Appl Math Optim (2018) 78:1–23
https://doi.org/10.1007/s00245-016-9392-y

Nonlinear Elliptic Inclusions with Unilateral Constraint and Dependence on the Gradient

Nikolaos S. Papageorgiou[1] · **Vicenţiu D. Rădulescu**[2,3] · **Dušan D. Repovš**[4]

Published online: 8 November 2016

Abstract We consider a nonlinear Neumann elliptic inclusion with a source (reaction term) consisting of a convex subdifferential plus a multivalued term depending on the gradient. The convex subdifferential incorporates in our framework problems with unilateral constraints (variational inequalities). Using topological methods and the Moreau-Yosida approximations of the subdifferential term, we establish the existence of a smooth solution.

Keywords Convex subdifferential · Moreau-Yosida approximation · Elliptic differential inclusion · Morse iteration technique · Pseudomonotone map · Variational inequality

Mathematics Subject Classification 35J60 · 35K85

✉ Vicenţiu D. Rădulescu
vicentiu.radulescu@imar.ro

Nikolaos S. Papageorgiou
npapg@math.ntua.gr

Dušan D. Repovš
dusan.repovs@guest.arnes.si

1 Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece

2 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

3 Department of Mathematics, University of Craiova, Street A.I. Cuza 13, 200585 Craiova, Romania

4 Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann elliptic differential inclusion

$$\left\{ \begin{array}{ll} \operatorname{div}(a(u(z))Du(z)) \in \partial\varphi(u(z)) + F(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{1}$$

In this problem, $\varphi \in \Gamma_0(\mathbb{R})$ (that is, $\varphi : \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous, see Sect. 2) and $\partial\varphi(x)$ is the subdifferential of $\varphi(\cdot)$ in the sense of convex analysis. Also $F(z, x, \xi)$ is a multivalued term with closed convex values depending on the gradient of u. So, problem (1) incorporates variational inequalities with a multivalued reaction term.

By a solution of problem (1), we understand a function $u \in H^1(\Omega)$ such that we can find $g, f \in L^2(\Omega)$ for which we have

$$g(z) \in \partial\varphi(u(z)) \text{ and } f(z) \in F(z, u(z), Du(z)) \text{ for almost all } z \in \Omega,$$
$$\int_\Omega a(u(z))(Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega (g(z) + f(z))h(z)dz = 0 \text{ for all } h \in H^1(\Omega).$$

The presence of the gradient in the multifunction F, precludes the use of variational methods in the analysis of (1). To deal with such problems, a variety of methods have been proposed. Indicatively, we mention the works of Amann and Crandall [1], de Figueiredo, Girardi and Matzeu [5], Girardi and Matzeu [8], Loc and Schmitt [13], Pohozaev [20]. All these papers consider problems with no unilateral constraint (that is, $\varphi = 0$) and the reaction term F is single-valued. Variational inequalities (that is, problems where φ is the indicator function of a closed, convex set), were investigated by Arcoya, Carmona and Martinez Aparicio [2], Matzeu and Servadei [15], Mokrane and Murat [17]. All have single valued source term.

Our method of proof is topological and it is based on a slight variant of Theorem 8 of Bader [3] (a multivalued alternative theorem). Also, our method uses approximations of φ and the theory of nonlinear operators of monotone type. In the next section, we recall the basic notions and mathematical tools which we will use in the sequel.

2 Mathematical Background

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). By $\Gamma_0(X)$ we denote the cone of all convex functions $\varphi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are proper (that is, not identically $+\infty$) and lower semicontinuous. By $\operatorname{dom}\varphi$ we denote the effective domain of φ, that is,

$$\operatorname{dom}\varphi := \{u \in X : \varphi(u) < +\infty\}.$$

Given $\varphi \in \Gamma_0(X)$, the subdifferential of φ at $u \in X$ is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, h\rangle \leqslant \varphi(u+h) - \varphi(u) \text{ for all } h \in X\}.$$

Evidently $\partial\varphi(u) \subseteq X^*$ is w^*-closed, convex and possibly empty. If φ is continuous at $u \in X$, then $\partial\varphi(u) \subseteq X^*$ is nonempty, w^*-compact and convex. Moreover, if φ is Gâteaux differentiable at $u \in X$, then $\partial\varphi(u) = \{\varphi'_G(u)\}$ ($\varphi'_G(u)$ being the Gâteaux derivative of φ at u). We know that the map $\partial\varphi : X \to 2^{X^*}$ is maximal monotone. If $X = H =$ a Hilbert space and $\varphi \in \Gamma_0(H)$, then for every $\lambda > 0$, the "Moreau-Yosida approximation" φ_λ of φ, is defined by

$$\varphi_\lambda(u) = \inf\left[\varphi(h) + \frac{1}{2\lambda}||h-u||^2 : h \in H\right] \text{ for all } u \in H.$$

We have the following properties:

- φ_λ is convex, $\operatorname{dom}\varphi_\lambda = H$;
- φ_λ is Fréchet differentiable and the Fréchet derivative φ'_λ is Lipschitz continuous with Lipschitz constant $1/\lambda$;
- if $\lambda_n \to 0,\ u_n \to u$ in H, $\varphi'_{\lambda_n}(u_n) \xrightarrow{w^*} u^*$ in H, then $u^* \in \partial\varphi(u)$.

We refer for details to Gasinski and Papageorgiou [6] and Papageorgiou and Kyritsi [19].

We know that if $\varphi \in \Gamma_0(X)$, then φ is locally Lipschitz in the interior of its effective domain (that is, on int dom φ). So, locally Lipschitz functions are the natural candidate to extend the subdifferential theory of convex functions.

We say that $\varphi : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ we can find U a neighborhood of u and a constant $k > 0$ such that

$$|\varphi(v) - \varphi(y)| \leqslant k||v-y|| \text{ for all } v, y \in U.$$

For such functions we can define the generalized directional derivative $\varphi^0(u; h)$ by

$$\varphi^\circ(u; h) = \limsup_{\substack{u' \to u \\ \lambda \downarrow 0}} \frac{\varphi(u' + \lambda h) - \varphi(u')}{\lambda}.$$

Then $\varphi^\circ(u; \cdot)$ is sublinear continuous and so we can define the nonempty w^*-compact set $\partial_c\varphi(u)$ by

$$\partial_c\varphi(u) = \{u^* \in X^* : \langle u^*, h\rangle \leqslant \varphi^\circ(u; h) \text{ for all } h \in X\}.$$

We say that $\partial_c\varphi(u)$ is the "Clarke subdifferential" of φ at $u \in X$. In contrast to the convex subdifferential, the Clarke subdifferential is always nonempty. Moreover, if φ is convex, continuous (hence locally Lipschitz on X), then the two subdifferentials coincide, that is, $\partial\varphi(u) = \partial_c\varphi(u)$ for all $u \in X$. For further details we refer to Clarke [4].

Suppose that X is a reflexive Banach space and $A : X \to X^*$ a map. We say that A is "pseudomonotone", if the following two conditions hold:

- A is continuous from every finite dimensional subspace V of X into X^* furnished with the weak topology;
- if $u_n \xrightarrow{w} u$ in X, $A(u_n) \xrightarrow{w} u^*$ in X^* and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0$, then for every $y \in X$, we have

$$\langle A(u), u - y\rangle \leqslant \liminf_{n\to\infty} \langle A(u_n), u_n - y\rangle .$$

If $A : X \to X^*$ is maximal monotone, then A is pseudomonotone.

A pseudomonotone map $A : X \to X^*$ which is strongly coercive, that is,

$$\frac{\langle A(u), u\rangle}{||u||} \to +\infty \text{ as } ||u|| \to \infty,$$

it is surjective (see Gasinski and Papageorgiou [6, p. 336]).

Let V be a set and let $G : V \to 2^{X^*}\backslash\{\emptyset\}$ be a multifunction. The graph of G is the set

$$\operatorname{Gr} G = \{(v, u) \in V \times X : u \in G(v)\}.$$

(a) If V is a Hausdorff topological space and $\operatorname{Gr} G \subseteq V \times X$ is closed, then we say that G is "closed".

(b) If there is a σ-field Σ defined on V and $\operatorname{Gr} G \subseteq \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X, then we say that G is "graph measurable".

