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Research Article

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Nonlinear Nonhomogeneous Robin Problems with Superlinear Reaction Term

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Abstract: We consider a nonlinear Robin problem driven by a nonlinear, nonhomogeneous differential operator, and with a Carathéodory reaction term which is $(p-1)$-superlinear near $\pm\infty$ without satisfying the Ambrosetti–Rabinowitz condition and which does not have a standard subcritical polynomial growth. Using a combination of variational methods and Morse theoretic techniques, we prove a multiplicity theorem producing three nontrivial solutions (two of which have constant sign). In the process we establish some useful facts about the boundedness of the weak solutions of critical equations and the relation of Sobolev and Hölder local minimizers for functionals with a critical perturbation term.

Keywords: Critical Growth, Almost Critical Growth, Boundedness of Weak Solutions, Nonlinear Regularity and Nonlinear Maximum Principle, Superlinear Reaction, Three Nontrivial Solutions, Critical Groups

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and consider the following semilinear Dirichlet problem:

$$-\Delta u(z) = f(u(z)) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1.1}$$

Suppose that the reaction term $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$f \in C^1(\mathbb{R}, \mathbb{R}), \qquad f(0) = f'(0) = 0, \qquad |f'(x)| \le c_1(1 + |x|^{r-2}) \quad \text{for all } x \in \mathbb{R},$$

where $c_1 > 0$ and $2 < r < 2^*$ with

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

and there exist $\mu > 2$ and $M > 0$ such that

$$0 < \mu F(x) \le f(x)x \quad \text{for all } |x| \ge M, \qquad \text{with } F(x) = \int_0^x f(s)\,ds. \tag{1.2}$$

In (1.2) we recognize the Ambrosetti–Rabinowitz condition (AR-condition for short). Integrating (1.2), we obtain the following weaker condition:

$$c_2|x|^\mu \le F(x) \quad \text{for all } |x| \ge M \text{ and some } c_2 > 0. \tag{1.3}$$

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From (1.2) and (1.3), it follows that $f(\cdot)$ is superlinear near $\pm\infty$, that is,

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = +\infty.$$

Under (1.2), in a well-known paper, Wang [42] proved that problem (1.1) admits at least three nontrivial solutions. The multiplicity result of Wang [42] was extended to Dirichlet problems driven by the p-Laplacian by Liu [25]. More recent works relaxed the AR-condition. In this direction, we mention the papers [22, 28, 38] for Dirichlet problems, and [2] for Neumann problems always with the p-Laplacian as differential operator. Very recently Mugnai and Papageorgiou [31] extended the aforementioned result of Wang to Dirichlet (p, q)-equations (that is, equations driven by the sum of a p-Laplacian and a q-Laplacian, $1 < q < p < \infty$), without assuming the AR-condition.

The aim of this paper is to prove such a "three solutions theorem" for a larger class of differential equations in which the differential operator need not be homogeneous and covers as a special case the p-Laplacian $(1 < p < \infty)$. So, as above, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. The problem under consideration is the following:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In this problem $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous and strictly monotone and satisfies certain other regularity and growth conditions. The precise requirements on the map $a(\cdot)$ are listed in hypotheses (Ha) below. These hypotheses are quite general and incorporate in our framework many differential operators of interest such as the p-Laplacian and the (p, q)-Laplacian. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the generalized normal derivative defined by

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This particular normal derivative is dictated by the nonlinear Green's identity (see, for example, [13, p. 210]) and is also used by Lieberman in [23]. The reaction term $f(z, x)$ is a Carathéodory function (that is, $z \mapsto f(z, x)$ is measurable for all $x \in \mathbb{R}$ and continuous for almost all $z \in \Omega$), which is $(p - 1)$-superlinear in the x-variable but without satisfying the AR-condition. In this way we can fit in our analysis superlinear nonlinearities with "slower" growth near $\pm\infty$ which fail to satisfy the AR-condition. In addition, $f(z, \cdot)$ needs not to satisfy a polynomial subcritical growth and it grows in an almost critical fashion (see hypothesis (Hf) (i)). The nonhomogeneity of the differential operator and the failure of the Poincaré inequality in the ambient Sobolev space $W^{1,p}(\Omega)$, as well as the almost critical growth of the reaction term, are sources of difficulties which require new methods and techniques in order to overcome them.

Our approach uses variational tools based on the critical point theory together with Morse theory (critical groups). Also, the almost critical growth of $f(z, \cdot)$ requires a careful analysis of the boundedness of the weak solutions of (1.4).

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Given $\varphi \in C^1(X, \mathbb{R})$ we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

- Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded, with

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ which compensates for the fact the ambient space X need not be locally compact (usually X is infinite dimensional). It is more general than the usual Palais–Smale condition. Nevertheless, the C-condition leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. Prominent in this theory is the so-called "mountain pass theorem", due to Ambrosetti and Rabinowitz [4]. Here we state it in a slightly more general form (see, for example, [13, p. 648]).

Theorem 2.1. *Let X be a Banach space, let $\varphi \in C^1(X, \mathbb{R})$ satisfy the C-condition, let $u_0, u_1 \in X$ be such that*

$$\|u_1 - u_0\| > \rho > 0, \quad \max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho,$$

and let

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

Then $c \ge m_\rho$ and c is a critical value of φ.

Let $\vartheta \in C^1(0, \infty)$ and assume that it satisfies the following growth conditions:

$$0 < \hat{c} \le \frac{t\vartheta'(t)}{\vartheta(t)} \le c_0 \quad \text{and} \quad c_1 t^{p-1} \le \vartheta(t) \le c_2(1 + t^{p-1}) \tag{2.1}$$

for all $t > 0$ and some $c_1, c_2 > 0$, $1 < p < \infty$.

We introduce the precise conditions on the map $y \to a(y)$, $y \in \mathbb{R}^N$, involved in the definition of the differential operator.

(Ha) We set $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, and assume the following:

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0, \infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1.$$

(ii) For some $c_3 > 0$ and all $y \in \mathbb{R}^N \setminus \{0\}$,

$$|\nabla a(y)| \le c_3 \frac{\vartheta(|y|)}{|y|}.$$

(iii) For all $y \in \mathbb{R}^N \setminus \{0\}$ and all $\xi \in \mathbb{R}^N$,

$$\frac{\vartheta(|y|)}{|y|}|\xi|^2 \le (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}.$$

(iv) If

$$G_0(t) = \int_0^t a_0(s)s\,ds \quad \text{for all } t \ge 0,$$

then we have

$$-\tilde{c} \le pG_0(t) - a_0(t)t^2 \quad \text{for all } t \ge 0 \text{ with } \tilde{c} > 0.$$

Remark 2.2. Hypotheses (Ha) (i)–(iii) are dictated by the nonlinear regularity theory of Lieberman [24] and the nonlinear maximum principle of Pucci and Serrin [36]. Hypothesis (Ha) (iv) corresponds to the particular features of our problem, but it is very mild and it is satisfied in all the major cases of interest as the examples below illustrate.

Set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}, \qquad \nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$.

Hypotheses (Ha) imply that the functions $G(\cdot)$, $G_0(\cdot)$ are both strictly convex and $G_0(\cdot)$ is also strictly increasing. The convexity of $G(\cdot)$ and the fact that $G(0) = 0$ imply

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$ and it is a straightforward consequence of hypotheses (Ha).

Lemma 2.3. *If hypotheses* (Ha) (i)–(iii) *hold, then so do the following:*
(a) $y \to a(y)$ *is continuous and strictly monotone, hence maximal monotone too,*
(b) $|a(y)| \leq c_4(1 + |y|^{p-1})$ *for some* $c_4 > 0$ *and all* $y \in \mathbb{R}^N$,
(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma, together with (2.1) and (2.2), leads to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.4. *If hypotheses* (Ha) (i)–(iii) *hold, then*

$$\frac{c_1}{p-1}|y|^p \leq G(y) \leq c_5(1 + |y|^p) \quad \textit{for all } y \in \mathbb{R}^N \textit{ and some } c_5 > 0.$$

Example 2.5. The following maps $a(\cdot)$ satisfy hypotheses (Ha):
(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < p < q < \infty$. This map corresponds to the (p, q)-differential operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such differential operators arise in many physical applications, see [6] (quantum physics) and [8] (plasma physics). Recently there have been some existence and multiplicity results for such equations, see [5, 9, 15, 27, 34, 35, 39, 40].
(c) $a(y) = (1 + |y|^2)^{(p-2)/2}y$ with $1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{(p-2)/2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y(1 + 1/(1 + |y|^{2p})^{1/2})$ with $1 < p < \infty$.

Our hypothesis on the boundary weight function $\beta(\cdot)$ is the following:
(Hβ) $\beta \in C^{1,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1)$, $\beta \geq 0$.

Remark 2.6. If $\beta \equiv 0$, then we have the Neumann problem.

In the analysis of problem (1.4), in addition to the Sobolev space $W^{1,p}(\Omega)$ we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we use the $(N-1)$-dimensional surface (Hausdorff) measure $\sigma(\cdot)$ and using this measure we can define the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leq p \leq \infty)$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the *trace map*, such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Recall that $\operatorname{im} \gamma_0 = W^{1/p',p}(\partial\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ and $\ker \gamma_0 = W_0^{1,p}(\Omega)$. Moreover, the trace map γ_0 is compact in $L^q(\partial\Omega)$ for all $q \in [1, \frac{Np-p}{N-p})$. In the sequel for the sake of notational simplicity, we will drop the use of the trace map γ_0. It is understood that all restrictions of Sobolev functions on the boundary $\partial\Omega$ are defined in the sense of traces.

In what follows, by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

For every $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We have

$$u = u^+ - u^-, \quad |u| = u^+ + u^- \quad \text{and} \quad u^+, u^- \in W^{1,p}(\Omega).$$

Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by $A_p\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ the nonlinear map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W^{1,p}(\Omega). \tag{2.3}$$

The next proposition is a special case of a more general result of Gasinski and Papageorgiou [14, 16].

Proposition 2.7. *The map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$, defined by (2.3), is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, if*

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$.

We consider the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f_0(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

In this problem, $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with critical growth in the x-variable, that is,

$$|f_0(z, x)| \le a_0(z)(1 + |x|^{p^*-1}) \quad \text{for almost all } z \in \Omega \text{ and all } z \in \mathbb{R}, \tag{2.5}$$

with $a_0 \in L^\infty(\Omega)_+$ and

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p. \end{cases}$$

By a *weak solution* of problem (2.4) we understand a function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\,d\sigma = \int_\Omega f_0(z, u)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

Next we establish the boundedness of weak solutions. Due to the critical growth of $f(z,\cdot)$, the Moser iteration technique used by Hu and Papageorgiou [18], and Winkert [43] does not work. Instead, we follow the approach of Garcia Azorero and Peral Alonso [11] (see also [41] for semilinear equations). An alternative method can be based on the work of Guedda and Véron [17].

Proposition 2.8. *If hypotheses* (Ha), (Hβ) *hold and $u \in W^{1,p}(\Omega)$ is a weak solution of* (2.4), *then $u \in L^q(\Omega)$ for all $q \in [1, \infty)$.*

Proof. Recalling that $u = u^+ - u^-$ and performing the argument on u^+ and u^- separately, we see that without any loss of generality, we may assume that $u \ge 0$.

