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Semilinear Neumann problems with indefinite and unbounded potential and crossing nonlinearity

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Dedicated to Professor Patrizia Pucci, with deep esteem and admiration, on the occasion of her 60th birthday

Abstract. We consider a semilinear Neumann problem with an indefinite and unbounded potential and an asymmetric reaction that crosses at least the principal eigenvalue of the operator $-\Delta + \beta I$ in $H^1(\Omega)$, β being the potential function. Using a combination of variational methods, with truncation and perturbation techniques and Morse theory, we prove multiplicity theorems providing precise sign information for all the solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2–boundary $\partial\Omega$. In this paper we study the following semilinear Neumann problem

$$(1) \qquad \begin{cases} \Delta u(z) + \beta(z)u(z) = f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\beta \in L^s(\Omega)$ with $s > N$ and, in general, it is indefinite (sign changing) and unbounded. We assume that the reaction $f(z, x)$ is a measurable function which is C^1 in the x–variable. The aim of this paper is to prove a multiplicity theorem for problem (1) providing information for all the solutions, provided that the reaction $x \to f(z, x)$ exhibits an asymmetric behavior at $+\infty$ and $-\infty$ (crossing and jumping nonlinearity).

The multiplicity of solutions for such semilinear elliptic equations was first studied by Hofer [**13**], who examined a Dirichlet problem with $\beta \equiv 0$. Assuming that $f(z, x) = f(x)$ with $f \in C^1(\mathbb{R})$, $f(0) = 0$, $f'(0) \in (\lambda_i, \lambda_{i+1})$ for some $i \geq 2$ (here $\{\lambda_i\}_{i \geq 1}$ denotes the sequence of distinct eigenvalues of $(-\Delta, H_0^1(\Omega))$ and that $\limsup_{x \to \pm\infty} f(x)/x < \lambda_1$, Hofer [**13**] proved that the equation has at least four nontrivial solutions, two of which have constant sign (one positive and the other negative). Later, Bartsch & Wang [**3**] proved that from the other two solutions, one is nodal (sign-changing). In fact, Dancer & Du [**7**] and Li & Wang [**16**] established that both solutions are nodal. In the aforementioned works it is assumed that asymptotically at $\pm\infty$ the quotient $f(z, x)/x$ stays below λ_1 and this makes the

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energy functional of the problem coercive. Problems with asymmetric (crossing) reaction, are usually studied using the so-called "Fučik spectrum". We refer to the works Các [**5**], Cuesta & Gossez [**6**], Magalhaes [**19**], and Perera & Schechter [**22**]. However, this approach has two serious limitations. First, the use of the Fučik spectrum requires that the limits $\lim_{x\to\pm\infty} f(z,x)/x$ do exist. Second, our knowledge of the Fučik spectrum is limited (see Schechter [**24**]). More recently, Liu & Sun [**18**] considered the asymmetric Dirichlet problem with $\beta = 0$ and without any use of the Fučik spectrum. Their method of proof is based on some elaborate flow invariance arguments. In fact, Liu & Sun [**18**, p. 1071] mention that alternatively "Morse theory could work, but then the techniques will be more complicated". In the present paper, working in the framework of Neumann problems (which in principle are more difficult to deal with, due to the failure of the Poincaré inequality) with an indefinite and unbounded potential $\beta(\cdot)$, using a combination of variational methods and Morse theory, we prove multiplicity results with precise sign information for all the solutions, under weaker conditions on the reaction $f(z,x)$ than in Liu & Sun [**18**]. Our approach is based on the critical point theory, together with suitable perturbation, truncation and comparison techniques and with the use of Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the sequel. Also, we examine the spectral properties of the operator $H^1(\Omega) \ni u \longmapsto -\Delta u + \beta u$.

2. Mathematical Background

In the study of problem (1), in addition to the Sobolev space $H^1(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega});\ u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+;\ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \longmapsto g(z,x)$ is measurable, and for a.a. $z \in \Omega$, the function $x \longmapsto f(z,x)$ is continuous) with subcritical growth in $x \in \mathbb{R}$, namely

$$|g(z,x)| \leq \alpha(z)\,(1 + |x|^{r-1}) \qquad \text{for a.a. } z \in \Omega \text{ and for all } x \in \mathbb{R}$$

with $\alpha \in L^\infty(\Omega)_+$ and $1 < r < 2^*$, where $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1,2$. Also, let $\beta \in L^s(\Omega)$ with $s > N$, $G(z,x) = \int_0^x g(z,t)dt$, and consider the C^1–functional $\Psi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Psi_0(u) = \frac{1}{2}\,\|Du\|_2^2 + \frac{1}{2}\int_\Omega \beta u^2 dz - \int_\Omega G(z,u)dz \qquad \text{for all } u \in H^1(\Omega)\,.$$

The next result was first proved by Brezis & Nirenberg [**4**] for the "Dirichlet" space $H_0^1(\Omega)$ and was later extended to the space $W_0^{1,p}(\Omega)$ (with $1 < p < \infty$) by Garcia-Azorero, Manfredi & Peral Alonso [**11**] and to the space $W^{1,p}(\Omega)$ (Neumann case) by Iannizzotto & Papageorgiou [**14**]. The proof of [**14**] applies in the present setting using the regularity results of Wang [**26**]. So, we have:

Proposition 2.1. *Assume that $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$–minimizer of Ψ_0, that is, there exists $\rho_0 > 0$ such that $\Psi_0(u_0) \leq \Psi_0(u_0 + h)$ for all $h \in C^1(\overline{\Omega})$ with*

$\|h\|_{C^1(\overline{\Omega})} \leq \rho_0$. *Then* $u_0 \in C^{1,\gamma}(\overline{\Omega})$ *for some* $\gamma \in (0,1)$ *and* u_0 *is a local* $H^1(\Omega)$–*minimizer of* Ψ_0, *that is, there exists* $\rho_1 > 0$ *such that* $\Psi_0(u_0) \leq \Psi_0(u_0 + h)$ *for all* $h \in H^1(\Omega)$ *with* $\|h\|_{H^1(\Omega)} \leq \rho_1$.

Here and in the sequel, we denote by $\|\cdot\|$ the norm in $H^1(\Omega)$, that is,

$$\|u\| = \left(\|u\|_2^2 + \|Du\|_2^2\right)^{1/2} \qquad \text{for all } u \in H^1(\Omega).$$

Next we recall some basic definitions and facts from critical point theory. For details, we refer to the books by Gasinski & Papageorgiou [**12**] and Kristaly, Rădulescu & Varga [**15**].

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the dual pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the "Cerami condition" (the "C–condition" for short), if the following is true:
"Every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subset \mathbb{R}$ is bounded and

$$(1 + \|x_n\|_X)\,\varphi'(x_n) \to 0 \ \text{ in } X^*,$$

admits a strongly convergent subsequence."

This compactness-type condition is in general weaker than the more usual Palais-Smale condition. Nevertheless it suffices to prove a deformation theorem and to deduce the minimax theory for certain critical values of φ. In particular, we have the following result, known in the literature as the "mountain pass theorem".

THEOREM 2.1. *Assume that* $\varphi \in C^1(X)$, $x_0, x_1 \in X$, $\|x_1 - x_0\|_X > \rho > 0$,

$$\max \|\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x);\ \|x - x_0\|_X = \rho\} = \eta_l$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0,1]; X);\ \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

and φ *satisfies the C–condition. Then* $c \geq \eta_l$ *and* c *is a critical value of* φ.

Next, from Morse theory, we recall the definition of critical groups and the Morse relation. So, let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We define the following sets:

$$\varphi^c = \{x \in X;\ \varphi(x) \leq c\}, \quad K_\varphi = \{x \in X;\ \varphi'(x) = 0\}, \quad K_\varphi^c = \{x \in K_\varphi;\ \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \geq 0$ be an integer. By $H_k(Y_1, Y_2)$ we denote the kth singular homology group for the topological pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $x \in K_\varphi^c$ are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \setminus \{x\}) \qquad \text{for all } k \geq 0,$$

where $\mathcal{U}$ is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the neighborhood $\mathcal{U}$ of x.

Suppose that $\varphi \in C^1(X)$ satisfies the C–condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \qquad \text{for all } k \geq 0.$$

The second deformation theorem (see for example Gasinski & Papageorgiou [**12**, p. 628]) implies that the above definition of critical groups of φ at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$M(t,x)=\sum_{k\geq 0}\operatorname{rank} C_k(\varphi,x)t^k \qquad \text{for all } t\in\mathbb{R} \text{ and all } x\in K_\varphi$$

$$P(t,\infty)=\sum_{k\geq 0}\operatorname{rank} C_k(\varphi,\infty)t^k \qquad \text{for all } t\in\mathbb{R}\,.$$

The Morse relation says

$$\sum_{x\in K_\varphi} M(t,x)=P(t,\infty)+(1+t)Q(t), \tag{2}$$

where $Q(t)=\sum_{k\geq 0}a_k t^k$ is a formal series with nonnegative integer coefficients a_k, $k\geq 0$.

Suppose that $X=H$ is a Hilbert space, $x\in H$, $\mathcal{U}$ is a neighborhood of x and $\varphi\in C^2(\mathcal{U})$. If $x\in K_\varphi$, then the Morse index of x denoted by $\mu=\mu(x)$, is defined as the supremum of the dimensions of vector subspaces of H in which $\varphi''(x)$ is negative definite. The nullity of φ at $x\in K_\varphi$, denoted by $\nu=\nu(x)$, is defined to be the dimension of $\operatorname{Ker}\varphi''(x)$. We say that $x\in K_\varphi$ is nondegenerate if $\nu(x)=0$, that is, $\varphi''(x)$ is invertible. If $x\in K_\varphi$ is nondegenerate with Morse index μ, then (see Mawhin & Willem [**20**])

$$C_k(\varphi,x)=\delta_{k,\mu}\mathbb{Z} \qquad \text{for all } k\geq 0, \tag{3}$$

where $\delta_{k,\mu}$ denotes the Kronecker symbol.

Now we develop the spectrum of $-\Delta u+\beta u$ for $u\in H^1(\Omega)$. We follow Willem [**27**], where the Dirichlet eigenvalue problem is examined. For completeness we provide the details. So, we examine the following liner eigenvalue problem

$$\begin{cases} -\Delta u(z)+\beta(z)u(z)=\lambda u(z) & \text{in } \Omega\\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Set

$$\sigma(u)=\frac{1}{2}\,\|Du\|_2^2+\frac{1}{2}\int_\Omega \beta u^2dz, \qquad u\in H^1(\Omega).$$

To analyze the linear eigenvalue problem (4) it suffices to impose the following condition on the potential β. Eventually, in order to deal with problem (1) we will have to strengthen this condition.