As we already mentioned in the Introduction, our approach uses a slight variant of Theorem 8 of Bader [3] in which the Banach space V is replaced by its dual V^* equipped with the w^*-topology. A careful reading of the proof of Bader [3], reveals that the result remains true if we make this change.

So, as above X is a Banach space, V^* is a dual Banach space, $G : X \to 2^{V^*}$ is a multifunction with nonempty, w^*-compact, convex values. We assume that $G(\cdot)$ is "upper semicontinuous" (usc for short), from X with the norm topology into V^* with the w^*-topology (denoted by $V^*_{w^*}$), that is, for all $U \subseteq V^*$ w^*-open, we have

$$G^-(U) = \{x \in X : G(x) \cap U \neq \emptyset\} \text{ is open.}$$

Note that if $\operatorname{Gr} G \subseteq X \times V^*_{w^*}$ is closed and $G(\cdot)$ is locally compact into $V^*_{w^*}$, that is, for all $u \in X$ we can find U a neighborhood of u such that $\overline{G(U)}^{w^*}$ is w^*-compact in V^*, then G is usc from X into $V^*_{w^*}$. Also, let $K : V^*_{w^*} \to X$ be a sequentially continuous map. Then the nonlinear alternative theorem of Bader [3], reads as follows.

Theorem 1 *Assume that G and K are as above and $S = K \circ G : X \to 2^X\backslash\{\emptyset\}$ maps bounded sets into relatively compact sets. Define*

$$E = \{u \in X : u \in tS(u) \text{ for some } t \in (0, 1)\}.$$

Then either E is unbounded or $S(\cdot)$ admits a fixed point.

3 Existence Theorem

In this section we prove an existence theorem for problem (1). We start by introducing the hypotheses on the data of problem (1).

$H(a)$: $a : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$|a(x) - a(y)| \leqslant k|x - y| \text{ for all } x, y \in \mathbb{R}, \text{ some } k > 0,$$
$$0 < c_1 \leqslant a(x) \leqslant c_2 \text{ for all } x \in \mathbb{R}.$$

$H(\varphi)$: $\varphi \in \Gamma_0(\mathbb{R})$ and $0 \in \partial\varphi(0)$.

Remark 1 We recall that in $\mathbb{R}\times\mathbb{R}$, every maximal monotone set is of the subdifferential type. In higher dimensions this is no longer true (see Papageorgiou and Kyritsi [19, p. 175]).

$H(F)$: $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to P_{f_c}(\mathbb{R})$ is a multifunction such that

(i) for all $(x, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $z \mapsto F(z, x, \xi)$ is graph measurable;
(ii) for almost all $z \in \Omega$, $(x, \xi) \mapsto F(z, x, \xi)$ is closed;
(iii) for almost all $z \in \Omega$ and all $(x, \xi, v) \in \text{Gr}\, F(z, \cdot, \cdot)$, we have

$$|v| \leqslant \gamma_1(z, |x|) + \gamma_2(z, |x|)|\xi|$$

with

$$\sup[\gamma_1(z, s) : 0 \leqslant s \leqslant k] \leqslant \eta_{1,k}(z) \text{ for almost all } z \in \Omega,$$
$$\sup[\gamma_2(z, s) : 0 \leqslant s \leqslant k] \leqslant \eta_{2,k}(z) \text{ for almost all } z \in \Omega,$$
$$\text{and } \eta_{1,k}, \eta_{2,k} \in L^\infty(\Omega);$$

(iv) there exists $M > 0$ such that if $|x_0| > M$, then we can find $\delta > 0$ and $\eta > 0$ such that

$$\inf[vx + c_1|\xi|^2 : |x-x_0|+|\xi| \leqslant \delta, v \in F(z, x, \xi)] \geqslant \eta > 0 \text{ for almost all } z \in \Omega,$$

with $c_1 > 0$ as in hypothesis $H(a)$;
(v) for almost all $z \in \Omega$ and all $(x, \xi, v) \in \text{Gr}\, F(z, \cdot, \cdot)$, we have

$$vx \geqslant -c_3|x|^2 - c_4|x||\xi| - \gamma_3(z)|x|$$

with $c_3, c_4 > 0$ and $\gamma_3 \in L^1(\Omega)_+$.

Remark 2 Hypothesis $H(F)(iv)$ is an extension to multifunctions of the Nagumo-Hartman condition for continuous vector fields (see Hartman [9, p. 433], Knobloch [11] and Mawhin [16]).

Let $\hat{a} : H^1(\Omega) \to H^1(\Omega)^*$ be the nonlinear continuous map defined by

$$\langle \hat{a}(u), h \rangle = \int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega). \tag{2}$$

Proposition 2 *If hypotheses $H(a)$ hold, then the map $\hat{a} : H^1(\Omega) \to H^1(\Omega)^*$ defined by (2) is pseudomonotone.*

Proof Evidently $\hat{a}(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), see hypotheses $H(a)$ and it is defined on all of $H^1(\Omega)$. So, in order to prove the desired pseudomonotonicity of $\hat{a}(\cdot)$, it suffices to show the following:

(GP): "If $u_n \xrightarrow{w} u$ in $H^1(\Omega)$, $\hat{a}(u_n) \xrightarrow{w} u^*$ in $H^1(\Omega)^*$ and $\limsup_{n\to\infty} \langle \hat{a}(u_n), u_n - u \rangle \leqslant 0$,
then $u^* = \hat{a}(u)$ and $\langle \hat{a}(u_n), u_n \rangle \to \langle \hat{a}(u), u \rangle$"

(see Gasinski and Papageorgiou [6], Proposition 3.2.49, p. 333).

So, according to (GP) above we consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega),\ \hat{a}(u_n) \xrightarrow{w} u^* \text{ in } H^1(\Omega)^* \text{ and } \limsup_{n\to\infty} \langle \hat{a}(u_n), u_n - u \rangle \leqslant 0. \tag{3}$$

We have

$$\begin{aligned} \langle \hat{a}(u_n), u_n - u \rangle &= \int_\Omega a(u_n)(Du_n, Du_n - Du)_{\mathbb{R}^N} dz \\ &= \int_\Omega a(u_n)|Du_n - Du|^2 dz + \int_\Omega a(u_n)(Du, Du_n - Du)_{\mathbb{R}^N} dz. \end{aligned} \tag{4}$$

Hypotheses $H(a)$ and (3) imply that

$$\int_\Omega a(u_n)(Du, Du_n - Du)_{\mathbb{R}^N} dz \to 0 \text{ as } n \to \infty. \tag{5}$$

Also we have

$$\begin{aligned} &\int_\Omega a(u_n)|Du_n - Du|^2 dz \geqslant c_1 ||Du_n - Du||_2^2 \text{ (see hypotheses } H(a)), \\ &\Rightarrow Du_n \to Du \text{ in } L^2(\Omega, \mathbb{R}^N) \text{ (see(3), (4), (5))} \\ &\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (see(3))}. \end{aligned} \tag{6}$$

For all $h \in H^1(\Omega)$, we have

$$\langle \hat{a}(u_n), h \rangle = \int_\Omega a(u_n)(Du_n, Dh)_{\mathbb{R}^N} dz \to \int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz = \langle \hat{a}(u), h \rangle$$

(see (3) and hypotheses $H(a)$),

$\Rightarrow \hat{a}(u_n) \xrightarrow{w} \hat{a}(u)$ in $H^1(\Omega)^*$,
$\Rightarrow \hat{a}(u) = u^*$ (see(3)).

From (6) and the continuity of $a(\cdot)$ (see hypotheses $H(a)$), we have

$$\langle \hat{a}(u_n), u_n \rangle \to \langle \hat{a}(u), u \rangle .$$

Therefore property (GP) is satisfied and so we conclude that $\hat{a}(\cdot)$ is pseudomonotone. □

Next we will approximate problem (1) using the Moreau-Yosida approximations of $\varphi \in \Gamma_0(\mathbb{R})$. For this approach to lead to a solution of problem (1), we need to have a priori bounds for the approximate solutions. The proposition which follows is a crucial step in this direction. Its proof is based on the so-called "Moser iteration technique".

