PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before May 31, 2009. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11404. *Proposed by Raimond Struble, North Carolina State at Raleigh, Raleigh, NC.* Any three non-concurrent cevians of a triangle create a subtriangle. Identify the sets of non-concurrent cevians which create a subtriangle whose incenter coincides with the incenter of the primary triangle. (A *cevian* of a triangle is a line segment joining a vertex to an interior point of the opposite edge.)

11405. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let P be an interior point of a tetrahedron $ABCD$. When X is a vertex, let X' be the intersection of the opposite face with the line through X and P. Let XP denote the length of the line segment from X to P.
(**a**) Show that $PA \cdot PB \cdot PC \cdot PD \geq 81 PA' \cdot PB' \cdot PC' \cdot PD'$, with equality if and only if P is the centroid of $ABCD$.
(**b**) When X is a vertex, let X'' be the foot of the perpendicular from P to the plane of the face opposite X. Show that $PA \cdot PB \cdot PC \cdot PD = 81 PA'' \cdot PB'' \cdot PC'' \cdot PD''$ if and only if the tetrahedron is regular and P is its centroid.

11406. *Proposed by A. A. Dzhumadil'daeva, Almaty, Republics Physics and Mathematics School, Almaty, Kazakhstan.* Let $n!!$ denote the product of all positive integers not greater than n and congruent to n mod 2, and let $0!! = (-1)!! = 1$. Thus, $7!! = 105$ and $8!! = 384$. For positive integer n, find

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2(n-i)-1)!!$$

in closed form.

11407. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, New York, NY.* Let p be prime greater than 3. Does there exists a ring with more than one element (not necessarily having a multiplicative identity) such that for all x in the ring, $\sum_{i=1}^{p} x^{2i-1} = 0$?

11408. *Proposed by Marius Cavachi, "Ovidius" University of Constanţa, Constanţa, Romania.* Let k be a fixed integer greater than 1. Prove that there exists an integer n greater than 1, and distinct integers $a_1, a_2, \ldots, a_n$, all greater than 1, such that both $\sum_{j=1}^{n} a_j$ and $\sum_{j=1}^{n} \phi(a_j)$ are kth powers of a positive integer. Here ϕ denotes Euler's totient function.

11409. *Proposed by Paolo Perfetti, Dept. Math, University "Tor Vergata", Rome, Italy.* For positive real α and β, let

$$S(\alpha, \beta, N) = \sum_{n=2}^{N} n \log(n)(-1)^n \prod_{k=2}^{n} \frac{\alpha + k \log k}{\beta + (k+1)\log(k+1)}.$$

Show that if $\beta > \alpha$, then $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists.

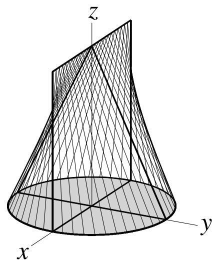
11410. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For $0 < \phi < \pi/2$, find

$$\lim_{x\to 0} x^{-2} \left(\frac{1}{2} \log\cos\phi + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{\sin^2(nx)}{(nx)^2} \sin^2(n\phi) \right).$$

SOLUTIONS

A Solid with the Rupert Property

11291 [2007, 451]. *Proposed by Richard Jerrard and John Wetzel, University of Illinois at Urbana-Champaign, Urbana, IL.*
The base of a solid P symmetric in the xz-plane is the unit disk $x^2 + y^2 \le 1$ in the xy-plane. The portion of P in the half-space $y \ge 0$ is bounded by the surface swept by a segment PQ as P moves uniformly from $(1, 0, 2)$ to $(-1, 0, 2)$ while Q moves uniformly around the unit semicircle from $(1, 0, 0)$ to $(-1, 0, 0)$. The solid is completed by reflecting this half through the xz-plane (see sketch).



The projections of P are the unit disk in the xy-plane, a square region of side two in the xz-plane, and an isosceles triangular region in the yz-plane with base and altitude two. Show that P has the *Rupert property*, that is, it is possible to cut a tunnel through P through which a second copy of P can be passed.

