyset$. Then $x \notin F^{+}([-\infty, \lambda])$, and therefore, $x \notin \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right)$. Put $U = X \setminus \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right)$, which is an open neighborhood of x. For every $x' \in U$, we have $x' \notin \mathrm{cl}\left(F^{+}([-\infty, \lambda])\right)$ and then, $x' \notin F^{+}([-\infty, \lambda])$. We conclude that $F\left(x'\right) \cap]\lambda, +\infty] \neq \emptyset$, for every $x' \in U$. $\square$

Example 2.5 Consider the extended real set-valued mapping $F : \mathbb{R} \rightrightarrows \overline{\mathbb{R}}$ defined by

$$F(x) := \begin{cases} [0, +\infty], & \text{if } x = 0, \\ \left[\frac{1}{|x|}, +\infty\right], & \text{otherwise.} \end{cases}$$

Clearly, F is l-lower semicontinuous on $\mathbb{R}$. However, F is not lower semicontinuous at 0. Indeed, take $V =]a, b[$, $a, b \in \mathbb{R}_{+}$ and $a < b$. We have $F(0) \cap V \neq \emptyset$, but any open neighborhood of 0 contains a small enough point x such that $\frac{1}{|x|} > b$.

We say that F is *l-upper semicontinuous* at $x \in X$, if for every $\lambda \in \mathbb{R}$ such that $F(x) \cap [-\infty, \lambda[\neq \emptyset$, there exists an open neighborhood U of x such that $F\left(x'\right) \cap [-\infty, \lambda[\neq \emptyset$, for every $x' \in U$.

Clearly, the notion of l-upper semicontinuous generalizes both upper semicontinuity of extended real single-valued mappings and lower semicontinuity of set-valued mappings. Also, an extended real set-valued mapping F is l-lower semicontinuous at $x \in X$ if and only if $-F$ is l-upper semicontinuous at x.

We say that F is l-upper semicontinuous on a subset S of X if F is l-upper semicontinuous at every point of S.

By a similar proof to that of Proposition 2.4, we obtain the following result for l-upper semicontinuous set-valued mappings.

Proposition 2.5 *Let X be a Hausdorff topological space, S a subset of X and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Then, F is l-upper semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{+}([\lambda, +\infty]) \cap S = \mathrm{cl}\left(F^{+}([\lambda, +\infty])\right) \cap S.$$

Example 2.6 Consider the extended real set-valued mapping $F : \mathbb{R} \rightrightarrows \overline{\mathbb{R}}$ defined by

$$F(x) := \begin{cases} [-\infty, 0], & \text{if } x = 0, \\ \left[\infty, -\frac{1}{|x|}\right], & \text{otherwise.} \end{cases}$$

Clearly F is l-upper semicontinuous on $\mathbb{R}$, but it is not lower semicontinuous at 0.

Now, we derive two other definitions from upper semicontinuity of set-valued mappings.

We say that F is *u-lower semicontinuous* at $x \in X$ if for every $\lambda \in \mathbb{R}$ such that $F(x) \subset]\lambda, +\infty]$, there exists an open neighborhood U of x such that $F\left(x'\right) \subset]\lambda, +\infty]$, for every $x' \in U$.

Clearly, the notion of u-lower semicontinuous generalizes both lower semicontinuity of extended real single-valued mappings and upper semicontinuity of set-valued mappings.

We say that F is u-lower semicontinuous on a subset S of X if F is u-lower semicontinuous at every point of S.

Proposition 2.6 *Let X be a Hausdorff topological space, S a subset of X and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Then, F is u-lower semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{-}([-\infty, \lambda]) \cap S = \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right) \cap S.$$

Proof Assume that F is u-lower semicontinuous on S and let $\lambda \in \mathbb{R}$. Let $x \in \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right) \cap S$. If $x \notin F^{-}([-\infty, \lambda])$, then $F(x) \subset]\lambda, +\infty]$. Since $x \in S$, then there exists an open neighborhood U of x such that $F\left(x'\right) \subset]\lambda, +\infty[$, for every $x' \in U$. It follows that $U \cap F^{-}([-\infty, \lambda]) = \emptyset$, which contradicts the fact that $x \in \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right)$.