For $\beta > 1$ and $\lambda > 0$, we introduce the following Lipschitz continuous functions:

$$H(t) = \begin{cases} t^\beta & \text{if } 0 \le t \le \lambda, \\ \beta\lambda^{\beta-1}(t-\lambda) + \lambda^\beta & \text{if } \lambda < t, \end{cases}$$

$$S(t) = \begin{cases} t^{(\beta-1)p+1} & \text{if } 0 \le t \le \lambda, \\ \beta((\beta-1)p+1)\lambda^{(\beta-1)p}(t-\lambda) + \lambda^{(\beta-1)p+1} & \text{if } \lambda < t. \end{cases}$$

It is easy to check that these two functions satisfy the following properties (see, e.g., [13, p. 194] or [11]):

(P1) $S(t) \le tS'(t)$ for all $t \ge 0$,

(P2) $c_5 H'(t) \le S'(t)$ for all $t \ge 0$ with $c_5 > 0$ independent of $\lambda > 0$,

(P3) $t^{p-1}S(t) \le c_7 H(t)^p$ for all $t \ge 0$ with $c_7 > 0$ independent of $\lambda > 0$, and $H(y), S(y) \in W^{1,p}(\Omega)$ for every $y \in W^{1,p}(\Omega)$,

We fix $\beta > 1$ such that $\beta p < p^*$, and let $\vartheta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ with $0 \le \vartheta \le 1$ to be fixed precisely in the process of the proof. We use the test function

$$h = \vartheta^p S(u) \in W^{1,p}(\Omega), \quad h \ge 0.$$

We have

$$\int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}h\,d\sigma = \int_\Omega f_0(z,u)h\,dz. \tag{2.6}$$

Note that

$$Dh = p\vartheta^{p-1}S(u)D\vartheta + \vartheta^p G'(u)Du,$$

and so

$$\int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz = p\int_\Omega \vartheta^{p-1}S(u)(a(Du), D\vartheta)_{\mathbb{R}^N}\,dz + \int_\Omega \vartheta^p G'(u)(a(Du), Du)_{\mathbb{R}^N}\,dz. \tag{2.7}$$

Using (2.7) in (2.6), we have

$$\int_\Omega \vartheta^p G'(u)(a(Du), Du)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}h\,d\sigma \le \int_\Omega f_0(z,u)h\,dz - p\int_\Omega \vartheta^{p-1}S(u)(a(Du), D\vartheta)_{\mathbb{R}^N}\,dz. \tag{2.8}$$

From Lemma 2.3 and since $\vartheta^p S'(u) \ge 0$ (see (P1)), we have

$$\frac{c_1}{p-1}\int_\Omega \vartheta^p S'(u)|Du|^p\,dz \le \int_\Omega \vartheta^p S'(u)(a(Du), Du)_{\mathbb{R}^N}\,dz. \tag{2.9}$$

Also, using (P1) and Young's inequality with $\epsilon > 0$, we have

$$\begin{aligned}
\Bigl|p\int_\Omega \vartheta^{p-1}S(u)(a(Du), D\vartheta)_{\mathbb{R}^N}\,dz\Bigr| &\le p\int_\Omega \vartheta^{p-1}S(u)|a(Du)||D\vartheta|\,dz \\
&= p\int_\Omega \vartheta^{p-1}S(u)^{1/p}S(u)^{(p-1)/p}|a(Du)||D\vartheta|\,dz \\
&\le p\int_\Omega \vartheta^{p-1}|a(Du)|S(u)^{1/p}(uS'(u))^{(p-1)/p}|D\vartheta|\,dz \\
&\le \epsilon\int_\Omega \vartheta^p|a(Du)|^{p/(p-1)}S'(u)\,dz + c_\epsilon\int_\Omega u^{p-1}S(u)|D\vartheta|\,dz \\
&\le \epsilon\int_\Omega c_8(1+|Du|^p)S'(u)\vartheta^p\,dz + c_\epsilon\int_\Omega u^{p-1}S(u)|D\vartheta|^p\,dz
\end{aligned} \tag{2.10}$$

for some $c_8 > 0$, with $c_\epsilon > 0$ (see Lemma 2.3).

We return to (2.8) and use (2.9) and (2.10). So, choosing $\epsilon \in (0,1)$ small and since $\int_{\partial\Omega} \beta(z)u^{p-1}h \, d\sigma \geq 0$, we have (see (2.5) and (P3))

$$\begin{aligned}\int_\Omega \vartheta^p S'(u)|Du|^p \, dz &\leq \epsilon c_9 \int_\Omega \vartheta^p S'(u)\, dz + c_9 c_\epsilon \int_\Omega u^{p-1}S(u)|D\vartheta|^p \, dz + c_9 \int_\Omega f_0(z,u)\vartheta^p S(u)\, dz \\ &\leq \epsilon c_9 \int_\Omega \vartheta^p S'(u)\, dz + c_9 c_\epsilon \int_\Omega H(u)^p |D\vartheta|^p \, dz \\ &\quad + c_9 \|a_0\|_\infty \int_\Omega \vartheta^p S(u)\, dz + c_9 \|a_0\|_\infty \int_\Omega u^{p^*-1}\vartheta^p S(u)\, dz \end{aligned} \tag{2.11}$$

for some $c_9 > 0$. Using (P2), we obtain

$$c_{10} \int_\Omega (\vartheta H'(u))^p |Du|^p \, dz \leq \int_\Omega \vartheta^p S'(u)|Du|^p \, dz \quad \text{for some } c_{10} > 0. \tag{2.12}$$

Then, on account of (P3), (2.11) and (2.12), we have the following estimate:

$$\begin{aligned}\int_\Omega |D(\vartheta H(u))|^p \, dz &= \int_\Omega |\vartheta H'(u)Du + H(u)D\vartheta|^p \, dz \\ &\leq c_{11}\Big[\int_\Omega |\vartheta H'(u)Du|^p \, dz + \int_\Omega H(u)^p |D\vartheta|^p \, dz\Big] \\ &\leq c_{12}\Big[\int_\Omega \vartheta^p S'(u)|Du|^p \, dz + \int_\Omega H(u)^p |D\vartheta|^p \, dz\Big] \\ &\leq c_{13}\Big[\int_\Omega H(u)^p |D\vartheta|^p \, dz + \int_\Omega u^{p^*-p}(\vartheta H(u))^p \, dz + \int_\Omega \vartheta^p S(u)\, dz\Big] \\ &\leq c_{14}\Big[\int_\Omega H(u)^p |D\vartheta|^p \, dz + \int_\Omega u^{p^*-p}(\vartheta H(u))^p \, dz + 1\Big] \end{aligned} \tag{2.13}$$

for some $c_{1i} > 0$, $i = 1,2,3,4$, since $0 \leq \vartheta \leq 1$ and $|D\vartheta(\cdot)|$ is bounded.

We choose $\rho > 0$ such that for any ball B_ρ of radius $\rho > 0$ with $B_\rho \cap \Omega \neq \varnothing$, we have

$$\|u\|^{p^*-p}_{L^{p^*}(B_\rho \cap \Omega)} \leq \frac{1}{\eta c_{14}} \quad \text{with } \eta > 0 \tag{2.14}$$

(recall that $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$).

Given $z_0 \in \Omega$ choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ with $0 \leq \vartheta \leq 1$, $\operatorname{supp} \vartheta = \bar{B}_\rho(z_0)$ and $\vartheta \equiv 1$ on $\bar{B}_{\rho/2}(z_0)$. Using Hölder's inequality and (2.14), we have

$$\begin{aligned}\int_\Omega u^{p^*-p}\vartheta^p H(u)^p \, dz &= \int_{B_\rho(z_0)\cap\Omega} u^{p^*-p}\vartheta^p H(u)^p \, dz \\ &\leq \Big(\int_\Omega \vartheta^{p^*} H(u)^{p^*} \, dz\Big)^{p/p^*} \Big(\int_{B_\rho(z_0)\cap\Omega} u^{p^*} \, dz\Big)^{(p^*-p)/p^*} \\ &\leq \frac{1}{\eta c_{14}} \Big(\int_\Omega \vartheta^{p^*} H(u)^{p^*} \, dz\Big)^{p/p^*}. \end{aligned} \tag{2.15}$$

Note that for $\delta > 0$, $u \to \delta\|u\|_{p^*} + \|Du\|_p$ is an equivalent norm on the Sobolev space $W^{1,p}(\Omega)$ (see, for example, [13, p. 227]). So, by choosing $\delta > 0$ small, we can find $c_{15} > 0$ such that

$$\Big(\int_\Omega \vartheta^{p^*} H(u)^{p^*} \, dz\Big)^{p/p^*} \leq c_{15} \int_\Omega |D(\vartheta H(u))|^p \, dz. \tag{2.16}$$

Using (2.16) in (2.15), we obtain

$$\int_\Omega u^{p^*-p}\vartheta^p H(u)^p \, dz \leq \frac{c_{15}}{\eta c_{14}} \int_\Omega |D(\vartheta H(u))|^p \, dz. \tag{2.17}$$

Returning to (2.13) and using (2.17) with $\eta > \frac{c_{15}}{c_{14}}$, we have

$$\int_\Omega |D(\vartheta H(u))|^p \, dz \le c_{16}\Big[\int_\Omega H(u)^p \, dz + 1\Big]$$

for some $c_{16} > 0$, and hence, by (2.16),

$$\Big(\int_\Omega \vartheta^{p^*} H(u)^{p^*} \, dz\Big)^{p/p^*} \le c_{17}\Big[\int_\Omega H(u)^p \, dz + 1\Big] \tag{2.18}$$

for some $c_{17} > 0$. Letting $\lambda \to +\infty$ in (2.18) (see the definition of $H(\cdot)$ in the beginning of the proof) yields

$$\Big(\int_{B_{\rho/2}(z_0)\cap\Omega} u^{\beta p^*} \, dz\Big)^{p/p^*} \le c_{17}\Big[\int_\Omega u^{\beta p} \, dz + 1\Big]. \tag{2.19}$$

Since $\beta p < p^*$ and $u \in W^{1,p}(\Omega)$, we have that $u \in L^{\beta p^*}(B_{\rho/2}(z_0) \cap \Omega)$. Then, from (2.19) and since Ω is totally bounded, we infer that $u \in L^{\beta p^*}(\Omega)$. Fix $\epsilon_0 > 0$ such that $\beta - \epsilon_0 > 1$. Then, by repeating the above argument, we can generate a sequence $\{\beta_n\}_{n\ge1}$ such that $\beta_1 p^* < \beta p^*$, $\beta_n \ge (\beta - \epsilon_0)^n$ and $u \in L^{\beta_n p^*}(\Omega)$ for all $n \in \mathbb{N}$. Since $(\beta - \epsilon_0)^n \to +\infty$, we conclude that $u \in L^q(\Omega)$ for all $q \in [1, \infty)$. □

Next we will establish the essential boundedness of u and produce a useful bound for its L^∞-norm. We start with a lemma, which is essentially [20, Lemma 5.1, p. 71]. For completeness in our argument we include it here.

Lemma 2.9. *If $u \in W^{1,p}(\Omega)$, $0 \le u$, $q \in (1, p^*)$, $k_0 > 1$ and $\bar{c} > 0$ are such that*

$$\int_{E_k} |Du|^p \, dz \le \bar{c} k^p |E_k|_N^{p/q} \quad \text{for all } k \ge k_0, \tag{2.20}$$

where $E_k = \{z \in \Omega : u(z) \ge k\}$, then there exists $M_1 = M_1(\Omega, \bar{c}, q, k_0) > 0$ such that $\|u\|_\infty \le M$.

