DE GRUYTER Adv. Nonlinear Stud. 2021; 21(3): 501–521

Research Article

Chao Ji and Vicenţiu D. Rădulescu*

Multiplicity and Concentration of Solutions for Kirchhoff Equations with Magnetic Field

https://doi.org/10.1515/ans-2021-2130
Received December 4, 2020; revised March 26, 2021; accepted March 27, 2021

Abstract: In this paper, we study the following nonlinear magnetic Kirchhoff equation:

$$\begin{cases} -(a\epsilon^2 + b\epsilon[u]^2_{A/\epsilon})\Delta_{A/\epsilon}u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$, $a, b > 0$ are constants, $V : \mathbb{R}^3 \to \mathbb{R}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ are continuous potentials, and $\Delta_A u$ is the magnetic Laplace operator. Under a local assumption on the potential V, by combining variational methods, a penalization technique and the Ljusternik–Schnirelmann theory, we prove multiplicity properties of solutions and concentration phenomena for ϵ small. In this problem, the function f is only continuous, which allows to consider larger classes of nonlinearities in the reaction.

Keywords: Kirchhoff Equation, Magnetic Field, Concentration, Multiple Solutions, Variational Methods

MSC 2010: 35J20, 35J60, 58E05

Communicated by: Laurent Veron

1 Introduction and Main Results

This paper is devoted to the qualitative analysis of solutions for the nonlinear magnetic Kirchhoff equation in $\mathbb{R}^3$. We are concerned with the existence and multiplicity of solutions, as well as with concentration properties of solutions for small values of the positive parameter. A feature of this paper is that the reaction has weak regularity, which allows to consider larger classes of nonlinearities. The main result is described in the final part of this section.

In this paper, we study the following nonlinear magnetic Kirchhoff equation:

$$\begin{cases} -(a\epsilon^2 + b\epsilon[u]^2_{A/\epsilon})\Delta_{A/\epsilon}u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases} \tag{1.1}$$

where $\epsilon > 0$, $a, b > 0$ are constants, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, the magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ is Hölder continuous with exponent $\alpha \in (0, 1]$, and $-\Delta_A u$ is the magnetic Laplace operator of the follow-

***Corresponding author: Vicenţiu D. Rădulescu,** Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland; and Department of Mathematics, University of Craiova, 200585 Craiova, Romania; and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania, e-mail: radulescu@inf.ucv.ro. https://orcid.org/0000-0003-4615-5537
Chao Ji, Department of Mathematics, East China University of Science and Technology, Shanghai, 200237, P. R. China, e-mail: jichao@ecust.edu.cn

ing form:

$$-\Delta_A u := \Big(\frac{1}{i}\nabla - A(x)\Big)^2 u = -\Delta u - \frac{2}{i}A(x)\cdot\nabla u + |A(x)|^2 u - \frac{1}{i}u\,\mathrm{div}(A(x)).$$

The definition of $[u]^2_{A/\epsilon}$ will be given in Section 2.

For problem (1.1), there is a vast literature concerning the existence and multiplicity of bound state solutions for the case $A \equiv 0$ and $a = b = 0$. The first result in this direction was given by Floer and Weinstein in [8], where the case $N = 1$ and $f = i_{\mathbb{R}}$ is considered. Later on, several authors generalized this result to larger values of N, using different methods. For instance, He and Zou [10] considered the following fractional Schrödinger equation:

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + u^{2^*_s-1}, \quad x \in \mathbb{R}^N,$$

where V is a positive continuous function and satisfies the local assumption $\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x)$, and $f \in C$ is a function having subcritical and superlinear growth. By using the Nehari manifold method and the Ljusternik–Schnirelmann category theory, they obtained the multiplicity of positive solutions. We note that f is only continuous, and the Nehari manifold is only a topological manifold. He and Zou [10] applied the method that Szulkin and Weth developed in [20]. He and Zou [11] also studied multiplicity of concentrating solutions for a class of fractional Kirchhoff equations when the potential satisfies a local assumption and the nonlinear term f is only continuous. We also note that Ji, Fang and Zhang [12] considered a multiplicity result for asymptotically linear Kirchhoff equations. For further results about Kirchhoff equations, see [9, 19, 22, 23] and the references therein.

On the other hand, when $a = b = 0$, the magnetic nonlinear Schrödinger equation (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [1, 3, 4, 7, 15, 16, 18, 24, 25] and the references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [7]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\epsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we want to mention [1] where Alves, Figueiredo and Furtado used the method of the Nehari manifold, the penalization method and Ljusternik–Schnirelmann category theory for subcritical nonlinearity $f \in C^1$. We point out that if f is only continuous, then the arguments developed in [1] fail. In [13, 14], Ji and Rădulescu used the method of the Nehari manifold, the penalization method and Ljusternik–Schnirelmann category theory to study the multiplicity and concentration results for a magnetic Schrödinger equation in which the nonlinearity f is only continuous and subcritical and critical nonlinear terms, respectively. We also note the recent contribution [2] where Ambrosio studied multiplicity and concentration of solutions for a fractional Kirchhoff equation with magnetic field and critical growth.

Motivated by [11, 13], in the present paper, our main goal is to study multiplicity and concentration of nontrivial solutions for problem (1.1) only when f is continuous. Comparing with the result in [13], due to the presence of the nonlocal term, it is not clear to show the weak convergence of a bounded (PS) sequence of problem (1.1) is a solution of problem (1.1). Moreover, as we will see later, due to the presence of the magnetic field $A(x)$, problem (1.1) cannot be changed into a pure real-valued problem, and hence we should deal with a complex-valued problem directly, which causes several new difficulties in employing the methods in dealing with our problem. Our problem is more complicated than the pattern without magnetic field and we need additional technical estimates.

Throughout the paper, we make the following assumptions on the potential V:

(V1) There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$.

(V2) There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 = \min_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the following conditions:

(f1) $f(t) = 0$ if $t \leq 0$, and $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$.

(f2) There exists $q \in (4, 6)$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0.$$

(f3) There is a positive constant $\theta > 4$ such that

$$0 < \frac{\theta}{2}F(t) \leq tf(t) \quad \text{for all } t > 0, \text{ where } F(t) = \int_0^t f(s)\,ds.$$

(f4) $\frac{f(t)}{t}$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that V satisfies* (V1), (V2) *and f satisfies* (f1)–(f4). *Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\epsilon_\delta > 0$ such that, for any $0 < \epsilon < \epsilon_\delta$, problem (1.1) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *nontrivial solutions. Moreover, for every sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ϵ_n} one of these solutions of problem* (1.1) *for $\epsilon = \epsilon_n$ and if $\eta_{\epsilon_n} \in \mathbb{R}^3$ is the global maximum point of $|u_{\epsilon_n}|$, then*

$$\lim_{\epsilon_n \to 0^+} V(\eta_{\epsilon_n}) = V_0.$$

The paper is organized as follows. In Section 2, we introduce the functional setting and give some preliminaries. In Section 3, we study the modified problem and prove the Palais–Smale condition for the modified functional, and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the autonomous problem associated. It allows us to show that the modified problem has multiple solutions. Finally, in Section 5, we give the proof of Theorem 1.1.

Notation.

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line.
- $B_R(y)$ denotes the open ball centered at $y \in \mathbb{R}^3$ with radius $R > 0$, and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$.
- $\|\cdot\|$, $\|\cdot\|_q$ and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R})$, $L^q(\mathbb{R}^3, \mathbb{R})$ and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$.

2 Abstract Setting

In this section, we introduce the function spaces and some useful preliminary remarks, which will be useful for our arguments.

For $u : \mathbb{R}^3 \to \mathbb{C}$, we set

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u.$$

Consider the function spaces

$$D_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R})\}$$

and

$$H_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in D_A^1(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C})\}.$$

The space $H_A^1(\mathbb{R}^3, \mathbb{C})$ is a Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \operatorname{Re} \int_{\mathbb{R}^3} (\nabla_A u \overline{\nabla_A v} + u\bar{v})\,dx \quad \text{for any } u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. We denote by $\|u\|_A$ the norm induced by this inner product, and $[u]_A^2 := \int_{\mathbb{R}^3} |\nabla_A u|^2\,dx$.

On $H_A^1(\mathbb{R}^3, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see, e.g., [17, Theorem 7.21]):

$$|\nabla_A u(x)| \geq |\nabla|u(x)|| \quad \text{for all } u \in H_A^1(\mathbb{R}^3, \mathbb{C}). \tag{2.1}$$

Moreover, making a simple change of variables, since

$$\Delta_{A_\epsilon} = \epsilon^2 \Delta_{A/\epsilon} \quad \text{and} \quad [u]^2_{A_\epsilon} = \frac{1}{\epsilon}[u]^2_{A/\epsilon},$$

we can see that problem (1.1) is equivalent to

$$-(a + b[u]^2_{A_\epsilon})\Delta_{A_\epsilon} u + V_\epsilon(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where $A_\epsilon(x) = A(\epsilon x)$ and $V_\epsilon(x) = V(\epsilon x)$.