Solution by Mark D. Meyerson, U.S. Naval Academy, Annapolis, MD. The circular base of P is inscribed in the square of side 2 with vertices at $\pm\sqrt{2}$ on the x and y axes. Let S denote this square. The four points of tangency are $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$. If we rotate the base disk about the x-axis by less than $90°$ and orthogonally project the result to the xy-plane, we get a ellipsoidal region that lies strictly inside S. Such a rotation applied to the top edge of P will move its projection away from the x-axis; if the rotation is small this projection will continue to lie strictly inside S. And so, for such a small rotation, since S is convex, all segments connecting the top edge of P to the boundary of the base of P will project into the interior of S. Since these segments, together with the base, form the boundary of P, the orthogonal projection to the xy-plane of this rotation of P lies strictly inside S. Since S is congruent to the xz-plane slice of P, we can drill a slightly smaller square tunnel in the y-axis direction through which P can pass.

As an example, after a rotation through $11.537°$ about the x-axis, all points of P will project to points of the xy-plane more than 0.01 units from S. So a square tunnel of side length 1.98 (leaving a border of 0.01) will allow P to pass (with a clearance of about 0.00005).

Also solved by R. Bagby, D. Chakerian, O. P. Lossers (Netherlands), J. Schaer (Canada), V. Schindler (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

An Infinite Product

11299 [2007, 547]. *Proposed by Pablo Fernàndez Refolio, UAM, Spain.* Show that

$$\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right)=\frac{e\sqrt{e}}{2\pi}.$$

Solution by Timothy Achenbach, Hillsborough Community College, Plant City, FL. Let

$$P_k=\prod_{n=2}^{k}\left(\frac{1}{e}\left(\frac{n^2}{(n-1)(n+1)}\right)^{n^2-1}\right)$$

be the partial product. For $2\le j\le k-1$, the factor j appears in the numerator with exponent $2(j^2-1)$ and in the denominator with exponents $j(j+2)$ and $j(j-2)$. After cancelling, only a factor of j^2 will remain in the denominator. The factor k occurs in the numerator with exponent $2(k^2-1)$ and in the denominator with exponent $k(k-2)$, hence after cancelling k occurs with exponent k^2+2k-2. The factor $k+1$ occurs only in the denominator with exponent k^2-1. Hence

$$P_k=\frac{k^{k^2+2k}}{e^{k-1}(k+1)^{k^2-1}(k!)^2}.$$

The desired infinite product is thus

$$\lim_{k\to\infty}P_k=\frac{e^2}{2\pi}\cdot\lim_{k\to\infty}\left(\frac{e^{k-1}k^{k^2-1}}{(k+1)^{k^2-1}}\right)\cdot\lim_{k\to\infty}\left(\frac{k^k\sqrt{2\pi k}}{e^k k!}\right)^2.$$

The second limit is 1 by Stirling's formula. For the first limit, take logarithms and use the Taylor series for $\log(1+z)$ to get

$$k-1-(k^2-1)\log\left(1+\frac{1}{k}\right)=k-1-(k^2-1)\left(\frac{1}{k}-\frac{1}{2k^2}+\mathrm{O}(k^{-3})\right),$$

which simplifies to $-1/2+O(1/k)$. Hence the first limit is $e^{-1/2}$ and the infinite product is $e\sqrt{e}/(2\pi)$.

Also solved by T. Amdeberhan & V. Moll, S. Amghibech (Canada), R. Bagby, D. Beckwith, B. Brandie, B. S. Burdick, R. Chapman (U. K.), K. Dale (Norway), P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, J. Grivaux (France), E. A. Herman, C. Hill, G. Keselman, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), K. McInturff, M. Omarjee (France), A. Plaza (Spain), G. T. Prajitura, M. A. Prasad (India), O. G. Ruehr, H.-J. Seiffert (Germany), N. C. Singer, A. Stadler (Switzerland), R. Tauraso (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), F. Wang (China), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

A Bernstein Polynomial Integral

11300 [2007, 547]. *Proposed by Ulrich Abel, University of Applied Sciences Giessen-Friedberg, Friedberg, Germany.* For integers k and n with $0\le k\le n$, let $p_{n,k}(t)=$

$\binom{n}{k}t^k(1-t)^{n-k}$. Let $K_n(x,y) = \sum_{k=0}^{n}(y-k/n)p_{n,k}(x)p_{n,k}(y)$. Prove that for $0 \le u \le 1$ and $0 \le y \le 1$, $\int_{x=0}^{u} K_n(x,y)\,dx \ge 0$.