Proof. From [20, p. 45], we know that

$$\Big(\int_{E_k} (u - k)^p \, dz\Big)^{1/p} \le c_{18}\Big(\int_{E_k} |Du|^p \, dz\Big)^{1/p} |E_k|_N^{1/p - 1/p^*} \tag{2.21}$$

for some $c_{18} > 0$. Using (2.20), (2.21) and Hölder's inequality, we have

$$\begin{aligned} \int_{E_k} (u - k) \, dz &\le \Big(\int_{E_k} (u - k)^p \, dz\Big)^{1/p} |E_k|_N^{1-1/p} \\ &\le c_{18}\Big(\int_{E_k} |Du|^p \, dz\Big)^{1/p} |E_k|_N^{1/p-1/p^*} |E_k|^{1-1/p} \\ &\le c_{19} k |E_k|^{1+1/q-1/p^*} \quad \text{for all } k \ge k_0. \end{aligned} \tag{2.22}$$

Let $\vartheta = \frac{1}{q} - \frac{1}{p^*} > 0$ (recall that $\vartheta \in (1, p^*)$). Then from (2.22) we have

$$\int_{E_k} (u - k) \, dz \le c_{19} k |E_k|_N^{1+\vartheta} \quad \text{for all } k \ge k_0. \tag{2.23}$$

We set (see Ziemer [44, p. 19])

$$\xi(k) = \int_{E_k} (u - k) \, dz = \int_k^\infty |E_s|_N \, ds,$$

and have

$$-\xi'(k) = |E_k|_N. \tag{2.24}$$

From (2.23) we have

$$\xi(k)^{-1/(1+\vartheta)} \geq (c_{19}k)^{-1/(1+\vartheta)}|E_k|_N^{-1},$$

and using (2.24) this yields

$$-\xi'(k)\xi(k)^{-1/(1+\vartheta)} \geq (c_{19}k)^{-1/(1+\vartheta)}. \tag{2.25}$$

Let $k^* = \operatorname{ess\,sup}_\Omega u$ and integrate (2.25) from k_0 to k^*. Then

$$(k^*)^{\vartheta/(1+\vartheta)} \leq k_0^{\vartheta/(1+\vartheta)} + c_{19}\xi(k_0)^{\vartheta/(1+\vartheta)} = M_1^{(1+\vartheta)/\vartheta}. \qquad \square$$

Now we are ready to establish the essential boundedness of the weak solutions of problem (2.4) and provide a useful description of their bound.

Proposition 2.10. *If hypotheses* (Ha), (Hβ) *hold and* $u \in W^{1,p}(\Omega)$ *is a weak solution of problem* (2.4), *then there exists* $M_2 = M_2(p, N, \|u\|_{p^*}, \Omega) > 0$ *such that* $\|u\|_\infty \leq M_2$.

Proof. As in the proof of Proposition 2.8, without any loss of generality, we may assume that $u \geq 0$.

Let $u_k = (u-k)^+ \in W^{1,p}(\Omega)$ and $E_k = \operatorname{supp} u_k$ $k \in \mathbb{N}$. Since $u \in W^{1,p}(\Omega)$ is a weak solution of the Robin problem (2.4), we have

$$\int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}h\,d\sigma = \int_\Omega f_0(z,u)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{2.26}$$

In (2.26), we choose $h = u_k \in W^{1,p}(\Omega)$ and, by hypothesis (Hβ), we obtain

$$\int_{E_k} (a(Du), Du)_{\mathbb{R}^N}\,dz \leq \int_{E_k} f_0(z,u)u\,dz,$$

which implies (see Lemma 2.3)

$$\frac{c_1}{p-1}\int_{E_k} |Du|^p\,dz \leq \int_{E_k} f_0(z,u)u\,dz. \tag{2.27}$$

Note that, using (2.5), we have

$$\begin{aligned}\left|\int_{E_k} f_0(z,u)u\,dz\right| &\leq \int_{E_k} |f_0(z,u)||u|\,dz \\ &\leq c_{20}\int_{E_k}(1+u^{p^*-1})u\,dz \\ &\leq 2c_{20}\int_{E_k} u^{p^*-1}u\,dz \quad (\text{since } k \in \mathbb{N}) \\ &= 2c_{20}\int_{E_k} u^p u^{p^*-p}\,dz \end{aligned} \tag{2.28}$$

for some $c_{20} > 0$,

We choose $q \in (p, p^*)$. Using Proposition 2.8, we have $u^p \in L^{q/p}(\Omega)$ and $u^{p^*-p} \in L^{q/(q-p)}(\Omega)$. Note that $\frac{p}{q} + \frac{q-p}{q} = 1$. So, using Hölder's inequality in (2.28), and in view of Proposition 2.8, (2.19) and [20, p. 45], we have

$$\begin{aligned}\left|\int_{E_k} f_0(z,u)u\,dz\right| &\leq 2c_{20}\Big(\int_{E_k} u^q\,dz\Big)^{p/q}\Big(\int_{E_k} u^{(p^*-p)q/(p-q)}\,dz\Big)^{(q-p)/q} \\ &\leq c_{21}\Big(\int_{E_k} u^q\,dz\Big)^{p/q} \\ &= c_{21}\Big(\int_{E_k} (u-k+k)^q\,dz\Big)^{p/q}\end{aligned}$$

$$\begin{aligned}&\leq c_{22}\Big(\int_{E_k}(u-k)^q\,dz\Big)^{p/q}+c_{22}k^p|E_k|_N^{p/q}\\&\leq c_{23}|E_k|_N^{1/q-1/p^*}\int_{E_k}|Du|^p\,dz+c_{22}k^p|E_k|_N^{p/q}\end{aligned}$$

for some $c_{21}=c_{21}(\|u\|_{p^*})>0$, $c_{22}>0$ and $c_{23}>0$.

Returning to (2.27) and choosing $k\in\mathbb{N}$ big so that $|E_k|_N$ is small, we have

$$\int_{E_k}|Du|^p\,dz\leq c_{24}k^p|E_k|_N^{p/q} \tag{2.29}$$

for some $c_{24}>0$ (note that all the above estimation constants depend only on $(p,N,\|u\|_{p^*},\Omega)$). Then, from (2.29) and Lemma 2.9, we see that we can find $M_2=M_2(p,N,\|u\|_{p^*},\Omega)>0$ such that

$$u\in L^\infty(\Omega)\quad\text{with }\|u\|_\infty\leq M_2.$$ □

Remark 2.11. As we already said, an alternative approach can be based on the work of Guedda and Véron, see [17]. Indeed, let

$$K(z)=\frac{\operatorname{sign}(u)f_0(z,u(z))}{1+|u(z)|^{p-1}}.$$

Then from (2.5) we have

$$|K(z)|\leq\frac{c_{25}(1+|u(z)|^{p^*}-1)}{1+|u(z)|^{p-1}}\leq c_{26}(1+|u(z)|^{p^*-p})\quad\text{for almost all }z\in\Omega,$$

for some $c_{25},c_{26}>0$. Note that $p^*-p=\frac{p^2}{N-p}$ for $p<N$ and recall that $u\in L^{p^*}(\Omega)$. Hence, $K\in L^{N/p}(\Omega)$. We have

$$\begin{aligned}-\operatorname{div}a(Du(z))&=K(z)|u(z)|^{p-2}u(z)+\operatorname{sign}(u)K(z)\\&=K(z)(|u(z)|^{p-2}u(z)+\operatorname{sign}(u))\\&=\frac{f_0(z,u(z))}{1+|u(z)|^{p-1}}(1+|u(z)|^{p-1})\\&=f_0(z,u(z)).\end{aligned}$$

So, keeping in mind that for every $\epsilon>0$, $u\mapsto\epsilon\|u\|_{p^*}+\|Du\|_p$ is an equivalent norm on $W^{1,p}(\Omega)$, we can follow the proof of [17, Proposition 2.1] (with suitable modifications to accommodate the more general differential operator and the boundary term), to prove that $u\in L^q(\Omega)$ for all $q\in[1,+\infty)$. Then we can continue with Lemma 2.9 and Proposition 2.10 to reach the desired conclusion.

We can use Proposition 2.10 to prove a result comparing Sobolev and Hölder local minimizers of certain C^1-functionals. Such a result was first proved by Brezis and Nirenberg [7] for functionals defined on $H_0^1(\Omega)$ and it was extended to functionals defined on $W_0^{1,p}(\Omega)$ by Garcia Azorero, Peral Alonso and Manfredi [12] and to functionals defined on $W^{1,p}(\Omega)$ by Motreanu and Papageorgiou [30], and Papageorgiou and Rădulescu [33]. All these works involve perturbation terms with subcritical growth. Our result here is more general, since the functional is more general and the perturbation has critical growth.

So, let $F_0(z,x)=\int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0\colon W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\varphi_0(u)=\int_\Omega G(Du)\,dz+\frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma-\int_\Omega F_0(z,u)\,dz\quad\text{for all }u\in W^{1,p}(\Omega).$$

Proposition 2.12. *If $u_0\in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$ minimizer of φ_0, that is, we can find $\rho_0>0$ such that*

$$\varphi_0(u_0)\leq\varphi_0(u_0+h)\quad\text{for all }h\in C^1(\overline{\Omega})\text{ with }\|h\|_{C^1(\overline{\Omega})}\leq\rho_0,$$

then $u_0\in C^{1,\eta}(\overline{\Omega})$ for some $\eta\in(0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, we can find $\rho_1>0$ such that

$$\varphi_0(u_0)\leq\varphi_0(u_0+h)\quad\text{for all }h\in W^{1,p}(\Omega)\text{ with }\|h\|\leq\rho_1.$$

Proof. Since by hypothesis u_0 is a local $C^1(\overline{\Omega})$-minimizer of φ_0, for every $h \in C^1(\overline{\Omega})$ and for $t > 0$ small, we have $\varphi_0(u_0) \le \varphi_0(u_0 + th)$, and hence

$$0 \le \langle \varphi_0'(u_0), h\rangle \quad \text{for all } h \in C^1(\overline{\Omega}). \tag{2.30}$$

Recalling that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, from (2.30) we infer that $\varphi_0'(u_0) = 0$, and therefore

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h \, d\sigma = \int_{\Omega} f_0(z, u_0)h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{2.31}$$

From the nonlinear Green's identity, we have

$$\langle A(u_0), h\rangle = \langle -\operatorname{div} a(Du_0), h\rangle + \left\langle \frac{\partial u_0}{\partial n_a}, h \right\rangle_{\partial\Omega} \quad \text{for all } h \in W^{1,p}(\Omega), \tag{2.32}$$

where by $\langle \cdot , \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair $(W^{-1/p',p'}(\partial\Omega), W^{1/p',p}(\partial\Omega))$. Note that

$$\operatorname{div} a(Du_0) \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*.$$

So, if by $\langle \cdot , \cdot \rangle_0$ we denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$, from (2.32), we have

$$\langle -\operatorname{div} a(Du_0), h\rangle_0 = \langle A(u_0), h\rangle_0 = \langle A(u_0), h\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

Hence, by (2.31),

$$\langle -\operatorname{div} a(Du_0), h\rangle_0 = \int_{\Omega} f_0(z, u_0)h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

and therefore

$$-\operatorname{div} a(Du_0(z)) = f_0(z, u_0(z)) \quad \text{for almost all } z \in \Omega. \tag{2.33}$$

From (2.31), (2.32) and (2.33), we obtain

$$\left\langle \frac{\partial u_0}{\partial n_a} + \beta(z)|u_0|^{p-2}u_0, h \right\rangle_{\partial\Omega} = 0 \quad \text{for all } h \in W^{1,p}(\Omega). \tag{2.34}$$

Recall that, if γ_0 is the trace map, then $\operatorname{im}\gamma_0 = W^{1/p',p}(\partial\Omega)$. So, from (2.34) it follows that

$$\frac{\partial u_0}{\partial n_a} + \beta(z)|u_0|^{p-2}u_0 = 0 \quad \text{in } W^{-1/p',p'}(\partial\Omega).$$

From Proposition 2.10 we have that $u_0 \in L^\infty(\Omega)$. So, the nonlinear regularity result of Lieberman [24, p. 320] implies that

$$u_0 \in C^{1,\eta}(\overline{\Omega}) \quad \text{for some } \eta \in (0, 1).$$

Next we show that u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0. We argue indirectly. So, we assume that u_0 is not a local $W^{1,p}(\Omega)$-minimizer of φ_0. Given $\epsilon > 0$, we consider the set

$$\bar{B}_\epsilon^* = \{h \in W^{1,p}(\Omega) : \|h\|_{p^*} \le \epsilon\},$$

and define

$$m_\epsilon^* = \inf\{\varphi_0(u_0 + h) : h \in \bar{B}_\epsilon^*\}. \tag{2.35}$$