Let H_ϵ be the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_\epsilon := \operatorname{Re} \int_{\mathbb{R}^3} (\nabla_{A_\epsilon} u \overline{\nabla_{A_\epsilon} v} + V_\epsilon(x) u\bar{v})\, dx$$

and let $\|\cdot\|_\epsilon$ denote the norm induced by this inner product.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_{A_\epsilon}(\mathbb{R}^3, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\|u\| \le C\|u\|_\epsilon$. Therefore, the embedding $H_\epsilon \hookrightarrow L^r(\mathbb{R}^3, \mathbb{C})$ is continuous for $2 \le r \le 6$ and the embedding $H_\epsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ is compact for $1 \le r < 6$.

3 The Modified Problem

As in [6], to study problem (1.1), or equivalently (2.2), we modify suitably the nonlinearity f so that, for $\epsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By (f4), there exists a unique number $a > 0$ verifying $Kf(a_0) = V_0$, where V_0 is given in (V1). Hence we consider the function

$$\tilde{f}(t) := \begin{cases} f(t), & t \le a_0, \\ \dfrac{V_0}{K}, & t > a_0. \end{cases}$$

Now we introduce the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$:

$$g(x, t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t),$$

where χ_Λ is the characteristic function on Λ. Set $G(x, t) := \int_0^t g(x, s)\, ds$.

In view of (f1)–(f4), we have that g is a Carathéodory function satisfying the following properties:

(g1) $g(x, t) = 0$ for each $t \le 0$.

(g2) $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$, and there exists $q \in (4, 6)$ such that

$$\lim_{t\to+\infty} \frac{g(x, t)}{t^{\frac{q-2}{2}}} = 0 \quad \text{uniformly in } x \in \mathbb{R}^3.$$

(g3) $g(x, t) \le f(t)$ for all $t \ge 0$ and uniformly in $x \in \mathbb{R}^3$.

(g4) $0 < \theta G(x, t) \le 2g(x, t)t$ for each $x \in \Lambda$, $t > 0$.

(g5) $0 < G(x, t) \le g(x, t)t \le V_0 t/K$ for each $x \in \Lambda^c$, $t > 0$.

(g6) For each $x \in \Lambda$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0, +\infty)$, and for each $x \in \Lambda^c$ the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0, a_0)$.

Next, we consider the *modified* problem

$$-(a + b[u]^2_{A_\epsilon})\Delta_{A_\epsilon} u + V_\epsilon(x)u = g(\epsilon x, |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

Note that, if u is a solution of problem (3.1) with

$$|u(x)|^2 \le a_0 \quad \text{for all } x \in \Lambda^c_\epsilon,\ \Lambda_\epsilon := \{x \in \mathbb{R}^3 : \epsilon x \in \Lambda\},$$

then u is a solution of problem (2.2).

The functional associated to (3.1) is

$$J_\epsilon(u) := \frac{a}{2}[u]_{A_\epsilon}^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx + \frac{b}{4}[u]_{A_\epsilon}^4 - \frac{1}{2}\int_{\mathbb{R}^3} G(\epsilon x, |u|^2)\,dx$$

defined in H_ϵ. It is standard to prove that $J_\epsilon \in C^1(H_\epsilon, \mathbb{R})$ and its critical points are the weak solutions of the modified problem (3.1).

We denote by $\mathcal{N}_\epsilon$ the Nehari manifold of J_ϵ, that is,

$$\mathcal{N}_\epsilon := \{u \in H_\epsilon \setminus \{0\} : J_\epsilon'(u)[u] = 0\},$$

and define the number c_ϵ by

$$c_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} J_\epsilon(u).$$

Let H_ϵ^+ be the open subset H_ϵ given by

$$H_\epsilon^+ = \{u \in H_\epsilon : |\operatorname{supp}(u) \cap \Lambda_\epsilon| > 0\},$$

and $S_\epsilon^+ = S_\epsilon \cap H_\epsilon^+$, where S_ϵ is the unit sphere of H_ϵ. Note that S_ϵ^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ϵ and contained in H_ϵ^+. Therefore, $H_\epsilon = T_u S_\epsilon^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_\epsilon^+$, where $T_u S_\epsilon^+ = \{v \in H_\epsilon : \langle u, v\rangle_\epsilon = 0\}$.

Now we show that the functional J_ϵ satisfies the mountain pass geometry.

Lemma 3.1. *For any fixed $\epsilon > 0$, the functional J_ϵ satisfies the following properties:*
(i) *There exist $\beta, r > 0$ such that $J_\epsilon(u) \geq \beta$ if $\|u\|_\epsilon = r$.*
(ii) *There exists $e \in H_\epsilon$ with $\|e\|_\epsilon > r$ such that $J_\epsilon(e) < 0$.*

Proof. (i) By (g2), (g4) and (g5), for any $\zeta > 0$ small, there exists $C_\zeta > 0$ such that

$$G(\epsilon x, |u|^2) \leq \zeta|u|^4 + C_\zeta|u|^q \quad \text{for all } x \in \mathbb{R}^3.$$

By the Sobolev embedding theorem, it follows that

$$\begin{aligned} J_\epsilon(u) &\geq \frac{a}{2}[u]_{A_\epsilon}^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx + \frac{b}{4}[u]_{A_\epsilon}^4 - \frac{\zeta}{2}\int_{\mathbb{R}^3}|u|^4\,dx - \frac{C_\zeta}{2}\int_{\mathbb{R}^3}|u|^q\,dx \\ &\geq \frac{1}{2}\|u_n\|_\epsilon^2 - C_1\zeta\|u_n\|_\epsilon^4 - C_2C_\zeta\|u_n\|_\epsilon^q. \end{aligned}$$

Hence we can choose some $\beta, r > 0$ such that $J_\epsilon(u) \geq \beta$ if $\|u\|_\epsilon = r$ since $q > 4$.

(ii) For each $u \in H_\epsilon^+$ and $t > 0$, by the definition of g and (f3), one has

$$\begin{aligned} J_\epsilon(tu) &\leq \frac{t^2}{2}\|u\|_\epsilon^2 + \frac{bt^4}{4}[u]_{A_\epsilon}^4 - \frac{1}{2}\int_{\Lambda_\epsilon} G(\epsilon x, t^2|u|^2)\,dx \\ &\leq \frac{t^2}{2}\|u\|_\epsilon^2 + \frac{bt^4}{4}[u]_{A_\epsilon}^4 - C_1t^\theta\int_{\Lambda_\epsilon}|u|^\theta\,dx + C_2|\operatorname{supp}(u) \cap \Lambda_\epsilon|. \end{aligned}$$

Since $\theta > 4$, we can get the conclusion. □

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\epsilon$ and the incompleteness of S_ϵ^+.

Lemma 3.2. *Assume that* (V1)–(V2) *and* (f1)–(f4) *are satisfied. Then the following properties hold:*
(A1) *For any $u \in H_\epsilon^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\epsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞).*
(A2) *There is some $\tau > 0$ independent of u such that $t_u \geq \tau$ for all $u \in S_\epsilon^+$. Moreover, for each compact $\mathcal{W} \subset S_\epsilon^+$ there is a constant $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

(A3) *The map* $\widehat{m}_\epsilon : H_\epsilon^+ \to \mathcal{N}_\epsilon$ *given by* $\widehat{m}_\epsilon(u) = t_u u$ *is continuous, and* $m_\epsilon = \widehat{m}_\epsilon|_{S_\epsilon^+}$ *is a homeomorphism between* S_ϵ^+ *and* $\mathcal{N}_\epsilon$. *Moreover,* $m_\epsilon^{-1}(u) = \frac{u}{\|u\|_\epsilon}$.

(A4) *If there is a sequence* $\{u_n\} \subset S_\epsilon^+$ *such that* $\operatorname{dist}(u_n, \partial S_\epsilon^+) \to 0$, *then* $\|m_\epsilon(u_n)\|_\epsilon \to \infty$ *and* $J_\epsilon(m_\epsilon(u_n)) \to \infty$.