H_0: $\beta\in L^{N/2}(\Omega)$ if $N\geq 3$, $\beta\in L^r(\Omega)$ with $r>1$ if $N=2$ and $\beta\in L^1(\Omega)$ if $N=1$.

Lemma 2.1. *If hypothesis H_0 holds, then*

$$\widehat{\lambda}_1=\inf\{\sigma(u);u\in H^1(\Omega),\|u\|_2=1\}>-\infty.$$

Proof. We treat the case $N\geq 3$, the other two cases being similar using the Sobolev embedding theorem.

We proceed by contradiction. So, suppose that the conclusion of Lemma is not true. Then we can find $\{u_n\}_{n\geq 1}\subset H^1(\Omega)$ such that $\|u_n\|_2=1$ for all $n\geq 1$ and $\sigma(u_n)\to-\infty$ as $n\to\infty$. So, we can find $n_0\geq 1$ such that

$$\sigma(u_n)\leq -1 \qquad \text{for all } n\geq n_0. \tag{5}$$

Suppose that $\|u_n\|\to\infty$ as $n\to\infty$ and let $y_n=\frac{u_n}{||u_n||}$. Then $||y_n||=1$ for all $n\geq 1$ and so we may assume that

$$y_n\to y \text{ weakly in } H^1(\Omega) \text{ and } y_n\to y \text{ in } L^2(\Omega). \tag{6}$$

Note that $\{y_n^2\}_{n\geq 1} \subset L^{\frac{N}{N-2}}(\Omega)$ is bounded (by the Sobolev embedding theorem) and so, by (6), we may assume that

$$y_n^2 \to y^2 \quad \text{weakly in } L^{\frac{N}{N-2}}(\Omega). \tag{7}$$

Therefore

$$\int_\Omega \beta y_n^2 dz \to \int_\Omega \beta y^2 dz \quad (\text{since } \tfrac{N-2}{N} + \tfrac{2}{N} = 1, \text{ see } H_0). \tag{8}$$

Thus, by (5) and (8) and passing at the limit as $n \to \infty$, we obtain $\sigma(y) \leq 0$.

If $y = 0$, then $y_n \to 0$ in $H^1(\Omega)$, a contradiction to the fact that $||y_n|| = 1$ for all $n \geq 1$. Hence $y \neq 0$. On the other hand

$$||y_n||_2 = \frac{||u_n||_2}{||u_n||} = \frac{1}{||u_n||} \to 0 \text{ as } n \to \infty,$$

which implies $||y||_2 = 0$ (see (6)), hence $y = 0$, a contradiction. This proves that $\{u_n\}_{n\geq 1} \subset H^1(\Omega)$ is bounded. So we may assume that

$$u_n \to u \text{ weakly in } H^1(\Omega)$$
$$u_n \to u \text{ weakly in } L^{2^*}(\Omega) \quad (\text{recall } 2^* = \tfrac{2N}{N-2})$$

Therefore

$$\int_\Omega \beta u_n^2 dz \to \int_\Omega \beta u^2 dz.$$

Taking the limit as $n \to \infty$ we obtain $\sigma(u) \leq \widehat{\lambda}_1 = -\infty$, a contradiction. So, we conclude that $\widehat{\lambda}_1 > -\infty$. $\square$

By virtue of Lemma 2.1, we see that we can find $\widehat{\gamma} > \max\{-\widehat{\lambda}_1, 0\}$ such that

$$\sigma(u) + \widehat{\gamma}||u||_2^2 \geq \widehat{c}||u||^2 \qquad \text{for all } u \in H^1(\Omega) \text{ with } \widehat{c} > 0. \tag{9}$$

Then relation (9) suggests the introduction of the following inner product on $H^1(\Omega)$:

$$(u, y)_* = \int_\Omega (Du, Dy)_{\mathbb{R}^N} dz + \int_\Omega (\beta(z) + \widehat{\gamma}) uy dz \qquad \text{for all } u, y \in H^1(\Omega).$$

Given $h \in L^2(\Omega)$, by the Riesz representation theorem, we can find a unique $u \in H^1(\Omega)$ such that

$$(u, v)_* = \int hv dz \qquad \text{for all } v \in H^1(\Omega). \tag{10}$$

So, we can define the continuous linear map $K_0 : L^2(\Omega) \to H^1(\Omega)$ which to each $h \in L^2(\Omega)$ assigns the unique $u \in H^1(\Omega)$ satisfying (10). Let $i : H^1(\Omega) \to L^2(\Omega)$ be the embedding map. By virtue of the Sobolev embedding theorem, i is compact and so $K_0 \circ i$ is compact, self-adjoint and positive. Then by the spectral theorem for such operators (see, for example, Gasinski & Papageorgiou [**12**, p. 296]), we can find $\{\mu_n\}_{n\geq 1}$ a sequence of eigenvalues of $K_0 \circ i$ such that $\mu_1 > \mu_2 > \ldots > \mu_n > \ldots > 0$ and $\mu_n \to 0$.

We set $\widehat{\lambda_n} = \frac{1}{\mu_n} - \widehat{\gamma}$ for all $n \geq 1$. Then $\{\widehat{\lambda_n}\}_{n\geq 1}$ is the sequence of distinct eigenvalues of (4). We have $-\infty < \widehat{\lambda_1} < \widehat{\lambda_2} < \ldots < \widehat{\lambda_n} < \ldots$, and $\widehat{\lambda_n} \to +\infty$ as $n \to \infty$.

To these eigenvalues corresponds a sequence $\{\widehat{u_n}\}_{n\geq 1} \subset H^1(\Omega)$ of eigenfunctions, which form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Moreover, if $\beta \in L^s(\Omega)$ with $s > N$, then the regularity results of Wang [**26**] imply $\{\widehat{u_n}\}_{n\geq 1} \subset C^1(\overline{\Omega})$. These eigenvalues admit variational characterizations of

Courant type using the Rayleigh quotient $\frac{\sigma(u)}{||u||_2^2}$ for all $u \in H^1(\Omega)\backslash\{0\}$. So, denoting by $E(\widehat{\lambda}_i)$ the eigenspace corresponding to $\widehat{\lambda}_i$, we have

$$\widehat{\lambda}_1 = \inf\left\{\frac{\sigma(u)}{||u||_2^2}; u \in H^1(\Omega), u \neq 0\right\}. \tag{11}$$

$$\begin{aligned}\widehat{\lambda}_k &= \inf\left\{\frac{\sigma(u)}{||u||_2^2}; u \in \overline{\underset{\mathrm{i}\geq \mathrm{k}}{\oplus} E(\widehat{\lambda}_i)}, u \neq 0\right\} \\ &= \sup\left\{\frac{\sigma(u)}{||u||_2^2}; u \in \overset{k}{\underset{\mathrm{i}=1}{\oplus}} E(\widehat{\lambda}_i), u \neq 0\right\} \text{ for } k \geq 2.\end{aligned} \tag{12}$$

The infimum in (11) and both the infimum and the supremum in (12) are realized on $E(\widehat{\lambda}_k)$, $k \geq 1$. The first eigenvalue $\widehat{\lambda}_1$ is simple (that is, $\dim E(\widehat{\lambda}_1) = 1$) and from (11) it is clear that the nontrivial elements of $E(\widehat{\lambda}_1)$ do not change sign. In fact $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign changing) eigenfunctions. By $\widehat{u}_1$ we denote the L^2-normalized (that is, $||\widehat{u}_1||_{L_2} = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. If $\beta \in L^s(\Omega)$ with $s > N$, then $\widehat{u}_1 \in C_+\backslash\{0\}$ and in fact by the Harnack inequality of Pucci & Serrin [**23**, p. 163], we have $\widehat{u}_1(z) > 0$ for all $z \in \Omega$. Finally, if $\beta^+ \in L^\infty(\Omega)$, then the boundary point theorem of Pucci & Serrin [**23**, p. 120] implies that $\widehat{u}_1 \in \operatorname{int} C_+$. When $\beta \in L^s(\Omega)$ with $s > N/2$, the eigenspaces $E(\widehat{\lambda}_k)$ have the so-called "Unique Continuation Property" (UCP for short). Namely, if $u \in E(\widehat{\lambda}_k)$ and u vanishes on a set of positive measure, then $u \equiv 0$ (see de Figueiredo & Gossez [**10**]).

A similar analysis can be conducted for a weighted version of the eigenvalue problem (4). So, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following linear eigenvalue problem

$$\left\{\begin{array}{rcll} -\Delta u(z) + \beta(z)u(z) &=& \lambda m(z)u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} &=& 0 & \text{on } \partial\Omega\,. \end{array}\right. \tag{13}$$

As for (4), the eigenvalue problem (13) has a strictly increasing sequence $\{\tilde{\lambda}_k(m)\}_{k\geq 1}$ of eigenvalues such that $\tilde{\lambda}_k(m) \to +\infty$ as $k \to \infty$. These eigenvalues admit variational characterizations in terms of Rayleigh quotient $\frac{\sigma(u)}{\int_\Omega mu^2 dz}$ for all $u \in H^1(\Omega), u \neq 0$ (see (11), (12)). The first eigenvalues $\tilde{\lambda}_j(m) > 0$ is simple and has eigenfunctions of constant sign. These eigenspaces $E(\tilde{\lambda}_k(m))$ have the UCP and this leads to the following monotonicity property for the eigenvalues:

PROPOSITION 2.2. *If $m_1, m_2 \in L^\infty(\Omega) \setminus \{0\}$, $m_1(z) \leq m_2(z)$ a.e. in Ω and $m_1 \neq m_2$, then $\tilde{\lambda}_k(m_2) < \tilde{\lambda}_k(m_1)$ for all $k \geq 1$.*

Also, as a consequence of the Harnack inequality (see Pucci & Serrin [**23**, p. 163]), we have the following useful inequality.