Conversely, let $x \in S$ and $\lambda \in \mathbb{R}$ be such that $F(x) \subset]\lambda, +\infty]$. Then $x \notin F^{-}([-\infty, \lambda])$, and therefore, we have $x \notin \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right)$. Put $U = X \setminus \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right)$, which is an open neighborhood of x. For every $x' \in U$, we have $x' \notin \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right)$ and then, $x' \notin F^{-}([-\infty, \lambda])$. We conclude that $F\left(x'\right) \subset]\lambda, +\infty]$, for every $x' \in U$. □

Example 2.7 Consider the extended real set-valued mapping $F : \mathbb{R} \rightrightarrows \overline{\mathbb{R}}$ defined by

$$F(x) := \begin{cases} \{0\}, & \text{if } x = 0, \\ \left[\frac{1}{|x|}, +\infty\right], & \text{otherwise.} \end{cases}$$

Clearly, F is u-lower semicontinuous on $\mathbb{R}$. However, F is not upper semicontinuous at 0. Indeed, take $V =]a, b[$, $a \in \mathbb{R}_-$ and $b \in \mathbb{R}_+$. We have $F(0) \subset V$, but any open neighborhood of 0 contains a small enough point x such that $\frac{1}{|x|} > b$.

We say that F is *u-upper semicontinuous* at $x \in X$ if for every $\lambda \in \mathbb{R}$ such that $F(x) \subset [-\infty, \lambda[$, there exists an open neighborhood U of x such that $F\left(x'\right) \subset [-\infty, \lambda[$, for every $x' \in U$.

Clearly, the notion of u-upper semicontinuous generalizes both upper semicontinuity of extended real single-valued mappings and upper semicontinuity of set-valued mappings. Also, an extended real set-valued mapping F is u-lower semicontinuous at $x \in X$ if and only if $-F$ is u-upper semicontinuous at x.

We say that F is u-upper semicontinuous on a subset S of X if F is u-upper semicontinuous at every point of S.

By a similar proof to that of Proposition 2.6, we obtain the following result for u-upper semicontinuous set-valued mappings.

Proposition 2.7 *Let X be a Hausdorff topological space, S a subset of X and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Then, F is u-upper semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^-([\lambda, +\infty]) \cap S = \mathrm{cl}\left(F^-([\lambda, +\infty])\right) \cap S.$$

Example 2.8 Consider the extended real set-valued mapping $F : \mathbb{R} \rightrightarrows \overline{\mathbb{R}}$ defined by

$$F(x) := \begin{cases} \{0\}, & \text{if } x = 0, \\ \left[-\infty, -\frac{1}{|x|}\right], & \text{otherwise.} \end{cases}$$

Clearly, F is u-upper semicontinuous on $\mathbb{R}$, but it is not upper semicontinuous at 0.

As a summary, we have the following characterizations for extended real single-valued mappings.

Proposition 2.8 *Let X be a Hausdorff topological space and $x_0 \in X$. For an extended real single-valued mapping $f : X \to \overline{\mathbb{R}}$, the following conditions are equivalent:*

1. *f is lower semicontinuous at x_0,*
2. *f is l-lower semicontinuous at x_0,*
3. *f is u-lower semicontinuous at x_0.*

Proposition 2.9 *Let X be a Hausdorff topological space and $x_0 \in X$. For an extended real single-valued mapping $f : X \to \overline{\mathbb{R}}$, the following conditions are equivalent:*

1. *f is upper semicontinuous at x_0,*
2. *f is l-upper semicontinuous at x_0,*
3. *f is u-upper semicontinuous at x_0.*

3 Existence of Solutions of Set-Valued Equilibrium Problems

In this section, we deal with the existence of solutions of both strong set-valued equilibrium problems and weak set-valued equilibrium problems. By using our notions of convexity and continuity introduced in the paper, these results are more general than both those obtained in [2] for set-valued equilibrium problems and most of the corresponding results in the literature for single-valued equilibrium problems, including our own results involving continuity on the set of coerciveness; see, for example, [9–12].

We need in the sequel the notion of KKM mappings and the well-known intersection lemma due to Ky Fan; see [13].

Let X be a real topological Hausdorff vector space and M a subset of X. Recall that a set-valued mapping $F : M \rightrightarrows X$ is said to be a KKM mapping if for every finite subset $\{x_1, \ldots, x_n\}$ of M, we have

$$\operatorname{conv}\{x_1, \ldots x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

It is well known by Ky Fan's lemma [13] that if

1. F is a KKM mapping,
2. $F(x)$ is closed for every $x \in M$ and
3. there exists $x_0 \in M$ such that $F(x_0)$ is compact,

then $\bigcap_{x\in M} F(x) \neq \emptyset$.