So, we consider the following nonlinear Neumann problem:

$$\left\{ \begin{array}{ll} -\text{div}\,(a(u(z))Du(z)) = g(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{7}$$

The conditions on the reaction term $g(z, x)$ are the following:

$H(g)$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto g(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto g(z, x)$ is continuous) and

$$|g(z, x)| \leqslant a(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+, 2 \leqslant r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geqslant 3 \\ +\infty & \text{if } N = 1, 2 \end{cases}$ (the critical Sobolev exponent).

By a weak solution of problem (7), we understand a function $u \in H^1(\Omega)$ such that

$$\int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz = \int_\Omega g(z, u)h dz \text{ for all } h \in H^1(\Omega).$$

Proposition 3 *If hypothesis $H(g)$ holds and $u \in H^1(\Omega)$ is a nontrivial weak solution of (7), then $u \in L^\infty(\Omega)$ and $||u||_\infty \leqslant M = M(||a||_\infty, N, 2, ||u||_{2^*})$.*

Proof Let $p_0 = 2^*$ and $p_{n+1} = 2^* + \frac{2^*}{2}(p_n - r)$ for all $n \in \mathbb{N}_0$. Evidently $\{p_n\}_{n \geqslant 0}$ is increasing. First suppose that $u \geqslant 0$. For every $k \in \mathbb{N}$ we set

$$u_k = \min\{u, k\} \in H^1(\Omega) . \tag{8}$$

Let $\vartheta = p_n - r > 0$ (note that $p_n \geqslant 2^* > r$). We have

$$\hat{a}(u) = N_g(u) \text{ in } H^1(\Omega)^* \tag{9}$$

with $N_g(u)(\cdot) = g(\cdot, u(\cdot)) \in L^{r'}(\Omega) \subseteq H^1(\Omega)^*$, $\frac{1}{r} + \frac{1}{r'} = 1$ (the Nemytskii map corresponding to g). On (9) we act with $u_k^{\vartheta+1}$ (see (8)). Then

$$\left\langle \hat{a}(u), u_k^{\vartheta+1} \right\rangle = \int_\Omega g(z,u) u_k^{\vartheta+1} dz. \tag{10}$$

Note that

$$\begin{aligned}
\left\langle \hat{a}(u), u_k^{\vartheta+1} \right\rangle &= \int_\Omega a(u) \left(Du, Du_k^{\vartheta+1} \right)_{\mathbb{R}^N} dz \\
&= (\vartheta+1) \int_\Omega u_k^\vartheta a(u) (Du, Du_k)_{\mathbb{R}^N} dz \\
&\geqslant (\vartheta+1) \int_\Omega u_k^\vartheta c_1 |Du_k|^2 dz \text{ (see hypothesis H(a) and recall that } u \geqslant 0) \\
&= c_1 (\vartheta+1) \frac{2}{\vartheta+2} \int_\Omega \left| Du_k^{\frac{\vartheta+2}{2}} \right|^2 dz.
\end{aligned} \tag{11}$$

Also we have

$$\begin{aligned}
&\int_\Omega g(z,u) u_k^{\vartheta+1} dz \\
&\quad \leqslant \int_\Omega a(z)(1+u^{r-1}) u^{\vartheta+1} dz \text{ (see hypothesis } H(g), \text{ (8) and recall } u \geqslant 0) \\
&\quad \leqslant c_3 \left(1 + \int_\Omega u^{p_n} dz \right) \text{ for some } c_3 > 0 \text{ (since } \vartheta + 1 < \vartheta + r = p_n).
\end{aligned} \tag{12}$$

We return to (10) and use (11) and (12). Then

$$\begin{aligned}
&c_1(\vartheta+1) \frac{2}{\vartheta+2} \int_\Omega \left[\left| Du_k^{\frac{\vartheta+2}{2}} \right|^2 + \left| u_k^{\frac{\vartheta+2}{2}} \right|^2 \right] dz \\
&\quad \leqslant c_4 \left(1 + \int_\Omega u^{p_n} dz \right) \text{ for some } c_4 > 0 \text{ (since } \vartheta + r = p_n) \\
&\Rightarrow \| u_k^{\frac{\vartheta+2}{2}} \|^2 \leqslant c_5 \left(1 + \int_\Omega u^{p_n} dz \right) \text{ for some } c_5 > 0, \text{ all } k \in \mathbb{N}, \text{ and } n \in \mathbb{N}_0.
\end{aligned}$$

Here $|| \cdot ||$ denotes the norm of $H^1(\Omega)$ (recall that $||v|| = [||v||_2^2 + ||Dv||_2^2]^{1/2}$ for all $v \in H^1(\Omega)$).

By the Sobolev embedding theorem (see (8) and note that $H^1(\Omega) \hookrightarrow L^{\frac{2p_{n+1}}{p_n}}(\Omega)$) we have

$$||u_k||_{p_{n+1}}^{p_n} \leqslant c_6 \left(1 + \int_\Omega u^{p_n} dz\right) \text{ for some } c_6 > 0, \text{ all } k \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}.$$

Let $k \to \infty$. Then $u_k(z) \uparrow u(z)$ for almost all $z \in \Omega$ (see (8)). So, by the monotone convergence theorem, we have

$$\left(\int_\Omega u^{p_{n+1}} dz\right)^{\frac{p_n}{p_{n+1}}} \leqslant c_6 \left(1 + \int_\Omega u^{p_n} dz\right) \text{ for all } n \in \mathbb{N}_0. \tag{13}$$

Recall that $p_0 = 2^*$ and by the Sobolev embedding theorem we have $u \in L^{2^*}(\Omega)$. So, from (13) and by induction we infer that $u \in L^{p_n}(\Omega)$ for all $n \in \mathbb{N}_0$. Also we have

$$||u||_{p_{n+1}}^{p_n} \leqslant c_6(1 + ||u||_{p_n}^{p_n}) \text{ for all } n \in \mathbb{N}_0 \text{ (see(13))}.$$

Since $p_n < p_{n+1}$, using the Hölder and Young inequalities (the latter with $\epsilon > 0$ small), we obtain

$$||u||_{p_n} \leqslant c_7 \text{ for some } c_7 > 0, \text{ all } n \in \mathbb{N}_0. \tag{14}$$

Claim 1.$p_n \to \infty$.

Arguing by contradiction, suppose that the Claim is not true. Since $\{p_n\}_{n\in\mathbb{N}_0}$ is increasing, we have

$$p_n \to p_* > 2^*. \tag{15}$$

By definition

$$\begin{aligned}
&p_{n+1} = 2^* + \frac{2^*}{2}(p_n - r),\\
\Rightarrow\ &p_* = 2^* + \frac{2^*}{2}(p_* - r) \text{ (see 15))}\\
\Rightarrow\ &p_*\left(\frac{2^*}{2} - 1\right) = 2^*\left(\frac{r}{2} - 1\right) < 2^*\left(\frac{2^*}{2} - 1\right) \text{ (since } 2 \leqslant r < 2^*\text{)},\\
\Rightarrow\ &p_* < 2^*, \text{ a contradiction (see 15).}
\end{aligned}$$

This proves the Claim.

So, passing to the limit as $n \to \infty$ in (14), it follows from Gasinski and Papageorgiou [7, p. 477] that

$$||u||_\infty \leqslant c_7, \text{ hence } u \in L^\infty(\Omega).$$

Moreover, it is clear from the above proof that $||u||_\infty \leqslant M = M(||a||_\infty, N, 2, ||u||_{2^*})$.