By our contradiction hypothesis, we have

$$m_\epsilon^* < \varphi_0(u_0). \tag{2.36}$$

Let $\{h_n\}_{n\ge1} \subseteq \bar{B}_\epsilon^*$ be a minimizing sequence for (2.35). Then, since $u \to \|u\|_{p^*} + \|Du\|$ is an equivalent norm on the Sobolev space $W^{1,p}(\Omega)$, we see that $\{h_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$\begin{aligned} h_n &\xrightarrow{w} \hat{h}_\epsilon \quad && \text{in } W^{1,p}(\Omega) \text{ and in } L^{p^*}(\Omega), \\ h_n(z) &\to \hat{h}_\epsilon(z) \quad && \text{for almost all } z \in \Omega. \end{aligned} \tag{2.37}$$

Using the extended Fatou's lemma, we see that φ_0 is sequentially weakly lower semicontinuous. So, we have

$$\varphi_0(u_0 + \hat{h}_\epsilon) \le \liminf_{n\to\infty} \varphi_0(u_0 + h_n).$$

Since $\|h_\epsilon\|_{p^*} \le \epsilon$ (see (2.37)), it follows that $m_\epsilon^* = \varphi(u_0 + \hat{h}_\epsilon)$, hence, by (2.36), $\hat{h}_\epsilon \neq 0$. By the Lagrange multiplier rule (see, for example, [32, p. 35]), we can find $\lambda_\epsilon \le 0$ such that

$$\langle \varphi_0'(u_0 + \hat{h}_\epsilon), v\rangle = \lambda_\epsilon \int_\Omega |\hat{h}_\epsilon|^{p^*-2}\hat{h}_\epsilon v\,dz \quad \text{for all } v \in W^{1,p}(\Omega),$$

which implies

$$\langle A(u_0, \hat{h}_\epsilon), v\rangle + \int_{\partial\Omega} \beta(z)|u_0 + \hat{h}_\epsilon|^{p-2}(u_0 + \hat{h}_\epsilon)v\,d\sigma = \int_\Omega f_0(z, u_0 + \hat{h}_\epsilon)v\,dz + \lambda_\epsilon \int_\Omega |\hat{h}_\epsilon|^{p^*-2}\hat{h}_\epsilon v\,dz \tag{2.38}$$

for all $v \in W^{1,p}(\Omega)$. From (2.38), as above using the nonlinear Green's identity, we obtain

$$\begin{cases} -\operatorname{div} a(D(u_0 + \hat{h}_\epsilon)(z)) = f_0(z, (u_0 + \hat{h}_\epsilon)(z)) + \lambda_\epsilon |\hat{h}_\epsilon(z)|^{p^*-2}\hat{h}_\epsilon(z) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial(u_0 + \hat{h}_\epsilon)}{\partial n_a} + \beta(z)|u_0 + \hat{h}_\epsilon|^{p-2}(u_0 + \hat{h}_\epsilon) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.39}$$

First assume that $\lambda_\epsilon \in [-1, 0]$ for all $\epsilon \in (0, 1]$. Then, from (2.39) and Proposition 2.10, we can find $M_3 > 0$ such that

$$\|u_0 + \hat{h}_\epsilon\|_\infty \le M_3 \quad \text{for all } \epsilon \in (0, 1]. \tag{2.40}$$

Invoking the regularity result of Lieberman [24], we can find $\eta \in (0, 1)$ and $M_4 > 0$ such that

$$u_0 + \hat{h}_\epsilon \in C^{1,\eta}(\overline{\Omega}), \quad \|u_0 + \hat{h}_\epsilon\|_{C^1(\overline{\Omega})} \le M_4 \quad \text{for all } \epsilon \in (0, 1]. \tag{2.41}$$

Next suppose that there exists $\epsilon_n \downarrow 0$ such that $\lambda_n = \lambda_{\epsilon_n} < -1$ for all $n \in \mathbb{N}$. From (2.39) with $\hat{h}_n = \hat{h}_{\epsilon_n}$, we have

$$-\frac{1}{|\lambda_n|}\operatorname{div} a(D(u_0 + \hat{h}_n)(z)) = \frac{1}{|\lambda_n|} f_0(z, (u_0 + \hat{h}_n)(z)) + |\hat{h}_n(z)|^{p^*-2}\hat{h}_n(z) \tag{2.42}$$

for almost all $z \in \Omega$. Also, from the first part of the proof, we have

$$-\frac{1}{|\lambda_n|}\operatorname{div} a(Du_0(z)) = \frac{1}{|\lambda_n|} f_0(z, u_0(z)) \quad \text{for almost all } z \in \Omega. \tag{2.43}$$

Let $\mu > 1$ and consider the function $|\hat{h}_n|^\mu \hat{h}_n$ $n \in \mathbb{N}$. We have

$$D(|\hat{h}_n|^\mu \hat{h}_n) = |\hat{h}_n|^\mu D\hat{h}_n + \mu \hat{h}_n \frac{\hat{h}_n}{|\hat{h}_n|}|\hat{h}_n|^{\mu-1} D\hat{h}_n = (\mu + 1)|\hat{h}_n|^\mu D\hat{h}_n,$$

which, by (2.41) and the fact that $u_0 \in C^{1,\eta}(\overline{\Omega})$, implies

$$|\hat{h}_n|^\mu \hat{h}_n \in W^{1,p}(\Omega).$$

Using this as test function, from (2.5), (2.40), (2.42) and (2.43), we have

$$\begin{aligned} 0 &\le \langle A(u_0 + h_n) - A(u_0), |\hat{h}_n|^\mu \hat{h}_n\rangle + \int_{\partial\Omega} \beta(z)[|u_0 + \hat{h}_n|^{p-2}(u_0 + \hat{h}_n) - |u_0|^{p-2}u_0]\,d\sigma \\ &= \int_\Omega [f_0(z, u_0 + \hat{h}_n) - f_0(z, u_0)]|\hat{h}_n|^\mu \hat{h}_n\,dz + \lambda_n \int_\Omega |\hat{h}_n|^{p^*+\mu}\,dz \\ &\le M_5 \int_\Omega |\hat{h}_n|^{\mu+1}\,dz + \lambda_n \int_\Omega |\hat{h}_n|^{p^*+\mu}\,dz \\ &\le M_5 |\Omega|_N^{(p^*-1)/(p^*+\mu)} \|\hat{h}_n\|_{p^*+\mu}^{\mu+1} + \lambda_n \|\hat{h}_n\|_{p^*+\mu}^{p^*+\mu} \end{aligned}$$

for some $M_5 > 0$ and all $n \in \mathbb{N}$, where we have used Hölder's inequality with the exponents $\frac{p^*+\mu}{\mu+1}$, $\frac{p^*+\mu}{p^*-1}$. Thus,

$$|\lambda_n| \|\hat{h}_n\|_{p^*+\mu}^{p^*-1} \leq M_5 |\Omega|_N^{(p^*-1)/(p^*+\mu)},$$

and hence

$$\|\hat{h}_n\|_{p^*+\mu}^{p^*-1} \leq M_6$$

for some $M_6 > 0$ (independent of $\mu > 1$) and all $n \in \mathbb{N}$ (recall that $|\lambda_n| > 1$). Since $\mu > 1$ is arbitrary, we let $\mu \to \infty$ and obtain that

$$\|\hat{h}_n\|_\infty \leq M_7$$

for some $M_7 > 0$ and all $n \in \mathbb{N}$. So, the nonlinear regularity theory of Lieberman [24] implies that for some $\eta \in (0, 1)$ and some $M_8 > 0$, we have (see (2.42) and recall that $u_0 \in C^{1,\eta}(\overline{\Omega})$)

$$\hat{h}_n \in C^{1,\eta}(\overline{\Omega}), \quad \|\hat{h}_n\|_{C^1(\overline{\Omega})} \leq M_8 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, in both cases (case 1: $\lambda_\epsilon \in [-1, 0]$ for all $\epsilon \in (0, 1]$ and case 2: $\lambda_{\epsilon_n} < -1$ for some $\epsilon_n \downarrow 0$), we reach the same uniform $C^{1,\eta}(\overline{\Omega})$ bounds for the sequence $\{\hat{h}_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that (see (2.36))

$$\varphi_0(u_0 + h_n) < \varphi_0(u_0) \quad \text{for all } n \in \mathbb{N}.$$

Recalling that $\|\hat{h}_n\|_{p^*} \leq \epsilon_n$ for all $n \in \mathbb{N}$ and exploiting the compact embedding of $C^{1,\eta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we have

$$\hat{h}_n \to 0 \text{ in } C^1(\overline{\Omega}),$$

hence

$$u_0 + \hat{h}_n \to u_0 \text{ in } C^1(\overline{\Omega}),$$

and therefore

$$\varphi_0(u_0) \leq \varphi_0(u_0 + \hat{h}_n) \quad \text{for all } n \geq n_0.$$

But recall that

$$\varphi_0(u_0 + \hat{h}_n) < \varphi_0(u_0) \quad \text{for all } n \in \mathbb{N},$$

a contradiction. This proves that $u_0 \in C^{1,\eta}(\overline{\Omega})$ is also a local $W^{1,p}(\Omega)$-minimizer of φ_0. □

Remark 2.13. A careful reading of the proof of Proposition 2.8 reveals that the result remains valid if instead we use the more general nonlinear boundary condition

$$\frac{\partial u}{\partial n_a} = \xi(z, u) \quad \text{on } \partial\Omega$$

with $\xi \in C^{0,\eta}(\partial\Omega \times \mathbb{R})$, $0 < \eta < 1$, such that

$$|\xi(z, x)| \leq c_{25}|x|^\tau \quad \text{for all } (z, x) \in \partial\Omega \times \mathbb{R},$$

with $c_{25} > 0$ and $\tau \in (1, p]$. For simplicity in our presentation, we have used in problem (2.4) the Robin boundary condition from problem (1.1), simplifying this way a little the necessary estimates.

As we already mentioned in the introduction, we will also use tools from Morse theory (critical groups). So, let us recall some basic definitions and facts from that theory.

Given a Banach space X, a function $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}, \quad K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group for the topological pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0.$$

Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is dependent of the choice of the neighborhood U of u.

Suppose that φ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and that K_φ is finite. We define

$$\begin{aligned} M(t, u) &= \sum_{k\in\mathbb{N}_0} \operatorname{rank} C_k(\varphi, u) t^k \quad && \text{for all } t \in \mathbb{R} \text{ and all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k\in\mathbb{N}_0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad && \text{for all } t \in \mathbb{R}. \end{aligned}$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.44}$$

where $Q(t) = \sum_{k\in\mathbb{N}_0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Finally, from [33] we recall that the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega \end{cases}$$

has a smallest eigenvalue $\hat{\lambda}_1(p, \beta) \geq 0$. If $\beta \not\equiv 0$, then $\hat{\lambda}_1(p, \beta) > 0$, while if $\beta \equiv 0$ then $\hat{\lambda}_1(p, 0) = \hat{\lambda}_1(p) = 0$ (Neumann problem). The eigenfunctions corresponding to this eigenvalue have constant sign and

$$\hat{\lambda}_1(p, \beta) = \inf\left\{ \frac{\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.$$

By $\hat{u}_1(p, \beta)$ we denote the L^p-normalized (that is, $\|\hat{u}_1(p, \beta)\|_p = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(p, \beta)$. We have

$$\hat{\lambda}_1(p, \beta) = \|D\hat{u}_1(p, \beta)\|_p^p + \int_{\partial\Omega} \beta(z)|\hat{u}_1(p, \beta)|^p \, d\sigma,$$

and from the nonlinear regularity theory and the nonlinear maximum principle, we have $\hat{u}_1(p, \beta) \in \operatorname{int} C_+$.