Proof. (A1) As in the proof of Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small, and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ verifying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\epsilon$. Now, we show that t_u is unique. Arguing by contradiction, suppose that there exist $t_1 > t_2 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i a[u]_{A_\epsilon}^2 + t_i \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx + t_i^3 b[u]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} g(\epsilon x, t_i^2|u|^2) t_i |u|^2\,dx.$$

Hence,

$$\frac{a[u]_{A_\epsilon}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx}{t_i^2} + b[u]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} \frac{g(\epsilon x, t_i^2|u|^2)|u|^2}{t_i^2}\,dx,$$

which implies that

$$\begin{aligned}
&\Big(\frac{1}{t_1^2} - \frac{1}{t_2^2}\Big)\Big(a[u]_{A_\epsilon}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx\Big)\\
&\quad = \int_{\mathbb{R}^3} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4\,dx\\
&\quad \geq \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4\,dx\\
&\qquad + \int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4\,dx\\
&\quad \geq \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4\,dx + \frac{1}{K}\Big(\frac{1}{t_1^2} - \frac{1}{t_2^2}\Big)\int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} V_0|u|^2\,dx.
\end{aligned}$$

Since $t_1 > t_2 > 0$, we have

$$\begin{aligned}
&\Big(a[u]_{A_\epsilon}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx\Big)\\
&\quad \leq \frac{t_1^2 t_2^2}{t_2^2 - t_1^2} \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4\,dx + \frac{1}{K}\int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} V_0|u|^2\,dx\\
&\quad \leq \frac{1}{K}\int_{\Lambda_\epsilon^c} V_0|u|^2\,dx\\
&\quad \leq \frac{1}{K}\|u\|_\epsilon^2,
\end{aligned}$$

which is a contradiction. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a unique $t = t_u$ so that $g_u'(t) = 0$ and $t_u u \in \mathcal{N}_\epsilon$.

(A2) For any $u \in S_\epsilon^+$, we have

$$t_u + t_u^3 b[u]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} g(\epsilon x, t_u^2|u|^2) t_u |u|^2\,dx.$$

From (g2), Sobolev embeddings and since $q > 4$, we get

$$t_u \leq \zeta t_u^3 \int_{\mathbb{R}^3} |u|^4\,dx + C_\zeta t_u^{q-1} \int_{\mathbb{R}^3} |u|^q\,dx \leq C_1 \zeta t_u^3 + C_2 C_\zeta t_u^{q-1},$$

which implies that $t_u \geq \tau$ for some $\tau > 0$. Suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ with $t_n := t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in H_ϵ. Moreover, using the proof of Lemma 3.1 (ii), we have that $J_\epsilon(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\epsilon$. From the definition of g and by (g4), (g5) and $\theta > 4$, it follows that

$$\begin{aligned} J_\epsilon(v_n) &= J_\epsilon(v_n) - \frac{1}{\theta} J'_\epsilon(v_n)[v_n] \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|v_n\|_\epsilon^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) b[v_n]_{A_\epsilon}^4 + \int_{\Lambda_\epsilon^c} \Big(\frac{1}{\theta} g(\epsilon x, |v_n|^2)|v_n|^2 - \frac{1}{2} G(\epsilon x, |v_n|^2)\Big)\, dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|v_n\|_\epsilon^2 - \frac{1}{K} \int_{\mathbb{R}^3} V(\epsilon x)|v_n|^2\, dx\Big) \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big)\|v_n\|_\epsilon^2. \end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\epsilon = t_n$, we obtain

$$0 < \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big) \leq \frac{J_\epsilon(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves (A2).

(A3) First of all, we note that $\widehat{m}_\epsilon$, m_ϵ and m_ϵ^{-1} are well defined. Indeed, by (A2), for each $u \in H_\epsilon^+$, there is a unique $\widehat{m}_\epsilon(u) \in \mathcal{N}_\epsilon$. On the other hand, if $u \in \mathcal{N}_\epsilon$, then $u \in H_\epsilon^+$. Otherwise, we have $|\mathrm{supp}(u) \cap \Lambda_\epsilon| = 0$ and by (g5), we have

$$\begin{aligned} a[u]_{A_\epsilon}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\, dx + b[u]_{A_\epsilon}^4 &= \int_{\mathbb{R}^3} g(\epsilon x, |u|^2)|u|^2\, dx \\ &= \int_{\Lambda_\epsilon^c} g(\epsilon x, |u|^2)|u|^2\, dx \\ &\leq \frac{1}{K} \int_{\mathbb{R}^3} V(\epsilon x)|u|^2\, dx \\ &\leq \frac{1}{K}\|u\|_\epsilon^2, \end{aligned}$$

which is impossible since $K > 2$ and $u \neq 0$. Therefore, $m_\epsilon^{-1}(u) = \frac{u}{\|u\|_\epsilon} \in S_\epsilon^+$ is well defined and continuous. From

$$m_\epsilon^{-1}(m_\epsilon(u)) = m_\epsilon^{-1}(t_u u) = \frac{t_u u}{t_u \|u\|_\epsilon} = u \quad \text{for all } u \in S_\epsilon^+,$$

we conclude that m_ϵ is a bijection.

Now we prove that $\widehat{m}_\epsilon : H_\epsilon^+ \to \mathcal{N}_\epsilon$ is continuous. Let $\{u_n\} \subset H_\epsilon^+$ and $u \in H_\epsilon^+$ such that $u_n \to u$ in H_ϵ. By (A2), there exists $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\epsilon$, that is,

$$t_n^2 a[u_n]_{A_\epsilon}^2 + t_n^2 \int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2\, dx + t_n^4 b[u_n]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} g(\epsilon x, t_n^2|u_n|^2) t_n^2 |u_n|^2\, dx \quad \text{for all } n \in N,$$

and passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$t_0^2 a[u]_{A_\epsilon}^2 + t_0^2 \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\, dx + t_0^4 b[u]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} g(\epsilon x, t_0^2|u|^2) t_0^2 |u|^2\, dx,$$

which implies that $t_0 u \in \mathcal{N}_\epsilon$ and $t_u = t_0$. This proves that $\widehat{m}_\epsilon(u_n) \to \widehat{m}_\epsilon(u)$ in H_ϵ^+. Thus, $\widehat{m}_\epsilon$ and m_ϵ are continuous functions and (A3) is proved.

(A4) Let $\{u_n\} \subset S_\epsilon^+$ be a subsequence such that $\mathrm{dist}(u_n, \partial S_\epsilon^+) \to 0$. Then, for each $v \in \partial S_\epsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ϵ. Therefore, by (V1), (V2) and the Sobolev embedding theorem, there exists a constant $C_r > 0$ such that

$$\begin{aligned}\|u_n\|_{L^r(\Lambda_\epsilon)} &\le \inf_{v\in\partial S_\epsilon^+}\|u_n - v\|_{L^r(\Lambda_\epsilon)}\\ &\le C_r\Big(\inf_{v\in\partial S_\epsilon^+}\int_{\Lambda_\epsilon}(|\nabla_{A_\epsilon}u_n - v|^2 + V_\epsilon(x)|u_n - v|^2)\,dx\Big)^{\frac12}\\ &\le C_r\,\mathrm{dist}(u_n,\partial S_\epsilon^+)\end{aligned}$$

for all $n \in N$ and $r \in [2, 6]$. By (g2), (g3) and (g5), for each $t > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} G(\epsilon x, t^2|u_n|^2)\,dx &\le \int_{\Lambda_\epsilon} F(t^2|u_n|^2)\,dx + \frac{t^2}{K}\int_{\Lambda_\epsilon^c} V(\epsilon x)|u_n|^2\,dx\\ &\le C_1t^4\int_{\Lambda_\epsilon}|u_n|^4\,dx + C_2t^q\int_{\Lambda_\epsilon}|u_n|^q\,dx + \frac{t^2}{K}\|u_n\|_\epsilon^2\\ &\le C_3t^4\,\mathrm{dist}(u_n,\partial S_\epsilon^+)^4 + C_4t^q\,\mathrm{dist}(u_n,\partial S_\epsilon^+)^q + \frac{t^2}{K}.\end{aligned}$$

Therefore,

$$\limsup_n\int_{\mathbb{R}^3} G(\epsilon x, t^2|u_n|^2)\,dx \le \frac{t^2}{K}\quad\text{for all } t > 0.$$

On the other hand, from the definition of m_ϵ and the last inequality, for all $t > 0$, one has

$$\begin{aligned}\liminf_n J_\epsilon(m_\epsilon(u_n)) &\ge \liminf_n J_\epsilon(tu_n)\\ &\ge \liminf_n \frac{t^2}{2}\|u_n\|_\epsilon^2 - \frac{t^2}{K}\\ &= \frac{K-2}{2K}t^2.\end{aligned}$$

This implies that

$$\liminf_n \frac12\|m_\epsilon(u_n)\|_\epsilon^2 \ge \frac{K-2}{2K}t^2\quad\text{for all } t > 0.$$

From the arbitrariness of $t > 0$, it is easy to see that $\|m_\epsilon(u_n)\|_\epsilon \to \infty$ and $J_\epsilon(m_\epsilon(u_n)) \to \infty$ as $n \to \infty$. This completes the proof of Lemma 3.2. □

Now we define the function

$$\widehat{\Psi}_\epsilon : H_\epsilon^+ \to \mathbb{R}$$

by $\widehat{\Psi}_\epsilon(u) = J_\epsilon(\widehat{m}_\epsilon(u))$ and set $\Psi_\epsilon := (\widehat{\Psi}_\epsilon)|_{S_\epsilon^+}$.

From Lemma 3.2, we have the following result.