Finally for the general case, we write $u = u^+ - u^-$, with $u^\pm = \max\{\pm u, 0\} \geqslant 0$ and work with each one separately as above, to conclude $u^\pm \in L^\infty(\Omega)$, hence $u \in L^\infty(\Omega)$. □

Now for $\lambda > 0$, let φ_λ be the Moreau-Yosida approximation of $\varphi \in \Gamma_0(\mathbb{R})$ and for $\vartheta \in L^\infty(\Omega)$, consider the following auxiliary Neumann problem:

$$\left\{ \begin{array}{ll} -\text{div}\,(a(u(z))Du(z)) + u(z) + \varphi'_\lambda(u(z)) = \vartheta(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right\} \tag{16}$$

Proposition 4 *If hypotheses $H(a)$, $H(\varphi)$ hold and $\vartheta \in L^\infty(\Omega)$, then problem (16) admits a unique solution $u \in C^1(\overline{\Omega})$.*

Proof Let $V_\lambda : H^1(\Omega) \to H^1(\Omega)^*$ be the nonlinear map defined by

$$V_\lambda(u) = \hat{a}(u) + u + N_{\varphi'_\lambda}(u) \text{ for all } u \in H^1(\Omega).$$

As before $N_{\varphi'_\lambda}(u)$ is the Nemytskii map corresponding to φ'_λ (that is, $N_{\varphi'_\lambda}(u)(\cdot) = \varphi'_\lambda(u(\cdot))$). We have

$$\begin{aligned} \langle V_\lambda(u), u\rangle &= \left\langle \hat{a}(u), u\right\rangle + ||u||_2^2 + \int_\Omega \varphi'_\lambda(u)udz \\ &\geqslant c_1||Du||_2^2 + ||u||_2^2 \\ &\quad \text{(see hypothesis } H(a) \text{ and recall that } \varphi'_\lambda \text{ is increasing, } \varphi'_\lambda(0) = 0), \\ \Rightarrow &\quad V_\lambda \text{ is strongly coercive.} \end{aligned} \tag{17}$$

Using the Sobolev embedding theorem we see that $u \mapsto N_{\varphi'_\lambda}(u)$ is completely continuous from $H^1(\Omega)$ into $H^1(\Omega)^*$ (that is, if $u_n \xrightarrow{w} u$ in $H^1(\Omega)$, then $N_{\varphi'_\lambda}(u_n) \to N_{\varphi'_\lambda}(u)$ in $H^1(\Omega)^*$), hence it is pseudomonotone. From Proposition 2 we know that $\hat{a}(\cdot)$ is pseudomonotone and of course the same is true for the embedding $H^1(\Omega) \hookrightarrow H^1(\Omega)^*$ (which is compact). So, from Gasinski and Papageorgiou [6], Proposition 3.2.51, p. 334, we infer that

$$u \mapsto V_\lambda(u) \text{ is pseudomonotone.} \tag{18}$$

Recall that a pseudomonotone strongly coercive map is surjective. So, from (17), (18) it follows that there exists $u \in H^1(\Omega)$ such that

$$\begin{aligned} V_\lambda(u) = \vartheta, \Rightarrow &\int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz \\ &+ \int_\Omega uhdz + \int_\Omega \varphi'_\lambda(u)hdz = \int_\Omega \vartheta hdz \text{ for all } h \in H^1(\Omega). \end{aligned} \tag{19}$$

From the nonlinear Green's identity (see Gasinski and Papageorgiou [6], Theorem 2.4.53, p. 210), we have

$$\int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz = \langle -\text{div}\,(a(u)Du), h\rangle + \left\langle a(u)\frac{\partial u}{\partial n}, h \right\rangle_{\partial\Omega} \text{ for all } h \in H^1(\Omega), \tag{20}$$

where by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair $(H^{-\frac{1}{2},2}(\partial\Omega), H^{\frac{1}{2},2}(\partial\Omega))$.

From the representation theorem for the elements of $H^{-1}(\Omega) = H_0^1(\Omega)^*$ (see Gasinski and Papageorgiou [6], Theorem 2.4.57, p. 212), we have

$$\text{div}\,(a(u)Du) \in H^{-1}(\Omega).$$

So, if by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$ we have

$$\begin{aligned} &\langle -\text{div}\,(a(u)Du), h\rangle_0 = \int_\Omega a(u)(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } h \in H_0^1(\Omega),\\ \Rightarrow\ &\langle -\text{div}\,(a(u)Du), h\rangle_0 = \int_\Omega (\vartheta - u - \varphi_\lambda'(u))h dz \text{ for all } h \in H_0^1(\Omega) \text{ (see (19))},\\ \Rightarrow\ &-\text{div}\,(a(u(z))Du(z)) = \vartheta(z) - u(z) - \varphi_\lambda'(u(z)) \text{ for almost all } z \in \Omega. \end{aligned} \tag{21}$$

Then from (19), (20), (21) it follows that

$$\left\langle a(u)\frac{\partial u}{\partial n}, h \right\rangle_{\partial\Omega} = 0 \text{ for all } h \in H^1(\Omega). \tag{22}$$

If by γ_0 we denote the trace map, we recall that

$$\text{im}\,\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega)$$

(see Gasinski and Papageorgiou [6], Theorem 2.4.50, p. 209). Hence from (22) we infer that

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \text{ (see hypothesis } H(a)).$$

Therefore we have

$$\left\{ \begin{array}{ll} -\text{div}\,(a(u)(z)Du(z)) + u(z) + \varphi_\lambda'(u(z)) = \vartheta(z) & \text{for almost all } z \in \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right\} \tag{23}$$

From (23) and Proposition 3, we infer that

$$u \in L^\infty(\Omega).$$

Then we can use Theorem 2 of Lieberman [12] and conclude that

$$u \in C^1(\overline{\Omega}).$$

We establish in what follows the uniqueness of this solution. So, suppose that $v \in C^1(\overline{\Omega})$ is another solution. We have

$$\hat{a}(u) + u + N_{\varphi'_\lambda}(u) = \vartheta \text{ in } H^1(\Omega)^*, \tag{24}$$

$$\hat{a}(v) + v + N_{\varphi'_\lambda}(v) = \vartheta \text{ in } H^1(\Omega)^*. \tag{25}$$

Let $k > 0$ be the Lipschitz constant in hypothesis $H(a)$. We introduce the following function

$$\eta_\epsilon(s) = \begin{cases} \int_\epsilon^s \frac{dt}{(kt)^2} & \text{is } s \geqslant \epsilon \\ 0 & \text{if } s < \epsilon \end{cases} \quad \text{with } \epsilon > 0. \tag{26}$$

Evidently η_ϵ is Lipschitz continuous. So, from Marcus and Mizel [14], we have

$$\eta_\epsilon(u - v) \in H^1(\Omega), \tag{27}$$

$$D(\eta_\epsilon(u - v)) = \eta'_\epsilon(u - v)D(u - v) \tag{28}$$

(see also Gasinski and Papageorgiou [6], Proposition 2.4.25, p. 195). Subtracting (25) from (24), we have

$$\hat{a}(u) - \hat{a}(v) + (u - v) + (N_{\varphi'_\lambda}(u) - N_{\varphi'_\lambda(v)}) = 0 \text{ in } H^1(\Omega)^*. \tag{29}$$

On (29) we act with $\eta_\epsilon(u - v) \in H^1(\Omega)$ (see (27)). Then

$$\left\langle \hat{a}(u) - \hat{a}(v), \eta_\epsilon(u - v)\right\rangle + \int_\Omega (u - v)\eta_\epsilon(u - v)dz + \int_\Omega (\varphi'_\lambda(u) - \varphi'_\lambda(v))(u - v)dz = 0. \tag{30}$$

We have

$$\int_\Omega (u - v)\eta_\epsilon(u - v)dz = \int_{\{u-v\geqslant\epsilon\}} (u - v)\eta_\epsilon(u - v)dz$$
$$\geqslant \frac{1}{k}\int_{\{u-v\geqslant\epsilon\}} \left(\frac{u - v}{\epsilon} - 1\right) dz \text{ (see (26))}. \tag{31}$$

Recall that φ'_λ is increasing. Therefore

$$\int_\Omega (\varphi'_\lambda(u) - \varphi'_\lambda(v))\eta_\epsilon(u - v)dz = \int_{\{u-v\geqslant\epsilon\}} (\varphi'_\lambda(u) - \varphi'_\lambda(v))\eta_\epsilon(u - v)dz \geqslant 0 \text{ (see (26))}. \tag{32}$$

We return to (30) and use (31), (32). Then

$$
\begin{aligned}
&\left\langle \hat{a}(u) - \hat{a}(v), \eta_\epsilon(u-v)\right\rangle \leqslant 0,\\
\Rightarrow &\int_\Omega (a(u)Du - a(v)Dv, D\eta_\epsilon(u-v))_{\mathbb{R}^N} dz \leqslant 0,\\
\Rightarrow &\int_\Omega a(u)(Du - Dv, D\eta_\epsilon(u-v))_{\mathbb{R}^N} dz \leqslant -\int_\Omega (a(u)\\
&-a(v))(Dv, D\eta_\epsilon(u-v))_{\mathbb{R}^N} dz.
\end{aligned}
\tag{33}
$$