Solution by the BSI Problems Group, Bonn, Germany. We use generating functions. Write $[s^k]P(s)$ for "the coefficient of s^k in the polynomial $P(s)$." Letting $f_n(s,x) = (x+s(1-x))^n$, we have $p_{n,k}(x) = [s^k](1-x+sx)^n = [s^{n-k}](x+s(1-x))^n = [s^{n-k}]f_n(s,x)$. Thus

$$\sum_{k=0}^{n} p_{n,k}(x)p_{n,k}(y) = [s^n](1-(1-s)x)^n f_n(s,y).$$

Compute

$$\left(y-\frac{k}{n}\right)p_{n,k}(y) = \frac{-y(1-y)}{n}p'_{n,k}(y).$$

Thus

$$\begin{aligned}
K_n(x,y) &= \sum_{k=0}^{n}\left(y-\frac{k}{n}\right)p_{n,k}(x)p_{n,k}(y)\\
&= \left(\frac{-y(1-y)}{n}\frac{\partial}{\partial y}\right)[s^n]\big((1-(1-s)x)^n f_n(s,y)\big)\\
&= -y(1-y)[s^n](1-s)(1-(1-s)x)^n f_{n-1}(s,y)\\
&= \frac{y(1-y)}{n+1}[s^n]\frac{\partial}{\partial x}(1-(1-s)x)^{n+1}f_{n-1}(s,y)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^u K_n(x,y)\,dx &= \frac{y(1-y)}{n+1}[s^n]\big((1-(1-s)u)^{n+1}-1\big)f_{n-1}(s,y)\\
&= \frac{y(1-y)}{n+1}[s^n]\left(\sum_{j=1}^{n+1}\binom{n+1}{j}s^ju^j(1-u)^{n+1-j}\right)f_{n-1}(s,y).
\end{aligned}$$

For $0 \le u \le 1$ and $0 \le y \le 1$, all coefficients of the two polynomials in s are nonnegative, so all coefficients of their product are also nonnegative.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), A. Stadler (Switzerland), GCHQ Problem Solving Group (U. K.), and Microsoft Research Problems Group.

A Quartic Inequality

11301 [2007, 547]. *Proposed by Finbarr Holland, University College Cork, Ireland.* Find the least real number M such that, for all complex a, b, and c,

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M(|a|^2+|b|^2+|c|^2)^2.$$

Solution by Byoung Tae Bae, Institute of Science Education, Yonsei University, Seoul, Korea. First note that

$$ab(a^2-b^2)+bc(b^2-c^2)+ac(c^2-a^2) = (b-c)(a-c)(a-b)(a+b+c).$$

It is required to find the smallest real number M such that

$$I = \frac{|(a-b)(b-c)(c-a)(a+b+c)|}{(|a|^2+|b|^2+|c|^2)^2} \le M.$$

Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, where a_j, b_j, and c_j are real numbers for $j \in \{1, 2\}$. Now

$$I = \frac{K}{(a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2)^2},$$

where K is given by

$$K = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}\sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2}\sqrt{(c_1 - a_1)^2 + (c_2 - a_2)^2} \\ \cdot \sqrt{(a_1 + b_1 + c_1)^2 + (a_2 + b_2 + c_2)^2}.$$

Now the AM-GM inequality states that for nonnegative numbers x_1, x_2, x_3, and x_4,

$$\sqrt[4]{x_1 x_2 x_3 x_4} \le \frac{1}{4}(x_1 + x_2 + x_3 + x_4),$$

with equality if and only if $x_1 = x_2 = x_3 = x_4$. Applying this to the case where $x_1 = |a - b|^2$, $x_2 = |b - c|^2$, $x_3 = |c - a|^2$, and $x_4 = |a + b + c|^2$ gives

$$K \le \frac{1}{16}\big[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (b_1 - c_1)^2 + (b_2 - c_2)^2 \\ + (c_1 - a_1)^2 + (c_2 - a_2)^2 + (a_1 + b_1 + c_1)^2 + (a_2 + b_2 + c_2)^2\big]^2,$$

with equality if and only if $(a_1 - b_1)^2 + (a_2 - b_2)^2 = (b_1 - c_1)^2 + (b_2 - c_2)^2 = (c_1 - a_1)^2 + (c_2 - a_2)^2 = (a_1 + b_1 + c_1)^2 + (a_2 + b_2 + c_2)^2$. However, since

$$(a_i - b_i)^2 + (b_i - c_i)^2 + (c_i - a_i)^2 + (a_i + b_i + c_i)^2 = 3(a_i^2 + b_i^2 + c_i^2),$$

holds for $i = 1, 2$, it follows that

$$K \le \frac{9}{16}(a_1^2 + b_1^2 + c_1^2 + a_2^2 + b_2^2 + c_2^2)^2 = \frac{9}{16}(|a|^2 + |b|^2 + |c|^2)^2.$$