Lemma 3.3. *Assume that* (V1)–(V2) *and* (f1)–(f4) *are satisfied. Then the following assertions hold:*

(B1) $\widehat{\Psi}_\epsilon \in C^1(H_\epsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\epsilon'(u)v = \frac{\|\widehat{m}_\epsilon(u)\|_\epsilon}{\|u\|_\epsilon}J_\epsilon'(\widehat{m}_\epsilon(u))[v]\quad\textit{for all } u \in H_\epsilon^+ \textit{ and all } v \in H_\epsilon.$$

(B2) $\Psi_\epsilon \in C^1(S_\epsilon^+, \mathbb{R})$ *and*

$$\Psi_\epsilon'(u)v = \|m_\epsilon(u)\|_\epsilon J_\epsilon'(\widehat{m}_\epsilon(u))[v]\quad\textit{for all } v \in T_uS_\epsilon^+.$$

(B3) *If* $\{u_n\}$ *is a* $(\mathrm{PS})_c$ *sequence of* Ψ_ϵ, *then* $\{m_\epsilon(u_n)\}$ *is a* $(\mathrm{PS})_c$ *sequence of* J_ϵ. *If* $\{u_n\} \subset \mathcal{N}_\epsilon$ *is a bounded* $(\mathrm{PS})_c$ *sequence of* J_ϵ, *then* $\{m_\epsilon^{-1}(u_n)\}$ *is a* $(\mathrm{PS})_c$ *sequence of* Ψ_ϵ.

(B4) u *is a critical point of* Ψ_ϵ *if and only if* $m_\epsilon(u)$ *is a critical point of* J_ϵ. *Moreover, the corresponding critical values coincide and*

$$\inf_{S_\epsilon^+}\Psi_\epsilon = \inf_{\mathcal{N}_\epsilon} J_\epsilon.$$

As in [21], we have the following variational characterization of the infimum of J_ϵ over $\mathcal{N}_\epsilon$:

$$c_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} J_\epsilon(u) = \inf_{u\in H_\epsilon^+} \sup_{t>0} J_\epsilon(tu) = \inf_{u\in S_\epsilon^+} \sup_{t>0} J_\epsilon(tu).$$

Lemma 3.4. *Let $c > 0$ and let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence for J_ϵ. Then $\{u_n\}$ is bounded in H_ϵ.*

Proof. Assume that $\{u_n\} \subset H_\epsilon$ is a $(\mathrm{PS})_c$ sequence for J_ϵ, that is, $J_\epsilon(u_n) \to c$ and $J'_\epsilon(u_n) \to 0$. By using (g4), (g5) and $\theta > 4$, we have

$$\begin{aligned}
c + o_n(1) + o_n(1)\|u_n\|_\epsilon &\geq J_\epsilon(u_n) - \frac{1}{\theta} J'_\epsilon(u_n)[u_n] \\
&= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) b[u_n]_{A_\epsilon}^4 + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |u_n|^2)\Big)\,dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \int_{\Lambda_\epsilon^c} \Big(\frac{1}{\theta} g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |u_n|^2)\Big)\,dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 - \frac{1}{2} \int_{\Lambda_\epsilon^c} G(\epsilon x, |u_n|^2)\,dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 - \frac{1}{2K} \int_{\mathbb{R}^3} V(\epsilon x)|u_n|^2\,dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2K}\Big)\|u_n\|_\epsilon^2.
\end{aligned}$$

Since $K > 2$, from the above inequalities we obtain that $\{u_n\}$ is bounded in H_ϵ. □

The following result is important to prove the $(\mathrm{PS})_{c_\epsilon}$ condition for the functional J_ϵ.

Lemma 3.5. *The functional J_ϵ satisfies the $(\mathrm{PS})_c$ condition at any level $c > 0$.*

Proof. Let $(u_n) \subset H_\epsilon$ be a $(\mathrm{PS})_c$ sequence for J_ϵ. By Lemma 3.4, (u_n) is bounded in H_ϵ. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ϵ and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ for all $1 \leq r < 6$ as $n \to +\infty$. Moreover, $J'_\epsilon(u) = 0$ and

$$a[u]_{A_\epsilon}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2\,dx + b[u]_{A_\epsilon}^4 = \int_{\mathbb{R}^3} g(\epsilon x, |u|^2)|u|^2\,dx.$$

For the fixed $\epsilon > 0$, let $R > 0$ be such that $\Lambda_\epsilon \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$, for R large enough,

$$\limsup_n \int_{B_R^c(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)\,dx \leq \zeta. \tag{3.2}$$

Let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \text{ for } x \in B_{R/2}(0), \quad \phi_R = 1 \text{ for } x \in B_R^c(0), \quad 0 \leq \phi_R \leq 1, \quad |\nabla\phi_R| \leq \frac{C}{R},$$

where $C > 0$ is a constant independent of R. Since the sequence $(\phi_R u_n)$ is bounded in H_ϵ, we have

$$J'_\epsilon(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$\begin{aligned}
&a \operatorname{Re} \int_{\mathbb{R}^3} \nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon}(\phi_R u_n)}\,dx + \int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2 \phi_R\,dx + b[u_n]_{A_\epsilon}^2 \operatorname{Re} \int_{\mathbb{R}^3} \nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon}(\phi_R u_n)}\,dx \\
&\quad = \int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R\,dx + o_n(1).
\end{aligned}$$

Since

$$\overline{\nabla_{A_\epsilon}(u_n\phi_R)} = i\overline{u_n}\nabla\phi_R + \phi_R \overline{\nabla_{A_\epsilon} u_n},$$

using (g5), we have

$$\begin{aligned}\int_{\mathbb{R}^3}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)\phi_R\,dx &\le \int_{\mathbb{R}^3} g(\epsilon x,|u_n|^2)|u_n|^2\phi_R\,dx-(a+b[u_n]_{A_\epsilon}^2)\operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R\,dx+o_n(1)\\ &\le \frac{1}{K}\int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2\phi_R\,dx+C\Big|\operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R\,dx\Big|+o_n(1).\end{aligned}$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of (u_n) in H_ϵ, we obtain

$$\Big(1-\frac{1}{K}\Big)\int_{\mathbb{R}^3}(a|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)\phi_R\,dx\le \frac{C}{R}\|u_n\|_2\|\nabla_{A_\epsilon}u_n\|_2+o_n(1)\le \frac{C_1}{R}+o_n(1),$$

and so (3.2) holds.

Now, we prove that for any $R>0$ the following limit holds:

$$\limsup_n \int_{B_R(0)}(|\nabla_{A_\epsilon}u_n|^2+V_\epsilon(x)|u_n|^2)\,dx=\int_{B_R(0)}(|\nabla_{A_\epsilon}u|^2+V_\epsilon(x)|u|^2)\,dx. \tag{3.3}$$

Let $\phi_\rho\in C^\infty(\mathbb{R}^3,\mathbb{R})$ be a cut-off function such that

$$\phi_\rho=1 \text{ for } x\in B_\rho(0),\quad \phi_\rho=0 \text{ for } x\in B_{2\rho}^c(0),\quad 0\le\phi_\rho\le 1,\quad |\nabla\phi_\rho|\le\frac{C}{\rho},$$

where $C>0$ is a constant independent of ρ. Let

$$P_n(x)=M(u_n)|\nabla_{A_\epsilon}u_n-\nabla_{A_\epsilon}u|^2+V_\epsilon(x)|u_n-u|^2,$$

where

$$M(u_n)=a+b\int_{\mathbb{R}^3}|\nabla_{A_\epsilon}u_n|^2\,dx.$$

For the fixed $R>0$, choosing $\rho>R>0$, we have

$$\begin{aligned}\int_{B_R}P_n(x)\,dx &\le \int_{\mathbb{R}^3}P_n(x)\phi_\rho(x)\,dx\\ &= M(u_n)\int_{\mathbb{R}^3}|\nabla_{A_\epsilon}u_n-\nabla_{A_\epsilon}u|^2\phi_\rho(x)\,dx+\int_{\mathbb{R}^3}V_\epsilon(x)|u_n-u|^2\phi_\rho(x)\,dx\\ &= J_{n,\rho}^1-J_{n,\rho}^2+J_{n,\rho}^3+J_{n,\rho}^4,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}J_{n,\rho}^1&=M(u_n)\int_{\mathbb{R}^3}|\nabla_{A_\epsilon}u_n|^2\phi_\rho(x)\,dx+\int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\phi_\rho(x)\,dx-\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)|u_n|^2\phi_\rho\,dx,\\ J_{n,\rho}^2&=M(u_n)\operatorname{Re}\int_{\mathbb{R}^3}\nabla_{A_\epsilon}u_n\overline{\nabla_{A_\epsilon}u}\phi_\rho(x)\,dx+\operatorname{Re}\int_{\mathbb{R}^3}V_\epsilon(x)u_n\overline{u}\phi_\rho(x)\,dx-\operatorname{Re}\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)u_n\overline{u}\phi_\rho(x)\,dx,\\ J_{n,\rho}^3&=-M(u_n)\operatorname{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n-\nabla_{A_\epsilon}u)\overline{\nabla_{A_\epsilon}u}\phi_\rho(x)\,dx+\operatorname{Re}\int_{\mathbb{R}^3}V_\epsilon(x)(u_n-u)\overline{u}\phi_\rho(x)\,dx,\\ J_{n,\rho}^4&=\operatorname{Re}\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}\phi_\rho(x)\,dx.\end{aligned}$$