3 Three Solutions Theorem

The hypotheses on the reaction $f(z, x)$ are as follows:

(Hf) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:

(i) We have

$$\lim_{x\to\pm\infty} \frac{f(z, x)}{|x|^{p^*-2}x} = 0 \quad \text{uniformly for almost all } z \in \Omega,$$

and for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) If $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x\to\pm\infty} \frac{F(z, x)}{|x|^p} = +\infty \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) If $\xi(z,x) = f(z,x)x - pF(z,x)$, then there exists $\eta \in L^1(\Omega)_+$ such that

$$\xi(z,x) \le \xi(z,y) + \eta(z) \quad \text{for almost all } z \in \Omega \text{ and all } 0 \le x \le y \text{ or } y \le x \le 0.$$

(iv) There exist $\delta > 0$ and $\gamma_\delta > 0$ such that

$$-\gamma_\delta |x|^p \le f(z,x)x \quad \text{for almost all } z \in \Omega \text{ and all } |x| \le \delta.$$

- If $\beta \not\equiv 0$, then there exists $\mathbb{J} \in L^\infty(\Omega)_+$ such that $\mathbb{J}(z) \le \hat{\lambda}_1(p,\hat{\beta})$ for almost all $z \in \Omega$, $\mathbb{J} \not\equiv \hat{\lambda}_1(p,\hat{\beta})$, with $\hat{\beta} = \frac{p-1}{c_1}\beta$ and

$$\limsup_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \mathbb{J}(x) \quad \text{uniformly for almost all } z \in \Omega.$$

- If $\beta \equiv 0$, then $f(z,x)x \le 0$ for almost all $z \in \Omega$ and all $|x| \le \delta$.

Remark 3.1. Hypothesis (Hf) (i) is more general than the usual polynomial subcritical growth condition which says that

$$|f(z,x)| \le c_{26}(1 + |x|^{r-1}) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}, \tag{3.1}$$

with $c_{26} > 0$ and $1 < r < p^*$. For example the function (for the sake of simplicity we drop the z-dependence)

$$f(x) = \frac{|x|^{p^*-2}x}{\ln((1+|x|^p))} - \frac{p}{p^*}\frac{|x|^{p^*}|x|^{p-2}x}{\ln(1+|x|^p)^2(1+|x|^p)},$$

with primitive

$$F(x) = \frac{1}{p^*}\frac{|x|^{p^*}}{\ln(1+|x|^p)},$$

satisfies hypothesis (Hf) (i) but fails to satisfy the subcritical polynomial growth (3.1). The lack of compactness in the embedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is a source of difficulties which we have to overcome. We do this without any appeal to the concentration-compactness principle (see Ambrosetti and Malchiodi [3, p. 252]). It is not clear how hypothesis (Hf) (i) can lead to concentration phenomena and for this reason our approach avoids the use of the concentration-compactness method of Lions. Instead we show that despite the almost critical growth of the reaction term $f(z,\cdot)$ (see hypothesis (Hf) (i)), the compactness condition is still valid for the energy functional of the problem and so we can proceed with the usual variational methods of critical point theory. Hypothesis (Hf) (iv) implies that

$$f(z,0) = 0 \quad \text{for almost all } z \in \Omega.$$

Then hypothesis (Hf) (iii) implies

$$\xi(z,0) = 0 \le \xi(z,x) + \eta(z) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

hence

$$pF(z,x) \le f(z,x)x + \eta(z) \quad \text{for almost all } z \in \Omega,$$

and therefore, from hypothesis (Hf) (ii), we obtain

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for almost all } z \in \Omega.$$

Hypotheses (Hf) (ii)–(iii) replace the AR-condition and allow in our framework superlinear reactions with “slower” growth near $\pm\infty$ which fail to satisfy the AR-condition (see the examples below). Hypothesis (Hf) (iii) is a quasimonotonicity condition on $\xi(z,\cdot)$ and it is satisfied if, for example, we can find $M_9 > 0$ such that for almost all $z \in \Omega$,

$$x \to \frac{f(z,x)}{x^{p-1}} \text{ is nondecreasing on } [M_9, +\infty) \quad \text{and} \quad x \to \frac{f(z,x)}{|x|^{p-2}x} \text{ is nonincreasing on } (-\infty, -M_9].$$

More restrictive versions of hypothesis (Hf) (iii) were used by Li and Yang [22], Liu [26], Miyagaki and Souto [28], and Sun [38]. We should mention that all these conditions originate from the important work of Jeanjean [19] (see also Struwe [37]), who was the first to employ an alternative to the AR-condition. So, Jeanjean [19] assumed (for $p = 2$) that there exists $\vartheta \geq 1$ such that

$$\xi(z, sx) \leq \vartheta\xi(z, x) \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and } s \in [0, 1].$$

The disadvantage of this condition is that it is global. In contrast, the previous remarks show that condition (Hf) (iii) avoids this global character and so it is a quite generic condition. For a further discussion and comparison of these extensions of the AR-condition, we refer to the paper by Li and Yang [22].

Example 3.2. The following primitive functions satisfy hypotheses (Hf) (for the sake of simplicity we drop the z-dependence):

$$F_1(x) = \frac{1}{q}|x|^q - \frac{1}{p}|x|^p,$$
$$F_2(x) = \frac{1}{p^*}\frac{|x|^{p^*}}{\ln(1+|x|^{p^*})} + \begin{cases} -\frac{1}{p}|x|^p & \text{if } |x| \leq 1, \\ \frac{1}{p}|x|^p \ln|x| - \frac{1}{p} & \text{if } 1 < |x|, \end{cases}$$

with $1 < p < q < p^*$. Note that $f_2(x) = \frac{d}{dx}F_2(x)$ fails to satisfy (3.1) and the AR-condition.

We introduce the following truncations-perturbations of the reaction term $f(z, \cdot)$:

$$\hat{f}_+(z, x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(z, x) + x^{p-1} & \text{if } x > 0, \end{cases} \tag{3.2}$$

$$\hat{f}_-(z, x) = \begin{cases} f(z, x) + |x|^{p-2}x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases} \tag{3.3}$$

Both are Carathéodory functions. We set

$$\hat{F}_\pm(z, x) = \int_0^x \hat{f}_\pm(z, s)\, ds,$$

and consider the C^1-functionals $\hat{\varphi}_\pm \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}\|u\|_p^p \pm \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^\pm)^p\, d\sigma - \int_\Omega \hat{F}_\pm(z, u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, let $\varphi \colon W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.4) defined by

$$\varphi(u) = \int_\Omega G(Du)\, dz - \int_\Omega F(z, u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently, $\varphi \in C^1(W^{1,p}(\Omega))$.

Proposition 3.3. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then the functionals* $\hat{\varphi}_\pm$ *satisfy the C-condition.*

Proof. We give the proof (similarly, in two other occurrences) for the functional $\hat{\varphi}_+$; the proof for $\hat{\varphi}_-$ is similar.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\hat{\varphi}_+(u_n)| \leq M_{10} \quad \text{for some } M_{10} > 0 \text{ and all } n \in \mathbb{N}, \tag{3.4}$$

$$(1 + \|u_n\|)\hat{\varphi}_+'(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{3.5}$$

From (3.5) we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega |u_n|^{p-2}u_n h\, dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p-1}h\, d\sigma - \int_\Omega \hat{f}_+(z, u_n)h\, dz\right| \leq \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \tag{3.6}$$

for all $h \in W^{1,p}(\Omega)$ with $\epsilon_n \to 0^+$. In (3.6) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then, by Lemma 2.3 and (3.2),

$$\frac{c_1}{p-1}\|Du_n^-\|_p^p + \|u_n^-\|_p^p \le \epsilon_n \quad \text{for all } n \in \mathbb{N},$$

hence

$$u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega). \tag{3.7}$$

We use (3.7) in (3.4). Then, because of Corollary 2.4 and (3.2), we have

$$\left| \int_\Omega pG(Du_n^+)\,dz + \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma - \int_\Omega pF(z,u_n^+)\,dz \right| \le M_{11} \quad \text{for all } n \in \mathbb{N}, \tag{3.8}$$

for some $M_{11} > 0$. In (3.6) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\,dz - \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma + \int_\Omega f(z,u_n^+)u_n^+\,dz \le \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

We add (3.8) and (3.9) and use hypothesis (Ha) (iv) to obtain

$$\int_\Omega \xi(z,u_n^+)\,dz \le M_{12} \quad \text{for all } n \in \mathbb{N}, \tag{3.10}$$

for some $M_{12} > 0$.

Claim 1. $\{u_n^+\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded.

We argue indirectly. So, suppose that Claim 1 is not true. By passing to a subsequence if necessary, we may assume that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega),\ y \ge 0. \tag{3.11}$$

Suppose that $y \ne 0$ and let $\Omega_+(y) = \{y > 0\}$. Then $|\Omega_+(y)|_N > 0$ and we have

$$u_n^+(z) \to +\infty \quad \text{for almost all } z \in \Omega_+(y).$$

Hypothesis (Hf) (ii) implies that

$$\frac{F(z,u_n^+(z))}{\|u_n^+\|^p} = \frac{F(z,u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \to +\infty \quad \text{for almost all } z \in \Omega_+(y).$$

From this fact and Fatou's lemma (see also hypothesis (Hf) (ii) and (3.11)), we have

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \to +\infty. \tag{3.12}$$

From (3.8) and in view of Corollary 2.4, hypothesis (Hβ) and (3.11) (recall also that $p > 1$), we have

$$\begin{aligned}\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p} &\le M_{11} + \frac{1}{\|u_n^+\|^p}\int_\Omega G(Du_n^+)\,dz + \int_{\partial\Omega} \beta(z) y_n^p\,d\sigma \\ &\le c_{27}(1 + \|y_n\|^p) \\ &\le c_{28} \quad \text{for all } n \in \mathbb{N},\end{aligned} \tag{3.13}$$

for some $c_{27}, c_{28} > 0$. Comparing (3.12) and (3.13), we reach a contradiction.