We define the following set-valued mappings $\Phi^+, \Phi^{++} : C \rightrightarrows C$ by

$$\Phi^+(y) := \{x \in C : \Phi(x, y) \cap \mathbb{R}_+ \neq \emptyset\} \quad \forall y \in C.$$

and

$$\Phi^{++}(y) := \{x \in C : \Phi(x, y) \subset \mathbb{R}_+\} \quad \forall y \in C.$$

We remark that $\Phi^{++}(y) \subset \Phi^+(y)$, for every $y \in C$ and

1. $x_0 \in C$ is a solution of the set-valued equilibrium problem (Wsvep) if and only if $x_0 \in \bigcap_{y\in C} \Phi^+(y)$,
2. $x_0 \in C$ is a solution of the set-valued equilibrium problem (Ssvep) if and only if $x_0 \in \bigcap_{y\in C} \Phi^{++}(y)$.

In the sequel, we set

$$\operatorname{cl}\Phi^+(y) = \operatorname{cl}\left(\Phi^+(y)\right) \quad \text{and} \quad \operatorname{cl}\Phi^{++}(y) = \operatorname{cl}\left(\Phi^{++}(y)\right),$$

the closure of $\Phi^+(y)$ and $\Phi^{++}(y)$, respectively, for every $y \in C$.

Lemma 3.1 *Let C be a nonempty and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

1. *$\Phi(x,x) \subset \mathbb{R}_+$, for every $x \in C$;*
2. *Φ is convexly quasi-convex in its second variable on C.*

Then, the set-valued mappings $\operatorname{cl}\Phi^{++} : C \rightrightarrows C$ *and* $\operatorname{cl}\Phi^{+} : C \rightrightarrows C$ *are KKM mappings.*

Proof It suffices to prove that the set-valued mapping $\Phi^{++} : C \rightrightarrows C$ is a KKM mapping. Let $\{y_1, \ldots, y_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ be such that $\sum_{i=1}^n \lambda_i = 1$. Put $\tilde{y} = \sum_{i=1}^n \lambda_i y_i$. By assumption (2), for $\{z_1, \ldots, z_n\}$ with $z_i \in \Phi(\tilde{y}, y_i)$ for every $i = 1, \ldots, n$, there exists $z \in \Phi(\tilde{y}, \tilde{y})$ such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

We have $z \geq 0$ since $\Phi(\tilde{y}, \tilde{y}) \subset \mathbb{R}_+$ by assumption (1). It follows that there exists $i_0 \in \{1, \ldots, n\}$ such that $\Phi\left(\tilde{y}, y_{i_0}\right) \cap \mathbb{R}_-^* = \emptyset$, which implies that $\Phi\left(\tilde{y}, y_{i_0}\right) \subset \mathbb{R}_+$. Otherwise, all the z_i can be taken in $\mathbb{R}_-^*$, and therefore, $z \in \mathbb{R}_-^*$, which is impossible. We conclude that

$$\sum_{i=1}^n \lambda_i y_i = \tilde{y} \in \Phi^{++}\left(y_{i_0}\right) \subset \bigcup_{i=1}^n \Phi^{++}(y_i),$$

which proves that Φ^{++} is a KKM mapping. □

The following result generalizes both [2, Theorem 3.1] obtained for set-valued equilibrium problems when the self-segment-dense set D is equal to C and [12, Theorem 3.1] for single-valued equilibrium problems.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

1. *$\Phi(x,x) \subset \mathbb{R}_+$, for every $x \in C$;*
2. *Φ is convexly quasi-convex in its second variable on C;*
3. *there exist a compact set K of C and a point $y_0 \in K$ such that $\Phi(x, y_0) \cap \mathbb{R}_-^* \neq \emptyset$, for every $x \in C \setminus K$;*
4. *Φ is l-upper semicontinuous in its first variable on K.*

Then, the set of solutions of the set-valued equilibrium problem (Ssvep) *is a nonempty and compact set.*

Proof Assumption (1) yields $\Phi^{++}(y)$ is nonempty, for every $y \in C$. Clearly, $\operatorname{cl}\Phi^{++}(y)$ is closed for every $y \in C$, and $\operatorname{cl}\Phi^{++}(y_0)$ is compact since it lies in K by assumption (3).

