PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before February 28, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11449. *Proposed by Michel Bataille, Rouen, France.* (correction) Find the maximum and minimum values of

$$\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}$$

given that $a+b \geq c > 0,\ b+c \geq a > 0$, and $c+a \geq b > 0$.

11453. *Proposed by Richard Stanley, Massachussetts Institute of Technology, Cambridge, MA.* Let Δ be a finite collection of sets such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Fix $k \geq 0$. Suppose that every F in Δ (including $F = \emptyset$) with $\#F \leq k$ satisfies

$$\sum_{G \in \Delta, G \supseteq F} (-1)^{\#G} = 0.$$

Show that $\#\Delta$ is divisible by 2^{k+1}.

11454. *Proposed by Azer Kerimov, Bilkent University, Ankara, Turkey.* Alice and Bob play a game based on a 2-connected graph G with n vertices, where $n > 2$. Alice selects two vertices u and v. Bob then orients up to $2n-3$ of the edges. Alice then orients the remaining edges and selects some edge e, which may have been oriented by her or by Bob. If the oriented graph contains a path from u to v through e, then Bob wins; otherwise, Alice wins. Prove that Bob has a winning strategy, while if he is granted only $2n-4$ edges to orient, on some graphs he does not. (A graph is 2-connected if it has at least three vertices and each subgraph obtained by deleting one vertex is connected.)

11455. *Proposed by Christian Blatter, Swiss Federal Institute of Technology, Zurich, Switzerland.* Determine the triangles T of maximal area in the Cartesian plane with the property that for all nonzero integer pairs (m,n) the interiors of T and $T+(m,n)$ are disjoint.

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11456. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France.* Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right).$$

11457. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* For real numbers a and b with $0 \le a \le b$, find

$$\int_{x=a}^{b} \arccos\left(\frac{x}{\sqrt{(a+b)x - ab}}\right) dx.$$

11458. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Let $a_1, \dots, a_n$ be nonnegative and let r be a positive integer. Show that

$$\left(\sum_{1\le i,j\le n} \frac{i^r j^r a_i a_j}{i+j-1}\right)^2 \le \sum_{m=1}^{n} m^{r-1} a_m \sum_{1\le i,j,k\le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

11459. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* Find all pairs (s, z) of complex numbers such that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \left(\prod_{j=1}^{k}(sj - z)\right)\left(\prod_{j=0}^{n-k-1}(sj + z)\right)$$

converges.

SOLUTIONS

Two Log Gamma Integrals

11329 [2007, 925]. *Proposed by T. Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA.* Let $f(t) = 2^{-t}\ln\Gamma(t)$, where Γ denotes the classical gamma function, and let γ be Euler's constant. Derive the following integral identities:

$$\int_0^\infty f(t)\,dt = 2\int_0^1 f(t)\,dt - \frac{\gamma + \ln\ln 2}{\ln 2},$$

$$\int_0^\infty t f(t)\,dt = 2\int_0^1 (t+1) f(t)\,dt - \frac{(\gamma + \ln\ln 2)(1 + 2\ln 2) - 1}{\ln^2 2}.$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. From

$$\int_0^\infty e^{-x}\ln x\,dx = \Gamma'(1) = -\gamma,$$

it follows that for $\alpha > 0$,

$$\int_0^\infty e^{-\alpha t}\ln t\,dt = -\frac{\gamma + \ln\alpha}{\alpha} \quad \text{and} \quad \int_0^\infty t\,e^{-\alpha t}\ln t\,dt = \frac{1 - (\gamma + \ln\alpha)}{\alpha^2}, \qquad (*)$$

where the second integral in $(*)$ is obtained by differentiating the first with respect to α. Set $\alpha = \ln 2$. By the well-known formula $\Gamma(t+1) = t\,\Gamma(t)$, we see that f satisfies

$$2f(t+1) = f(t) + 2^{-t}\ln t = f(t) + e^{-\alpha t}\ln t.$$

Hence

$$\int_0^\infty 2f(t)\,dt - \int_0^1 2f(t)\,dt = \int_1^\infty 2f(t)\,dt = \int_0^\infty 2f(t+1)\,dt$$

$$= \int_0^\infty f(t)\,dt + \int_0^\infty e^{-\alpha t}\ln t\,dt,$$

and therefore

$$\int_0^\infty f(t)\,dt = 2\int_0^1 f(t)\,dt - \frac{\gamma + \ln\alpha}{\alpha}.$$