So, we assume that $y \equiv 0$. Let $k > 0$ and set $v_n = (kp)^{1/p} y_n$ for all $n \in \mathbb{N}$. From (3.11) we have

$$v_n \xrightarrow{w} 0 \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad v_n \to 0 \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.14}$$

Let $c_{29} = \sup_{n\geq 1} \|v_n\|_{p^*}^{p^*} < +\infty$ (see (3.14)). Hypothesis (Hf) (i) implies that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$|F(z,x)| \leq \frac{\epsilon}{2c_{29}}|x|^{p^*} + c_\epsilon \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.15}$$

From (3.15), for every measurable set $E \subseteq \Omega$ with $|E_N| \leq \frac{\epsilon}{2c_\epsilon}$, we have

$$\Big|\int_E F(z,v_n)\,dz\Big| \leq \int_E |F(z,v_n)|\,dz \leq \frac{\epsilon}{2c_{29}}\|v_n\|_{p^*}^{p^*} + c_\epsilon |E|_N \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \in \mathbb{N},$$

hence $\{F(\cdot, v_n(\cdot))\}_{n\geq 1} \subseteq L^1(\Omega)$ is uniformly integrable. Since $F(z, v_n(z)) \to 0$ for almost all $z \in \Omega$, from the extended dominated convergence theorem (Vitali's theorem), we have

$$\int_\Omega F(z,v_n)\,dz \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Recall that we have assumed that $\|u_n^+\| \to \infty$. So, we can find $n_0 \in \mathbb{N}$ such that

$$0 < (kp)^{1/p}\frac{1}{\|u_n^+\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{3.17}$$

Consider the C^1-functional $\hat{\psi}_+ \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,d\sigma - \int_\Omega \hat{F}_+(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Let $t_n \in [0,1]$ be such that

$$\hat{\psi}_+(t_n u_n^+) = \max_{0\leq t\leq 1} \hat{\psi}_+(tu_n^+) \quad \text{for all } n \in \mathbb{N}.$$

From (3.16) we see that we can find $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that

$$\int_\Omega F(z,v_n)\,dz \leq \frac{c_1}{2(p-1)}k \quad \text{for all } n \geq n_1. \tag{3.18}$$

Using (3.17), (3.18) and hypothesis (Hβ), we have

$$\hat{\psi}_+(t_n u_n^+) \geq \hat{\psi}_+(v_n) \geq \frac{c_1 k}{p-1} - \frac{c_1 k}{2(p-1)} - \|v_n\|_p^p = \frac{c_1 k}{3(p-1)} \quad \text{for all } n \geq n_1. \tag{3.19}$$

Recall that $k > 0$ is arbitrary. So, from (3.19) it follows that

$$\hat{\psi}_+(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{3.20}$$

From (3.4) and (3.7) and since $\hat{\psi}_+ \leq \hat{\varphi}_+$ (see Corollary 2.4), we see that

$$\{\hat{\psi}_+(u_n^+)\}_{n\geq 1} \subseteq \mathbb{R} \text{ is bounded.} \tag{3.21}$$

Also, we have

$$\hat{\psi}_+(0) = 0. \tag{3.22}$$

From (3.20)–(3.22), it follows that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0,1) \quad \text{for all } n \geq n_2. \tag{3.23}$$

Then, for $n \geq n_2$, we have

$$\frac{d}{dt}\hat{\psi}_+(tu_n^+)\Big|_{t=t_n} = 0,$$

which, by the chain rule, yields

$$\frac{c_1}{p-1}\langle A_p(t_n u_n^+), u_n^+\rangle + \int_{\partial\Omega} \beta(z)(t_n u_n^+)^{p-1} u_n^+\,d\sigma = \int_\Omega f(z, t_n(u_n^+)u_n^+)\,dz,$$

and therefore

$$\frac{c_1}{p-1}\|D(t_n u_n^+)\|_p^p + \int_{\partial\Omega} \beta(z)(t_n u_n^+)^p \, d\sigma = \int_\Omega f(z, t_n u_n^+)(t_n u_n^+) \, dz \quad \text{for all } n \geq n_2. \tag{3.24}$$

From hypothesis (Hf) (iii) and (3.23), we have

$$\int_\Omega f(z, t_n u_n^+)(t_n u_n^+) \, dz \leq \int_\Omega \xi(z, u_n^+) \, dz + \int_\Omega pF(z, t_n u_n^+) \, dz + \|\eta\|_1 \quad \text{for all } n \geq n_2. \tag{3.25}$$

Using (3.25) in (3.24), from (3.10) we obtain

$$\hat{\psi}_+(t_n u_n^+) \leq M_{12} + \|\eta\|_1 = M_{13} \quad \text{for all } n \geq n_2. \tag{3.26}$$

Comparing (3.20) and (3.26), we reach a contradiction. This proves Claim 1. □

From (3.7) and Claim 1, it follows that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we way assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.27}$$

Let $c_{30} = \sup_{n\geq 1} \|u_n\|_{p^*}^{p^*} < +\infty$ (see (3.27)). Hypothesis (Hf) (i) implies that given $\epsilon > 0$, we can find $\hat{c}_\epsilon > 0$ such that

$$|f(z,x)| \leq \frac{\epsilon}{2c_{30}}|x|^{p^*-1} + \hat{c}_\epsilon \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}.$$

For $E \subseteq \Omega$ measurable, we have

$$\begin{aligned} \left|\int_E f(z, u_n)(u_n - u) \, dz\right| &\leq \int_E |f(z, u_n)||u_n - u| \, dz \\ &\leq \frac{\epsilon}{2c_{30}} \int_E |u_n|^{p^*-1}|u_n - u| \, dz + \hat{c}_\epsilon \int_E |u_n - u| \, dz. \end{aligned} \tag{3.28}$$

Using Hölder's inequality, we have (recall that $\frac{1}{p^*} + \frac{1}{(p^*)'} = 1$)

$$\hat{c}_\epsilon \int_E |u_n - u| \, dz \leq \hat{c}_\epsilon |E|_N^{1/(p^*)'} \|u_n - u\|_{p^*} \leq 2\hat{c}_\epsilon |E|_N^{1/(p^*)'} c_{30}^{1/p^*}. \tag{3.29}$$

Thus,

$$\frac{\epsilon}{2c_{30}} \int_E |u_n|^{p^*-1}|u_n - u| \, dz \leq \frac{\epsilon}{2c_{30}} \|u_n\|_{p^*}^{p^*-1} \|u_n - u\|_{p^*} \leq \frac{\epsilon}{2} \quad \text{for all } n \in \mathbb{N}. \tag{3.30}$$

Choose $E \subseteq \Omega$ measurable with

$$|E|_N \leq \frac{\epsilon}{2(2\hat{c}_\epsilon)^{(p^*)'} c_{30}^{1/p^*-1}}.$$

Then from (3.29) we have

$$\hat{c}_\epsilon \int_E |u_n - u| \, dz \leq \frac{\epsilon}{2} \quad \text{for all } n \in \mathbb{N}. \tag{3.31}$$

From (3.28), (3.30) and (3.31), it follows that

$$\sup_{n\geq 1} \int_E |f(z, u_n)||u_n - u| \, dz \leq \epsilon,$$

hence $\{f(\cdot, u_n(\cdot))(u_n - u)(\cdot)\}_{n\geq 1} \subseteq L^1(\Omega)$ is uniformly integrable. From (3.27) we have (at least for a subsequence) that

$$f(z, u_n(z))(u_n - u)(z) \to 0 \quad \text{for almost all } z \in \Omega.$$

So, employing the extended dominated convergence theorem (Vitali's theorem), we have

$$\int_\Omega f(z, u_n)(u_n - u) \, dz \to 0 \quad \text{as } n \to \infty. \tag{3.32}$$

In (3.6), we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$, and use (3.27), (3.32) and hypothesis (Hβ). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

and by Proposition 2.7,

$$u_n \to u \quad \text{in } W^{1,p}(\Omega),$$

which implies that $\hat{\varphi}_+$ satisfies the C-condition. Similarly for $\hat{\varphi}_-$ using (3.3). □

A careful reading of the above proof shows with minor and straightforward changes, we can have the same result for the energy functional φ. Therefore, we can state the following proposition.

Proposition 3.4. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then the energy functional φ satisfies the C-condition.*

Hypothesis (Hf) (ii) leads easily to the following result.

Proposition 3.5. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold and $u \in \operatorname{int} C_+$, then $\hat{\varphi}_\pm(tu) \to -\infty$ as $t \to +\infty$.*

The next result establishes the mountain pass geometry (see Theorem 2.1) for the functionals $\hat{\varphi}_\pm$. Also, this result will be useful in generating a third nontrivial solution for problem (1.4), since it identifies the nature of $u = 0 \in K_\varphi$.

Proposition 3.6. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then $u = 0$ is a local minimizer of the functional $\hat{\varphi}_\pm$ and φ.*

Proof. We do the proof for the functional $\hat{\varphi}_+$; the proofs for $\hat{\varphi}_-$ and φ are similar.

First suppose $\beta \not\equiv 0$. Hypothesis (Hf) (iv) implies that given $\epsilon > 0$, we can find $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$F(z,x) \le \frac{1}{p}(\mathfrak{I}(z) + \epsilon)|x|^p \quad \text{for almost all } z \in \Omega \text{ and all } |x| \le \delta_1. \tag{3.33}$$

Let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta_1$. Then, in view of (3.2), (3.33), [33] and (Hf) (iv), we have

$$\begin{aligned}
\hat{\varphi}_+(u) &= \int_\Omega G(Du)\,dz + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p\,d\sigma - \int_\Omega F(z,u^+)\,dz \\
&\ge \frac{c_1}{p(p-1)}\Big[\|Du^+\|_p^p + \int_{\partial\Omega}\hat{\beta}(z)(u^+)^p\,d\sigma - \int_\Omega \mathfrak{I}(z)(u^+)^p\,dz - \epsilon\|u^+\|^p\Big] + \frac{1}{p}\Big[\frac{c_1}{p-1}\|Du^-\|_p^p + \|u^-\|_p^p\Big] \\
&\ge (c_{31} - \epsilon)\|u^+\|^p + c_{32}\|u^-\|^p
\end{aligned}$$

for some $c_{31}, c_{32} > 0$, with $\hat{\beta} = \frac{p-1}{c_1}\beta$. Choosing $\epsilon \in (0, c_{31})$, from (3.33) we infer that

$$\hat{\varphi}_+(u) \ge c_{33}\|u\|^p \quad \text{for all } u \in C^1(\overline{\Omega}) \text{ with } \|u\|_{C^1(\overline{\Omega})} \le \delta_1,$$

hence $u = 0$ is a local $C^1(\overline{\Omega})$-minimizer of $\hat{\varphi}_+$, and therefore, by Proposition 2.12, $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\hat{\varphi}_+$.

Next suppose that $\beta \equiv 0$. Let $\delta > 0$ be as postulated by hypothesis (Hf) (iv) and let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta$. Then hypothesis (H$f$) (iv) implies

$$-\int_\Omega F(z,u)\,dz \ge 0.$$

So, we have

$$\hat{\varphi}_+(u) \ge 0 = \hat{\varphi}_+(0) \quad \text{for all } u \in C^1(\overline{\Omega}) \text{ with } \|u\|_{C^1(\overline{\Omega})} \le \delta,$$

and again by Proposition 2.12, $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\hat{\varphi}_+$.

Similarly for the functionals $\hat{\varphi}_-$ and φ. □

Proposition 3.7. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then $K_{\hat{\varphi}_+} \subseteq C_+$ and $K_{\hat{\varphi}_-} \subseteq C_+$.*

Proof. Let $u \in K_{\hat{\varphi}_+}$. Then $\hat{\varphi}_+'(u) = 0$ and (3.2) imply

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uh\,dz + \int_{\partial\Omega} \beta(z)(u^+)^{p-1}h\,d\sigma = \int_\Omega [f(z,u^+) + (u^+)^{p-1}]h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.34}$$

In (3.34) we choose $h = -u^- \in W^{1,p}(\Omega)$. Then, by Lemma 2.3,

$$\frac{c_1}{p-1}\|Du^-\|_p^p + \|u^-\|_p^p \le 0,$$

hence $u \ge 0$. From Proposition 2.10 we have that $u \in L^\infty(\Omega)$. So, we can use the regularity theory of Lieberman [24, p. 320] and have that $u \in C_+$. Therefore,

$$K_{\hat{\varphi}_+} \subseteq C_+.$$

Similarly, for the functional $\hat{\varphi}_-$, using this time (3.3), we show that $K_{\hat{\varphi}_-} \subseteq -C_+$. □

Now we are ready to produce two constant sign solutions for problem (1.4).

Proposition 3.8. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then problem* (1.4) *has at least two constant sign solutions* $u_0 \in \operatorname{int} C_+$ *and* $v_0 \in -\operatorname{int} C_+$.

Proof. Proposition 3.7 together with (3.2) and (3.3) indicate that we may assume that $K_{\hat{\varphi}_+}$ and $K_{\hat{\varphi}_-}$ are infinite or, otherwise, we already have a whole sequence of distinct solutions of constant sign.