Therefore, $I \le 9/16$. To conclude that $M = 9/16$, it suffices that there be complex a, b, and c for which $I = 9/16$. Breaking it down by real and imaginary parts, we take $(a_1, a_2) = (-\sqrt{3} + \sqrt{2/3}, 1 + \sqrt{2/3})$, $(b_1, b_2) = (\sqrt{3} + \sqrt{2/3}, 1 + \sqrt{2/3})$, and $(c_1, c_2) = (\sqrt{2/3}, -2 + \sqrt{2/3})$.

Also solved by R. Bagby, M. Bataille (France), D. Beckwith, D. R. Bridges, R. Chapman (U. K.), P. P. Dályay (Hungary), P. De (India), O. Kouba (Syria), O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), T. L. Radulescu (Romania), H.-J. Seiffert (Germany), N. C. Singer, A. Stadler (Switzerland), F. Wang (China), H. Widmer (Switzerland), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, Northeastern University Math Problem Solving Group, and the proposer.

A Series with Harmonic Numbers

11302 [2007, 547]. *Proposed by Horst Alzer, Waldbröl, Germany.* Find

$$\sum_{k=2}^{\infty} \frac{(2k + 1)H_k^2}{(k - 1)k(k + 1)(k + 2)},$$

where H_k is the kth harmonic number, defined to be $\sum_{j=1}^{k} 1/j$.

Solution by Michael Vowe, Fachschule Nordwestschweiz, Muttenz, Switzerland. Denote the proposed sum by S. Since

$$\frac{2k + 1}{(k - 1)k(k + 1)(k + 2)} = \frac{1}{(k - 1)k(k + 1)} - \frac{1}{k(k + 2)},$$

summation by parts gives

$$S=\frac{H_2^2}{3}+\sum_{k=2}^{\infty}\frac{1}{k(k+2)}\left(H_{k+1}^2-H_k^2\right).$$

Since

$$H_{k+1}^2-H_k^2=(H_{k+1}+H_k)(H_{k+1}-H_k)=\left(2H_k+\frac{1}{k+1}\right)\frac{1}{k+1},$$

we obtain

$$\begin{aligned}
S&=\frac{3}{4}+\sum_{k=2}^{\infty}\frac{2H_k}{k(k+1)(k+2)}+\sum_{k=2}^{\infty}\frac{1}{k(k+1)^2(k+2)}\\
&=\frac{3}{4}+\sum_{k=2}^{\infty}\left(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right)H_k+\sum_{k=2}^{\infty}\frac{1}{k(k+1)^2(k+2)}\\
&=\frac{3}{4}+\frac{H_2}{6}+\sum_{k=2}^{\infty}\frac{1}{(k+1)(k+2)}(H_{k+1}-H_k)+\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2(k+2)}\\
&=1+\sum_{k=2}^{\infty}\left(\frac{1}{(k+1)^2(k+2)}+\frac{1}{k(k+1)^2(k+2)}\right)\\
&=1+\sum_{k=2}^{\infty}\left(\frac{1}{2k(k+1)}-\frac{1}{2(k+1)(k+2)}\right)=1+\frac{1}{12}=\frac{13}{12}.
\end{aligned}$$

Also solved by T. Amdeberhan & T. V. Angelis, B. T. Bae (Korea), R. Bagby, M. Bataille (France), D. Beckwith, P. Bracken, M. A. Carlton, R. Chapman (U. K.), K. Dale (Norway), P. P. Dályay (Hungary), M. N. Deshpande, C. R. Diminnie, M. J. Englefield (Australia), W. Fosheng (China), M. Goldenberg and M. Kaplan, J. Grivaux (France), E. A. Herman, G. Keselman, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), G. T. Prajitura, M. A. Prasad (India), R. Pratt, O. G. Ruehr, H.-J. Seiffert (Germany), N. C. Singer, A. Stadler (Switzerland), R. Tauraso (Italy), M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