The set-valued mapping $\operatorname{cl}\Phi^{++} : C \rightrightarrows C$ is a KKM mapping by Lemma 3.1. Then, by using Ky Fan lemma, we have

$$\bigcap_{y \in C} \operatorname{cl}\Phi^{++}(y) \neq \emptyset.$$

Since the subset $\Phi^{++}(y_0)$ is contained in the compact K, then

$$\bigcap_{y\in C} \Phi^{++}(y) = \bigcap_{y\in C} \left(\Phi^{++}(y) \cap K\right).$$

and

$$\bigcap_{y\in C} \operatorname{cl} \Phi^{++}(y) = \bigcap_{y\in C} \left(\operatorname{cl} \Phi^{++}(y) \cap K\right).$$

We remark that for all $y \in C$, $\Phi^{++}(y)$ is the upper inverse set $\Phi^{+}([0, +\infty[, y)$ of $[0, +\infty[$ by the set-valued mapping $\Phi(., y)$ which is l-upper semicontinuous on K. Then, by Proposition 2.5, we have $\operatorname{cl} \Phi^{++}(y) \cap K = \Phi^{++}(y) \cap K$. Therefore,

$$\bigcap_{y\in C} \operatorname{cl} \Phi^{++}(y) = \bigcap_{y\in C} \left(\operatorname{cl} \Phi^{++}(y) \cap K\right) = \bigcap_{y\in C} \left(\Phi^{++}(y) \cap K\right) = \bigcap_{y\in C} \Phi^{++}(y).$$

It follows that the set of solutions of the strong set-valued equilibrium problem (Ssvep) is nonempty and compact since it is closed and contained in the compact set K. □

Now, we turn to weak set-valued equilibrium problems. First, we obtain the following result which also generalizes [12, Theorem 3.1] for single-valued equilibrium problems by using u-upper semicontinuity which is derived from upper semicontinuity of set-valued mappings rather than l-upper semicontinuity in Theorem 3.1 which is derived from lower semicontinuity.

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

1. *$\Phi(x, x) \subset \mathbb{R}_+$, for every $x \in C$;*
2. *Φ is convexly quasi-convex in its second variable on C;*
3. *there exist a compact set K of C and a point $y_0 \in K$ such that $\Phi(x, y_0) \subset \mathbb{R}_-^*$, for every $x \in C \setminus K$;*
4. *Φ is u-upper semicontinuous in its first variable on K.*

Then, the set of solutions of the set-valued equilibrium problem (Wsvep) is a nonempty and compact set.

Proof Assumption (1) yields that $\Phi^{+}(y)$ is nonempty, for every $y \in C$. Clearly, $\operatorname{cl} \Phi^{+}(y)$ is closed for every $y \in C$, and $\operatorname{cl} \Phi^{+}(y_0)$ is compact since it lies in K, by assumption (3).

The set-valued mapping $\operatorname{cl} \Phi^{+} : C \rightrightarrows C$ is a KKM mapping by Lemma 3.1. Then, by the Ky Fan lemma, we have

$$\bigcap_{y\in C} \operatorname{cl} \Phi^{+}(y) \neq \emptyset.$$

Since the subset $\Phi^{+}(y_0)$ is contained in the compact K, then

$$\bigcap_{y\in C} \Phi^{+}(y) = \bigcap_{y\in C} \left(\Phi^{+}(y) \cap K\right).$$

and

$$\bigcap_{y\in C} \operatorname{cl} \Phi^{+}(y) = \bigcap_{y\in C} \left(\operatorname{cl} \Phi^{+}(y) \cap K\right).$$

We remark that for all $y \in C$, $\Phi^{+}(y)$ is the lower inverse set $\Phi^{-}([0,+\infty[, y)$ of $[0,+\infty[$ by the set-valued mapping $\Phi(., y)$ which is u-upper semicontinuous on K. Then, by Proposition 2.7, we have $\operatorname{cl} \Phi^{+}(y) \cap K = \Phi^{+}(y) \cap K$. Therefore,

$$\bigcap_{y\in C} \operatorname{cl} \Phi^{+}(y) = \bigcap_{y\in C} \left(\operatorname{cl} \Phi^{+}(y) \cap K\right) = \bigcap_{y\in C} \left(\Phi^{+}(y) \cap K\right) = \bigcap_{y\in C} \Phi^{+}(y).$$

It follows that the set of solutions of the set-valued equilibrium problem (Wsvep) is nonempty and compact, since it is closed and contained in the compact set K. □

In many applications, the set-valued Φ is concave in its second variable. We give here the following result for concavely quasi-convex set-valued mappings.

Lemma 3.2 *Let C be a nonempty convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

1. *$\Phi(x, x) \cap \mathbb{R}_{+} \neq \emptyset$, for every $x \in C$;*
2. *Φ is concavely quasi-convex in its second variable on C.*

Then, the set-valued mapping $\operatorname{cl} \Phi^{+} : C \rightrightarrows C$ is a KKM mapping.