PROPOSITION 2.3. *If $\vartheta \in L^s(\Omega)$ with $s > \frac{N}{2}$, $\vartheta(z) \leq \widehat{\lambda}_1$ a.e. in Ω and $\vartheta \neq \widehat{\lambda}_1$ then there exists $c_0 > 0$ such that*

$$\eta(u) = \sigma(u) - \int_\Omega \vartheta\ u^2 dz \geq c_0||u||^2 \qquad \textit{for all } u \in H^1(\Omega).$$

Proof. Evidently $\eta \geq 0$. Suppose that the result is not true. Then by virtue of the 2-homogeneity of the functional η, we can find $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that $||u_n|| = 1$ for all $n \geq 1$ and $\eta(u_n) \downarrow 0$ as $n \to \infty$. We may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega). \tag{14}$$

The functional $\sigma(\cdot)$ is sequentially weakly lower semi-continuous. Thus, for some $\xi \in \mathbb{R}$,

$$\sigma(u) \leq \int_\Omega \vartheta u^2 dz \quad (\text{see } (14)) \Rightarrow \sigma(u) = \widehat{\lambda}_1 ||u||_2^2 \quad (\text{see } (11)) \Rightarrow u = \xi \widehat{u}_1\,. \tag{15}$$

If $\xi = 0$ then $u_n \to 0$ in $H^1(\Omega)$, which contradicts the fact that $||u_n|| = 1$ for all $n \geq 1$. So $\xi \neq 0$. By the Harnack inequality $\widehat{u}_1(z) > 0$ for all $z \in \Omega$ (see Pucci & Serrin [**23**, p. 163]). Hence $|u(z)| > 0$ for all $z \in \Omega$ and so from (15) we have $\sigma(u) < \widehat{\lambda}||u||_2^2$, which contradicts (11). ☐

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and for $u \in H^1(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in H^1(\Omega)$, $|u| = u^+ + u^-$, and $u = u^+ - u^-$.

Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \qquad \text{for all } u \in H^1(\Omega)$$

(the Nemytsky map corresponding to h). Also, $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} dz \qquad \text{for all } u, y \in H^1(\Omega).$$

Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3. Solutions of Constant Sign

In this section we produce solutions of constant sign for problem (1). In section 4 we have the full multiplicity theorems.

The hypotheses on the data of (1), are the following:

H_1 : *$\beta \in L^s(\Omega)$ with $s > N$ and $\beta^+ \in L^\infty(\Omega)$.*

H_2 : *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable such that for a.a. $z \in \Omega, f(z, 0) = 0, f(z, \cdot) \in C^1(\mathbb{R})$ and (i) $|f_x'(z, x)| \leq a(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega), 2 \leq r < 2^*$;*

(ii) there exist functions $\theta, \widehat{\theta} \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}, \text{ a.e. in } \Omega, \ \theta \neq \widehat{\lambda}_1 \text{ and}$$
$$\widehat{\theta}(z) \leq \liminf_{x\to+\infty} \frac{f(z,x)}{x} \leq \limsup_{x\to+\infty} \frac{f(z,x)}{x} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there are functions $\eta, \widehat{\eta} \in L^\infty(\Omega)$ such that

$$\eta(z) \geq \widehat{\lambda}, \text{ a.e. in } \Omega, \ \eta \neq \widehat{\lambda}_1 \text{ and}$$
$$\eta(z) \leq \liminf_{x\to-\infty} \frac{f(z,x)}{x} \leq \limsup_{x\to-\infty} \frac{f(z,x)}{x} \leq \widehat{\eta}(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exists integer $\ell \geq 2$ such that

$$f_x'(z,0) \in [\widehat{\lambda}_\ell, \widehat{\lambda}_{\ell+1}] \text{ a.e. in } \Omega \ \ f_x'(\cdot,0) \neq \widehat{\lambda}_\ell, \ f_x'(\cdot,0) \neq \widehat{\lambda}_{\ell+1} \text{ and}$$
$$f_x'(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

Remark: Hypotheses $H_2(ii)$, (iii) classify this nonlinearity as "crossing" or "jumping" since as we move from $-\infty$ to $+\infty$ the quotient $\frac{f(z,x)}{x}$ crosses at least the principal eigenvalue $\widehat{\lambda}_1$. This asymmetric behavior of $f(z,\cdot)$ makes it impossible to use the methods and techniques of the papers mentioned in the Introduction.

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\sigma(u) - \int_\Omega F(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently $\varphi \in C^2(H^1(\Omega))$. Also let $\widehat{\gamma} > 0$ be as in (9). We introduce the following perturbations-truncations of the reaction $\rho(z,\cdot)$:

$$\widehat{f}_+(z,x) = f(z,x^+) + \widehat{\gamma}x^+ \qquad \text{and} \qquad \widehat{f}_-(z,x) = f(z,-x^-) + \widehat{\gamma}(-x^-).$$

Both are Carathéodory functions. We set $\widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)ds$ and consider the C^1- functionals $\widehat{\varphi}_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{\gamma}}{2}||u||_2^2 - \int_\Omega \widehat{F}_\pm(z,u)dz \qquad \text{for all } u \in H^1(\Omega).$$

PROPOSITION 3.1. *If hypotheses H_1 and H_2 hold, then φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ be a sequence such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi^{'}(u_n) \to 0 \text{ in } H^1(\Omega)^*. \tag{16}$$

From (16) we have

$$\begin{aligned} &|\left\langle \varphi^{'}(u_n), h \right\rangle| \leq \tfrac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in H^1(\Omega) \text{ with } \epsilon_n \to 0 \\ &\Rightarrow |\textstyle\int_\Omega u_n h dz + \int_\Omega \beta u_n h dz - \int_\Omega f(z,u_n)h dz| \leq \tfrac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } n \geq 1. \end{aligned} \tag{17}$$

In (17) we choose $h = u_n^+ \in H^1(\Omega)$. Then

$$||Du_n^+||_2^2 + \int_\Omega \beta(u_n^+)^2 dz - \int_\Omega f(z,u_n^+)u_n^+ dz \leq \epsilon_n \qquad \text{for all } n \geq 1. \tag{18}$$

Suppose that $||u_n^+|| \to \infty$. We set $y_n = \frac{u_n^+}{||u_n^+||}$, $n \geq 1$. Then $||y_n|| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^{2s^{'}}(\Omega) \ (\frac{1}{s} + \frac{1}{s^{'}} = 1). \tag{19}$$

From (18) we have

$$\sigma(y_n) - \int_\Omega \frac{f(z,u_n^+)}{||u_n^+||} y_n dz \ \leq \ \frac{\epsilon_n}{||u_n^+||^2} \text{ for all } n \geq 1. \tag{20}$$

From hypotheses $H_2(i), (ii)$ we obtain

$$\left\{ \frac{N_f(u_n^+)}{||u_n^+||} \right\}_{n\geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\frac{N_f(u_n^+)}{||u_n^+||} \xrightarrow{w} g \text{ in } L^2(\Omega). \tag{21}$$

Moreover, hypothesis $H_2(ii)$ implies that

$$g = \theta_0 y \text{ with } \widehat{\theta}(z) \leq \theta_0(z) \leq \theta(z) \text{ a.e. in } \Omega. \tag{22}$$

So, if in (20) we pass to the limit as $n \to \infty$ and use (19), (21), (22), then

$$\sigma(y) - \int_\Omega \theta_0 y^2 dz \leq 0 \Rightarrow \; C_0||y||^2 \leq 0 \text{ (see Proposition 2.3)} \Rightarrow \; y = 0.$$

Then

$$\sigma(y_n) \to 0 \Rightarrow \; Dy_n \to 0 \text{ in } L^2(\Omega,\, \mathbb{R}) \Rightarrow \; y_n \to 0 \text{ in } H^1(\Omega) \text{ (see (19))},$$

which contradicts the fact that $||y_n|| = 1$ for all $n \geq 1$. This proves that

$$\{u_n\}_{n \leq 1} \subset H^1(\Omega) \text{ is bounded.} \tag{23}$$

From (17) and (23), we have for all $n \geq 1$ and for some $M_1 > 0$,

$$|\left\langle A(-u_n^-), h \right\rangle + \int_\Omega \beta(-u_n^-)hdz - \int_\Omega f(z, -u_n^-)hdz| \leq M_1||h||\,. \tag{24}$$

Suppose that $||u_n^-|| \to \infty$. We set $v_n = \frac{u_n^-}{||u_n^-||}$, $n \geq 1$. Then $||v_n|| = 1$ for all $n \geq 1$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } H^1(\Omega) \text{ and } v_n \to v \text{ in } L^2(\Omega). \tag{25}$$

From (24), we have for all $n \geq 1$,

$$|\left\langle A(-v_n^-), h \right\rangle + \int_\Omega \beta(-v_n)hdz - \int_\Omega \frac{f(z, -u_n^-)}{||u_n^-||}hdz| \leq \frac{M_1||h||}{||u_n^-||}\,. \tag{26}$$

Again we have that $\left\{\frac{N_f(-u_n^-)}{||u_n^-||}\right\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded and by virtue of hypothesis $H_2(iii)$ we have

$$\frac{N_f(-u_n^-)}{||u_n^-||} \xrightarrow{w} g_* = \eta_0 v \text{ in } L^2(\Omega) \text{ with } \eta(z) \leq \eta_0(z) \leq \widehat{\eta}(z) \text{ for a.a. } z \in \Omega. \tag{27}$$

So, if in (26) we pass to the limit as $n \to \infty$ and use (25) and (27), then

$$\begin{aligned} &\langle A(-v), h\rangle + \textstyle\int_\Omega \beta(-v)hdz = \int_\Omega \eta_0(-v)hdz \text{ for all } h \in H^1(\Omega) \\ &\Rightarrow A(v) + \beta v = \eta_0 v, \\ &\Rightarrow -\Delta v(z) + \beta(z)v(z) = \eta_0(z)v(z) \text{ a.e. in } \Omega,\; \tfrac{\theta v}{\theta n} = 0 \text{ on } \theta\Omega. \end{aligned} \tag{28}$$

Also, if in (26) we choose $h = v_n - v \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (25) and (27), then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(-v_n), v_n - v\rangle = 0 \Rightarrow ||Dv_n||_2 \to ||Dv||_2 \\ &\Rightarrow v_n \to v \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property, see (25))} \\ &\Rightarrow ||v|| = 1, v \geq 0. \end{aligned} \tag{29}$$

By virtue of Proposition 2.2, we have

$$\tilde{\lambda}_1(\eta_0) \leq \tilde{\lambda}_1(\eta) < \tilde{\lambda}_1(\widehat{\lambda}_1) = 1. \tag{30}$$