Let $\Omega_\epsilon = \{z \in \Omega : (u-v)(z) \geqslant \epsilon\}$. Then

$$
\begin{aligned}
&\int_\Omega a(u)(Du - Dv, D\eta_\epsilon(u-v))_{\mathbb{R}^N} dz\\
&= \int_{\Omega_\epsilon} a(u)\eta_\epsilon'(u-v)|Du - Dv|^2 dz \text{ (see (26), (28))}\\
&\geqslant c_1 \int_{\Omega_\epsilon} \frac{|Du - Dv|^2}{k^2(u-v)^2} dz \text{ (see hypothesis} H(a) \text{ and (26)).}
\end{aligned}
\tag{34}
$$

Also we have

$$
\begin{aligned}
&-\int_\Omega (a(u) - a(v))(Dv, D\eta_\epsilon(u-v))_{\mathbb{R}^N} dz\\
&\leqslant \int_{\Omega_\epsilon} k(u-v)\eta_\epsilon'(u-v)(Dv, Du - Dv)_{\mathbb{R}^N} dz \text{ (see hypothesis } H(a) \text{ and (28))}\\
&= \int_{\Omega_\epsilon} \frac{1}{k(u-v)}(Dv, Du - Dv)_{\mathbb{R}^N} dz \text{ (see (26))}\\
&\leqslant ||Dv||_2 \left(\int_{\Omega_\epsilon} \frac{|Du - Dv|^2}{k^2|u-v|^2} dz\right)^{1/2} \text{ (by the Cauchy-Schwarz inequality).}
\end{aligned}
\tag{35}
$$

Returning to (33) and using (34), (35) we obtain

$$
\int_{\Omega_\epsilon} \frac{|Du - Dv|^2}{|u-v|^2} dz \leqslant \frac{k^2}{c_1^2} ||Dv||_2^2.
$$

Let Ω_ϵ^* be a connected component of $\hat{\Omega} = \{z \in \Omega;\ (u-v)(z) > 0\}$, $\hat{\Omega} \neq \Omega$ (see (31)). We have

$$
\int_{\Omega_\epsilon^*} \frac{|Du - Dv|^2}{|u-v|^2} dz \leqslant \frac{k^2}{c_1^2} ||Dv||_2^2 \quad \text{with } \Omega_\epsilon^* = \Omega_\epsilon \cap \Omega^*.
\tag{36}
$$

Consider the function

$$
\gamma_\epsilon(y) = \begin{cases} \int_\epsilon^y \frac{dt}{t} & \text{if } t \geqslant \epsilon \\ 0 & \text{if } t < \epsilon. \end{cases}
\tag{37}
$$

This function is Lipschitz continuous and as before from Marcus and Mizel [14], we have

$$\gamma_\epsilon(u-v) \in H^1(\Omega) \tag{38}$$

$$\begin{aligned} D\gamma_\epsilon(u-v) &= \gamma_\epsilon'(u-v)(Du-Dv) \\ &= \frac{1}{u-v}(Du-Dv) \text{ for almost all } z \in \Omega_\epsilon \text{ (see (37))}. \end{aligned} \tag{39}$$

Returning to (36) and using (38), (39), we obtain

$$\int_{\Omega^*} |D\gamma_\epsilon(u-v)|^2 dz \leqslant \frac{k^2}{c_1^2}||Dv||_2^2. \tag{40}$$

Note that $u = v$ on $\partial\Omega^*$ (that is, $u - v \in H_0^1(\Omega^*)$; recall that $u, v \in C^1(\overline{\Omega})$). Hence

$$\gamma_\epsilon(u-v) \in H_0^1(\Omega^*). \tag{41}$$

From (40), (41) and the Poincaré inequality, we have

$$\int_{\Omega^*} |\gamma_\epsilon(u-v)|^2 dz \leqslant c_8||v||^2 \text{ for some } c_8 > 0, \text{ all } \epsilon > 0.$$

If $|\Omega^*|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$), then letting $\epsilon \to 0^+$, we reach a contradiction (see (37)). So, every connected component of the open set

$$\hat{\Omega} = \{z \in \Omega : u(z) > v(z)\}$$

is Lebesgue-null. Hence $|\hat{\Omega}|_N = 0$ and so

$$u \leqslant v. \tag{42}$$

Interchanging the roles of u, v in the above argument, we also obtain

$$v \leqslant u. \tag{43}$$

From (42) and (43) we conclude that

$$u = v.$$

This prove the uniqueness of the solution $u \in C^1(\overline{\Omega})$ of the auxiliary problem (16). □

Let $C_n^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$ and for every $\lambda > 0$ let $K_\lambda : L^\infty(\Omega) \to C_n^1(\overline{\Omega})$ be the map which to each $\vartheta \in L^\infty(\Omega)$ assigns the unique solution $u =$

$K_\lambda(\vartheta) \in C^1_n(\overline{\Omega})$ of the auxiliary problem (16) (see Proposition 4). The next proposition establishes the continuity properties of this map.

Proposition 5 *If hypotheses $H(a)$, $H(\varphi)$ hold then the map $K_\lambda : L^\infty(\Omega) \to C^1_n(\overline{\Omega})$ is sequentially continuous from $L^\infty(\Omega)$ furnished with the w^*-topology into $C^1_n(\overline{\Omega})$ with the norm topology.*

Proof Suppose that $\vartheta_n \xrightarrow{w^*} \vartheta$ in $L^\infty(\Omega)$ and let $u_n = K_\lambda(\vartheta_n)$, $u = K_\lambda(\vartheta)$.

For every $n \in \mathbb{N}$, we have

$$\hat{a}(u_n) + u_n + N_{\varphi'_\lambda}(u_n) = \vartheta_n$$
$$\Rightarrow -\text{div}\,(a(u_n(z))Du_n(z)) + u_n(z) + \varphi'_\lambda(u_n(z)) = \vartheta_n(z) \tag{44}$$
$$\text{for almost all } z \in \Omega, \ \frac{\partial u_n}{\partial n} = 0 \text{ on } \partial\Omega. \tag{45}$$

On (44) we act with $u_n \in C^1_n(\overline{\Omega})$. Then

$$\int_\Omega a(u_n)|Du_n|^2 dz + ||u_n||_2^2 + \int_\Omega \varphi'_\lambda(u_n)u_n dz = \int_\Omega \vartheta_n u_n dz$$
$$\Rightarrow c_1||Du_n||_2^2 + ||u_n||_2^2 \leqslant c_9||u_n|| \text{ for some } c_9 > 0, \text{ all } n \in \mathbb{N}$$
$$(\text{see hypothesis } H(a) \text{and recall that } \varphi'_\lambda \text{ is increasing with } \varphi'_\lambda(0) = 0)$$
$$\Rightarrow ||u_n|| \leqslant c_{10} \text{ for some } c_{10} > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

By passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } H^1(\Omega) \text{ and } u_n \to \hat{u} \text{ in } L^2(\Omega). \tag{46}$$

Then for every $h \in H^1(\Omega)$ we have

$$\langle \hat{a}(u_n), h\rangle = \int_\Omega a(u_n)(Du_n, Dh)_{\mathbb{R}^N} dz \to \int_\Omega a(\hat{u})(D\hat{u}, Dh)_{\mathbb{R}^N} dz = \langle \hat{a}(\hat{u}), h\rangle$$
$$(\text{see (46) and hypothesis } H(a)),$$
$$\Rightarrow \hat{a}(u_n) \xrightarrow{w} \hat{a}(\hat{u}) \text{ in } H^1(\Omega)^*. \tag{47}$$

Therefore, if in (44) we pass to the limit as $n \to \infty$ and use (46), (47), then

$$\hat{a}(\hat{u}) + \hat{u} + N_{\varphi'_\lambda}(\hat{u}) = \vartheta,$$
$$\Rightarrow \hat{u} = u \in C^1(\overline{\Omega}) = \text{the unique solution of (16) (see Proposition 4).}$$

From (45) and Proposition 3, (recall that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded), we see that we can find $c_{11} > 0$ such that

$$||u_n||_\infty \leqslant c_{11} \text{ for all } n \in \mathbb{N}. \tag{48}$$

Then (48) and Theorem 2 of Lieberman [12] imply that we can find $\alpha \in (0, 1)$ and $c_{12} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}),\ ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{12} \text{ for all } n \in \mathbb{N}. \tag{49}$$

From (49), the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (46), we have

$$\begin{aligned} &u_n \to u \text{ in } C^1(\overline{\Omega}),\\ \Rightarrow\ &K_\lambda(\vartheta_n) \to K_\lambda(\vartheta) \text{ in } C^1(\overline{\Omega}). \end{aligned}$$

This proves that K_λ is sequentially continuous from $L^\infty(\Omega)$ with the w^*-topology into $C^1_n(\overline{\Omega})$ with the norm topology. □

We consider the following approximation to problem (1):

$$\left\{ \begin{array}{ll} \operatorname{div}(a(u(z))Du(z)) \in \varphi'_\lambda(u(z)) + F(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \lambda > 0. \end{array} \right\} \tag{50}$$

Proposition 6 *If hypotheses* $H(a)$, $H(\varphi)$, $H(F)$ *hold and* $\lambda > 0$, *then problem* (50) *admits a solution* $u_\lambda \in C^1(\overline{\Omega})$.