Proof Let $\{y_1, \ldots, y_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_{+}$ be such that $\sum_{i=1}^{n} \lambda_i = 1$. Put $\tilde{y} = \sum_{i=1}^{n} \lambda_i y_i$. By assumption (2), for $z \in \Phi(\tilde{y}, \tilde{y})$, there exist $\{z_1, \ldots, z_n\}$ with $z_i \in \Phi(\tilde{y}, y_i)$ for every $i = 1, \ldots, n$, such that

$$z \leq \max\{z_i : i = 1, \ldots, n\}.$$

It follows that there exists $i_0 \in \{1, \ldots, n\}$ such that $\Phi\left(\tilde{y}, y_{i_0}\right) \cap \mathbb{R}_{+} \neq \emptyset$. Otherwise, all the z_i are in $\mathbb{R}_{-}^{*}$, and therefore, $z \in \mathbb{R}_{-}^{*}$. Since $z \in \Phi(\tilde{y}, \tilde{y})$, then $\Phi(\tilde{y}, \tilde{y}) \subset \mathbb{R}_{-}^{*}$, which yields a contradiction since $\Phi(\tilde{y}, \tilde{y}) \cap \mathbb{R}_{+} \neq \emptyset$ by assumption (1). We conclude that

$$\sum_{i=1}^{n} \lambda_i y_i = \tilde{y} \in \Phi^{+}\left(y_{i_0}\right) \subset \bigcup_{i=1}^{n} \Phi^{+}(y_i),$$

which proves that $\operatorname{cl} \Phi^{+}$ is a KKM mapping. □

The following result generalizes both [2, Theorem 3.2] obtained for set-valued equilibrium problems when the self-segment-dense set D is equal to C, and [2, Theorem 3.1] for single-valued equilibrium problems. Here, we remark that the conclusion is the same as in Theorem 3.2 for convexly quasi-convex set-valued mappings, but with weaker first condition.

Theorem 3.3 *Let C be a nonempty, closed and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R}$ be a set-valued mapping, and assume that the following conditions hold:*

1. *$\Phi(x, x) \cap \mathbb{R}_+ \neq \emptyset$, for every $x \in C$;*
2. *Φ is concavely quasi-convex in its second variable on C;*
3. *there exist a compact set K of C and a point $y_0 \in K$ such that $\Phi(x, y_0) \subset \mathbb{R}^*_-$, for every $x \in C \setminus K$;*
4. *Φ is u-upper semicontinuous in its first variable on K.*

Then, the set of solutions of the set-valued equilibrium problem (Wsvep) *is a nonempty and compact set.*

Proof By using Lemma 3.2 instead of Lemma 3.1, the proof follows step by step that of Theorem 3.2. □

4 Browder Variational Inclusions

In this section, we deal with Browder variational inclusions under mild conditions on the involved data. Browder variational inclusions appear as a generalization of Browder–Hartman–Stampacchia variational inequalities and have many applications, including applications to nonlinear elliptic boundary value problems and the surjectivity of set-valued mappings; see, for example, [5,14] and the references therein.

Let C be a nonempty, closed and convex subset of a real normed vector space X. In the literature, a notion of coerciveness for set-valued operators exists and generalizes that for linear operators and bilinear forms on Hilbert spaces. A set-valued operator $F : C \rightrightarrows X^*$ is said to be coercive on C if there exists $y_0 \in C$ such that

$$\lim_{\substack{\|x\| \to +\infty \\ x \in C}} \frac{\inf_{x^* \in F(x)} \langle x^*, x - y_0 \rangle}{\|x\|} = +\infty.$$

It is not hard to see that if F is coercive on C, then there exists $R > 0$ such that $y_0 \in K_R$ and $\langle x^*, y_0 - x \rangle \subset \mathbb{R}^*_-$, for every $x \in C \setminus K_R$ and every $x^* \in F(x)$, where $K_R = \{x \in C : \|x\| \le R\}$. Clearly, K_R is weakly compact whenever X is reflexive. The set K_R (which may not be unique) is called a set of coerciveness. In what follows, we will need a compact set of coerciveness. Unfortunately, closed balls in X are not compacts except if X is finite dimensional space.

In the sequel, for $x \in X$ and a subset A of X^*, we set

$$\langle A, x \rangle = \left\{ \langle x^*, x \rangle : x^* \in A \right\}.$$

Problems of the form: "find $x_0 \in C$ such that $\langle A, x_0 \rangle \subset \mathbb{R}_+$" or "find $x_0 \in C$ such that $\langle A, x_0 \rangle \cap \mathbb{R}_+ \neq \emptyset$" are called Browder variational inclusions.