From (28) and (30) it follows that $v \neq 0$ (see (29)) is nodal, a contradiction. This means that

$$\{u_n^-\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded} \Rightarrow \; \{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded (see (23)).}$$

Hence we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega). \tag{31}$$

In (17) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (31). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \Rightarrow ||Du_n||_2 \to ||Du||_2 \Rightarrow u_n \to u \text{ in } H^1(\Omega)\,.$$

This proves that φ satisfies the C-condition. □

PROPOSITION 3.2. *If hypotheses H_1 and H_2 hold, then problem* (1) *has a solution u_0 in* $\operatorname{int} C_+$ *which is a local minimizer of the functional φ.*

Proof. Hypotheses $H_2(i),(ii)$ imply that given $\epsilon > 0$, we can find $C_\epsilon > 0$ such that

$$F(z,x) \le \frac{1}{2}(\theta(z)+\epsilon)x^2 + C_\epsilon \text{ for a.a. } z \in \Omega, \quad \text{and for all } x \ge 0. \tag{32}$$

Then for all $u \in H^1(\Omega)$ we have

$$\begin{aligned}\widehat{\varphi}_+(u) = \quad & \frac{1}{2}\sigma(u) + \frac{\widehat{\gamma}}{2}||u||_2^2 - \int_\Omega \widehat{F}_+(z,u)dz \\ & \ge \frac{1}{2}[\sigma(u) - \int_\Omega \theta u^2 dz] - \frac{\epsilon}{2}||u||^2 - C_\epsilon|\Omega|_N \text{ (see (32))} \\ & \ge \frac{C_0-\epsilon}{2}||u||^2 - C_\epsilon|\Omega|_N \text{ (see Prop. 2.3).}\end{aligned} \tag{33}$$

Choosing $\epsilon \in (0, C_0)$, from (33) we infer that $\widehat{\varphi}_+$ is coercive. Also, it is sequentially weakly lower semi-continuous. So, by the Weierstrass theorem, we can find $v_0 \in H^1(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf\{\widehat{\varphi}_+(u) : u \in H^1(\Omega)\}. \tag{34}$$

Hypothesis $H_2(iv)$ implies that we can find $\delta > 0$ and $\epsilon > \widehat{\lambda}_1$ (recall $\ell \ge 2$, see $H_2(iv)$) such that

$$F(z,x) \ge \frac{\epsilon}{2}x^2 \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]\,. \tag{35}$$

Since $\widehat{u}_1 \in \operatorname{int} C_+$ (see Section 2), we can find $t \in (0,1)$ small such that $t\widehat{u}_1(z) \in [0,\delta]$ for all $z \in \bar{\Omega}$. Then

$$\begin{aligned}\widehat{\varphi}_+(t\widehat{u}_1) = \quad & \frac{t^2}{2}\sigma(\widehat{u}_1) - \int_\Omega F(z,t\widehat{u}_1) \text{ (recall the definition of } \widehat{f}_+(z,x)) \\ & \le \frac{t^2}{2}[\widehat{\lambda}_1 - \epsilon] \text{ (see (35) and recall that } ||\widehat{u}_1||_2 = 1) \\ & < 0 \text{ (since } \epsilon > \widehat{\lambda}_1) \Rightarrow \widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+ \text{ (see (34)),}\end{aligned}$$

hence $u_0 \ne 0$. Therefore

$$\widehat{\varphi}_+^{'}(u_0) = 0 \Rightarrow A(u_0) + (\beta + \widehat{\gamma})u_0 = N_{\widehat{f}_+}(u_0). \tag{36}$$

On (36) we act with $-u_0^- \in H^1(\Omega)$. Then

$$\sigma(u_0^-) + \widehat{\gamma}||u_0^-||_2^2 = 0 \Rightarrow \widehat{c}||u_0^-||^2 \le 0 \text{ (see (9)), hence } u_0 \ge 0,\ u_0 \ne 0.$$

Then relation (36) becomes

$$A(u_0) + \beta u_0 = N_f(u_0) \Rightarrow -\Delta u_0 + \beta u_0 = f(z,u_0) \text{ in } \Omega, \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega\,. \tag{37}$$

Hypotheses H_2 imply

$$|f(z,x)| \le c_1|x| \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_1 > 0\,. \tag{38}$$

We set

$$\zeta(s) = \begin{cases} \frac{f(z,u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $\zeta \in L^\infty(\Omega)$ (see (38)). From (37) we have

$$-\Delta u_0(z) = (\zeta - \beta)(z) u_0(z) \text{ a.e. in } \Omega, \ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega\,. \tag{39}$$

Note that $\zeta - \beta \in L^s(\Omega)$ with $s > N$ (see H_1). Lemma 5.1 of Wang [**26**] implies that $u_0 \in L^\infty(\Omega)$. Then from (39) we have $\Delta u_0 \in L^s(\Omega)$. Invoking Lemma 5.2 of Wang [**26**], we have $u_0 \in W^{2,s}(\Omega)$. Since $s > N$, from the Sobolev embedding theorem we have $W^{2,s}(\Omega) \subset C^{1+\alpha}(\bar{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$. Therefore $u_0 \in C_+ \setminus \{0\}$. From (39) we have

$$\begin{aligned} &\Delta u_0(z) \leq \ (||\zeta||_\infty + ||\beta^+||_\infty)\ u_0(z) \text{ a.e. in } \Omega \text{ (see } H_1) \\ &\Rightarrow\ u_0 \ \in\ \operatorname{int} C_+ \text{ (see Pucci \& Serrin [23, p. 120] and Vázquez [25]).} \end{aligned}$$

Note that $\widehat{\varphi}_+|_{C_+} = \varphi|_{C_+}$. So, $u_0 \in \operatorname{int} C_+$ is a local $C^1(\bar{\Omega})$–minimizer of φ, hence by virtue of Proposition 2.1 $u_o \in \operatorname{int} C_+$ is a local $H^1(\Omega)$–minimizer of φ. □

In fact we can show that problem (1) has a smallest nontrivial positive solution.

PROPOSITION 3.3. *Assume that hypotheses H_1 and H_2 hold. Then problem* (1) *has a smallest nontrivial positive solution $u_+ \in int C_+$ (that is, if u is a nontrivial positive solution of* (1)*, then $u_+ \leq u$).*

Proof: Let S_+ be the set of nontrivial positive solutions of (1). From Proposition 3.2 and its proof, we have $S_+ \neq \emptyset$ and $S_+ \subseteq \operatorname{int} C_+$.

We know that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1, u \leq u_2$, see Aizicovici, Papageorgiou & Staicu [**1**, p. 703]). So, without any loss of generality, we may assume that there exists $M_2 > 0$ such that $u(z) \leq M_2$ for all $z \in \bar{\Omega}$, all $u \in S_+$.

Let $C \subseteq S_+$ be a chain (a totally ordered subset of S_+). From Dunford & Schwartz [**8**, p. 336], we know that we can find $\{u_n\}_{n\geq 1} \subseteq C$ such that $\inf C = \inf_{n\geq 1} u_n$.

We have for all $n \geq 1$,

$$A(u_n) + \beta u_n \ = \ N_f(u_n) \Rightarrow \{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.} \tag{40}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega). \tag{41}$$

Moreover, acting on (40) with $u_n - u \in H^1(\Omega)$, passing to the limit as $n \to \infty$ and using (41) and the Kadec-Klee property of Hilbert spaces, we obtain

$$u_n \to u \text{ in } H^1(\Omega)\,. \tag{42}$$

Then passing to the limit as $n \to \infty$ in (40) and using (41), we have

$$A(u) + \beta u \ = \ N_f(u) \Rightarrow\ u \in C_+ \text{ is a solution of (1).}$$

If we show that $u \neq 0$, then $u \in S_+$. Suppose that $u = 0$ and let $y_n = \frac{u_n}{||u_n||}$, $n \geq 1$. Then $||y_n|| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \ \to\ y \text{ in } L^2(\Omega),\ y \geq 0. \tag{43}$$

From (40) we have

$$A(y_n) + \beta y_n = \frac{N_f(u_n)}{||u_n||} \text{ for all } n \geq 1. \tag{44}$$

Since $u_n \to 0$ in $H^1(\Omega)$ (see (42)), by virtue of hypothesis $H_2(iv)$, we have

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \mu_0 = my \text{ in } L^2(\Omega), \text{ where } m(\cdot) = f_x^{'}(\cdot, 0) \in L^\infty(\Omega). \tag{45}$$

So, from (43) and (44) it follows that

$$\begin{aligned} & A(y) + \beta y = my \\ \Rightarrow\ & -\Delta y(z) + \beta(z) y(z) = m(z) y(z) \text{ a.e. in } \Omega, \ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{46}$$

From Proposition 2.2 and hypothesis $H_2(iv)$, we have

$$\tilde{\lambda}_l(m) < \tilde{\lambda}_l(\tilde{\lambda}_l) = 1 \text{ and } 1 = \tilde{\lambda}_{l+1}(\tilde{\lambda}_{l+1}) < \tilde{\lambda}_{l+1}(m). \tag{47}$$

From (46) and (47) it follows that $y = 0$. On the other hand, acting on (44) with $y_n - y \in H^1(\Omega)$ and using (43) and (45), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (as before)} \Rightarrow ||y|| = 1,$$

a contradiction. Therefore $u \in S_+$ and $u \in \operatorname{int} C$.

Since C is an arbitrary chain, from the Kuratowski-Zara lemma it follows that S_+ has a minimal element $u_+ \in S_+ \subseteq \operatorname{int} C_+$. If $u \in S_+$, then since S_+ is downward directed, we can find $\tilde{u} \in S_+$ such that $\tilde{u} \leq u_t, \tilde{u} \leq u$. The minimality of u_+ implies that $\tilde{u} = u_+$ and so $u_+ \leq u$. Therefore u_+ is the smallest nontrivial positive solution of problem (1). □

Let S_- be the set of nontrivial negative solutions of problem (1).

In general hypotheses H_2 do not guarantee that $S_- \neq \emptyset$. If $S_- \neq \emptyset$, then $S_- \subseteq -\operatorname{int} C_+$ and it is upward directed (that is, $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \leq v_1,\ v_2 \leq v$, see [**1**]).