Proof Consider the multifunction $N : C^1_n(\overline{\Omega}) \to 2^{L^\infty(\Omega)}$ defined by

$$N(u) = \left\{ f \in L^\infty(\Omega) : f(z) \in F(z, u(z), Du(z)) \text{ for almost all } z \in \Omega \right\}.$$

Hypotheses $H(F)(i), (ii)$ imply that the multifunction $z \mapsto F(z, u(z), Du(z))$ admits a measurable selection (see Hu and Papageorgiou [10, p. 21]) and then hypothesis $H(F)(iii)$ implies that this measurable selection belongs in $L^\infty(\Omega)$ and so $N(\cdot)$ has nonempty values, which is easy to see that they are w^*-compact (Alaoglu's theorem) and convex. Let

$$N_1(u) = u - N(u) \text{ for all } u \in C^1_n(\overline{\Omega}).$$

We consider the following fixed point problem

$$u \in K_\lambda N_1(u). \tag{51}$$

Let $E = \{u \in C^1_n(\overline{\Omega}) : u \in tK_\lambda N_1(u) \text{ for some } t \in (0, 1)\}$.
Claim 2. The set $E \subseteq C^1_n(\overline{\Omega})$ is bounded.
Let $u \in E$. Then from the definitions of K_λ and N_1 we have

$$\hat{a}\left(\frac{1}{t}u\right) + \frac{1}{t}u + N_{\varphi'_\lambda}\left(\frac{1}{t}u\right) = u - f \text{ with } f \in N(u). \tag{52}$$

On (52) we act with $u \in H^1(\Omega)$. Using hypothesis $H(a)$, we obtain

$$\frac{c_1}{t}||Du||_2^2 + \frac{1}{t}||u||_2^2 \leqslant ||u||_2^2 - \int_\Omega f u dz$$
$$\text{(recall that } \varphi'_\lambda \text{ is increasing and } \varphi'_\lambda(0) = 0\text{)},$$
$$\Rightarrow c_1||Du||_2^2 \leqslant (t-1)||u||_2^2 - t\int_\Omega f u dz \leqslant -t\int_\Omega f u dz \text{ (recall that } t \in (0,1)\text{)}. \tag{53}$$

Hypothesis $H(F)(v)$ implies that

$$-t\int_\Omega f u dz \leqslant tc_3||u||_2^2 + tc_4\int_\Omega |u|^2\,|Du|dz + \int_\Omega \gamma_3(z)|u|^2 dz. \tag{54}$$

Let $M > 0$ be as postulated by hypothesis $H(F)(iv)$. We will show that

$$||u||_\infty \leqslant M.$$

To this end let $\hat{\sigma}_0(z) = |u(z)|^2$. Let $z_0 \in \overline{\Omega}$ be such that

$$\hat{\sigma}_0(z_0) = \max_{\overline{\Omega}} \hat{\sigma}_0 \text{ (recall that } u \in E \subseteq C_n^1(\overline{\Omega})\text{)}.$$

Suppose that $\hat{\sigma}_0(z_0) > M^2$. First assume that $z_0 \in \Omega$. Then

$$0 = D\hat{\sigma}_0(z_0) = 2u(z_0)Du(z_0),$$
$$\Rightarrow Du(z_0) = 0 \text{ (since } |u(z_0)| > M\text{)}.$$

Let $\delta, \eta > 0$ be as in hypothesis $H(F)(iv)$. Since $\hat{\sigma}_0(z_0) > M^2$ and $u \in C_n^1(\overline{\Omega})$ we can find $\delta_1 > 0$ such that

$$z \in \overline{B}_{\delta_1}(z_0) = \{z \in \Omega : |z - z_0| \leqslant \delta_1\} \Rightarrow |u(z) - u(z_0)| + |Du(z)| \leqslant \delta$$
$$\text{(recall that } Du(z_0) = 0\text{)},$$
$$\Rightarrow tf(z)u(z) + tc_1|Du(z)|^2 \geqslant t\eta > 0 \text{ for almost all } z \in \overline{B}_{\delta_1}(z_0)$$
$$\text{(see hypothesis } H(F)(iv)\text{)}. \tag{55}$$

From (52) as before (see the proof of Proposition 4), we have

$$-\operatorname{div}\left(a\left(\frac{1}{t}u(z)\right)D\left(\frac{1}{t}u\right)(z)\right) + \varphi'_\lambda\left(\frac{1}{t}u(z)\right)$$
$$= \left(1 - \frac{1}{t}\right)u(z) - f(z) \text{ for almost all } z \in \Omega. \tag{56}$$

Using (56) in (55), we obtain

$$\left[\operatorname{div}\left(a\left(\frac{1}{t}u(z)\right)Du(z)\right) - t\varphi_\lambda'\left(\frac{1}{t}u(z)\right) + (t-1)u(z)\right]u(z)$$
$$+tc_1|Du(z)|^2 \geqslant t\eta \quad \text{for almost all } z \in \overline{B}_{\delta_1}(z_0). \tag{57}$$

We integrate over $\overline{B}_{\delta_1}(z_0)$ and use the fact that $t \in (0, 1)$. Then

$$\int_{\overline{B}_{\delta_1}(z_0)} \operatorname{div}\left(a\left(\frac{1}{t}u\right)Du\right)udz - t\int_{\overline{B}_{\delta_1}(z_0)} \varphi_\lambda'\left(\frac{1}{t}u\right)udz$$
$$+tc_1\int_{\overline{B}_{\delta_1}(z_0)} |Du|^2dz \geqslant \mu\eta|\overline{B}_{\delta_1}(z_0)|_N$$
$$\Rightarrow \int_{\overline{B}_{\delta_1}(z_0)} \operatorname{div}\left(a\left(\frac{1}{t}u\right)Du\right)udz + tc_1\int_{\overline{B}_{\delta_1}(z_0)} |Du|^2dz > 0$$
$$\text{(recall that } \varphi_\lambda' \text{ is increasing and } \varphi_\lambda'(0) = 0\text{).}$$

Using the nonlinear Green's identity (see Gasinski and Papageorgiou [6], Theorem 2.4.53, p. 210), we obtain

$$0 < -\int_{\overline{B}_{\delta_1}(z_0)} a\left(\frac{1}{t}u\right)|Du|^2dz + \int_{\partial\overline{B}_{\delta_1}(z_0)} a\left(\frac{1}{t}u\right)\frac{\partial u}{\partial n}ud\sigma + tc_1\int_{\overline{B}_{\delta_1}(z_0)} |Du|^2dz.$$

Here by $\sigma(\cdot)$ we denote the $(N-1)$-dimensional Hausdorff (surface) measure defined on $\partial\Omega$. Hence we ahve

$$0 < -c_1\int_{\overline{B}_{\delta_1}(z_0)} |Du|^2dz + \int_{\partial\overline{B}_{\delta_1}(z_0)} a\left(\frac{1}{t}u\right)\frac{\partial u}{\partial n}ud\sigma + tc_1\int_{\overline{B}_{\delta_1}(z_0)} |Du|^2dz$$
$$\text{(see hypothesis } H(a)\text{)},$$
$$\Rightarrow 0 < \int_{\partial\overline{B}_{\delta_1}(z_0)} a\left(\frac{1}{t}u\right)\frac{\partial u}{\partial n}ud\sigma \text{ (recall that } t \in (0,1)\text{)},$$
$$\Rightarrow 0 < c_2\int_{\partial\overline{B}_{\delta_1}(z_0)} \frac{\partial u}{\partial n}ud\sigma \quad \text{(see hypothesis } H(a)\text{)}.$$

Thus we can find a continuous path $\{c(t)\}_{t\in[0,1]}$ in $\overline{B}_{\delta_1}(z_0)$ with $c(0) = z_0$ such that

$$a < \int_0^1 u(c(t))(Du(c(t)), c'(t))_{\mathbb{R}^N}dt = \int_0^1 \frac{1}{2}\frac{d}{dt}u(c(t))^2dt$$
$$= \frac{1}{2}[u(c(1)) - u(z_0)],$$
$$\Rightarrow u(z_0) < u(c(1)),$$

which contradicts the choice of z_0. So, we cannot have $z_0 \in \Omega$.