It is easy to see that

$$J_{n,\rho}^1=J_\epsilon'(u_n)[\phi_\rho u_n]-M(u_n)\operatorname{Re}\int_{\mathbb{R}^3}i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_\rho\,dx$$

and

$$J^2_{n,\rho} = J'_\epsilon(u_n)[\phi_\rho u] - M(u_n)\,\mathrm{Re}\int_{\mathbb{R}^3} i\bar{u}\,\nabla_{A_\epsilon}u_n \nabla\phi_\rho\,dx.$$

Then

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^1_{n,\rho}| = 0, \qquad \lim_{\rho\to\infty}\limsup_{n\to\infty}|J^2_{n,\rho}| = 0.$$

On the other hand, since the sequence (u_n) is bounded in H_ϵ, we assume that

$$\int_{\mathbb{R}^3}|\nabla_{A_\epsilon}u_n|^2\,dx \to l^2.$$

Then

$$\begin{aligned}
J^3_{n,\rho} &= -(a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)\overline{\nabla_{A_\epsilon}(u\phi_\rho(x))}\,dx - \mathrm{Re}\int_{\mathbb{R}^3}V_\epsilon(x)(u_n-u)\overline{(u\phi_\rho(x))}\,dx\\
&\quad + (a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)i\bar{u}\,\nabla\phi_\rho\,dx + o_n(1)\\
&= -(a+bl^2)\langle u_n-u, u\phi_\rho(x)\rangle + (a+bl^2)\,\mathrm{Re}\int_{\mathbb{R}^3}(\nabla_{A_\epsilon}u_n - \nabla_{A_\epsilon}u)i\bar{u}\,\nabla\phi_\rho\,dx + o_n(1),
\end{aligned}$$

and thus

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^3_{n,\rho}| = 0.$$

Now we prove that

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^4_{n,\rho}| = 0. \tag{3.5}$$

It is easy to see that

$$J^4_{n,\rho} \le \int_{(\mathbb{R}^3\setminus\Lambda_\epsilon)\cap B_{2\rho}(0)}|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}|\,dx + \int_{\Lambda_\epsilon\cap B_{2\rho}(0)}|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}|\,dx.$$

Using the Sobolev compact embedding $H_\epsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ for $1\le r<6$, (g5), (f1) and (f2) imply that

$$\int_{(\mathbb{R}^3\setminus\Lambda_\epsilon)\cap B_{2\rho}(0)}|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}|\,dx \to 0 \quad \text{as } n\to\infty$$

and

$$\int_{\Lambda_\epsilon\cap B_{2\rho}(0)}|g(\epsilon x,|u_n|^2)u_n\overline{(u_n-u)}|\,dx \to 0 \quad \text{as } n\to\infty.$$

Thus, (3.5) holds. Moreover, by (3.4), it follows that

$$0 \le \limsup_n \int_{B_R} P_n(x)\,dx \le \limsup_n(|J^1_{n,\rho}| + |J^2_{n,\rho}| + |J^3_{n,\rho}| + |J^4_{n,\rho}|) = 0.$$

Then

$$\limsup_n \int_{B_R} P_n(x)\,dx = 0.$$

Thus, (3.3) holds. Finally, from (3.2) and (3.3), we have

$$\begin{aligned}
\|u\|^2_\epsilon &\le \liminf_n \|u_n\|^2_\epsilon\\
&\le \limsup_n \|u_n\|^2_\epsilon\\
&\le \limsup_n\Big\{\int_{B_R(0)}(a|\nabla_{A_\epsilon}u_n|^2 + V_\epsilon(x)|u_n|^2)\,dx + \int_{B^c_R(0)}(a|\nabla_{A_\epsilon}u_n|^2 + V_\epsilon(x)|u_n|^2)\,dx\Big\}\\
&\le \int_{B_R(0)}(a|\nabla_{A_\epsilon}u|^2 + V_\epsilon(x)|u|^2)\,dx + \zeta.
\end{aligned}$$

Passing to the limit as $\zeta \to 0$, we have $R \to \infty$, which implies that

$$\|u\|_\epsilon^2 \le \liminf_n \|u_n\|_\epsilon^2 \le \limsup_n \|u_n\|_\epsilon^2 \le \|u\|_\epsilon^2.$$

Then $u_n \to u$ in H_ϵ, and we complete the proof of this theorem. □

Since f is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

Corollary 3.6. *The functional Ψ_ϵ satisfies the* $(\mathrm{PS})_c$ *condition on S_ϵ^+ at any level $c > 0$.*

Proof. Let $\{u_n\} \subset S_\epsilon^+$ be a $(\mathrm{PS})_c$ sequence for Ψ_ϵ. Then $\Psi_\epsilon(u_n) \to c$ and $\|\Psi_\epsilon'(u_n)\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n} S_\epsilon^+)^*$. By Lemma 3.3 (B3), we know that $\{m_\epsilon(u_n)\}$ is a $(\mathrm{PS})_c$ sequence for J_ϵ in H_ϵ. From Lemma 3.5, we know that there exists a $u \in S_\epsilon^+$ such that, up to a subsequence, $m_\epsilon(u_n) \to m_\epsilon(u)$ in H_ϵ. By Lemma 3.2 (A3), we obtain

$$u_n \to u \quad \text{in } S_\epsilon^+,$$

and the proof is complete. □

Proposition 3.7. *Assume that* (V1)–(V2) *and* (f1)–(f4) *hold. Then problem* (3.1) *has a ground state solution for any $\epsilon > 0$.*

Proof. From Lemma 3.1 and Lemma 3.5, we can obtain the existence of a ground state $u \in H_\epsilon$ for problem (3.1). □

4 Multiple Solutions for the Modified Problem

4.1 The Autonomous Problem

For our scope, we also need to study the following *limit* problem:

$$-(a + b[u]^2)\Delta u + V_0 u = f(|u|^2)u, \quad u : \mathbb{R}^3 \to \mathbb{R}, \tag{4.1}$$

whose associated C^1-functional, defined on $H^1(\mathbb{R}^3, \mathbb{R})$, is

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_0 u^2)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)\,dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_0(u).$$

Let S_0 be the unit sphere of $H_0 := H^1(\mathbb{R}^3, \mathbb{R})$ and let it be a complete and smooth manifold of codimension 1. Therefore, $H_0 = T_u S_0 \bigoplus \mathbb{R}u$ for each $u \in T_u S_0$, where $T_u S_0 = \{v \in H_0 : \langle u, v\rangle_0 = 0\}$.

Lemma 4.1. *Let V_0 be given in* (V1) *and suppose that* (f1)–(f4) *are satisfied. Then the following properties hold:*

(a1) *For any $u \in H_0 \setminus \{0\}$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞).*

(a2) *There is a $\tau > 0$ independent of u such that $t_u > \tau$ for all $u \in S_0$. Moreover, for each compact $\mathcal{W} \subset S_0$ there exists a t_u such that $t_u \le C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

(a3) *The map $\widehat{m} : H_0 \setminus \{0\} \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous, and $m_0 = \widehat{m}_0|_{S_0}$ is a homeomorphism between S_0 and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$.*

(a4) *If there is a sequence $\{u_n\} \subset S_0$ such that* $\operatorname{dist}(u_n, \partial S_0) \to 0$, *then $\|m(u_n)\|_0 \to \infty$ and $I_0(m(u)) \to \infty$ as $n \to \infty$.*

Lemma 4.2. *Let V_0 be given in* (V1) *and suppose that* (f1)–(f4) *are satisfied. Then the following assertions hold:*

(b1) $\widehat{\Psi}_0 \in C^1(H_0 \setminus \{0\}, \mathbb{R})$ *and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_0'(\widehat{m}(u))[v] \quad \text{for all } u \in H_0 \setminus \{0\} \text{ and all } v \in H_0.$$

(b2) $\Psi_0 \in C^1(S_0, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(\widehat{m}(u))[v] \quad \text{for all } v \in T_u S_0.$$

(b3) *If $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence of Ψ_0, then $\{m(u_n)\}$ is a $(\mathrm{PS})_c$ sequence of I_0. If $\{u_n\} \subset \mathcal{N}_0$ is a bounded $(\mathrm{PS})_c$ sequence of I_0, then $\{m^{-1}(u_n)\}$ is a $(\mathrm{PS})_c$ sequence of Ψ_0.*

(b4) *We have that u is a critical point of Ψ_0 if and only if $m(u)$ is a critical point of I_0. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

Similarly to the previous argument, we have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in H_0 \setminus \{0\}} \sup_{t>0} I_0(tu) = \inf_{u \in S_0} \sup_{t>0} I_0(tu).$$

The next result is useful in later arguments.