Reasoning as in the proof of Proposition 3.3, we have:

PROPOSITION 3.4. *Assume that hypotheses H_1 and H_2 hold and $S_- \neq \emptyset$. Then problem (1) has a biggest nontrivial negative solution $v_- \in -int C_+$ (that is, if v is a nontrivial negative solution of (1), then $v \leq v_-$).*

If we strengthen the conditions on $f(z,\cdot)$ we can guarantee that $S_- \neq \emptyset$. These stronger conditions on the reaction f are the following:

H_3: *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$, hypotheses $H_3(i) \to (iv)$ are the same as the corresponding hypotheses $H_2(i) \to (iv)$ and(v) there exists $\xi_x > 0$ such that $f(z, -\xi_x) \geq (-\xi_x)\beta(z)$ a.e. in Ω.*

REMARK 3.1. *If $\beta \equiv 0$, then condition $H_3(v)$ implies that for a.a. $z \in \Omega$, $f(z,\cdot)$
has a zero in $(-\infty, 0)$.*

PROPOSITION 3.5. *If hypotheses H_1 and H_3 hold, then $S_- \neq \emptyset$, $S_- \subseteq -int C_+$.*

Proof. We consider the following perturbation-truncation of the reaction $f(z,\cdot)$:

$$\tau(z,x)=\begin{cases} f(z,-\xi_x)+\widehat{\gamma}(-\xi_x) & \text{if } x<-\xi_x\\ f(z,x)+\widehat{\gamma}x & \text{if } -\xi_x\le x\le 0\\ 0 & \text{if } 0<x.\end{cases} \tag{48}$$

Clearly $\tau(\cdot,\cdot)$ is a Carathéodory function. We set $T(z,x)=\int_0^x\tau(z,s)ds$ and consider the C^1-functional $\chi_0: H^1(\Omega)\to\mathbb{R}$ defined by

$$\chi_0(u)=\frac{1}{2}\,\sigma(u)\;+\;\frac{\widehat{\gamma}}{2}\,||u||_2^2\;-\;\int_\Omega T(z,u)dz \qquad \text{for all } u\in H^1(\Omega).$$

From (48) and (9), we see that χ_0 is coercive. Also, it is sequentially weakly lower semi-continuous. Se, we can find $v_0\in H^1(\Omega)$ such that

$$\chi_0(v_0)\;=\inf\{\chi_0(v): v\in H^1(\Omega)\}. \tag{49}$$

As in the proof of Proposition 3.2, for $t\in(0,1)$ small such that at least we have $t\widehat{u}_1(z)\in[0,\xi_x]$ for all $z\in\bar{\Omega}$, we have $\chi_0(-t\widehat{u}_1)\;<\;0$. Therefore $\chi_0(v_0)<0=\chi_0(0)$ (see (49)), hence $v_0\neq 0$.

From (49) we have

$$\chi_0^{'}(v_0)=0\Rightarrow\; A(v_0)\;+\;(\beta+\widehat{\gamma})v_0\;\;=\;\;N_\tau(v_0)\,. \tag{50}$$

On (50) we act with $(-\xi_x-v_0)^+\in H^1(\Omega)$. Then

$$\begin{aligned}
&\left\langle A(v_0),\;(-\xi_x-v_0)^+\right\rangle\;+\;\int_\Omega(\beta(z)+\widehat{\gamma})\,v_0\,(-\xi_x-v_0)^+\;dz\\
&=\;\int_\Omega\;\tau(z,v_0)\,(-\xi_x-v_0)^+\;dz\\
&=\;\int_\Omega\;[f(z_1-\xi_x)\;+\;\widehat{\gamma}-\xi_x]\;(-\xi_z-v_0)^+\;dz\quad(\text{see }(48))\\
&\geq\;\left\langle A(-\xi_x),(-\xi_x-v_0)^+\right\rangle\;+\;\int_\Omega(\beta(z)+\widehat{\gamma})(-\xi_x)(-\xi_x-v_0)^+\;dz,\\
&\Rightarrow\;\left\langle A(-\xi_x-v_0),\;(-\xi_x-v_0)^+\right\rangle\;+\;\int_\Omega\;(\beta(z)+\widehat{\gamma})(-\xi_x-v_0)(-\xi_x-v_0)^+\;dz\;\leq\;0,\\
&\Rightarrow\;\sigma((-\xi_x-v_0)^+)\;+\;\widehat{\gamma}\,||(-\xi_x-v_0)^+||_2^2\;\leq\;0,\\
&\Rightarrow\;-\xi_x\;\leq\;v_0\quad(\text{see }(9)).
\end{aligned}$$

Also, acting on (50) with $v_0^+\in H^1(\Omega)$, we obtain $v_0\le 0, v_0\neq 0$. Therefore

$$v_0\;\in\;[-\xi_x,0]\;\;=\;\;\{v\in H^1(\Omega):\;-\xi_x\;\leq\;v(z)\;\leq\;0\text{ a.e. in }\Omega\}.$$

Therefore (50) yields $A(v_0)+\beta v_0=N_f(v_0)$, hence v_0 is a nontrivial negative solution of problem (1). As before, from Wang [**26**] and the strong maximum principle, we deduce that $v_0\in -\mathrm{int}\,C_+$. $\square$

4. Nodal Solutions

In this section we present the full multiplicity theorems for problem (1) by producing nodal solutions.

First we treat the case $S_-\neq\varnothing$.

THEOREM 4.1. *Assume that hypotheses H_1 and H_2 hold and $S_-\neq\varnothing$. Then problem (1) has at least four nontrivial solutions: $u_0\in intC_+$, $v_0\in -intC_+$ and $y_0,\,\widehat{y}\in int_{C^1(\bar{\Omega})}[v_0,u_0]$ both nodal.*

Proof. From Proposition 3.2 we already have a nontrivial positive solution $u_0 \in \operatorname{int} C_+$. In fact, by virtue of Proposition 3.3, we may assume that u_0 is the smallest nontrivial positive solution of (1) (that is, $u_0 = u_+ \in \operatorname{int} C_+$). Similarly, since by hypothesis $S_- \neq \emptyset$, from Proposition 3.4 we can have a nontrivial negative solution $v_0 \in -\operatorname{int} C_+$ which can be taken to be the biggest such solution of (1) (that is, $v_0 = v_- \in -\operatorname{int} C_+$). We introduce the following perturbation-truncation of $f(z, \cdot)$:

$$g(z,x) = \begin{cases} f(z, v_0(z)) + \widehat{\gamma} v_0(z) & \text{if } x < v_0(z) \\ f(z,x) + \widehat{\gamma} x & \text{if } v_0(z) \leq x \leq u_0(z) \\ f(z, u_0(z)) + \widehat{\gamma} u_0(z) & \text{if } u_0(z) < x. \end{cases} \tag{51}$$

This is a Carathéodory function. We set $G(z,x) = \int_0^x g(z,s)ds$ and introduce the C^1-functional $\Psi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) \;=\; \frac{1}{2}\sigma(u) \;+\; \frac{\widehat{\gamma}}{2}||u||_2^2 \;-\; \int_\Omega G(z,u)dz \qquad \text{for all } u \in H^1(\Omega).$$

Also, let $g_\pm(z,x) = g(z, \pm x^\pm)$ and $G_\pm(z,x) = \int_0^x g_\pm(z,s)ds$. We introduce the C^1–functional $\Psi_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\pm(u) \;=\; \frac{1}{2}\sigma(u) \;+\; \frac{\widehat{\gamma}}{2}||u||_2^2 \;-\; \int_\Omega G_\pm(z,u)dz \qquad \text{for all } u \in H^1(\Omega).$$

Claim 1: $K_\Psi \subseteq [v_0, u_0], K_{\Psi_+} = \{0, u_0\}, K_{\Psi_-} = \{0, v_0\}$.

Let $u \in K_\Psi$. Then

$$A(u) \;+\; (\beta + \widehat{\gamma})u \;=\; N_g(u). \tag{52}$$

On (52) we act with $(u - u_0)^+ \in H^1(\Omega)$. Then

$$\begin{aligned} &\left\langle A(u), (u-u_0)^+ \right\rangle \;+\; \int_\Omega (\beta(z) + \widehat{\gamma})\, u(u-u_0)^+ \, dz \\ &= \int_\Omega g(z,u)(u-u_0)^+ \, dz \\ &= \int_\Omega [f(z,u_0) + \widehat{\gamma} u_0](u-u_0)^+ \, dz \quad \text{(see (51))} \\ &= \left\langle A(u_0), (u-u_0)^+ \right\rangle \;+\; \int_\Omega (\beta(z) + \widehat{\gamma}) u_0 (u-u_0)^+ \, dz \\ &\Rightarrow \sigma((u-u_0)^+) + \widehat{\gamma}||(u-u_0)^+||_2^2 \;=\; 0 \\ &\Rightarrow \widehat{c}||(u-u_0)^+||^2 \;\leq\; 0 \quad \text{(see (9))} \\ &\Rightarrow u \;\leq\; u_0. \end{aligned}$$

In a similar manner acting on (52) with $(v_0 - u)^+ \in H^1(\Omega)$, we obtain $v_0 \leq u$. Hence

$$u \in [v_0, u_0] \;=\; \{y \in H^1(\Omega) : v_0(z) \;\leq\; y(z) \;\leq\; u_0(z) \;\text{ a.e. in } \Omega\} \Rightarrow \; K_\Psi \;\subseteq\; [v_0, u_0].$$

Similarly we show that

$$K_{\Psi_+} \;\subseteq\; [0, u_0] \;=\; \{y \in H^1(\Omega) :\; 0 \leq y(z) \leq u_0(z) \text{ a.e. in } \Omega\}$$

and

$$K_{\Psi_-} \subseteq [v_0, 0] \;=\; \{y \in H^1(\Omega) :\; v_0(z) \;\leq\; y(z) \;\leq\; 0 \text{ a.e. in } \Omega\}.$$

Recall that u_0 and v_0 are extremal constant sign solutions. So, it follows that $K_{\Psi_+} \;=\; \{0, u_0\}$ and $K_{\Psi_-} \;=\; \{0, v_0\}$. This proves Claim 1.

CLAIM 2: $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are local minimizers of Ψ.

From (9) and (51) it follows that the functional Ψ_+ is coercive. Also, it is sequentially weakly lower semi-continuous. So, by the Weierstrass theorem, we can find $\tilde{u} \in H^1(\Omega)$ such that

$$\Psi_+(\tilde{u}) \ = \inf\{\Psi_+(u) : \ u \in H^1(\Omega)\}. \tag{53}$$

As before (see the proof of Proposition 3.2), for $t \in (0,1)$ small (at least such that $t\widehat{u}_1 \in [0, u_0]$, recall $u_0 \in \operatorname{int} C_+$), we have $\Psi_+(t\widehat{u}_1) < 0$. Therefore $\Psi_+(\tilde{u}) < 0 = \Psi_+(0)$ (see (53)), hence $\tilde{u} \neq 0$.