Therefore we assume that $z_0 \in \partial\Omega$. Since $u \in C^1_n(\overline{\Omega})$, again we have $Du(z_0) = 0$ and so the above argument applies with $\partial\overline{B}_{\delta_1}(z_0)$ replaced by $\partial\overline{B}_{\delta_1}(z_0) \cap \Omega$.

Hence we have proved that

$$||u||_\infty \leqslant M \text{ for all } u \in E \text{ (here } M > 0 \text{ is as in hypothesis } H(F)(iv)). \tag{58}$$

We use (58) in (54) and have

$$\begin{aligned} & -t\int_\Omega fudz \leqslant tc_{13}(1+||Du||_2) \text{ for some } c_{13} > 0, \\ \Rightarrow\ & c_1||Du||_2^2 \leqslant c_{13}(1+||Du||_2) \text{ (see (53) and recall } t \in (0,1)), \\ \Rightarrow\ & ||Du||_2 \leqslant c_{14} \text{ for some } c_{14} > 0, \text{ all } u \in E. \end{aligned} \tag{59}$$

Then (58), (59) imply that $E \subseteq H^1(\Omega)$ is bounded. Invoking Theorem 2 of Lieberman [12], we can find $c_{15} > 0$ such that

$$\begin{aligned} & ||u||_{C^1(\overline{\Omega})} \leqslant c_{15} \text{ for all } u \in E, \\ \Rightarrow\ & E \subseteq C^1_n(\overline{\Omega}) \text{ is bounded.} \end{aligned}$$

This proves the Claim.

Recall that hypotheses $H(F)(i)$, (ii), (iii) imply that N_1 is a multifunction which is usc from $C^1_n(\overline{\Omega})$ with the norm topology into $L^\infty(\Omega)$ with the w^*-topology (see Hu and Papageorgiou [10, p. 21]). This fact, Proposition 5 and the Claim, permit the use of Theorem 1. So, we can find $u_\lambda \in C^1_n(\overline{\Omega})$ such that

$$\begin{aligned} & u_\lambda \in K_\lambda N_1(u_\lambda), \\ \Rightarrow\ & u_\lambda \in C^1_n(\overline{\Omega}) \text{ is a solution of problem(50).} \end{aligned}$$

□

Now we are ready for the existence theorem concerning problem (1).

Theorem 7 *If hypotheses $H(a)$, $H(\varphi)$, $H(F)$ hold, then problem* (1) *admits a solution $u \in C^1_n(\overline{\Omega})$.*

Proof Let $\lambda_n \to 0^+$. From Proposition 6, we know that problem (50) (with $\lambda = \lambda_n$) has a solution $u_n = u_{\lambda_n} \in C^1_n(\overline{\Omega})$. Moreover, from the proof of that proposition, we have

$$||u_n||_\infty \leqslant M \text{ for all } n \in \mathbb{N} \text{ (see (58)).} \tag{60}$$

For every $n \in \mathbb{N}$, we have

$$\hat{a}(u_n) + N_{\varphi'_{\lambda_n}}(u_n) + f_n = 0 \text{ with } f_n \in N(u_n) \text{ (see the proof of Proposition 6).} \tag{61}$$

On (61) we act with u_n and obtain

$$\begin{aligned}
&c_1||Du_n||_2^2 \leqslant ||f_n||_2||u_n||_2 \\
&\quad \text{(see hypothesis } H(a) \text{and recall that } \varphi'_\lambda(s)s \geqslant 0 \text{ for all } s \in \mathbb{R}), \\
\Rightarrow\ &||Du_n||_2 \leqslant c_{16} \text{ for some } c_{16} > 0, \text{ all } n \in \mathbb{N} \\
&\text{(see (60) and hypothesis } H(F)(iii)).
\end{aligned} \tag{62}$$

From (60) and (62) it follows that

$$\{u_n\}_{n \geqslant 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

So, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega).$$

Acting on (61) with $N_{\varphi'_{\lambda_n}}(u_n)(\cdot) = \varphi'_{\lambda_n}(u_n(\cdot)) \in C(\overline{\Omega}) \cap H^1(\Omega)$ (recall that $\varphi'_{\lambda_n}(\cdot)$ is Lipschitz continuous and see Marcus and Mizel [14]), we have

$$\int_\Omega a(u_n)(Du_n, D\varphi'_\lambda(u_n))_{\mathbb{R}^N} dz + ||N_{\varphi'_{\lambda_n}}(u_n)||_2^2 = -\int_\Omega f_n \varphi'_{\lambda_n}(u_n) dz. \tag{63}$$

From the chain rule of Marcus and Mizel [14], we have

$$D\varphi'_{\lambda_n}(u_n) = \varphi''_{\lambda_n}(u_n) Du_n. \tag{64}$$

Since $\varphi'_{\lambda_n}(\cdot)$ is increasing (recall that φ_{λ_n} is convex), we have

$$\varphi''_{\lambda_n}(u_n(z)) \geqslant 0 \text{ for almost all } z \in \Omega. \tag{65}$$

Using (64), (65) and hypothesis $H(a)$, we see that

$$0 \leqslant \int_\Omega a(u_n)(Du_n, D\varphi'_{\lambda_n}(u_n))_{\mathbb{R}^N} dz. \tag{66}$$

Using (66) in (63), we obtain

$$\begin{aligned}
&||N_{\varphi'_{\lambda_n}}(u_n)||_2^2 \leqslant ||f_n||_2 ||N_{\varphi'_{\lambda_n}}(u_n)||_2 \text{ for all } n \in \mathbb{N}, \\
\Rightarrow\ &||N_{\varphi'_{\lambda_n}}(u_n)||_2 \leqslant ||f_n||_2 \leqslant c_{17} \text{ for some } c_{17} > 0, \text{ all } n \in \mathbb{N} \\
&\text{(see (60) and hypothesis H(F)(iii))} \\
\Rightarrow\ &\{N_{\varphi'_{\lambda_n}}(u_n)\}_{n \geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.}
\end{aligned}$$

So, we may assume that

$$N_{\varphi'_{\lambda_n}} \xrightarrow{w} g \text{ and } f_n \xrightarrow{w} f \text{ in } L^2(\Omega). \tag{67}$$

As in the proof of Proposition 5 (see (47)), we show that

$$\hat{a}(u_n) \xrightarrow{w} \hat{a}(u) \text{ in } H^1(\Omega)^*. \tag{68}$$

So, if in (61) we pass to the limit as $n \to \infty$ and use (67) and (68), we obtain

$$\begin{aligned} &\hat{a}(u) + g + f = 0, \\ \Rightarrow\ &-\text{div}\,(a(u(z))Du(z)) + g(z) + f(z) = 0 \text{ for almost all } z \in \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ &\text{(see the proof of Proposition 4).} \end{aligned} \tag{69}$$

Because of (60) and Theorem 2 of Lieberman [12], we know that there exist $\alpha \in (0, 1)$ and $c_{18} > 0$ such that

$$\begin{aligned} &u_n \in C^{1,\alpha}(\overline{\Omega}),\ ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_{18} \text{ for all } n \in \mathbb{N}, \\ \Rightarrow\ &u_n \to u \text{ in } C^1(\overline{\Omega}) \text{ (recall that } C^{1,\alpha}(\overline{\Omega}) \text{ is embedded compactly into } C^1(\overline{\Omega})). \end{aligned} \tag{70}$$

Recall that

$$\begin{aligned} &f_n(z) \in F(z, u_n(z), Du_n(z)) \text{ for almost all } z \in \Omega,\ \text{ all } n \in \mathbb{N}, \\ \Rightarrow\ &f(z) \in F(z, u(z), Du(z)) \\ &\text{(see (67), (70), hypothesis H(F)(ii) and Proposition 6.6.33, p. 521 of [19]),} \\ \Rightarrow\ &f \in N(u). \end{aligned} \tag{71}$$

Also, from (67), (70) and Corollary 3.2.51, p. 179 of [19], we have

$$g(z) \in \partial\varphi(u(z)) \text{ for almost all } z \in \Omega. \tag{72}$$

So, from (69), (71), (72) we conclude that $u \in C_n^1(\overline{\Omega})$ is a solution of problem (1). □

4 Examples

In this section we present two concrete situations illustrating our result.