In order to generalize and make more precise [2, Theorem 4.1], we note that the condition of either X is a Banach space or the set-valued operator F has convex values on the set of coerciveness is required. This is because in the proof, we need that these values are norm bounded. It is well known that a weak* compact set S of X^* is norm bounded whenever X is a Banach space or S is convex; see, for example, [15].

In our next result, we will never need the weak* compactness of the values of the set-valued operator. We will say that a subset S of X^* *attains its pairwise upper bounds* on a subset A if for every $z \in A$, the set $\{\langle x^*, z \rangle : x^* \in S\}$ has a maximum in $\mathbb{R}$. We can easily see that if a subset S of X^* is weak* compact, then the set $\{\langle x^*, z \rangle : x^* \in S\}$ is compact, and therefore, it attains its minimum and a maximum, for every $z \in X$.

By using our notions of semicontinuity and convexity of real set-valued mappings, we obtain the following result which generalizes [2, Theorem 4.1] and make it more precise. Here, Theorem 3.3 will be used because the constructed real set-valued bifunction in the proof is concave in its second variable.

Theorem 4.1 *Let X be a real normed vector space, C a nonempty, closed and convex subset of X. Suppose that $F : C \rightrightarrows X^*$ has the following conditions:*

1. *there exist a compact subset K of C and $y_0 \in K$ such that $\langle F(x), y_0 - x \rangle \subset \mathbb{R}^*_-$, for every $x \in C \setminus K$;*
2. *F is upper semicontinuous on K;*
3. *for every $x \in K$, $F(x)$ is norm bounded and attains its pairwise upper bounds on $C - x$.*

Then, there exists $\overline{x} \in K$ such that $\langle F(\overline{x}), y - \overline{x} \rangle \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$.

Proof Define the set-valued mapping $\Phi : C \times C \rightrightarrows \mathbb{R}$ by

$$\Phi(x, y) := \langle F(x), y - x \rangle.$$

We will show that all the conditions of Theorem 3.3 are satisfied. We remark that Φ is concave in its second variable, and then it is concavely quasi-convex in its second variable. Except the last condition, all the other conditions hold easily from our assumptions.

To prove that Φ is u-upper semicontinuous in its first variable on K, fix $y \in C$, and let $x \in K$ and $\lambda \in \mathbb{R}$ be such that $\Phi(x, y) \subset]-\infty, \lambda[$. That is, $\langle F(x), y - x \rangle \subset]-\infty, \lambda[$. Put $\lambda_x := \max(\langle F(x), y - x \rangle) < \lambda$, $\delta := \frac{\lambda - \lambda_x}{2}$ and

$$\delta_1 := \min\left\{\frac{\delta}{3(\|x\| + 1)}, \frac{\delta}{3(\|y\| + 1)}\right\},$$

where $\|.\|$ denotes the norm of X. Put $O := \bigcup_{x^* \in F(x)} B_{X^*}(x^*, \delta_1)$, where $B_{X^*}(x^*, \delta_1) = \{z \in X^* : \|z - x^*\|_* < \delta_1\}$, and $\|.\|_*$ denotes the norm of X^*. Clearly, $F(x)$ is contained in the open set O, and by the upper semicontinuity of F on K, let $\eta > 0$ be such that $F(w) \subset O$ for every $w \in B_X(x, \eta) \cap C$, where

$B_X(x,\eta) = \{w \in X \colon \|w - x\| < \eta\}$. Since $F(x)$ is norm bounded, put $\|F(x)\|_* := \sup\{\|x^*\|_* \colon x^* \in F(x)\}$ which is in $\mathbb{R}$. Finally, put

$$\eta_1 := \min\left\{\frac{\delta}{3(\|F(x)\|_* + 1)}, \eta, 1\right\},$$

and $U = B_X(x, \eta_1) \cap C$ which is an open subset of C containing x.

We will show that $\Phi(z, y) \subset]-\infty, \lambda[$, for every $z \in U$. To do this, let $z \in U$ and $z^* \in F(z)$. Let $x_0^* \in F(x)$ be such that $z^* \in B_{X^*}\left(x_0^*, \delta_1\right)$. We have

$$\begin{aligned}
\left|\langle z^*, y - z\rangle - \langle x_0^*, y - x\rangle\right| &= \left|\langle x_0^* - z^*, z\rangle + \langle x_0^*, x - z\rangle - \langle x_0^* - z^*, y\rangle\right| \\
&\le \|x_0^* - z^*\|_*\|z\| + \|x_0^*\|_*\|x - z\| + \|x_0^* - z^*\|_*\|y\| \\
&< \frac{\delta(\|x\| + \eta_1)}{3(\|x\| + 1)} + \frac{\delta\|x_0^*\|_*}{3(\|F(x)\|_* + 1)} + \frac{\delta\|y\|}{3(\|y\| + 1)} \\
&< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned}$$