Similarly,

$$\int_0^\infty 2(t-1)f(t)\,dt - \int_0^1 2(t-1)f(t)\,dt = \int_1^\infty 2(t-1)f(t)\,dt$$

$$= \int_0^\infty 2tf(t+1)\,dt = \int_0^\infty tf(t)\,dt + \int_0^\infty t\,e^{-\alpha t}\ln t\,dt,$$

which yields

$$\int_0^\infty 2(t-1)f(t)\,dt = \int_0^1 2(t-1)f(t)\,dt + \int_0^\infty tf(t)\,dt + \frac{1-(\gamma+\ln\alpha)}{\alpha^2}.$$

Combining this with $2\int_0^\infty f(t)\,dt = 4\int_0^1 f(t)\,dt - 2\,\frac{\gamma+\ln\alpha}{\alpha}$ shows that

$$\int_0^\infty tf(t)\,dt = 2\int_0^2 (t+1)f(t)\,dt - \frac{(\gamma+\ln\alpha)(1+2\alpha)-1}{\alpha^2}.$$

Also solved by S. Amghibech (Canada), R. Bagby, K. Boyadzhiev, P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Furdui, M. L. Glasser, G. C. Greubel, J. Grivaux (France), G. Keselman, K. Kneile, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Minkus, A. Nijenhuis, P. Perfetti (Italy), X. Retnam, A. Stadler (Switzerland), A. Stenger, R. Stong, Y. Yu, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers with O. Espinosa (Chile).

A New Lower Bound for the Sum of the Altitudes

11330 [2007, 925]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* For a triangle with semiperimeter s, inradius r, circumradius R, and heights h_a, h_b, and h_c, show that

$$h_a + h_b + h_c - 9r \ge 2s\sqrt{\frac{2r}{R}} - 6\sqrt{3}\,r.$$

Solution by João Guerreiro, student, Instituto Superior Técnico, Lisboa, Portugal. The area A of the triangle satisfies $A = rs = h_a a/2 = h_b b/2 = h_c c/2$. Also recall $abc = 4srR$. Upon dividing by $2A$, the proposed inequality becomes

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{9}{a+b+c} \ge 2\sqrt{\frac{a+b+c}{abc}} - \frac{6\sqrt{3}}{a+b+c}.$$

Let $u = ka$, $v = kb$, and $w = kc$, and choose k so that $uvw = u + v + w$. The proposed inequality then becomes

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} - \frac{9}{u+v+w} \geq 2 - \frac{6\sqrt{3}}{u+v+2}.$$

When this is multiplied by $u + v + w = uvw$, it becomes $uv + vw + wu \geq 2u + 2v + 2w + 9 - 6\sqrt{3}$, which is equivalent to $(u-1)(v-1) + (v-1)(w-1) + (w-1)(u-1) \geq 12 - 6\sqrt{3}$. Now let $x = u - 1$, $y = v - 1$, and $z = w - 1$, so

$$xyz + xy + yz + zx = 2, \qquad (*)$$

and the desired inequality becomes $xy + yz + zx \geq 12 - 6\sqrt{3}$ or $6\sqrt{3} - 10 \geq xyz$. If $xyz \leq 0$ this is true ($108 > 100$, so $6\sqrt{3} > 10$). Therefore, we may assume $xyz > 0$. We cannot have exactly two of x, y, z negative, since then exactly two of u, v, w would be less than 1, and that is incompatible with our assumptions about u, v, w. For example, $w \leq u + v + w = uvw$, which implies $uv \geq 1$, which could not happen if both $u < 1$ and $v < 1$. Therefore all three of x, y, z are positive.