For the first, let $\mu \leqslant 0$ and consider the function

$$\varphi(x) = \begin{cases} +\infty & \text{if } x < \mu \\ 0 & \text{if } \mu \leqslant x. \end{cases}$$

Evidently we have

$$\varphi \in \Gamma_0(\mathbb{R}) \text{ and } 0 \in \partial\varphi(0).$$

In fact note that

$$\partial\varphi(x) = \begin{cases} \emptyset & \text{if } x < \mu \\ \mathbb{R}_- & \text{if } x = \mu \\ \{0\} & \text{if } \mu < x. \end{cases}$$

Also consider a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfies hypotheses $H(F)(iii)$, (iv), (v). For example, we can have the following function (for the sake of simplicity we drop the z-dependence):

$$f(x, \xi) = c \sin x + x - \ln(1 + |\xi|) + \vartheta \text{ with } c_1\vartheta > 0.$$

Then according to Theorem 7, we can find a solution $u_0 \in C^1(\overline{\Omega})$ for the following problem:

$$\left\{ \begin{array}{l} \text{div}\,(a(u(z))Du(z)) \leqslant f(z, u(z), Du(z)) \text{ for almost all } z \in \{u = \mu\}, \\ \text{div}\,(a(u(z))Du(z)) = f(z, u(z), Du(z)) \text{ for almost all } z \in \{\mu < u\}, \\ u(z) \geqslant \mu \text{ for all } z \in \overline{\Omega},\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

For the second example, we consider a variational-hemivariational inequality. Such problems arise in mechanics, see Panagiotopoulos [18]. So, let $j(z, x)$ be a locally Lipschitz integrand (that is, for all $x \in \mathbb{R}$, $z \mapsto j(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto j(z, x)$ is locally Lipschitz). By $\partial_c j(z, x)$ we denote the Clarke subdifferential of $j(z, \cdot)$. We impose the following conditions on the integrand $j(z, x)$:

(a) for almost all $z \in \Omega$, all $x \in \mathbb{R}$ and all $v \in \partial j(z, x)$

$$|v| \leqslant \hat{c}_1(1 + |x|) \text{ for almost all } z \in \Omega,\ \text{ all } x \in \mathbb{R},\ \text{ with } \hat{c}_1 > 0;$$

(b) $0 < \hat{c}_2 \leqslant \liminf_{x\to\pm\infty} \frac{v}{x} \leqslant \limsup_{x\to\pm\infty} \frac{v}{x} \leqslant \hat{c}_3$ uniformly for almost all $z \in \Omega$, all $v \in \partial j(z, x)$

(c) $-\hat{c}_4 \leqslant \liminf_{x\to 0} \frac{v}{x} \leqslant \limsup_{x\to 0} \frac{v}{x} \leqslant \hat{c}_5$ uniformly for almost all $z \in \Omega$, all $v \in \partial j(z, x)$ and with $\hat{c}_4, \hat{c}_5 > 0$.

A possible choice of j is the following (as before for the sake of simplicity we drop the z-dependence):

$$j(x) = \begin{cases} \frac{1}{p}|x|^p - \cos(\frac{\pi}{2}|x|) & \text{if } |x| \leqslant 1 \\ \frac{1}{2}x^2 - \ln|x| + c & \text{if } 1 < |x| \end{cases} \text{ with } c = \frac{1}{p} - \frac{1}{2},\ 1 < p.$$

We set

$$F(z, x, \xi) = \partial j(z, x) + x|\xi| + \vartheta(z) \text{ with } \vartheta \in L^\infty(\Omega).$$

Using (a),(b),(c) above, we can see that hypotheses $H(F)$ are satisfied.

Also, suppose that φ satisfies hypothesis $H(\varphi)$. Two specific choices of interest are

$$\varphi(x) = |x| \text{ and } \varphi(x) = i_{[-1,1]}(x) = \begin{cases} 0 & \text{if } |x| \leqslant 1 \\ +\infty & \text{if } 1 < |x|. \end{cases}$$

Then the following problem admits a solution $u_0 \in C^1(\overline{\Omega})$:

$$\left\{ \begin{array}{ll} \operatorname{div}(a(u(z))Du(z)) \in \partial\varphi(u(z)) + F(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right\}$$

The case $\varphi \equiv 0$ (hemivariational inequalities) incorporates problems with discontinuities in which we fill-in the gaps at the jump discontinuities.

Acknowledgements V. Rădulescu was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, Project Number PN-II-PT-PCCA-2013-4-0614. D. Repovš was supported by the Slovenian Research Agency Grants P1-0292-0101, J1-6721-0101, J1-7025-0101 and J1-5435-0101.

References

1. Amann, H., Crandall, M.: On some existence theorems for semi-linear elliptic equations. Indiana Univ. Math. J. **27**, 779–790 (1978)
2. Arcoya, D., Carmona, J., Martinez Aparicio, P.J.: Elliptic obstacle problems with natural growth on the gradient and singular nonlinear terms. Adv. Nonlinear Stud. **7**, 299–318 (2007)
3. Bader, R.: A topological fixed-point index theory for evolution inclusions. Z. Anal. Anwend. **20**, 3–15 (2001)
4. Clarke, F.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
5. de Figueiredo, D., Girardi, M., Matzeu, M.: Semilinear singular elliptic equations with dependence on the gradient via mountain pass techniques. Differ. Integral Equ. **17**, 119–126 (2004)
6. Gasinski, L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman & Hall/CRC, Boca Raton, FL (2006)
7. Gasinski, L., Papageorgiou, N.S.: Exercises in Analysis: Part 1. Springer, New York (2014)
8. Girardi, M., Matzeu, M.: Positive and negative solutions of a quasilinear elliptic equation by a mountain pass method and truncature techniques. Nonlinear Anal. **59**, 199–210 (2004)
9. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
10. Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis. Part II: Applications. Kluwer, Dordrecht (2000)
11. Knobloch, W.: On the existence of periodic solutions for second order vector differential equations. J. Differ. Equ. **9**, 67–85 (1971)
12. Lieberman, G.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)
13. Loc, N.H., Schmitt, K.: Bernstein-Nagumo conditions and solutions to nonlinear differential inequalities. Nonlinear Anal. **75**, 4664–4671 (2012)
14. Marcus, M., Mizel, V.: Absolute continuity on tracks and mappings of Sobolev spaces. Arch. Ration. Mech. Anal. **45**, 294–320 (1972)
15. Matzeu, M., Servadei, R.: Semilinear elliptic variational inequalities with dependence on the gradient via mountain pass techniques. Nonlinear Anal. **72**, 4347–4359 (2010)
16. Mawhin, J.: Boundary value problems for nonlinear second order differential equations. J. Differ. Equ. **16**, 257–269 (1974)
17. Mokrane, A., Murat, F.: The Lewy-Stampacchia inequality for the obstacle problem with quadratic growth in the gradient. Ann. Mat. Pura. Appl. **184**, 347–360 (2005)
18. Panagiotopoulos, P.D.: Hemivariational Inequalities and Applications in Mechanics and Engineering. Springer-Verlag, New York (1993)
19. Papageorgiou, N.S., Kyritsi, S.: Handbook of Applied Analysis. Springer, New York (2009)
20. Pohozaev, S.I.: Equations of the type $\Delta u = f(x, u, Du)$ (Russian). Mat. Sb. (NS) **113**(155), 321–338 (1980)