Lemma 4.3. *Let $\{u_n\} \subset H_0$ be a $(\mathrm{PS})_c$ sequence for I_0 such that $u_n \rightharpoonup 0$. Then one of the following alternatives occurs:*

(i) *$u_n \to 0$ in H_0 as $n \to +\infty$.*

(ii) *There are a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx \geq \beta.$$

Proof. Assume that (ii) does not hold. Then, for every $R > 0$, we have

$$\lim_n \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, dx = 0.$$

Since $\{u_n\}$ is bounded in H_0, by the Lions lemma it follows that

$$u_n \to 0 \quad \text{in } L^r(\mathbb{R}^3),\ 2 < r < 6.$$

From the subcritical growth of f, we have

$$\int_{\mathbb{R}^3} F(u_n^2) \, dx = o_n(1) = \int_{\mathbb{R}^3} f(u_n^2) u_n^2 \, dx.$$

Moreover, from $I_0'(u_n)[u_n] \to 0$, it follows that

$$\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V_0 u_n^2) \, dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx\Big)^2 = \int_{\mathbb{R}^3} f(u_n^2) u_n^2 \, dx + o_n(1) = o_n(1).$$

Thus (i) holds. □

Remark 4.4. From Lemma 4.3 we see that if u is the weak limit of a $(\mathrm{PS})_{c_{V_0}}$ sequence $\{u_n\}$ of the functional I_0, then we have $u \neq 0$. Otherwise, we have that $u_n \rightharpoonup 0$ and if $u_n \not\to 0$, from Lemma 4.3 it follows that there are a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + z_n)$. It is easy to see that $\{v_n\}$ is also a $(\mathrm{PS})_{c_{V_0}}$ sequence for the functional I_0, it is bounded, and there exists $v \in H_0$ such that $v_n \rightharpoonup v$ in H_0 with $v \neq 0$.

Lemma 4.5. *Assume that* $V_0 > 0$ *and* f *satisfies* (f1)–(f4). *Then problem* (4.1) *has a positive ground state solution.*

Proof. First of all, it is easy to show that $c_{V_0} > 0$. Moreover, if $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of Ψ_0, so that u_0 is a critical point of I_0 by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and S_0 is a C^1 manifold, by Ekeland's variational principle, there exists a sequence $\omega_n \subset S_0$ with $\Psi_0(\omega_n) \to c_{V_0}$ and $\Psi_0'(\omega_n) \to 0$ as $n \to \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in N$. Then $I_0(u_n) \to c_{V_0}$ and $I_0'(u_n) \to 0$ as $n \to \infty$ by Lemma 4.2 (b3). Similar to the proof of Lemma 3.4, it is easy to know that $\{u_n\}$ is bounded in H_0. Thus, we have $u_n \rightharpoonup u$ in H_0, $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$, $1 \le r < 6$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, and thus $I_0'(u) = 0$. From Remark 4.4, we know that $u \neq 0$. Moreover,

$$\begin{aligned}
c_{V_0} \le I_0(u) &= I_0(u) - \frac{1}{\theta} I_0'(u)[u] \\
&= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u\|_0^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} f(u^2)u^2 - \frac{1}{2}F(u^2)\Big)\,dx \\
&\le \liminf_n \Big\{\Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_0^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2 + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} f(u_n)u_n^2 - \frac{1}{2}F(u_n^2)\Big)\,dx\Big\} \\
&= \liminf_n \Big\{I_0(u_n) - \frac{1}{\theta} I_0'(u_n)[u_n]\Big\} \\
&= c_{V_0},
\end{aligned}$$

Thus, u is a ground state solution. From the assumption on f, we have $u \ge 0$, and thus $u(x) > 0$ for all $x \in \mathbb{R}^N$. The proof is complete. □

Arguing as in [5, Proposition 4], there exists a positive radial ground state solution of problem (4.1), which implies that this solution decays exponentially at infinity with its gradient; moreover, this ground state solution is of class $C^2(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$.

Lemma 4.6. *Let* $(u_n) \subset \mathcal{N}_0$ *be such that* $I_0(u_n) \to c_{V_0}$. *Then* (u_n) *has a convergent subsequence in* H_0.

Proof. Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.1 (a3), Lemma 4.2 (b4) and the definition of c_{V_0}, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0 \quad \text{for all } n \in N,$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{u \in S_0} \Psi_0(u).$$

Since S_0 is a complete C^1 manifold, by the Ekeland's variational principle, there exists a sequence $\{\tilde{v}_n\} \subset S_0$ such that $\{\tilde{v}_n\}$ is a $(\mathrm{PS})_{c_{V_0}}$ sequence for Ψ_0 on S_0 and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Similar to Lemma 4.5, we may obtain the conclusion of this lemma. □

4.2 The Technical Results

In this subsection, we prove a multiplicity result for the modified problem (3.1) using the Ljusternik–Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, let $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and let $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \le t \le \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \ge \delta$.

For any $y \in M$, let us introduce the function

$$\Psi_{\epsilon,y}(x) := \eta(|\epsilon x - y|)\omega\Big(\frac{\epsilon x - y}{\epsilon}\Big) \exp\Big(i\tau_y\Big(\frac{\epsilon x - y}{\epsilon}\Big)\Big),$$

where

$$\tau_y(x) := \sum_{i=1}^{3} A_i(y)x_i.$$

Let $t_\epsilon > 0$ be the unique positive number such that

$$\max_{t\geq 0} J_\epsilon(t\Psi_{\epsilon,y}) = J_\epsilon(t_\epsilon \Psi_{\epsilon,y}).$$

Note that $t_\epsilon \Psi_{\epsilon,y} \in \mathcal{N}_\epsilon$.

Let us define $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by

$$\Phi_\epsilon(y) := t_\epsilon \Psi_{\epsilon,y}.$$

By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$. Moreover, the energy of the above functions has the following behavior as $\epsilon \to 0^+$.

Lemma 4.7. *The limit*

$$\lim_{\epsilon\to 0^+} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that the statement is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0^+$ satisfying

$$|J_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - c_{V_0}| \geq \delta_0.$$

For simplicity, we write Φ_n, Ψ_n and t_n for $\Phi_{\epsilon_n}(y_n)$, Ψ_{ϵ_n,y_n} and t_{ϵ_n}, respectively.

By the Lebesgue dominated convergence theorem, we have that

$$\|\Psi_n\|_{\epsilon_n}^2 \to \int_{\mathbb{R}^3} (|\nabla\omega|^2 + V_0\omega^2)\,dx \quad \text{as } n \to +\infty, \tag{4.2}$$

$$[\Psi_n]_{A_{\epsilon_n}}^4 \to [\omega]^4 \quad \text{as } n \to +\infty. \tag{4.3}$$

Since $J'_{\epsilon_n}(t_n\Psi_n)(t_n\Psi_n) = 0$, by the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$, observe that, if $z \in B_{\delta/\epsilon_n}(0)$, then $\epsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. We have

$$\begin{aligned}
\|\Psi_n\|_{\epsilon_n}^2 + t_n^2 b[\Psi_n]_{A_{\epsilon_n}}^4 &= \int_{\mathbb{R}^3} g(\epsilon_n z + y_n, t_n^2\eta^2(|\epsilon_n z|)\omega^2(z))\eta^2(|\epsilon_n z|)\omega^2(z)\,dz \\
&= \int_{\mathbb{R}^3} f(t_n^2\eta^2(|\epsilon_n z|)\omega^2(z))\eta^2(|\epsilon_n z|)\omega^2(z)\,dz \\
&\geq \int_{B_{\delta/(2\epsilon_n)}(0)} f(t_n^2\omega^2(z))\omega^2(z)\,dz \\
&\geq \int_{B_{\delta/2}(0)} f(t_n^2\omega^2(z))\omega^2(z)\,dz \\
&\geq f(t_n^2\gamma^2) \int_{B_{\delta/2}(0)} \omega^4(z)\,dz
\end{aligned}$$

for all n large enough and where $\gamma = \min\{\omega(z) : |z| \leq \frac{\delta}{2}\}$. Moreover, we have

$$t_n^{-2}\|\Psi_n\|_{\epsilon_n}^2 + b[\Psi_n]_{A_{\epsilon_n}}^4 \geq \frac{f(t_n^2\gamma^2)}{t_n^2\gamma^2}\gamma^2 \int_{B_{\delta/2}(0)} \omega^4(z)\,dz.$$

If $t_n \to +\infty$, by (f4) we derive a contradiction.