It follows that $\langle z^*, y - z\rangle < \langle x_0^*, y - x\rangle + \delta \le \lambda_x + \delta = \frac{\lambda+\lambda_x}{2} < \lambda$. Since z is arbitrary in U and z^* is arbitrary in $F(z)$, then $\Phi(z, y) \subset]-\infty, \lambda[$, for every $z \in U$. That is, Φ is u-upper semicontinuous in its first variable on K. $\square$

It is well known that beside the semicontinuity of set-valued mappings, there are, in the literature, different notions of demicontinuous set-valued operators which may be equivalent under suitable conditions. Now, we obtain the following results on the existence of solutions of Browder variational inclusions under assumptions of demicontinuity.

Recall that an open half-space in a real Hausdorff topological vector space E is a subset of the form

$$\{u \in E \colon \varphi(u) < r\}$$

for some continuous linear functional φ on E, not identically zero, and for some real number r.

Let X be a Hausdorff topological space and E a real Hausdorff topological vector space. Following [16], a set-valued operator $F : X \rightrightarrows E$ is said to be *upper demicontinuous at* $x \in X$ if for every open half-space H containing $F(x)$, there exists a neighborhood U of x such that $F\left(x'\right) \subset H$ for all $x' \in U$. It said to be upper demicontinuous on X if it is upper demicontinuous at every point of X. We say that F is upper demicontinuous on a subset S of X if it is upper demicontinuous at every point of S.

By applying a similar proof to that in Theorem 4.1 above and [2, Theorem 4.1], we obtain the following result.

Proposition 4.1 *Let X be a real normed vector space, C a nonempty, closed and convex subset of X and $S \subset C$. Suppose that $F : C \rightrightarrows X^*$ has the following conditions:*

1. *F is upper semicontinuous on S to X^* endowed with the weak* topology;*
2. *F has weak* compact values on S.*

Then, F is upper demicontinuous on S to X^* endowed with the weak topology.*

Now, by using the notion of demicontinuous set-valued operators, we obtain the following result on the existence of solutions of Browder variational inclusions.

Theorem 4.2 *Let X be a real normed vector space, C a nonempty, closed and convex subset of X. Suppose that $F : C \rightrightarrows X^*$ has the following conditions:*

1. *there exist a compact subset K of C and $y_0 \in C$ such that $\langle F(x), y - x\rangle \subset \mathbb{R}^*_-$, for every $x \in C \setminus K$;*
2. *F is upper demicontinuous on K to X^* endowed with the weak* topology.*

Then, there exists $\overline{x} \in K$ such that $\langle F(\overline{x}), y - \overline{x}\rangle \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$.

Proof Define the set-valued mapping $\Phi : C \times C \rightrightarrows \mathbb{R}$ by $\Phi(x, y) := \langle F(x), y - x\rangle$. It remains just to prove that Φ is u-upper semicontinuous in its first variable on K. Fix $y \in C$, and let $x \in K$ and $\lambda \in \mathbb{R}$ be such that $\Phi(x, y) \subset]-\infty, \lambda[$. That is, $\langle F(x), y - x\rangle \subset]-\infty, \lambda[$. Consider φ defined on X^* by $\varphi(u) := \langle u, y - x\rangle$, for very $u \in X^*$. Clearly, φ is not identically zero linear functional on X^* which is weak* continuous. It follows that $F(x)$ is in the open half-space $H = \{u \in X^* : \varphi(u) < \lambda\}$ in X^*. Since F is upper demicontinuous on K, let U be an open neighborhood of x such that $F(x') \subset H$, for every $x' \in U$. That is, $\Phi(x', y) \subset]-\infty, \lambda[$ for every $x' \in U$, which proves that Φ is u-upper semicontinuous in its first variable on K. □

5 Perspectives

Our investigations in this paper have led us to carry out some notions and obtain different results which seem to be important, not only for set-valued equilibrium problems, but for other areas of mathematics.

Firstly, the concepts of convexity and semicontinuity for (extended) real set-valued mappings introduced in the paper and involving half intervals rather than open sets are similar to those for single-valued mappings but produce results on set-valued equilibrium problems generalizing those in the literature on single-valued and set-valued equilibrium problems. We really think that these concepts will be also employed in other problems involving (extended) real set-valued mappings.