Signed Series Terms

11304 [2007, 548]. *Proposed by Teodora-Liliana Rădulescu, Fraţii Buzeşti College, Craiova, and Vicenţiu Rădulescu, University of Craiova, Romania.*
(**a**) Find a sequence $\langle z_n\rangle$ of distinct complex numbers, and a sequence $\langle \alpha_n\rangle$ of nonzero real numbers, such that for almost all complex numbers z (excluding a set of measure zero), $\sum_{n=1}^{\infty}\alpha_n|z-z_n|^{-1}$ diverges to $+\infty$ yet not all α_n are positive.
(**b**) Let $\langle \beta_n\rangle$ be a sequence of real numbers such that $\sum_{n=1}^{\infty}|\beta_n|$ is finite and such that, for almost all z in $\mathbb{C}$, $\sum_{n=1}^{\infty}\beta_n|z-z_n|^{-1}$ converges to a nonnegative real number. Prove that $\beta_n\geq 0$ for all n.
(**c***) Can there be a sequence $\langle \alpha_n\rangle$ of real numbers, not all positive, and a sequence $\langle z_n\rangle$ of distinct complex numbers, such that for almost all complex z, $\sum_{n=1}^{\infty}\alpha_n|z-z_n|^{-1}$ converges to a positive real number?

Solution by the GCHQ Problem Solving Group, Cheltenham, U. K. The problem statement uses metric and measure properties of C but not the algebraic ones, so the setting can be thought of as R^2 rather than C. It is then natural to extend it to R^d. Our example for part (**a**) works for $d\geq 1$ and our proof for part (**b**) works for $d\geq 2$.

(**a**) Let $d \geq 1$. Let $\langle z_n \rangle$ be any convergent sequence in R^d whose limit is, say, z_0, and let $\langle \alpha_n \rangle$ be the sequence $\langle 2, -1, 2, -1, \cdots \rangle$. Let $z \in R^d$ be any point except one of the z_n. There exists n_0 such that for all $n \geq n_0$, $(4/5)|z - z_0| < |z - z_n| < (4/3)|z - z_0|$. When $n \geq n_0$ is even, $\alpha_n |z - z_n|^{-1} + \alpha_{n+1}|z - z_{n+1}|^{-1} \geq (1/4)|z - z_0|^{-1}$, so $\sum_{n=1}^{\infty} \alpha_n |z - z_n|^{-1}$ diverges to $+\infty$.

(**b**) Let $d \geq 2$, and let S be the $d - 1$ dimensional surface area of the unit sphere in R^d. If $r > 0$ and $D = \{z \in R^d\colon\ |z| \leq r\}$, then $\int_D |z|^{-1}\, dz = Sr^{d-1}/(d-1)$, where dz is d-dimensional Lebesgue measure. The volume of D is Sr^d/d, so the mean value of $|z|^{-1}$ on D is $d/(r(d-1))$. It follows (on pairing off points in the symmetric difference of D and D') that the mean value of $|z|^{-1}$ on any spherical domain D' of radius r is at most $d/(r(d-1))$.

If $\beta_k < 0$ for some k, then there exists n_0 with $n_0 > k$ such that $\sum_{n=n_0}^{\infty} |\beta_n| < -\beta_k(d-1)/(2d)$. There exists $r > 0$ such that for all z with $0 < |z - z_k| < r$,

$$\sum_{\substack{n=1\\ n\neq k}}^{n_0-1} \frac{|\beta_n|}{|z - z_n|} < \frac{-\beta_k}{2|z - z_k|}.$$

For $N > n_0$, the mean value of $\sum_{n=n_0}^{N} |\beta_n|\, |z - z_n|^{-1}$ on $D = \{z\colon\ |z - z_k| \leq r\}$ is at most $\sum_{n=n_0}^{N} |\beta_n| d/(r(d-1)) \leq -\beta_k/(2r)$.

It follows that there is a subdomain D_1 of D, with D_1 having positive measure, on which

$$\sum_{n=n_0}^{\infty} \frac{|\beta_n|}{|z - z_n|} \leq \frac{-\beta_k}{2r} \leq \frac{-\beta_k}{2|z - z_k|}.$$

Therefore, on the subdomain D_1 the series $\sum_{n=1}^{\infty} \beta_n |z - z_n|^{-1}$ converges to a negative number. The result follows from this contradiction.

Editorial comment. More than one solver noted that part (**c**) is trivial unless it is required that $\alpha_n \neq 0$, but no solution was received for that nontrivial problem.