Therefore, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$. If $t_n \to 0$, using the fact that f is increasing and using the Lebesgue dominated convergence theorem, we obtain that

$$\|\Psi_n\|_{\epsilon_n}^2 + t_n^2 b[\Psi_n]_{A_{\epsilon_n}}^4 = \int_{\mathbb{R}^3} f(t_n^2\eta^2(|\epsilon_n z|)\omega^2(z))\eta^2(|\epsilon_n z|)\omega^2(z)\,dz \to 0 \quad \text{as } n \to +\infty,$$

which contradicts (4.2). Thus, from (4.2) and (4.3), we have $t_0 > 0$ and

$$\int_{\mathbb{R}^3} (|\nabla \omega|^2 + V_0 \omega^2)\, dx + t_0^2 b[\omega]^4 = \int_{\mathbb{R}^3} f(t_0 \omega^2)\omega^2\, dx,$$

so that $t_0 \omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue dominated convergence theorem, we get

$$\lim_n \int_{\mathbb{R}^3} F(|t_n \Psi_n|^2)\, dx = \int_{\mathbb{R}^3} F(\omega^2)\, dx.$$

Hence

$$\lim_n J_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = I_0(\omega) = c_{V_0},$$

which is a contradiction and the proof is complete. □

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$\Upsilon(x) := \begin{cases} x & \text{if } |x| < \rho, \\ \dfrac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^3$ is defined by

$$\beta_\epsilon(u) := \frac{1}{\|u\|_4^4} \int_{\mathbb{R}^3} \Upsilon(\epsilon x)|u(x)|^4\, dx.$$

We have the following lemma.

Lemma 4.8. *The limit*

$$\lim_{\epsilon \to 0^+} \beta_\epsilon(\Phi_\epsilon(y)) = y$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that there exist $\kappa > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| \geq \kappa. \tag{4.4}$$

Using the change of variable $z = (\epsilon_n x - y_n)/\epsilon_n$, we can see that

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3} (\Upsilon(\epsilon_n z + y_n) - y_n)\eta^4(|\epsilon_n z|)\omega^4(z)\, dz}{\int_{\mathbb{R}^3} \eta^4(|\epsilon_n z|)\omega^4(z)\, dz}.$$

Taking into account $(y_n) \subset M \subset M_\delta \subset B_\rho$ and the Lebesgue dominated convergence theorem, we can obtain that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.4). □

Now, we prove the following useful compactness result.

Proposition 4.9. *Let $\epsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $J_{\epsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that the sequence $(|v_n|) \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \epsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.*

Proof. Since $J'_{\epsilon_n}(u_n)[u_n] = 0$ and $J_{\epsilon_n}(u_n) \to c_{V_0}$, arguing as in Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\epsilon_n} \leq C$ for all $n \in \mathbb{N}$.

Arguing as in the proof of Lemma 3.2 and recalling that $c_{V_0} > 0$, we have that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2\, dx \geq \beta. \tag{4.5}$$

Now, let us consider the sequence $\{|v_n|\} \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$. Using (4.5), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ for some $v \neq 0$.

Let $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \epsilon_n \tilde{y}_n$.

By the diamagnetic inequality (2.1), we have

$$c_{V_0} \leq I_0(\tilde{v}_n) \leq \max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.

Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^3, \mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, we have that (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, that is, $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$.

Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, and, by Lemma 4.6, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$.

Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then, for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\epsilon_n}(0)$,

$$|\epsilon_n x + y_n| \geq |y_n| - \epsilon_n|x| > R.$$

Since $u_n \in \mathcal{N}_{\epsilon_n}$, using (V1) and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}\int_{\mathbb{R}^3} (a|\nabla|v_n||^2 + V_0|v_n|^2)\,dx &\leq \int_{\mathbb{R}^3} g(\epsilon_n x + y_n, |v_n|^2)|v_n|^2\,dx \\ &\leq \int_{B_{R/\epsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2\,dx + \int_{B^c_{R/\epsilon_n}(0)} f(|v_n|^2)|v_n|^2\,dx. \end{aligned} \tag{4.6}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\tilde{f}(t) \leq V_0/K$, we can see that (4.6) yields

$$\min\Big\{1, V_0\Big(1 - \frac{1}{K}\Big)\Big\} \int_{\mathbb{R}^3} (a|\nabla|v_n||^2 + V_0|v_n|^2)\,dx = o_n(1),$$

that is, $|v_n| \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which contradicts to $v \not\equiv 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\epsilon_n}(0)$, we have that $|\epsilon_n x + y_n - y_0| < 2r$ so that $\epsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n)$, we obtain

$$\begin{aligned} c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla\tilde{v}|^2 + V(y_0)|\tilde{v}|^2)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla\tilde{v}|^2\,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}|^2)\,dx \\ &\leq \liminf_n \Big(\frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla\tilde{v}_n|^2 + V(\epsilon_n x + y_n)|\tilde{v}_n|^2)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla\tilde{v}_n|^2\,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}_n|^2)\,dx\Big) \\ &= \liminf_n \Big(\frac{t_n^2}{2}\int_{\mathbb{R}^3} (a|\nabla|u_n||^2 + V(\epsilon_n z)|u_n|^2)\,dx + \frac{t_n^4 b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} F(|t_n u_n|^2)\,dx\Big) \\ &\leq \liminf_n J_{\epsilon_n}(t_n u_n) \\ &\leq \liminf_n J_{\epsilon_n}(u_n) = c_{V_0}, \end{aligned}$$

which is impossible and the proof is complete. □

Let now

$$\tilde{\mathcal{N}}_\epsilon := \{u \in \mathcal{N}_\epsilon : J_\epsilon(u) \leq c_{V_0} + h(\epsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\epsilon) \to 0$ as $\epsilon \to 0^+$.

For fixed $y \in M$, since, by Lemma 4.7, $|J_\epsilon(\Phi_\epsilon(y)) - c_{V_0}| \to 0$ as $\epsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\epsilon \neq \emptyset$ for any $\epsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\epsilon$ and the barycenter map.

Lemma 4.10. *We have*

$$\lim_{\epsilon \to 0^+} \sup_{u \in \tilde{\mathcal{N}}_\epsilon} \operatorname{dist}(\beta_\epsilon(u), M_\delta) = 0.$$

Proof. Let $\epsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ such that

$$\sup_{u \in \tilde{\mathcal{N}}_{\epsilon_n}} \inf_{y \in M_\delta} |\beta_{\epsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\epsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\epsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_0(t|u_n|) \leq J_{\epsilon_n}(tu_n)$ for any $t \geq 0$. Therefore, recalling that $\{u_n\} \subset \tilde{\mathcal{N}}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, we can deduce that

$$c_{V_0} \leq \max_{t \geq 0} I_0(t|u_n|) \leq \max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) \leq c_{V_0} + h(\epsilon_n),$$

which implies that $J_{\epsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$. Then Proposition 4.9 implies that there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \epsilon_n \tilde{y}_n \in M_\delta$ for n large enough.

Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\epsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\Upsilon(\epsilon_n z + y_n) - y_n)|u_n(z + \tilde{y}_n)|^4 \, dz}{\int_{\mathbb{R}^3} |u_n(z + \tilde{y}_n)|^4 \, dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\epsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^3$, we conclude the proof. □

4.3 Multiplicity of Solutions for Problem (3.1)

Finally, we present a relation between the topology of M and the number of solutions of the modified problem (3.1).

Theorem 4.11. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\epsilon}_\delta > 0$ such that, for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$, problem* (3.1) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *nontrivial solutions.*

Proof. For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S_\epsilon^+$ by

$$\pi_\epsilon(y) = m_\epsilon^{-1}(\Phi_\epsilon(y)) \quad \text{for all } y \in M.$$

By Lemma 4.7 and Lemma 3.3 (B4), we obtain

$$\lim_{\epsilon \to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon \to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0} \quad \text{uniformly in } y \in M.$$

Hence, there is a number $\hat{\epsilon} > 0$ such that the set $\tilde{S}_\epsilon^+ := \{u \in S_\epsilon^+ : \Psi_\epsilon(u) \leq c_{V_0} + h(\epsilon)\}$ is nonempty for all $\epsilon \in (0, \hat{\epsilon})$ since $\pi_\epsilon(M) \subset \tilde{S}_\epsilon^+$. Here h is given in the definition of $\tilde{\mathcal{N}}_\epsilon$.

Given $\delta > 0$, by Lemma 4.7, Lemma 3.2 (A3), Lemma 4.8, and Lemma 4.10, we can find $\tilde{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$ the diagram

$$M \xrightarrow{\Phi_\epsilon} \Phi_\epsilon(M) \xrightarrow{m_\epsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\epsilon} \Phi_\epsilon(M) \xrightarrow{\beta_\epsilon} M_\delta$$

is well defined and continuous. From Lemma 4.8, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$ for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\epsilon(\Phi_\epsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define

$$H(t, z) = z + (1 - t)\Theta(\epsilon, z).$$

Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, z) = \beta_\epsilon(\Phi_\epsilon(z))$ and $H(1, z) = z$ for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\epsilon \circ \Phi_\epsilon = (\beta_\epsilon \circ m_\epsilon) \circ \pi_\epsilon$ and the embedding $\iota : M \to M_\delta$. Thus, this fact implies that

$$\operatorname{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \geq \operatorname{cat}_{M_\delta}(M). \tag{4.7}$$

By Corollary 3.6 and the abstract category theorem [21], Ψ_ϵ has at least $\operatorname{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on S_ϵ^+. Therefore, from Lemma 3.3 (B4) and (4.7), we have that J_ϵ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\epsilon$, which implies that problem (3.1) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions. □

5 Proof of Theorem 1.1

In this section, we prove our main result. The idea is to show that the solutions u_ϵ obtained in Theorem 4.11 satisfy

$$|u_\epsilon(x)|^2 \leq a_0 \quad \text{for } x \in \Lambda_\epsilon^c$$

for $\epsilon > 0$ small. Arguing as in [26], the following uniform result holds.