From (53) we have $\tilde{u} \in K_{\Psi_+} = \{0, u_0\}$ (see Claim 1). So, $\tilde{u} = u_0$. Note that $\Psi_+|_{C_+} \ = \ \Psi|_{C_+}$ and $u_0 \in \operatorname{int} C_+$.

It follows that u_0 is a local $C^1(\bar{\Omega})$–minimizer of Ψ. From Proposition 2.1 we infer that u_0 is a local $H^1(\Omega)$–minimizer of Ψ. Similarly for $v_0 \in -\operatorname{int} C_+$ using this time the functional Ψ_-. This proves Claim 2.

Without any loss of generality, we may assume that $\Psi(v_0) \leq \Psi(u_0)$ (the analysis is similar if the opposite inequality holds). By virtue of Claim 2, we can find $\rho \in (0,1)$ such that

$$\Psi(v_0) \ \leq \ \Psi(u_0) < \inf\{\Psi(u) : \ ||u - u_0|| = \rho\} = \eta_\rho\,. \tag{54}$$

The functional Ψ is coercive (see (9) and (52)), hence it satisfies the C-condition. This fact and (54) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \ \in \ K_\Psi \ \subseteq \ [v_0, u_0] \text{ (see Claim 1) and } \eta_\rho \leq \Psi(y_0). \tag{55}$$

From (54) and (55) we have $y_0 \notin \{v_0, u_0\}$.

Hypothesis $H_2(iv)$ via the UCP implies that $u = 0$ is a nondegenerate critical point of $\varphi \in C^2(H^1(\Omega))$. Therefore

$$C_k(\varphi, 0) \ = \ \delta_{k,d_l}\mathbb{Z} \text{ for all } k \geq 0 \text{ with } d_l \ = \dim \bigoplus_{i=1}^{l} E(\widehat{\lambda}_i) \geq 2. \tag{56}$$

Note that

$$\begin{aligned}
&\Psi|_{[v_0,u_0]} \ = \ \varphi|_{[v_0,u_0]} \ \text{ (see (51)) and } v_0 \in -\operatorname{int} C_+, \ u_0 \in \operatorname{int} C_+, \\
&\Rightarrow \ C_k(\Psi|_{C^1(\bar{\Omega})}, 0) \ = \ C_K(\varphi|_{C^1(\bar{\Omega})}, 0) \text{ for all } k \geq 0, \\
&\Rightarrow \ C_k(\Psi, 0) \ = \ C_k(\varphi, 0) \text{ for all } k \geq 0 \ \text{ (see Palais [}\mathbf{21}\text{] and Bartsch [}\mathbf{2}\text{])}, \\
&\Rightarrow \ C_k(\Psi, 0) \ = \ \delta_{k,d_l}\mathbb{Z} \text{ for all } k \geq 0 \ \text{ (see (56))}. \qquad (57)
\end{aligned}$$

Recall that y_0 is a critical point of mountain pass type for Ψ. Hence

$$C_1(\Psi, y_0) \ \neq \ 0. \tag{58}$$

Since $d_l \geq 2$, comparing (57) and (58), we infer that $y_0 \neq 0$. Since $y_0 \in [v_0, u_0]$, the extremality of the solutions u_0, v_0 and (51) implies that y_0 is a solution of (1). Moreover, the regularity results of Wang [**26**] imply $y_0 \in C^1(\bar{\Omega})$.

We have

$$-\Delta y_0(z) \ + \ \beta(z) y_0(z) \ = \ f(z, y_0(z)) \ \text{ a.e. in } \Omega, \ \frac{\partial y_0}{\partial n} \ = \ 0 \text{ on } \partial\Omega.$$

Hypotheses $H(i)$ and the mean value theorem imply that if $\rho = \max\{||u_0||_\infty, ||v_0||_\infty\}$, then we can find $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z, x) + \xi_\rho x$ is nondecreasing on $[-\rho,\rho]$. We have

$$
\begin{aligned}
& -\Delta y_0(z) + (\beta(z) + \xi_\rho)y_0(z) = f(z,y_0(z)) + \xi_\rho y_0(z) \\
& \leq f(z,u_0(z)) + \xi_\rho u_0(z) \quad (\text{since } y_0 \leq u_0) \\
& = -\Delta u_0(z) + (\beta(z) + \xi_\rho)u_0(z) \quad \text{a.e. in } \Omega \\
& \Rightarrow \Delta(u_0 - y_0)(z) \leq (\|\beta^+\|_\infty + \xi_\rho)u_0(z) \text{ a.e. in } \Omega \quad (\text{see } H_1) \\
& \Rightarrow u_0 - y_0 \in \operatorname{int} C_+ \ (\text{see Vázquez } [\mathbf{25}]).
\end{aligned}
\tag{59}
$$

Similarly we show that

$$y_0 - v_0 \in \operatorname{int} C_+ \Rightarrow y_0 \in \operatorname{int}_{C^1(\bar\Omega)}[v_0,u_0].$$

Recall that $\Psi|_{[v_0,u_0]} = \varphi|_{[v_0,u_0]}$ (see (51)). Hence

$$
\begin{aligned}
& C_k(\Psi|_{C^1(\bar\Omega)}, y_0) = C_k(\varphi|_{C^1(\bar\Omega)}, y_0) \text{ for all } k \geq 0, \\
(60)\quad & \Rightarrow C_k(\Psi, y_0) = C_k(\varphi, y_0) \text{ for all } k \geq 0 \quad (\text{see } [\mathbf{21}], [\mathbf{2}]), \\
(61)\quad & \Rightarrow C_1(\varphi, y_0) \neq 0 \quad (\text{see } (58)).
\end{aligned}
$$

Hypotheses H_2 imply that $\varphi \in C^2(H^1(\Omega))$ and for all $u,v \in H^1(\Omega)$

$$\langle \varphi''(y_0)u,v\rangle = \int_\Omega (Du,Dv)_{\mathbb{R}^N}dz + \int_\Omega \beta(z)uv\,dz - \int_\Omega \widehat m_0\,uv\,dz,$$

where $\widehat m_0(z) = f_x'(z,y_0(z))$, $\widehat m_0 \in L^\infty(\Omega)_+$ (see $H_2(i)$). Therefore $\varphi''(y_0)$ is a Fredholm operator. Let $\sigma(\varphi''(y_0))$ denote the spectrum of $\varphi''(y_0)$ and suppose that $\sigma(\varphi''(y_0)) \subseteq [0,\infty)$. Then we have

$$\int_\Omega \widehat m_0 u^2\,dz \leq \sigma(u) \qquad \text{for all } u \in H^1(\Omega). \tag{62}$$

Let $u \in \operatorname{Ker}\varphi''(y_0)$. Then

$$-\Delta\, u(z) = (\widehat m_0 - \beta)(z)u(z) \ \text{ a.e. in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{63}$$

We have $\widehat m_0 - \beta \in L^s(\Omega)$ with $s > N$ (see H_1). If $(\widehat m - \beta)^+ = 0$, then from (63) it follows that $u = 0$. If $(\widehat m_0 - \beta)^+ \neq 0$, then from (62) and de Figueiredo [**9**], we have $\operatorname{Ker}\varphi''(y_0) \leq 1$. So, we can apply Proposition 2.5 of Bartsch [**2**] and obtain

$$
\begin{aligned}
(64)\quad & C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \\
& \Rightarrow C_k(\Psi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \ (\text{see } (60)).
\end{aligned}
$$

From Claim 2 we know that v_0 and u_0 are local minimizers of Ψ. Hence

$$C_k(\Psi,u_0) = C_k(\Psi,v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{65}$$

Finally recall that Ψ is coercive. Therefore

$$C_k(\Psi,\infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{66}$$

Suppose that $K_\Psi = \{0,u_0,v_0,y_0\}$. Then from (57), (64), (65), (66) and the Morse relation (see (2)) with $t=-1$, we have $(-1)^{d_l} + 2(-1)^0 + (-1)^1 = (-1)^0$, a contradiction.

This means that there exists $\widehat y \in K_\Psi, \widehat y \notin \{0,u_0,v_0,y_0\}$. From Claim 1 and Wang [**26**], we have $\widehat y \in [v_0,u_0] \cap C^1(\bar\Omega)$, hence $\widehat y$ is the second nodal solution of problem (1). As we did for y_0, using the strong maximum principle of Vázquez [**25**], we have $\widehat y \in \operatorname{int}_{C^1(\Omega)}[v_0,y_0]$.

Next we deal with the case when $S_- = \emptyset$. In this case there is no extremal negative solution. So, we introduce the following "unilateral" perturbation–truncation of the reaction $f(z,\cdot)$:

$$g_*(z,x) = \begin{cases} f(z,x) + \widehat{\gamma} x & \text{if } x \le u_+(z) \\ f(z,u_+(z)) + \widehat{\gamma} u_+(z) & \text{if } u_+(z) < x. \end{cases} \tag{67}$$

This is a Carathéodory function. We set $G_*(z,x) = \int_0^x g_*(z,s)ds$ and consider the C^1-functional $\varphi_* : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_*(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{\gamma}}{2}||u||_2^2 - \int_\Omega G_*(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

PROPOSITION 4.1. *Assume that hypotheses H_1 and H_2 hold. Then the functional φ_* satisfies the C–condition.*

Proof. Let $\{u_n\}_{n\ge1} \subseteq H^1(\Omega)$ be a sequence such that $\{\varphi_*(u_n)\}_{n\ge1} \subseteq \mathbb{R}$ is bounded and

$$(1+||u_n||)\varphi_*^{'}(u_n) \to 0 \quad \text{in } H^1(\Omega)^*. \tag{68}$$

From (67) we have for all $n \ge 1$,

$$|\langle \varphi_*'(u_n), h\rangle| \le \frac{\epsilon_n||h||}{1+||u_n||} \quad \text{for all } h \in H^1(\Omega) \text{ with } \epsilon_n \to 0^+$$
$$\Rightarrow |\langle A(u_n), h\rangle + \int_\Omega (\beta(z)+\widehat{\gamma})u_n\, hdz - \int_\Omega g_*(z,u_n)hdz| \le \frac{\epsilon_n||h||}{1+||u_n||}. \tag{69}$$