By the AM–GM inequality, $xy + yz + zx \geq 3(xyz)^{2/3}$. Let $d = (xyz)^{1/3}$. Using $(*)$, we have $2 \geq d^3 + 3d^2$. The polynomial p given by $p(t) = t^3 + 3t^2$ is increasing for positive t and takes the value 2 at $t = \sqrt{3} - 1$. So $p(d) \leq 2$ implies $d \leq \sqrt{3} - 1$, and $d^3 \leq (\sqrt{3} - 1)^3 = 6\sqrt{3} - 10$. That is, $xyz \leq 6\sqrt{3} - 10$, as required.

Editorial comment. The proposer noted that the inequality improves the well-known lower bound of $9r$ for $h_a + h_b + h_c$. He notes that the lemma "if u, v, w are three positive numbers whose sum and product are equal, then $(u-1)(v-1)(w-1) \leq 6\sqrt{3} - 10$" can be used to prove *Blundon's inequality:* $s \leq 2R + (3\sqrt{3} - 4)r$. P. P. Dályay proved a stronger inequality: $h_a + h_b + h_c - 9r \geq 2\sqrt{3}s^2/9R - 3\sqrt{3}r$.

Also solved by A. Alt, M. Cipu (Romania), P. P. Dályay (Hungary), J.-P. Grivaux (France), S. Hitotumatu (Japan), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, A. Nakhash, C. R. Pranesachar (India), V. Schindler (Germany), R. Stong, D. Vacaru (Romania), Microsoft Research Problems Group, and the proposer.

A Bisector Inequality

11337 [2008, 71]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* Suppose in triangle ABC we have opposite sides of lengths a, b, and c, respectively, with $a \leq b \leq c$. Let w_a and w_b be the lengths of the bisectors of the angles A and B respectively. Show that $a + w_a \leq b + w_b$.

Solution by Peter Nüesch, Ecole Polytechnique Fédérale de Lausanne, Switzerland. Let α, β, and γ be the angles of the triangle, I the incenter, r the inradius, and R the circumradius. We have $AI = r/\sin(\alpha/2) = 4R\sin(\beta/2)\sin(\gamma/2)$, and hence

$$AI - BI = 4R\sin\frac{\gamma}{2}\left(\sin\frac{\beta}{2} - \sin\frac{\alpha}{2}\right).$$

Since $\alpha \leq \beta$ and

$$b - a = 2R(\sin\beta - \sin\alpha) = 4R\sin\frac{\gamma}{2}\sin\left(\frac{\beta-\alpha}{2}\right),$$

we have $b - a \geq AI - BI$ or $a + AI \leq b + BI$. Furthermore, since $\alpha \leq \beta$, we have $\alpha + \beta/2 \leq \beta + \alpha/2$, and therefore

$$w_a - AI = \frac{r}{\sin(\beta + \frac{\alpha}{2})} \leq \frac{r}{\sin(\alpha + \frac{\beta}{2})} = w_b - BI.$$

Summing these two inequalities gives the desired result.

Also solved by A. Alt, P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, B.-T. Iordache (Romania), J. H. Lindsey II, J. Oelschlager, É. Pité (France), C. R. Pranesechar (India), M. A. Prasad (India), R. E. Prather, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), L. Zhou, Microsoft Research Problems Group, and the proposer.

A Gamma Limit

11338 [2008, 71]. *Proposed by Ovidiu Furdui, Cluj, Romania.* Let Γ denote the classical gamma function, and let $G(n) = \prod_{k=1}^{n} \Gamma(1/k)$. Find

$$\lim_{n\to\infty} G(n+1)^{1/(n+1)} - G(n)^{1/n}.$$

Solution by O. P. Lossers, Eindhoven, The Netherlands. First note that $\Gamma(1/k) = k\Gamma(1+1/k)$, so that $G(n) = n! \prod_{k=1}^{n} \Gamma(1+1/k)$. Let γ be Euler's constant, and recall that $\Gamma'(1) = -\gamma$. Thus $\Gamma(1+1/k) = (1-\gamma/k)(1+O(k^{-2}))$ as $k \to \infty$. Hence,