Lemma 5.1. *Let $\epsilon_n \to 0^+$ and let $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ be a solution of problem (3.1) for $\epsilon = \epsilon_n$. Then $J_{\epsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^3, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \hat{\epsilon}_\delta)$ and any solution $u_\epsilon \in \tilde{\mathcal{N}}_\epsilon$ of problem (3.1), it holds

$$\|u_\epsilon\|^2_{L^\infty(\Lambda_\epsilon^c)} \leq a_0. \tag{5.1}$$

We argue by contradiction and assume that there is a sequence $\epsilon_n \to 0$ such that for every n there exists $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ which satisfies $J'_{\epsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda_{\epsilon_n}^c)} > a_0. \tag{5.2}$$

As in Lemma 5.1, we have that $J_{\epsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.9 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n := \epsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then we can find $r > 0$ such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\epsilon_n}(\tilde{y}_n) \subset \Lambda_{\epsilon_n}$ for all n large enough.

By using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \leq a_0$ in $B_R^c(0)$ and n large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \leq a_0$ in $B_R^c(\tilde{y}_n)$ and n large enough. Moreover, if n is so large that $r/\epsilon_n > R$, then

$$\Lambda_{\epsilon_n}^c \subset B_{r/\epsilon_n}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n),$$

which gives $|u_n|^2 \leq a_0$ for any $x \in \Lambda_{\epsilon_n}^c$. This contradicts (5.2) and proves the claim.

Let now $\epsilon_\delta := \min\{\hat{\epsilon}_\delta, \tilde{\epsilon}_\delta\}$, where $\tilde{\epsilon}_\delta > 0$ is given by Theorem 4.11. Then we have $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.1). If $u_\epsilon \in \tilde{\mathcal{N}}_\epsilon$ is one of these solutions, then, by (5.1) and the definition of g, we conclude that u_ϵ is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\epsilon|$, where $\hat{u}_\epsilon(x) := u_\epsilon(\frac{x}{\epsilon})$ is a solution to problem (1.1) as $\epsilon \to 0^+$.

Take $\epsilon_n \to 0^+$ and the sequence (u_n) where each u_n is a solution of (3.1) for $\epsilon = \epsilon_n$. From the definition of g, there exists $\gamma \in (0, a)$ such that

$$g(\epsilon x, t^2)t^2 \leq \frac{V_0}{K} t^2 \quad \text{for all } x \in \mathbb{R}^N,\ |t| \leq \gamma.$$

Arguing as above, we can take $R > 0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{5.3}$$

Up to a subsequence, we may also assume that, for n large enough,

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.4}$$

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < \gamma$. Thus, since $J'_{\epsilon_n}(u_{\epsilon_n}) = 0$, using (g5) and the diamagnetic inequality (2.1), we obtain

$$\int_{\mathbb{R}^3} (a|\nabla|u_n||^2 + V_0|u_n|^2)\,dx + b\Big(\int_{\mathbb{R}^3} (|\nabla|u_n||^2)\,dx\Big)^2 \leq \int_{\mathbb{R}^3} g(\epsilon_n x, |u_n|^2)|u_n|^2\,dx \leq \frac{V_0}{K}\int_{\mathbb{R}^3} |u_n|^2\,dx.$$

Since $K > 2$, we obtain $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points p_n of $|u_{\epsilon_n}|$ belong to $B_R(\tilde{y}_n)$, that is, $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of problem (1.1) is $\hat{u}_n(x) = u_n(x/\epsilon_n)$, we can see that a maximum point η_{ϵ_n} of $|\hat{u}_n|$ is $\eta_{\epsilon_n} = \epsilon_n\tilde{y}_n + \epsilon_n q_n$. Since $q_n \in B_R$, $\epsilon_n\tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\epsilon_n}) = V_0.$$

The proof is now complete. □

Acknowledgment: The authors thank the referees for several valuable comments and remarks.

Funding: C. Ji was supported by Natural Science Foundation of Shanghai (20ZR1413900, 18ZR1409100). The work of V. D. Rădulescu was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI - UEFISCDI, project number PCE 137 / 2021, within PNCDI III. Moreover, V. D. Rădulescu acknowledges the support of the Slovenian Research Agency grant P1-0292.

References

[1] C. O. Alves, G. M. Figueiredo and M. F. Furtado, Multiple solutions for a nonlinear Schrödinger equation with magnetic fields, *Comm. Partial Differential Equations* **36** (2011), no. 9, 1565–1586.
[2] V. Ambrosio, Multiplicity and concentration of solutions for a fractional Kirchhoff equation with magnetic field and critical growth, *Ann. Henri Poincaré* **20** (2019), no. 8, 2717–2766.
[3] V. Ambrosio and V. D. Rădulescu, Fractional double-phase patterns: Concentration and multiplicity of solutions, *J. Math. Pures Appl. (9)* **142** (2020), 101–145.
[4] G. Arioli and A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, *Arch. Ration. Mech. Anal.* **170** (2003), no. 4, 277–295.
[5] J. Byeon, L. Jeanjean and M. Mariş, Symmetry and monotonicity of least energy solutions, *Calc. Var. Partial Differential Equations* **36** (2009), no. 4, 481–492.
[6] M. del Pino and P. L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* **4** (1996), no. 2, 121–137.
[7] M. J. Esteban and P.-L. Lions, Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, in: *Partial Differential Equations and the Calculus of Variations. Vol. I*, Progr. Nonlinear Differential Equations Appl. 1, Birkhäuser, Boston (1989), 401–449.
[8] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* **69** (1986), no. 3, 397–408.
[9] X. He and W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, *J. Differential Equations* **252** (2012), no. 2, 1813–1834.
[10] X. He and W. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, *Calc. Var. Partial Differential Equations* **55** (2016), no. 4, Article ID 91.
[11] X. He and W. Zou, Multiplicity of concentrating solutions for a class of fractional Kirchhoff equation, *Manuscripta Math.* **158** (2019), no. 1–2, 159–203.

[12] C. Ji, F. Fang and B. Zhang, A multiplicity result for asymptotically linear Kirchhoff equations, *Adv. Nonlinear Anal.* **8** (2019), no. 1, 267–277.
[13] C. Ji and V. D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation, *Calc. Var. Partial Differential Equations* **59** (2020), no. 4, Paper No. 115.
[14] C. Ji and V. D. Rădulescu, Concentration phenomena for nonlinear magnetic Schrödinger equations with critical growth, *Israel J. Math.* **241** (2021), 465–500.
[15] C. Ji and V. D. Rădulescu, Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$, *Manuscripta Math.* **164** (2021), no. 3–4, 509–542.
[16] C. Ji and V. D. Rădulescu, Multi-bump solutions for the nonlinear magnetic Choquard–Schrödinger equation with deepening potential well, preprint.
[17] E. H. Lieb and M. Loss, *Analysis*, Grad. Stud. in Math. 14, American Mathematical Society, Providence, 2001.
[18] X. Mingqi, V. D. Rădulescu and B. Zhang, A critical fractional Choquard–Kirchhoff problem with magnetic field, *Commun. Contemp. Math.* **21** (2019), no. 4, Article ID 1850004.
[19] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Nonlinear analysis—Theory and Methods*, Springer Monogr. Math., Springer, Cham, 2019.
[20] A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.* **257** (2009), no. 12, 3802–3822.
[21] A. Szulkin and T. Weth, The method of Nehari manifold, *Handbook of Nonconvex Analysis and Applications*, International Press, Boston (2010), 2314–2351.
[22] X. H. Tang and S. Chen, Ground state solutions of Nehari–Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var. Partial Differential Equations* **56** (2017), no. 4, Paper No. 110.
[23] X. H. Tang and B. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, *J. Differential Equations* **261** (2016), no. 4, 2384–2402.
[24] T. von Puttkamer and X. Wu, Linear groups, conjugacy growth, and classifying spaces for families of subgroups, *Int. Math. Res. Not. IMRN* (2019), no. 10, 3130–3168.
[25] Y. Zhang, X. Tang and V. D. Rădulescu, Small perturbations for nonlinear Schrödinger equations with magnetic potential, *Milan J. Math.* **88** (2020), no. 2, 479–506.
[26] A. Zhu and X. Sun, Multiple solutions for Schrödinger–Poisson type equation with magnetic field, *J. Math. Phys.* **56** (2015), no. 9, Article ID 091504.