Let $h = u_n^+ \in H^1(\Omega)$. We have

$$\begin{aligned} &\sigma(u_n^+) + \widehat{\gamma}||u_n^+||_2^2 \le M_3 \text{ for some } M_3 > 0 \text{ (see (67))} \\ &\Rightarrow \widehat{c}||u_n^+||^2 \le M_3 \text{ for all } n \ge 1 \text{ (see (84))} \\ &\Rightarrow \{u_n^+\}_{n\ge1} \subseteq H^1(\Omega) \text{ is bounded.} \end{aligned} \tag{70}$$

We assume that $||u_n^-|| \to \infty$ and set $y_n = \frac{u_n^-}{||u_n^-||}$. Then $||y_n|| = 1$ for all $n \ge 1$. So we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^{2s'}(\Omega)\ (\frac{1}{s}+\frac{1}{s'}=1) \tag{71}$$

From (69) and (70), we have

$$|\langle A(-y_n), h\rangle - \int_\Omega (\beta(z)+\widehat{\gamma})y_n hdz - \int_\Omega \frac{g_*(z,-u_n^-)}{||u_n^-||}hdz| \le M_4\frac{||h||}{||u_n^-||}. \tag{72}$$

Hypotheses H_2 (i),(ii),(iii) imply that

$$\left\{\frac{N_{g_*}(-u_n^-)}{||u_n^-||}\right\}_{n\ge1} \subset L^2(\Omega) \text{ is bounded.}$$

So, if in (72) we choose $h = y_n - y \in H^1(\Omega)$ and pass to the limit as $n \to \infty$, then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(-y_n), y_n - y\rangle = 0 \Rightarrow ||Dy_n||_2 \to ||Dy||_2 \\ &\Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (by the Kadec - Klee property)} \\ &\Rightarrow ||y|| = 1,\ y \ge 0. \end{aligned} \tag{73}$$

Then using hypothesis $H_2(iii)$ and by passing to a suitable suitable subsequence if necessary we have

$$\frac{Ng_*(-u_n^-)}{||u_n^-||} \xrightarrow{w} -(\eta_0 + \widehat{\gamma})y \text{ in } L^2(\Omega) \text{ with } \eta(z) \leq \eta_0(z) \leq \widehat{\eta}(z) \text{ a.e. in } \Omega\,. \tag{74}$$

So, if in (72) we pass to the limit as $n \longrightarrow \infty$, and use (73), then

$$\begin{aligned} &\langle A(y), h\rangle + \int_\Omega \beta y\, hdz = \int_r \eta_0 y\, hdz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow A(y) + \beta y = \eta_0 y, \\ &\Rightarrow -\Delta y(z) + \beta(z)y(z) = \eta_0(z)y(z) \text{ a.e. in } \Omega,\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{75}$$

From Proposition 2.2 we have $\tilde{\lambda}_1(\eta_0) \leq \tilde{\lambda}_1(\eta) < \tilde{\lambda}_1(\widehat{\lambda}_1) = 1$. So, from (75) it follows that y is nodal, witch contradicts (73). It follows that

$$\{u_n^-\}_{n\geq 1} \subset H^1(\Omega) \text{ is bounded} \Rightarrow \{u_n\}_{n\geq 1} \subset H^1(\Omega) \text{ is bounded (see (70)).}$$

Therefore, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^{2s'}(\Omega). \tag{76}$$

In (69) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (76). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \Rightarrow u_n \to u \text{ in } H^1(\Omega) \Rightarrow \varphi_* \text{ satisfies the C–condition.}$$

□

The next result permits the computation of the critical groups of φ_* at infinity.

PROPOSITION 4.2. *If hypotheses H_1 and H_2 hold, then $C_k(\varphi_*, \infty) = 0$ for all $k \geq 0$.*

Proof. Let $\lambda > \max\{\widehat{\lambda}_1, \frac{1}{|\Omega|_N}\int_\Omega \beta dz\}$ and consider the C^1–functional $\Psi_* : H^1(\Omega) \to R$ defined by

$$\Psi_*(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{\gamma}}{2}||u||_2^2 - \frac{1}{2}(\lambda + \widehat{\gamma})||u^-||_2^2 \quad \text{for all } u \in H^1(\Omega)\,.$$

We consider the homotopy

$$h(t,u) = h_t(u) = (1-t)\varphi_*(u) + t\Psi_*(u) \quad \text{for all } (t,u) \in [0,1] \times H^1(\Omega).$$

CLAIM 1. There exist $\epsilon \in \mathbb{R}$ and $\delta > 0$ such that

$$h_t(u) \leq \epsilon \Rightarrow (1 + ||u||)||(h_t)'(u)||_* \geq \delta||u||^2 \quad \text{for all } t \in [0,1].$$

We argue by contradiction. So, suppose that the Claim is not true. Since h maps bounded sets to bounded sets, we can find $\{t_n\}_{n\geq 1} \subset [0,1]$ and $\{u_n\}_{n\geq 1} \subset H^1(\Omega)$ such that

$$t_n \to t, ||u_n|| \to \infty, h_{t_n}(u_n) \to -\infty \text{ and } ||(h_{t_n})'(u_n)|| < \frac{1}{n}||u_n||\,. \tag{77}$$

From (77) we have

(78)

$$\begin{aligned} &|\langle A(u_n), h\rangle + \int_\Omega (\beta(z) + \widehat{\gamma})u_n hdz - (1-t_n)\int_\Omega g_*(z,u_n)hdz - (\lambda + \widehat{\gamma})t_n \int_\Omega u_n^- hdz| \\ &< \tfrac{1}{n}\,||u_n|| \cdot ||h|| \quad \text{for all } n \geq 1. \end{aligned}$$

In (78) first we choose $h = u_n^+ \in H^1(\Omega)$ then

$$\sigma(u_n^+) + \widehat{\gamma}||u_n^+||_2^2 \leq \frac{1}{n}\,||u_n|| \cdot ||u_n^+|| \quad \text{for all } n \geq 1. \tag{79}$$

Let $y_n = \frac{u_n}{||u_n||}$. Then from (79) we have

$$\begin{aligned} &\sigma(y_n^+) + \widehat{\gamma}||y_n^+||_2^2 \le \tfrac{1}{n}\,||y_n||\cdot||y_n^+|| \\ &\Rightarrow \widehat{c}||y_n^+||^2 \le \tfrac{1}{n}\,||y_n^+|| \text{ (see (9) and recall that } ||y_n||=1) \\ &\Rightarrow y_n^+ \to 0 \text{ in } H^1(\Omega). \end{aligned} \tag{80}$$

From (78) and (80), we have

(81)
$$|\langle A(-y_n^-),h\rangle + \int_\Omega(\beta(z)+\widehat{\gamma})(-y_n^-)dz - (1-t_n)\int_\Omega \frac{q_*(z,-u_n^-)}{||u_n||}hdz + (\lambda+\widehat{\gamma})t_n\int_\Omega y_n^- hdz|$$
$$\le \epsilon_n||h|| \text{ with } \epsilon_n \downarrow 0.$$

Since $||y_n|| = 1$ for all $n \ge 1$, by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} &y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^{2s'}(\Omega), \\ &\Rightarrow y_n^- \xrightarrow{w} y^- \text{ in } H^1(\Omega) \text{ and } y_n^- \to y^- \text{ in } L^{2s'}(\Omega). \end{aligned} \tag{82}$$

In (81) we choose $h = y_n^- - y^- \in H^1(\Omega)$ and pass to the limit as $n \to \infty$. Recalling that $\left\{\frac{Ng_*(-u_n^-)}{||u_n^-||}\right\}_{n\ge 1} \subseteq L^2(\Omega)$ is bounded and using (82), we obtain

$$\lim_{n\to\infty}\left\langle A(-y_n^-), y_n^- - y^-\right\rangle = 0 \Rightarrow y_n^- \to y^- \text{ in } H^1(\Omega). \tag{83}$$

Note that $y^- \neq 0$ or otherwise from (80) and (83), we have $y_n \to 0$ in $H^1(\Omega)$ which contradicts the fact that $||y_n|| = 1$ for all $n \ge 1$. Now, using hypothesis $H_2(iii)$, we have

$$\frac{Ng_*(-u_n^-)}{||u_n||} \xrightarrow{w} \xi = -(\eta_0+\widehat{\gamma})y^- \text{ in } L^2(\Omega), \text{ with } \eta \le \eta_0 \le \widehat{\eta}. \tag{84}$$

So, if in (81) we pass to the limit as $n \to \infty$ and use (83) and (84), then

$$\begin{aligned} &\langle A(y^-),h\rangle + \int_\Omega \beta y^- hdz = \int_\Omega \eta_0 y^- hdz \text{ for all } h \in H^1(\Omega) \\ &\Rightarrow A(y^-) + \beta y^- = \eta_0 y^- \\ &\Rightarrow -\Delta y^-(z) + \beta(z)y^-(z) = \eta_0(z)y^-(z) \text{ a.e. in } \Omega,\ \tfrac{\partial y^-}{\partial n} = 0 \text{ on } \partial\Omega\,. \end{aligned} \tag{85}$$

By virtue of Proposition 2.2, we have $\tilde{\lambda}_1(\eta_0) \le \tilde{\lambda}_1(\eta) < \tilde{\lambda}_1(\widehat{\lambda}_1) = 1$.

Therefore (85) implies that y^- must be nodal (recall $y^- \neq 0$), contradiction. This proves Claim 1.

Note that

$$|\partial_t h_t(u)| = |\Psi_*(u) - \varphi_*(u)| \le C_1||u||^2 \text{ for some } c_1 > 0 \text{ and } u \in H^1(\Omega).$$

Finally note that $h_0(\cdot) = \varphi_*$ satisfies the C-condition (see Proposition 4.1) while $h_1(\cdot) = \Psi_*$ also satisfies the C-condition since $\lambda > \widehat{\lambda}_1$. So, we have

$$C_k(\varphi_*,\infty) = C_k(\Psi_*,\infty) \quad \text{for all } k \ge 0 \text{ (see Liang \& Su [\textbf{17}])}. \tag{86}$$

It is easy to check that $K_{\Psi_*} \subset -C_+$ and since $\lambda > \widehat{\lambda_1}$, it follows that $K_{\Psi_*} = \{0\}$ (recall that $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign). Hence

$$C_k(\Psi_*,\infty) = C_k(\Psi_*,0) \text{ for all } k \ge 0. \tag{87}$$

Let $\zeta \in L^\infty(\Omega), \zeta \le 0,\ \zeta \neq\ 0$ and consider the homotopy

$$\tilde{h}_t(u) = \Psi_*(u) - t\zeta u \text{ for all } t \in [0,1], \text{ and } u \in H^1(\Omega).$$

Claim 2: $(\tilde{h}_t)'(u) = 0$ for all $t \in [0,1]$ and for all $u \in H^1(\Omega)\setminus\{0\}$.