Secondly, we have studied Browder variational inclusions in the realm of real normed vector spaces and weakened the conditions on the involved set-valued operator. When looking at the proof of Theorem 4.1 involving a upper semicontinuous set-valued operator F, we remark that we construct by open balls an open set O containing $F(x)$ in order to apply the upper semicontinuity of F. We wonder whether it is possible or not to find a definition of lower and upper semicontinuity of set-valued operators involving half balls, like that for real set-valued mapping introduced in the paper, in such a way that we obtain the same conclusion. On the other hand, our results on Browder variational inclusions involving demicontinuous set-valued operators provide us with an other way to further investigations and may be improved in the future.

Finally, we point out that the constructed set-valued bifunction Φ in Theorems 4.1 and 4.2 is concave in its second variable. Applications where the constructed set-valued bifunction Φ is convex in its second variable draw especially our attention. The important case involving pseudo-monotone operators represents a challenge to our future investigations.

6 Conclusions

In this paper, we have considered and studied set-valued equilibrium problems. Intrigued by the level of the existing results on set-valued equilibrium problems, we have decided to investigate the special case of (extended) real set-valued mappings and brought to light various concepts of convexity and semicontinuity involving half intervals rather than open sets. We have applied our new notions and obtained different results on the existence of solutions of set-valued equilibrium problems generalizing both those for set-valued equilibrium problems and those for single-valued equilibrium problems. Two results for set-valued equilibrium problems involving a convex set-valued bifunction in the second variable and a result for set-valued equilibrium problems involving a concave set-valued bifunction in the second variable are obtained. As applications, we have considered Browder variational inclusions in the realm of real normed vector spaces and weakened different conditions including the weak* compactness of the involved set-valued operator. Results on the existence of solutions of Browder variational inclusions involving upper semicontinuous and upper demicontinuous set-valued operators are obtained.

Acknowledgements V.D. Rădulescu acknowledges the support through a grant of the Ministry of Research and Innovation, CNCS–UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130, within PNCDI III.

References

1. Alleche, B.: Multivalued mixed variational inequalities with locally Lipschitzian and locally cocoercive multivalued mappings. J. Math. Anal. Appl. **399**, 625–637 (2013)
2. Alleche, B., Rădulescu, V.: Set-valued equilibrium problems with applications to Browder variational inclusions and to fixed point theory. Nonlinear Anal. Real World Appl. **28**, 251–268 (2016)
3. Kristály, A., Varga, C.: Set-valued versions of Ky Fan's inequality with application to variational inclusion theory. J. Math. Anal. Appl. **282**, 8–20 (2003)
4. Lásló, S., Viorel, A.: Densely defined equilibrium problems. J. Optim. Theory Appl. **166**, 52–75 (2015)
5. Shih, M.H., Tan, K.K.: Browder–Hartman–Stampacchia variational inequalities for multi-valued monotone operators. J. Math. Anal. Appl. **134**, 431–440 (1988)
6. Lásló, S., Viorel, A.: Generalized monotone operators on dense sets. Numer. Funct. Anal. Optim. **36**(7), 901–929 (2015)
7. Cambini, A., Martein, L.: Generalized Convexity and Optimization. Theory and Applications. Lecture Notes in Economics and Mathematical Systems, vol. 616. Springer, Berlin (2009)
8. Papageorgiou, N.S., Kyritsi-Yiallourou, S.T.H.: Handbook of Applied Analysis. Advances in Mechanics and Mathematics, vol. 19. Springer, Dordrecht (2009)
9. Alleche, B.: Semicontinuity of bifunctions and applications to regularization methods for equilibrium problems. Afr. Mat. **26**(7), 1637–1649 (2015)
10. Alleche, B., Rădulescu, V.: Equilibrium problem techniques in the qualitative analysis of quasi-hemivariational inequalities. Optimization **64**(9), 1855–1868 (2015)

11. Alleche, B., Rădulescu, V.: The Ekeland variational principle for equilibrium problems revisited and applications. Nonlinear Anal. Real World Appl. **23**, 17–25 (2015)
12. Alleche, B., Rădulescu, V.: Solutions and approximate solutions of quasi-equilibrium problems in Banach spaces. J. Optim. Theory Appl. **170**, 629–649 (2016)
13. Fan, K.: A generalization of Tychonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1961)
14. Browder, F.E.: The fixed point theory of multi-valued mappings in topological vector spaces. Math. Ann. **177**, 283–301 (1968)
15. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic, New York (2003)
16. Lee, C., Tan, K.K.: On Fan's extensions of Browder's fixed point theorems for multi-valued inward mappings. J. Aust. Math. Soc. **26**(2), 169–174 (1978)