$$\log\Gamma\left(1+\frac{1}{k}\right) = -\gamma\log\left(1+\frac{1}{k}\right) + c_k,$$

where $c_k = O(k^{-2})$. Therefore

$$\log\prod_{k=1}^{n}\Gamma\left(1+\frac{1}{k}\right) = -\gamma\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right) + \sum_{k=1}^{\infty} c_k - \sum_{k=n+1}^{\infty} c_k,$$

which simplifies to $-\gamma\log(n+1) + C + O(1/n)$ as $n \to \infty$ for some constant C. Hence

$$\left(\prod_{k=1}^{n}\Gamma\left(1+\frac{1}{k}\right)\right)^{1/n} = \exp\left[\frac{1}{n}(C-\gamma\log n) + O\left(\frac{1}{n^2}\right)\right].$$

By Stirling's approximation,

$$(n!)^{1/n} = \frac{n}{e}\left(\sqrt{2\pi n}\left(1+O\left(\frac{1}{n}\right)\right)\right)^{1/n} = \frac{n}{e}\exp\left[\frac{1}{n}\log\sqrt{2\pi n} + O\left(\frac{1}{n^2}\right)\right],$$

and we conclude that for appropriate constants a and b we have

$$G(n)^{1/n} = \frac{n}{e}\exp\left[\frac{1}{n}(a\log n + b) + O\left(\frac{1}{n^2}\right)\right] = \frac{n}{e} + \frac{a}{e}\log n + \frac{b}{e} + O\left(\frac{\log^2 n}{n}\right).$$

From this it follows that $\lim_{n\to\infty}\left(G(n+1)^{1/(n+1)} - G(n)^{1/n}\right) = 1/e$.

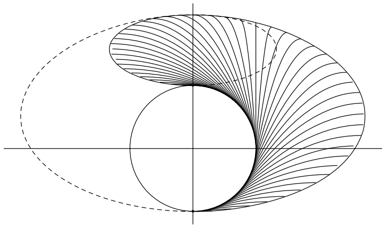
Also solved by S. Amghibech (Canada), R. Bagby, D. & J. Borwein (Canada), P. Bracken, D. R. Bridges, P. P. Dályay (Hungary), J.-P. Grivaux (France), E. A. Herman, G. Keselman, O. Kouba (Syria), P. Perfetti (Italy), M. A. Prasad (India), X. Ros (Spain), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, M. Vowe (Switzerland), B. Ward (Canada), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Raindrops Flying from a Spinning Umbrella

11340 [2008, 72]. *Proposed by Óscar Ciaurri and Luz Roncal, Universidad de la Rioja, Logroño, Spain.* An umbrella of radius 1 meter is spun with angular velocity ρ in the xz-plane about an axis (call it the y-axis) parallel to the ground. It is wet, and drops of water crawl along the ribs and fly off as they reach their ends.

Each drop leaves the umbrella with a velocity vector equal to the velocity of the tip of the rib at the point where it exited. It then follows a parabolic trajectory. If a drop spins off while on the downspin, then the high point in its arc will be the point of departure. Otherwise, the high point is the vertex of a parabolic arc in the xz-plane. Determine a parameterized family P_ρ of polynomials in two variables such that whenever $\rho^2 > g$, the various arc vertices reached by the water droplets all lie on the curve $P_\rho(x, z) = 0$. (Here g denotes the magnitude of the downward acceleration due to gravity.)

The figure shows the case $\rho^2/g = 4$, with the umbrella spinning counterclockwise.



Solution by Nora Thornber, Raritan Valley Community College, Somerville, NJ. With $\beta = g/\rho^2$, the required polynomial $P_\rho(x, z)$ is given by

$$P_\rho(x, z) = \left(\frac{\beta x^2}{2} + (\beta - z)(\beta^2 - 2\beta z + 1)\right)^2 - (\beta^2 - 2\beta z + 1)^3. \tag{1}$$

Proof. Let the point of departure of the droplet have coordinates (x_0, z_0). For initial departure angle β, we have

$$x_0 = \cos\beta, \quad z_0 = \sin\beta, \quad v_{0x} = -\rho\sin\beta = -\rho z_0, \quad v_{0z} = \rho\cos\beta = \rho x_0.$$