Again we argue indirectly. So suppose that we can find $t \in [0,1]$ and $u \in H^1(\Omega), u \neq 0$ such that

$$(\tilde{h}_t)'(u) = 0 \Rightarrow A(u) + (\beta + \widehat{\gamma})u = -(\lambda + \widehat{\gamma})u^- + t\zeta. \tag{88}$$

On (88) we act with $u^+ \in H^1(\Omega)$. Then $\sigma(u^+) + \widehat{\gamma}||u^+||_2^2 = t\int_\Omega \zeta u^+ dz \leq 0$. Thus, by (9), $\widehat{c}||u^+||^2 \leq 0$, hence $u \leq 0$, $u \neq 0$. Then relation (88) becomes

$$\begin{aligned} & A(u) + \beta u = \lambda u - t\zeta \\ & \Rightarrow -\Delta u(z) + \beta(z)u(z) = \lambda u(z) - t\zeta \text{ a.e. in } \Omega,\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ & \Rightarrow u \in -\operatorname{int} C_+ \text{ (as before using [26], [25]).} \end{aligned} \tag{89}$$

Let $v \in \operatorname{int} C_+$ and consider the function

$$R(v, -u)(z) = ||Dv(z)||^2 - (-Du(z), D(\frac{v^2}{-u})(z))_{\mathbb{R}^N}.$$

From Picone's identity (see, for example, Gasinski & Papageorgiou [**12**, p. 785]), we have

$$R(v, -u)(z) \geq 0 \quad \text{for a.a. } z \in \Omega.$$

Therefore

$$\begin{aligned} 0 \quad & \leq \quad \int_\Omega R(v,-u)dz = ||Dv||_2^2 - \int_\Omega (-\Delta u)\frac{v^2}{u}dz \text{ (by Green's identity)} \\ & = \quad ||Dv||_2^2 - \int_\Omega (\lambda - \beta(z))v^2 dz + t\int_\Omega \zeta v^2 dz \\ & \leq \quad ||Dv||_2^2 - \int_\Omega (\lambda - \beta(z))v^2 dz \text{ (since } \zeta \leq 0). \end{aligned}$$

Choose $v \equiv 1 \in \operatorname{int} C_+$. Then

$$0 \leq -\lambda|\Omega|_N + \int_\Omega \beta(z)dz < 0 \text{ (recall the choice of } \lambda),$$

a contradiction. This proves Claim 2.

Then Claim 2 and the homotopy invariance of critical groups imply that for all $k \geq 0$,

$$C_k(\Psi_*, 0) = 0 \Rightarrow C_k(\varphi_*, \infty) = 0 \quad \text{(see (86), (87))}.$$

$\square$

Now we are ready for the second multiplicity theorem for problem (1), taking care of the case $S_- = \Omega$.

THEOREM 4.2. *Assume that hypotheses H_1 and H_2 hold and $S_- = \Omega$. Then problem (1) has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$ and $y_0, \widehat{y} \in C^1(\bar{\Omega})$ nodal solutions such that $u_0 - y_0$, $u_0 - \widehat{y} \in \operatorname{int} C_+$.*

Proof. From Proposition 3.2 we already have a nontrivial positive solution $u_0 \in \operatorname{int} C_+$. By virtue of Proposition 3.3 we can always assume that u_0 is the smallest nontrivial positive solution of (1) (that is, $u_0 = u_+ \in \operatorname{int} C_+$). We still consider the C^1-functional $\varphi_* : H^1(\Omega) \to \mathbb{R}$ introduced in the beginning of this section. Let

$$g_*^+(z,x) = g_*(z,x^+) \text{ (see (67)) and } G_*^+(z,x) = \int_0^x g_*^+(z,s)ds.$$

We consider the C^1-functional $\varphi_*^+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_*^+(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{\gamma}}{2}||u||_2^2 - \int_\Omega G_*^+(z,u)dz \quad \text{for all } u \in H^1(\Omega)\,.$$

CLAIM 1. We have

$$K_{\varphi_*} \subset (u_0] = \{u \in H^1(\Omega) : u(z) \leq u_0(z) \text{ a.e. in } \Omega\} \text{ and } K_{\varphi_*^+} = \{0, u_0\}.$$

Let $u \in K_{\varphi_*}$. Then

$$A(u) + (\beta + \widehat{\gamma})u = N_{g_*}(u). \tag{90}$$

On (90) we act with $(u-u_0)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\langle A(u), (u-u_0)^+\rangle + \textstyle\int_\Omega (\beta(z)+\widehat{\gamma})u(u-u_0)^+dz = \int_\Omega g_*(z,u)(u-u_0)^+dz \\
&= \textstyle\int_\Omega [f(z,u_0) + \widehat{\gamma}u_0](u-u_0)^+dz \text{ (see (67) and recall } u_0 = u_+) \\
&= \langle A(u_0), (u-u_0)^+\rangle + \textstyle\int_\Omega (\beta(z)+\widehat{\gamma})u_0(u-u_0)^+dz \\
&\Rightarrow \langle A((u-u_0)^+), (u-u_0)^+\rangle + \textstyle\int_\Omega (\beta(z)+\widehat{\gamma})[(u-u_0)^+]^2dz = 0 \\
&\Rightarrow \widehat{c}\,||(u-u_0)^+||^2 \leq 0 \text{ (see (9))},
\end{aligned}$$

hence $u \leq u_0$. This proves that $K_{\varphi_*} \subseteq (u_0] = \{u \in H^1(\Omega) : u(z) \leq u_0(z) \text{ a.e. in } \Omega\}$. In a similar fashion, we show that $K_{\varphi_*^+} \subseteq [0, u_0] = \{u \in H^1(\Omega) : 0 \leq u(z) \leq u_0(z) \text{ a.e. in } \Omega\}$.

The extremality of $u_0 = u_+ \in \operatorname{int} C_+$ (see Proposition 3.3) implies $K_{\varphi_*^+} = \{0, u_0\}$. This proves Claim 1.

CLAIM 2: $u_0 \in \operatorname{int} C_+$ is a local minimizer of the functional φ_*.

Evidently φ_*^+ is coercive (see (67)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in H^1(\Omega)$ such that

$$\varphi_*^+(\widehat{u}_0) = \inf\{\varphi_*^+(u) : u \in H^1(\Omega)\}. \tag{91}$$

For $t \in (0,1)$ small and using hypothesis $H_2(iv)$, we have

$$\varphi_*^+(t\widehat{u}_1) < 0 \text{ (see the proof of Proposition 3.2)} \Rightarrow \varphi_*^+(\widehat{u}) < 0 = \varphi_k^+(0) \text{ (see (91))},$$

hence $\widehat{u}_0 \neq 0$. Then from (91) and Claim 1, we have $\widehat{u}_0 = u_0 \in \operatorname{int} C_+$. Since $\varphi_*|_{C_+} = \varphi_*^t|_{C_+}$ (see (67)), it follows that $u_0 \in \operatorname{int} C_+$ is a local $C^1(\bar{\Omega})$-minimizer of φ_*. Invoking Proposition 2.1, we have that $u_0 \in \operatorname{int} C_+$ in a local $H^1(\Omega)$-minimizer of φ_*. This proves Claim 2.

Claim 2 implies that we can find $\rho \in (0,1)$ small such that

$$\varphi_*(u_0) < \inf\{\varphi_*(u) : ||u-u_0|| = \rho\} = \eta_\rho^*,\ ||u_0|| > \rho\,. \tag{92}$$

Hypothesis $H_2(iii)$ implies that

$$\varphi_*(t\widehat{u}_1) \to -\infty \text{ as } t \to -\infty\,. \tag{93}$$

Finally recall that φ_* satisfies the C-condition (see Proposition 4.1). This fact together with (92) and (93), implies that we can apply Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_{\varphi_*} \subseteq (u_0] \text{ and } \eta_\rho^* \leq \varphi_*(y_0)\,. \tag{94}$$

From (92) and (94), we see that $y_0 \neq u_0$. Since y_0 is a critical point of φ_* of mountain pass type, we have

$$C_1(\varphi_*, y_0) \neq 0. \tag{95}$$

On the other hand, as in the proof of Theorem 4.1 (see (57)), using hypothesis $H_2(iv)$, we have

$$C_k(\varphi_*, 0) = \delta_{k_1 d_l}\mathbb{Z} \quad \text{for all } k \geq 0 \text{ and } d_l \geq 2. \tag{96}$$

Comparing (95) and (96), we conclude that $y_0 \neq 0$. Since $y_0 \in (u_0]$ (see Claim 1) and due to the extremality of u_0, we infer that y_0 is nodal and $y_0 \in C^1(\bar{\Omega})$ (see Wang [**26**]). Moreover, we have

$$u_0 - y_0 \in \operatorname{int} C_+ \ (\text{that is, } y_0 \in \operatorname{int}_{C^1(\bar{\Omega})}(u_0])$$
$$\text{and} \qquad C_k(\varphi_*, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{97}$$

From Claim 2 we have

$$C_k(\varphi_*, u_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0 \tag{98}$$

while from Proposition 4.2 we have

$$C_k(\varphi_*, \infty) = 0 \text{ for all } k \geq 0. \tag{99}$$

Suppose $K_{\varphi_*} = \{0, u_0, y_0\}$. From (96), (97), (98), (99) and the Morse relation (see (2)), we have $(-1)^{d_l} + (-1)^0 + (-1)^1 = 0$, a contradiction. So we can find $\widehat{y} \in K_{\varphi_*}, \widehat{y} \notin \{0, u_0, y_0\}$. We have $\widehat{y} \in (u_0]$ (see Claim 1), hence $\widehat{y} \in C^1(\bar{\Omega})$ (see Wang [**26**]) is nodal. Moreover, as for y_0, we have $u_0 - \widehat{y} \in \operatorname{int} C_+$ (that is, $\widehat{y} \in \operatorname{int}_{C^1(\bar{\Omega})}(u_0])$. ☐

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