Parts (**a**) and (**b**) also solved by J. H. Lindsey II, A. Stadler (Switzerland), and the proposer. Part (**a**) also solved by the Microsoft Research Problems Group.

An Inequality for Triangles

11306 [2007, 640]. *Proposed by Alexandru Rosoiu, University of Bucharest, Bucharest, Romania.* Let a, b, and c be the lengths of the sides of a nondegenerate triangle, let $p = (a + b + c)/2$, and let r and R be the inradius and circumradius of the triangle, respectively. Show that

$$\frac{a}{2} \cdot \frac{4r - R}{R} \leq \sqrt{(p - b)(p - c)} \leq \frac{a}{2},$$

and determine the cases of equality.

Solution by Victor Pambuccian, Arizona State University, Pheonix, AZ. Write S for the area of the triangle. The following are well known:

$$S^2 = p(p - a)(p - b)(p - c), \qquad S = \frac{abc}{4R} = rp.$$

Let x, y, and z denote the lengths of the tangent segments to the incircle from the vertices opposite c, a, and b, respectively. Now $a = z + x$, $b = x + y$, $c = y + z$,

$p = x + y + z$, and the inequalities to be proved become:

$$\frac{x+z}{2} \cdot \frac{14xyz - x^2(y+z) - z^2(x+y) - y^2(x+z)}{(x+y)(y+z)(z+x)} \le \sqrt{zx} \le \frac{x+z}{2}.$$

The second inequality follows from the AM–GM inequality, with equality when $x = z$ (that is, $b = c$). The first inequality may be rewritten

$$2(x+y)(y+z)\sqrt{zx} + x^2(y+z) + z^2(x+y) + y^2(x+z) \ge 14xyz. \tag{1}$$

Now by the AM–GM inequality, $x + y \ge 2\sqrt{xy}$ and $y + z \ge 2\sqrt{yz}$, so

$$2(x+y)(y+z)\sqrt{zx} \ge 8xyz. \tag{2}$$

Further applications of the AM–GM inequality yield

$$\begin{aligned} x^2(y+z) + y^2(z+x) + z^2(x+y) &\ge 2\left(x^2(yz)^{1/2} + y^2(zx)^{1/2} + z^2(xy)^{1/2}\right) \\ &= 2(xyz)^{1/2}\left(x^{3/2} + y^{3/2} + z^{3/2}\right) \\ &\ge 6(xyz)^{1/2}\left(x^{3/2}y^{3/2}z^{3/2}\right)^{1/3} = 6xyz. \end{aligned} \tag{3}$$

Inequality (1) is the sum of (2) and (3), and is thus established. Equality holds in (1) if and only if it holds in both (2) and (3). Since these were based on AM–GM inequalities for the pairs $\{x, y\}$, $\{y, z\}$, and $\{z, x\}$, equality occurs if and only if $x = y = z$, i.e., if and only if $a = b = c$ and the triangle is equilateral.

Editorial comment. Li Zhou proved a slightly stronger result in place of the first inequality, namely

$$\frac{a}{2} \cdot \frac{4r - R}{R} \le \frac{2(p-b)(p-c)}{a} \le \sqrt{(p-b)(p-c)}.$$

V. V. García rediscovered and used a lemma that strengthens the inequality $R \ge 2r$ of Euler: If m and h are the lengths of the median and altitude from the same vertex, respectively, then $R/(2r) \ge m/h$. (See IX.10.22, p 216, in D. Mitrinović, J. Pečarić, and V. Volonec's *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Boston, 1989.)

Also solved by S. Amghibech (Canada), M. Bataille (France), D. Beckwith, E. Braune (Austria), R. Chapman (U. K.), P. P. Dályay (Hungary), A. & A. Darbinyan (Armenia), P. De (India), J. Fabrykowski & T. Smotzer, W. Fosheng (China), V. V. García (Spain), C. Grosu (Romania), J. G. Heuver (Canada), R. A. Kopas, K.-W. Lau (China), J. Minkus, D. J. Moore, J. H. Nieto (Venezuela), Á. Plaza (Spain), J. Posch, C. R. Pranesachar (India), M. A. Prasad (India), H.-J. Seiffert (Germany), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), N. T. Tuan (Vietnam), M. Vowe (Switzerland), J. B. Zacharias, L. Zhou, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, Northwestern University Problem Solving Group, Princeton Problem Solving Group, and the proposer.