The droplet's path in the air is determined by the equations

$$x = v_{0x}t + x_0, \quad z = -(1/2)gt^2 + v_{0z}t + z_0.$$

Hence

$$z = -\left(\frac{g}{2}\right)\left(\frac{x - x_0}{v_{0x}}\right)^2 + v_{0z}\frac{x - x_0}{v_{0x}} + z_0.$$

To find the maximum point of the trajectory, set $dz/dx = 0$:

$$-g\left(\frac{x - x_0}{v_{0x}^2}\right) + \frac{v_{0z}}{v_{0x}} = 0.$$

Thus

$$x - x_0 = \frac{v_{0x}v_{0z}}{g} = -\frac{\rho^2 x_0 z_0}{g}. \tag{2}$$

The maximum height of the trajectory is

$$z = -\frac{g}{2}\left(\frac{v_{0z}}{g}\right)^2 + \frac{v_{0z}^2}{g} + z_0,$$

so that $z - z_0 = v_{0z}^2/(2g) = \rho^2 x_0^2/(2g)$. The droplets originate on a circle of radius 1, so $z - z_0 = \rho^2(1 - z_0^2)/(2g)$. Solving this quadratic equation for z_0 yields

$$z_0 = \beta \pm \sqrt{\beta^2 - 2\beta z + 1}, \tag{3}$$

where $\beta = g/\rho^2$. From this we obtain

$$x_0^2 = 1 - z_0^2 = -2\beta(\beta - z \pm \sqrt{\beta^2 - 2\beta z + 1}). \tag{4}$$

Since $x = x_0(1 - z_0/\beta)$ from (2), squaring both sides and substituting from (3) and (4) yields (1) as claimed.

Editorial comment. In the original, the caption incorrectly had $\rho^2/g = 2$.

Also solved by R. Bagby, N. Caro (Brazil), R. Chapman (U. K.), J. A. Grzesik, K. McInturff, E. I. Verriest, GCHQ Problem Solving Group, and the proposers.

An Application of Popoviciu's Inequality

11341 [2008, 166]. *Proposed by Cezar Lupu, University of Bucharest, Bucharest, Romania (student), and Tudorel Lupu, Decebal High School, Constanza, Romania.* Consider an acute triangle with side-lengths a, b, and c, with inradius r and semiperimeter p. Show that

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C \left(2 - \frac{3\sqrt{3}r}{p}\right).$$

Solution by Marian Tetiva, Bîrlad, Romania. We have

$$\frac{1 - \cos A}{\cos A} = \frac{2\sin^2(A/2)}{\cos A} = \frac{2\sin(A/2)\cos(A/2)\sin(A/2)}{\cos A \cos(A/2)} = \frac{\tan A}{\cot(A/2)}.$$

Also

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{p}{r}.$$

With some algebra, the inequality to be proved becomes

$$\tan A \tan B \tan C + 3\sqrt{3} \geq 2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$

or equivalently,

$$\tan A + \tan B + \tan C + 3\sqrt{3} \geq 2\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right), \tag{1}$$

since $\tan A \tan B \tan C = \tan A + \tan B + \tan C$.

Let I be an interval and $f : I \to \mathbb{R}$ be convex. For x, y, and z in I, let $x' = (y + z)/2$, $y' = (x + z)/2$, and $z' = (x + y)/2$. With this notation, Popoviciu's inequality states that

$$f(x) + f(y) + f(z) + 3f\left(\frac{x + y + z}{3}\right) \geq 2(f(x') + f(y') + f(z')).$$

Using the convex function tan on the interval $(0, \pi/2)$ and the three numbers A, B, C, we get (1) if we use

$$\tan\frac{A + B + C}{3} = \tan\frac{\pi}{3} = \sqrt{3}, \qquad \cot\frac{A}{2} = \tan\frac{B + C}{2},$$

and similar relations.

Editorial comment. Popoviciu's inequality may be found, with proof, in *Mathematical Miniatures,* by Svetoslav Savchev & Titu Andreescu (Mathematical Association of America, 2003).