From Proposition 3.6 we know that $u = 0$ is a local minimizer of $\hat{\varphi}_+$. So, we can find $\rho \in (0,1)$ small such that (see the proof of [1, Proposition 29])

$$\hat{\varphi}_+(0) = 0 < \inf\{\hat{\varphi}_+(u) : \|u\| = \rho\} = \hat{m}_\rho^+. \tag{3.35}$$

Combining (3.35) with Propositions 3.3 and 3.5, we see that we can apply Theorem 2.1 (the mountain pass theorem). So, by Proposition 3.7, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_+} \subseteq C_+ \quad \text{and} \quad \hat{m}_\rho^+ \le \hat{\varphi}_+(u_0). \tag{3.36}$$

From (3.35) and (3.36), we have that $u_0 \ne 0$. Also, since $u_0 \ge 0$ (see (3.36)), by (3.2), we have

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z)u_0^{p-1}h\,d\sigma = \int_\Omega f(z,u_0)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

Thus,

$$\begin{cases} -\operatorname{div} a(Du_0(z)) = f(z,u_0(z)) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_a} + \beta(z)u_0^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.37}$$

Hypothesis (Hf) (iv) implies that given $\rho > 0$, we can find $\hat{\xi}_\rho > 0$ such that

$$f(z,x)x + \hat{\xi}_\rho|x|^p \ge 0 \quad \text{for almost all } z \in \Omega \text{ and all } |x| \le \rho. \tag{3.38}$$

Let $\rho = \|u_0\|_\infty$ (recall that $u_0 \in C_+ \setminus \{0\}$) and let $\hat{\xi}_\rho > 0$ as in (3.38). Then from (3.37) we have

$$\operatorname{div} a(Du_0(z)) \le \hat{\xi}_\rho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega. \tag{3.39}$$

Let $\gamma(t) = a_0(t)t$ for $t > 0$. Then (1.2) and hypothesis (Ha) (ii) ensure that

$$\gamma'(t)t = a_0'(t)t^2 + a_0(t)t \ge c_1 t^{p-1}.$$

By integration, we obtain

$$\int_0^t \gamma'(s)s\,ds = \gamma(t)t - \int_0^t \gamma(s)\,ds = a_0(t)t^2 - G_0(t) \ge \frac{c_1}{p}t^p \quad \text{for all } t > 0. \tag{3.40}$$

Let

$$\hat{d}(t) = a_0(t)t^2 - G_0(t) \quad \text{and} \quad \hat{d}_0(t) = \frac{c_1}{p}t^p \quad \text{for all } t > 0.$$

Let $s > 0$ and consider the following two sets:

$$C_1 = \{t \in (0,1) : \hat{d}(t) \geq s\}, \quad C_2 = \{t \in (0,1) : \hat{d}_0(t) \geq s\}.$$

From (3.40) we see that $C_2 \subseteq C_1$ and so $\inf C_1 \leq \inf C_2$. Therefore, $\hat{d}^{-1}(s) \leq \hat{d}_0^{-1}(s)$ (see, e.g., [21, p. 6]). Then for $\delta > 0$ we have

$$\int_0^\delta \frac{1}{\hat{d}^{-1}(\frac{\hat{\xi}_\rho}{p}s^p)}\,ds \geq \int_0^\delta \frac{1}{\hat{d}_0^{-1}(\frac{\hat{\xi}_\rho}{p}s^p)}\,ds = \frac{\hat{\xi}_\rho}{p}\int_0^\delta \frac{ds}{s} = +\infty.$$

Hence, because of (3.39), we can apply the nonlinear strong maximum principle of Pucci and Serrin [36, p. 111] and have that

$$u_0(z) > 0 \quad \text{for all } z \in \Omega.$$

Then the boundary point theorem of Pucci and Serrin [36, p. 120] implies that $u_0 \in \operatorname{int} C_+$.

Similarly, working with the functional $\hat{\varphi}_-$, we produce a second constant sign solution $v_0 \in -\operatorname{int} C_+$. ☐

To produce a third nontrivial solution, we will use Morse theoretical tools (critical groups). To this end we compute the critical groups of φ at infinity.

Proposition 3.9. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold and* $\inf \varphi(K_\varphi) > -\infty$, *then* $C_k(\varphi, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. From hypotheses (Hf) (i)–(ii) we see that given $\gamma > 0$, we can find $c_{34} = c_{34}(\gamma) > 0$ such that

$$F(z,x) \geq \gamma|x|^p - c_{34} \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.41}$$

Let $u \in \partial B_1 = \{u \in W^{1,p}(\Omega) : \|u\| = 1\}$ and $t > 0$. On account of Corollary 2.4, (3.41) and hypothesis (Hβ), we have

$$\varphi(tu) \leq t^p\bigl[c_{35}\|Du\|_p^p + c_{36}\|u\|_{L^p(\partial\Omega)}^p - \gamma\|u\|_p^p\bigr] + c_{37} \tag{3.42}$$

for some $c_{35}, c_{36}, c_{37} > 0$. Because $\gamma > 0$ is arbitrary, from (3.42) we see that

$$\varphi(tu) \to -\infty \quad \text{as } t \to -\infty. \tag{3.43}$$

Also, using the chain rule, and hypotheses (Ha) (iv) and (Hf) (iii), we have

$$\begin{aligned}
\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u\rangle \\
&= \frac{1}{t}\langle \varphi'(tu), tu\rangle \\
&= \frac{1}{t}\Bigl[\int_\Omega (a(tDu), tDu)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)|tu|^p\,d\sigma - \int_\Omega f(z,tu)tu\,dz\Bigr] \\
&\leq \frac{1}{t}\Bigl[\int_\Omega pG(tDu)\,dz + \int_{\partial\Omega} \beta(z)|tu|^p\,d\sigma - \int_\Omega pF(z,tu)\,dz + c_{38}\Bigr] \\
&= \frac{1}{t}[p\varphi(tu) + c_{38}]
\end{aligned}$$

for some $c_{38} > 0$. Then (3.43) implies that for large $t > 0$ we have $\varphi(tu) \leq \mathfrak{I}_0 < -c_{38}$, and thus

$$\frac{d}{dt}\varphi(tu) < 0 \quad \text{for large } t > 0.$$

Therefore, we can find a unique $r(u) > 0$ such that $\varphi(r(u)u) = \mathfrak{I}_0$. The implicit function theorem implies that $r \in C(\partial B_1)$. We extend $r(\cdot)$ to all of $W^{1,p}(\Omega) \setminus \{0\}$ by

$$r_0(u) = \frac{1}{\|u\|}r\Bigl(\frac{u}{\|u\|}\Bigr) \quad \text{for all } u \in W^{1,p}(\Omega) \setminus \{0\}.$$

Then $r_0 \in C(W^{1,p}(\Omega) \setminus \{0\})$ and $\varphi(r_0(u)u) = \mathfrak{I}_0$. Also, if $\varphi(u) = \mathfrak{I}_0$, then $r_0(u) = 1$. So, we set

$$\hat{r}_0(u) = \begin{cases} 1 & \text{if } \varphi(u) \le \mathfrak{I}_0, \\ r_0(u) & \text{if } \mathfrak{I}_0 < \varphi(u). \end{cases} \tag{3.44}$$

Evidently, $\hat{r}_0 \in C(W^{1,p}(\Omega) \setminus \{0\})$. Consider the deformation $h(t,u)$ defined by

$$h(t,u) = (1-t)u + t\hat{r}_0(u)u \quad \text{for all } (t,u) \in [0,1] \times (W^{1,p}(\Omega) \setminus \{0\}).$$

We have

$$h(0,u) = u, \quad h(1,u) = \hat{r}_0(u)u \in \varphi^{\mathfrak{I}_0}$$

and (see (3.44))

$$h(t,\cdot)|_{\varphi^{\mathfrak{I}_0}} = \mathrm{id}|_{\varphi^{\mathfrak{I}_0}} \quad \text{for all } t \in [0,1].$$

It follows that

$$\varphi^{\mathfrak{I}_0} \text{ is a strong deformation retract of } W^{1,p}(\Omega) \setminus \{0\}. \tag{3.45}$$

We consider the radial retraction $\tilde{r}\colon W^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R}$ defined by

$$\tilde{r}(u) = \frac{u}{\|u\|} \quad \text{for all } u \in W^{1,p}(\Omega) \setminus \{0\}.$$

This map is continuous and $\tilde{r}|_{\partial B_1} = \mathrm{id}|_{\partial B_1}$. Therefore, ∂B_1 is a retract of $W^{1,p}(\Omega) \setminus \{0\}$. We consider the deformation $\tilde{h}(t,u)$ defined by

$$\tilde{h}(t,u) = (1-t)u + t\tilde{r}(u) \quad \text{for all } (t,u) \in [0,1] \times (W^{1,p}(\Omega) \setminus \{0\}).$$

Then

$$\tilde{h}(0,u) = u, \quad \tilde{h}(1,u) = \tilde{r}(u) \in \partial B_1 \quad \text{and} \quad \tilde{h}(1,\cdot)|_{\partial B_1} = \mathrm{id}|_{\partial B_1}.$$

Hence, we infer that

$$\partial B_1 \text{ is a deformation retract of } W^{1,p}(\Omega) \setminus \{0\}. \tag{3.46}$$

From (3.45) and (3.46), it follows that $\varphi^{\mathfrak{I}_0}$ and ∂B_1 are homotopy equivalent, hence

$$H_k(W^{1,p}(\Omega), \varphi^{\mathfrak{I}_0}) = H_k(W^{1,p}(\Omega), \partial B_1) \quad \text{for all } k \in \mathbb{N}_0,$$

and therefore, by choosing $\mathfrak{I}_0 < 0$ even more negative if necessary, we have

$$C_k(\varphi, \infty) = H_k(W^{1,p}(\Omega), \partial B_1) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.47}$$

The space $W^{1,p}(\Omega)$ is infinite dimensional and so ∂B_1 is contractible. Hence, from [29, p. 147], we have

$$H_k(W^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

and therefore, by (3.47),

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

With suitable changes in the above proof, we can also compute the critical groups at infinity for the functionals $\hat{\varphi}_\pm$. So, we have the following proposition.

Proposition 3.10. *Assume that hypotheses* (Ha), (Hβ) *and* (Hf) *hold and also that* $\inf \hat{\varphi}_\pm(K_{\hat{\varphi}_\pm}) > -\infty$. *Then* $C_k(\hat{\varphi}_\pm, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. We do the proof for $\hat{\varphi}_+$ the proof for the functional $\hat{\varphi}_-$ being similar.