Also solved by R. Chapman (U. K.), M. Cipu, P. P. Dályay (Hungary), Y. Dumont (France), C. Pohoata (Romania), C. R. Pranasechar (India), V. Schindler (Germany), R. Stong, S. Vandervelde, Z. Vörös (Hungary), J. B. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposers.

A Functional Equation

11345 [2008, 166]. *Proposed by Roger Cuculière, France.* Find all nondecreasing functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x + f(y)) = f(f(x)) + f(y)$ for all real x and y.

Solution by Richard Stong, San Diego, CA. Write T for the range of f. Taking $x = y = 0$, we get $f(0) = 0$. Taking just $x = 0$, we get $f(f(y)) = f(y)$. Taking $x = f(z)$ and $x = -f(y)$, we get $f(f(y) + f(z)) = f(y) + f(z)$ and $f(-f(y)) = -f(y)$. Therefore, T is an additive subgroup of $\mathbb{R}$, and f is the identity on T. There are three possibilities.

If $T = \{0\}$, then f is identically 0.

If T is dense in $\mathbb{R}$, then $f(x) = x$ on that dense set and f is nondecreasing, so $f(x) = x$ for all $x \in \mathbb{R}$.

If $T = \mathbb{Z}a$ for some $a > 0$, then $f(na) = na$ for all $n \in \mathbb{Z}$. Let $S = \{x : f(x) = 0\}$. Since f is nondecreasing, $S \subseteq (-a, a)$. For $n \in \mathbb{Z}$, taking $x \in S$ and $y = na$ in the functional equation yields $f(na + x) = na$. Conversely, if $f(z) = na$, then taking $x = z$ and $y = -na$ in the functional equation yields $z - na \in S$. Thus $f^{-1}(na) = na + S$. Hence the sets $na + S$ must partition $\mathbb{R}$; since f is nondecreasing, S must be a half-open interval of length a containing 0. Hence either

$$f(x) = a\left\lfloor \frac{x+b}{a} \right\rfloor \quad \text{or} \quad f(x) = a\left\lceil \frac{x-b}{a} \right\rceil$$

for some $b \in [0, a)$.

Editorial comment. Roman Ger (Katowice, Poland) analyzed solutions of this functional equation that are not necessarily nondecreasing. The Lebesgue measurable solutions are: (i) constant 0; (ii) identity function $f(x) = x$; or (iii) $f(x) = an(x)$, where a is any positive real and $n : \mathbb{R} \to \mathbb{Z}$ is defined by

$$n(x) = n_0(x - ka) + k \quad \text{for } x \in \big[ka, (k+1)a\big), k \in \mathbb{Z},$$

and $n_0 : [0, a) \to \mathbb{Z}$ is an arbitrary Lebesgue measurable function vanishing at 0. There are also solutions that are not Lebesgue measurable (assuming the axiom of choice).

Also solved by R. Bagby, M. Bataille (France), D. Beckwith, M. Bello-Hernandez & M. Benito (Spain), D. R. Bridges, R. Chapman (U. K.), R. Crise, C. Curtis, P. P. Dályay (Hungary), B. Ebanks, N. Eldredge, J. Freeman, R. Ger (Poland), J. Guerreiro & J. Matias (Portugal), J. W. Hagood, C. C. Heckman, E. A. Herman, K. K. Heuvers, N. Khachatryan (Armenia), K. Kneile, T. Konstantopoulos (U. K.), O. Kouba (Syria), J. H. Lindsey II, J. Lobo (Costa Rica), O. P. Lossers (Netherlands), B. Madore, F. B. Miles, A. Nakhash, J. H. Nieto (Venezuela), M. A. Prasad (India), J. B. Robertson, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), N. Thornber, B. Tomper, D. J. Troy, S. Vandervelde, V. Verdiyan (Armenia), E. I. Verriest, B. Ward (Canada), S. Y. Xiao (Canada), BSI Problems Group (Germany), Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Houghton College Problem Solving Group, NSA Problems Group, and the proposer.