Let $\partial B_1^+ = \{u \in \partial B_1 : u^+ \neq 0\}$. Consider the deformation $h_+\colon [0,1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\hat{u}_1(p,\beta)}{\|(1-t)u + t\hat{u}_1(p,\beta)\|} \quad \text{for all } (t,u) \in [0,1] \times \partial B_1^+.$$

We have

$$h_+(1,u) = \frac{\hat{u}_1(p,\beta)}{\|\hat{u}_1(p,\beta)\|} \in \partial B_1^+,$$

hence ∂B_1^+ is contractible. Hypotheses (Hf) (ii)–(iii) imply that for every $u \in \partial B_1^+$, we have

$$\hat{\varphi}_+(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.48}$$

For $u \in \partial B_1^+$ and $t > 0$, using the chain rule, (3.2), and hypotheses (Ha) (iv) and(Hf) (iii), we have

$$\begin{aligned}\frac{d}{dt}\hat{\varphi}_+(tu) &= \langle \hat{\varphi}_+'(tu), u\rangle \\ &= \frac{1}{t}\langle \hat{\varphi}_+'(tu), tu\rangle \\ &= \frac{1}{t}\Big[\int_\Omega (a(tDu), tDu)_{\mathbb{R}^N}\,dz + \|tu^-\|_p^p + \int_{\partial\Omega} \beta(z)(tu^+)^p\,d\sigma - \int_\Omega f(z,tu^+)tu^+\,dz\Big] \\ &\le \frac{1}{t}\Big[pG(tDu)\,dz + \|tu^-\|_p^p + \int_{\partial\Omega} \beta(z)(tu^+)^p\,dz - \int_\Omega pF(z,tu^+)\,dz + c_{39}\Big] \\ &= \frac{1}{t}[p\hat{\varphi}_+(tu) + c_{39}].\end{aligned} \tag{3.49}$$

From (3.48) and (3.49), it follows that

$$\frac{d}{dt}\hat{\varphi}_+(tu) < -\frac{c_{39}}{p} < 0 \quad \text{for large } t > 0. \tag{3.50}$$

Choose

$$\xi_0 < \min\Big\{-\frac{c_{39}}{p},\ \inf_{\bar{B}_1}\hat{\varphi}_+\Big\}$$

(recall that $\bar{B}_1 = \{u \in W^{1,p}(\Omega) : \|u\| \le 1\}$). Given $u \in \partial B_1^+$, because of (3.50) we see that there is unique $s_0(u) \ge 1$ such that

$$\begin{cases}\hat{\varphi}_+(tu) > \xi_0 & \text{if } t \in [0, s_0(u)), \\ \hat{\varphi}_+(tu) = \xi_0 & \text{if } t = s_0(u), \\ \hat{\varphi}_+(tu) < \xi_0 & \text{if } s_0(u) < t.\end{cases} \tag{3.51}$$

The implicit function theorem implies that $s_0 \in C(\partial B_1^+)$. Note that (see (3.51))

$$\hat{\varphi}_+^{\xi_0} = \{tu : u \in \partial B_1^+,\ t \ge \gamma_0(u)\}.$$

We define $E_+ = \{tu : u \in \partial B_1^+, t \ge 1\}$. We have $\hat{\varphi}_+^{\xi_0} \subseteq E_+$. We consider the deformation $\hat{h}_+(r,tu)$ defined by

$$\hat{h}_+(r,tu) = \begin{cases}(1-r)tu + rs_0(u)u & \text{if } t \in [0, s_0(u)], \\ tu & \text{if } s_0(u) < t,\end{cases} \quad \text{for all } (r,tu) \in [0,1] \times E_+.$$

We have (see (3.51))

$$\hat{h}_+(0,tu) = tu, \qquad \hat{h}_+(1,tu) \in \hat{\varphi}_+^{\xi_0} \qquad \text{and} \qquad \hat{h}_+(r,\cdot)|_{\hat{\varphi}_+^{\xi_0}} = \mathrm{id}|_{\hat{\varphi}_+^{\xi_0}} \quad \text{for all } r \in [0,1].$$

Therefore, $\hat{\varphi}_+^{\xi_0}$ is a strong deformation retract of E_+. Hence,

$$H_k(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_0}) = H_k(W^{1,p}(\Omega), E_+) \quad \text{for all } k \in \mathbb{N}_0,$$

and thus (by choosing $\xi_0 < 0$ even more negative if necessary)

$$C_k(\hat{\varphi}_+, \infty) = H_k(W^{1,p}(\Omega), E_+) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.52}$$

Consider the deformation

$$h_+^*(r, tu) = (1-r)tu + r\frac{tu}{\|tu\|} \quad \text{for all } (r, tu) \in [0,1] \times E_+.$$

We see that

$$h_+^*(0, tu) = tu, \quad h_+^*(1, tu) \in \partial B_1^+ \quad \text{and} \quad h_+^*(1, \cdot)|_{\partial B_1^+} = \mathrm{id}|_{\partial B_1^+}.$$

Therefore, ∂B_1^+ is a deformation retract of E_+. Hence,

$$H_k(W^{1,p}(\Omega), \partial B_1^+) = H_k(W^{1,p}(\Omega), E_+) \quad \text{for all } k \in \mathbb{N}_0,$$

which implies (recall that ∂B_1^+ is contractible)

$$H_k(W^{1,p}(\Omega), E_+) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Thus, by (3.52),

$$C_k(\hat{\varphi}_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Similarly for the functional $\hat{\varphi}_-$. □

Using Propositions 3.9 and 3.10, we can compute precisely the critical groups of the energy functional φ at the two constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ produced in Proposition 3.8.

First, we relate the critical groups of φ with those of $\hat{\varphi}_\pm$. In what follows we assume that the critical sets K_φ and $K_{\hat{\varphi}_\pm}$ are finite. Otherwise, we already have a whole sequence of distinct solutions of (1.4) (see Proposition 3.7, (3.2) and (3.3)).

Proposition 3.11. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then*

$$C_k(\varphi, u_0) = C_k(\hat{\varphi}_+, u_0) \quad \text{and} \quad C_k(\varphi, u_0) = C_k(\hat{\varphi}_-, v_0) \quad \text{for all } k \in \mathbb{N}_0.$$

Proof. We do the proof for the triple $(\varphi, \hat{\varphi}_+, u_0)$, the proof for the other triple $(\varphi, \hat{\varphi}_-, v_0)$ being similar.

We consider the homotopy

$$h(t, u) = (1-t)\varphi(u) + t\hat{\varphi}_+(u) \quad \text{for all } (t, u) \in [0,1] \times W^{1,p}(\Omega).$$

Suppose we can find $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t, \quad u_n \to u_0 \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad h'_u(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.53}$$

Then, from the equation in (3.53) and (3.2), we have

$$\begin{aligned} \langle A(u_n), v\rangle + \int_{\partial\Omega} \beta(z)(u_n^+)^{p-1} v \, d\sigma - t_n \int_{\partial\Omega} \beta(z)(u_n^-)^{p-1} v \, d\sigma - t_n \int_\Omega (u_n^-)^{p-1} v \, dz \\ = \int_\Omega f(z, u_n^+) v \, dz + (1-t_n) \int_\Omega f(z, -u_n^-) \, dz \quad \text{for all } v \in W^{1,p}(\Omega), \end{aligned}$$

which implies

$$\begin{cases} -\operatorname{div} a(Du_n(z)) - t_n u_n^-(z)^{p-1} = f(z, u_n^+(z)) + (1-t_n) f(z, -u_n^-(z)) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u_n}{\partial n_a} + \beta(z)((u_n^+)^{p-1} - t_n(u_n^-)^{p-1}) = 0 & \text{on } \partial\Omega. \end{cases}$$

From Proposition 2.10 we know that there exists $M_{14} > 0$ such that

$$\|u_n\|_\infty \le M_{14} \quad \text{for all } n \in \mathbb{N}.$$

So, from Lieberman [24] we know that there exist $\alpha \in (0,1)$ and $M_{15} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le M_{15} \quad \text{for all } n \in \mathbb{N}. \tag{3.54}$$

Because of (3.53) and since $C^{1,\alpha}(\overline{\Omega})$ is embedded compactly into $C^1(\overline{\Omega})$, from (3.54) we have

$$u_n \to u_0 \quad \text{in } C^1(\overline{\Omega}).$$

Recall that $u_0 \in \operatorname{int} C_+$ (see Proposition 3.8). So, we can find $n_0 \in \mathbb{N}$ such that

$$u_n \in \operatorname{int} C_+ \quad \text{for all } n \geq n_0,$$

hence $\{u_n\}_{n \geq n_0}$ are distinct (positive) solutions of (1.4) (see (3.53)), a contradiction (recall that we have assumed $K_{\hat{\varphi}_+}$ is finite). Therefore (3.53) can not happen. Then, invoking [10, Theorem 5.2] (the homotopy invariance of critical groups), we have

$$C_k(\varphi, u_0) = C_k(\hat{\varphi}_+, u_0) \quad \text{for all } k \in \mathbb{N}_0.$$

In a similar fashion we show that

$$C_k(\varphi, v_0) = C_k(\hat{\varphi}_-, v_0) \quad \text{for all } k \in \mathbb{N}_0.$$ □

Proposition 3.12. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then* $C_k(\hat{\varphi}_+, u_0) = C_k(\hat{\varphi}_-, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$.

Proof. We do the proof for the pair $(\hat{\varphi}_+, u_0)$, the proof for the pair $(\hat{\varphi}_-, v_0)$ being similar.

From Proposition 3.7 we know that $K_{\hat{\varphi}_+} \subseteq C_+$. So, we may assume that

$$K_{\hat{\varphi}_+} = \{0, u_0\} \tag{3.55}$$

or, otherwise, we already have a third nontrivial solution for problem (1.4) which in fact is positive. From the proof of Proposition 3.8 (see (3.35) and (3.36)) we have

$$0 = \hat{\varphi}_+(0) < m_\rho^+ \leq \hat{\varphi}_+(u_0).$$

Let $\xi_- < 0 < \xi_+ < m_\rho^+$, and consider the triple of sets

$$\hat{\varphi}_+^{\xi_-} \subseteq \hat{\varphi}_+^{\xi_+} \subseteq W^{1,p}(\Omega).$$

For this triple of sets, we consider the following corresponding long exact sequence of singular homology groups (see [29, p. 143]):

$$\cdots \to H_k(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_-}) \xrightarrow{i_*} H_k(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_+}) \xrightarrow{\partial_*} H_{k-1}(\hat{\varphi}_+^{\xi_+}, \hat{\varphi}_+^{\xi_-}) \to \cdots, \tag{3.56}$$

with i_* being the homomorphism induced by the inclusion $i\colon (W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_-}) \to (W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_+})$ and ∂_* is the boundary homomorphism. From (3.55) and since $\xi_- < 0 = \hat{\varphi}_+(0)$, we have (see Proposition 3.10)

$$H_k(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_-}) = C_k(\hat{\varphi}_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{3.57}$$

Also, we have

$$0 = \hat{\varphi}_+(0) < \xi_+ < \hat{\varphi}_+(u_0).$$

Then from (3.55) we have

$$H_k(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_+}) = C_k(\hat{\varphi}_+, u_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.58}$$

Similarly, we have (see Proposition 3.6)

$$H_{k-1}(\hat{\varphi}_+^{\xi_+}, \hat{\varphi}_+^{\xi_-}) = C_{k-1}(\hat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.59}$$

From (3.57)–(3.59) and the exactness of (3.56), we see that only the tail of that chain (that is, $k = 1$) is nontrivial. From the rank theorem, the exactness of (3.56), and using (3.57) and (3.59), we have

$$\operatorname{rank} H_1(W^{1,p}(\Omega), \hat{\varphi}_+^{\xi_+}) = \operatorname{rank} \ker \partial_* + \operatorname{rank} \operatorname{im} \partial_* = \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} \partial_* \leq 1. \tag{3.60}$$

From the proof of Proposition 3.8 we know that u_0 is a critical point of $\hat{\varphi}_+$ of mountain pass type. Therefore,

$$C_1(\hat{\varphi}_+, u_0) \neq 0. \tag{3.61}$$

From (3.58), (3.60), (3.61) and recalling that only for $k = 1$ the chain (3.56) is nontrivial, we conclude that

$$C_k(\hat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Similarly, for the pair $(\hat{\varphi}_-, v_0)$. □

From Propositions 3.11 and 3.12, we infer the following corollary.

Corollary 3.13. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then* $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$.

Now we ready for the "three solutions theorem" for problem (1.4).

Theorem 3.14. *If hypotheses* (Ha), (Hβ) *and* (Hf) *hold, then problem* (1.4) *has at least three nontrivial solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $y_0 \in C^1(\overline{\Omega})$.

Proof. From Proposition 3.8 we already have two constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$. Suppose that these are the only nontrivial solutions of problem (1.4) (that is, $K_\varphi = \{0, u_0, v_0\}$). From Corollary 3.13 we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.62}$$

From Proposition 3.6 we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.63}$$

Finally, Proposition 3.9 implies that

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{3.64}$$

From (3.62)–(3.64) and the Morse relation with $t = -1$ (see (2.44)), we have $2(-1)^1 + (-1)^0 = 0$, which implies $(-1)^1 = 0$, a contradiction. So, we can find $y_0 \in K_\varphi$, $y_0 \notin \{0, u_0, v_0\}$. This is the third nontrivial solution of problem (1.4) and, as before, the nonlinear regularity theory implies $y_0 \in C^1(\overline{\Omega})$. □

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